Assignment 6

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Download latex-tikz codes from

https://github.com/RaghavJuyal/AI1103/tree/main/Assignment6/Assignment6.tex

Statistics 2015 Paper I, Q.2 (a)

Let C be a circle of unit area with centre at origin and let S be a square of unit area with $\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{-1}{2},\frac{1}{2}\right)$, $\left(\frac{-1}{2},\frac{-1}{2}\right)$ and $\left(\frac{1}{2},\frac{-1}{2}\right)$ as the four vertices. If X and Y be two independent standard variates, show that

$$\iint_C \phi(x)\,\phi(y)\,dx\,dy \geq \iint_S \phi(x)\,\phi(y)\,dx\,dy$$

where $\phi(.)$ is the pdf of $N(0,1)$ distribution.

Solution

Definition 1. *PDF of normal distribution is given as*

$$\phi_Z(z) = N\left(\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(z-\mu)^2}{2\sigma^2}} \tag{0.0.1}$$

Corollary 0.1.

$$\phi_X(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}} \tag{0.0.2}$$

$$\phi_Y(y) = \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}} \tag{0.0.3}$$

Proof. Since $\phi(.)$ is the pdf of $N(0,1)$ distribution (given in question), $\mu = 0$ and $\sigma^2 = 1$.

$$\implies \phi_X(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}} \text{ and } \phi_Y(y) = \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}$$

□

Lemma 0.1. *For circle C,*

$$\iint_C \phi(x)\,\phi(y)\,dx\,dy = 1 - e^{\frac{-1}{2\pi}} \tag{0.0.4}$$

Proof. C has unit area with centre at origin.

$$\implies \pi \times r^2 = 1 \tag{0.0.5}$$

$$\implies |r| = \frac{1}{\sqrt{\pi}} \tag{0.0.6}$$

For the area inside circle C we get,

$$x^2 + y^2 \leq \frac{1}{\pi} \implies |y| \leq \sqrt{\frac{1}{\pi} - x^2} \tag{0.0.7}$$

From (0.0.7) we get,

$$\iint_C \phi(x)\,\phi(y)\,dx\,dy = \int_{x=\frac{-1}{\sqrt{\pi}}}^{x=\frac{1}{\sqrt{\pi}}}\int_{y=-\sqrt{\frac{1}{\pi}-x^2}}^{y=\sqrt{\frac{1}{\pi}-x^2}} \phi(x)\,\phi(y)\,dy\,dx \tag{0.0.8}$$

Using (0.0.2) and (0.0.3) in (0.0.8) we get,

$$\int_{x=\frac{-1}{\sqrt{\pi}}}^{x=\frac{1}{\sqrt{\pi}}}\int_{y=-\sqrt{\frac{1}{\pi}-x^2}}^{y=\sqrt{\frac{1}{\pi}-x^2}} \frac{1}{2\pi}e^{\frac{-\left(x^2+y^2\right)}{2}}\,dy\,dx \tag{0.0.9}$$

Converting it to polar coordinates we get,

$$\int_{r=0}^{r=\frac{1}{\sqrt{\pi}}}\int_{\theta=0}^{\theta=2\pi} \frac{1}{2\pi}e^{\frac{-r^2}{2}}\,r\,d\theta\,dr \tag{0.0.10}$$

$$= \int_0^{\frac{1}{\sqrt{\pi}}} e^{\frac{-r^2}{2}}\,r\,dr \tag{0.0.11}$$

$$= 1 - e^{\frac{-1}{2\pi}} \tag{0.0.12}$$

□

Definition 2. *The Q function is defined as*

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-u^2}{2}}\,du \tag{0.0.13}$$

Lemma 0.2.

$$\int_{-a}^{a} e^{\frac{-x^2}{2}}\,dx = \sqrt{2\pi}\,(1 - 2Q(a)) \tag{0.0.14}$$

Proof. Since $e^{\frac{-x^2}{2}}$ is an even function,

$$\int_{-a}^{a} e^{\frac{-x^2}{2}}\,dx = 2\int_{0}^{a} e^{\frac{-x^2}{2}}\,dx \tag{0.0.15}$$

$$\implies 2\int_{0}^{a} e^{\frac{-x^2}{2}}\,dx = 2\left(\int_{0}^{\infty} e^{\frac{-x^2}{2}}\,dx - \int_{a}^{\infty} e^{\frac{-x^2}{2}}\,dx\right)$$

$$= 2\sqrt{2\pi}\times\frac{1}{\sqrt{2\pi}}\left(\int_{0}^{\infty} e^{\frac{-x^2}{2}}\,dx - \int_{a}^{\infty} e^{\frac{-x^2}{2}}\,dx\right) \tag{0.0.16}$$

Comparing (0.0.16) with (0.0.13) we get,

$$\int_{-a}^{a} e^{\frac{-x^2}{2}}\,dx = 2\sqrt{2\pi}\left(Q(0) - Q(a)\right) \tag{0.0.17}$$

$$= 2\sqrt{2\pi}\left(\frac{1}{2} - Q(a)\right) \tag{0.0.18}$$

$$= \sqrt{2\pi}\,\left(1 - 2Q(a)\right) \tag{0.0.19}$$

□

Lemma 0.3. *For square S,*

$$\iint_S \phi(x)\,\phi(y)\,dx\,dy = \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \tag{0.0.20}$$

Proof. For square S (given in question),

$$\frac{-1}{2} \le x \le \frac{1}{2} \tag{0.0.21}$$

$$\frac{-1}{2} \le y \le \frac{1}{2} \tag{0.0.22}$$

From this we get,

$$\iint_S \phi(x)\,\phi(y)\,dx\,dy =$$

$$\int_{x=\frac{-1}{2}}^{x=\frac{1}{2}}\int_{y=\frac{-1}{2}}^{y=\frac{1}{2}} \phi(x)\,\phi(y)\,dy\,dx \tag{0.0.23}$$

Using (0.0.2) and (0.0.3) in (0.0.23) we get,

$$\int_{x=\frac{-1}{2}}^{x=\frac{1}{2}}\int_{y=\frac{-1}{2}}^{y=\frac{1}{2}} \frac{1}{2\pi}e^{\frac{-\left(x^2+y^2\right)}{2}}\,dy\,dx \tag{0.0.24}$$

Using (0.0.14) twice we get,

$$\int_{x=\frac{-1}{2}}^{x=\frac{1}{2}}\int_{y=\frac{-1}{2}}^{y=\frac{1}{2}} \frac{1}{2\pi}e^{\frac{-\left(x^2+y^2\right)}{2}}\,dy\,dx = \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \tag{0.0.25}$$

□

Solution: Calculating the values of (0.0.4) and (0.0.20) we get,

$$1 - e^{\frac{-1}{2\pi}} = 0.147136 \tag{0.0.26}$$

$$\left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 = 0.146631 \tag{0.0.27}$$

This proves that

$$1 - e^{\frac{-1}{2\pi}} \ge \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \tag{0.0.28}$$

$$\implies \iint_C \phi(x)\,\phi(y)\,dx\,dy \ge \iint_S \phi(x)\,\phi(y)\,dx\,dy \tag{0.0.29}$$