Assignment 5

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Download latex-tikz codes from

https://github.com/RaghavJuyal/AI1103/tree/main/Assignment5/Assignment5.tex

Question 113, CSIR UGC NET EXAM (Dec 2014)

Let $X_1, X_2, ..., X_n$ be independent and identically distributed Bernoulli(θ), where $0 < \theta < 1$ and $n > 1$. Let the prior density of θ be proportional to $\frac{1}{\sqrt{\theta(1-\theta)}}$, $0 < \theta < 1$. Define $S = \sum_{i=1}^{n} X_i$.
Then valid statements among the following are:

1. The posterior mean of θ does not exist;
2. The posterior mean of θ exists;
3. The posterior mean of θ exists and it is larger than the maximum likelihood estimator for all values of S.
4. The posterior mean of θ exists and it is larger than the maximum likelihood estimator for some values of S.

Solution

Definition 1. *Posterior mean is the mean of the posterior distribution of θ, i.e.,*

$$E(\theta|X) = \int \theta\, f(\theta|X)\, d\theta \tag{0.0.1}$$

Let $f(\theta)$ be the prior density of θ and $B(x, y)$ is the beta function.

$$f(\theta) \propto \frac{1}{\sqrt{\theta\,(1-\theta)}} \tag{0.0.2}$$

$$\int_0^1 f(\theta)\, d\theta = 1$$

$$\therefore f(\theta) = \frac{\theta^{-\frac{1}{2}}\,(1-\theta)^{-\frac{1}{2}}}{B(\frac{1}{2}, \frac{1}{2})} \tag{0.0.3}$$

Let $f(X|\theta)$ be the likelihood function.

$$\begin{aligned} f(X|\theta) &= \prod_{i=1}^{n} \theta^{X_i}\,(1-\theta)^{1-X_i} \\ &= \theta^{S}\,(1-\theta)^{n-S} \end{aligned} \tag{0.0.4}$$

Definition 2. *The maximum likelihood estimator is the value which maximizes the likelihood function, i.e.,*

$$MLE = arg\ max(f(X|\theta)) \tag{0.0.5}$$

$$\ln f(X|\theta) = S\,\ln\theta + (n-S)\ln(1-\theta) \tag{0.0.6}$$

$$\frac{\partial \ln f(X|\theta)}{\partial\theta} = \frac{S}{\theta} + \frac{S-n}{1-\theta} = 0$$

$$\therefore \text{MLE} = \frac{S}{n} \tag{0.0.7}$$

From (0.0.3) and (0.0.4) we get,

$$\begin{aligned} f(\theta|X) &\propto f(X|\theta)\, f(\theta) \\ &\propto \theta^{S-\frac{1}{2}}\,(1-\theta)^{n-S-\frac{1}{2}} \end{aligned} \tag{0.0.8}$$

where $f(\theta|X)$ is the posterior density of θ.

$$\int_0^1 f(\theta|X)\, d\theta = 1$$

$$\therefore f(\theta|X) = \frac{\theta^{S-\frac{1}{2}}\,(1-\theta)^{n-S-\frac{1}{2}}}{B(S+\frac{1}{2}, n-S+\frac{1}{2})} \tag{0.0.9}$$

From definition of beta function we get

$$\begin{aligned} B(x, y) &= \int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt \\ &= \frac{x+y}{xy} \times \frac{1}{{}^{x+y}C_x} \end{aligned} \tag{0.0.10}$$

From (0.0.1) we get,

$$\begin{aligned} E(\theta|X) &= \int_0^1 \theta\, f(\theta|X)\, d\theta \\ &= \int_0^1 \frac{\theta^{S+\frac{1}{2}}\,(1-\theta)^{n-S-\frac{1}{2}}}{B(S+\frac{1}{2}, n-S+\frac{1}{2})} \\ &= \frac{B(S+\frac{3}{2}, n-S+\frac{1}{2})}{B(S+\frac{1}{2}, n-S+\frac{1}{2})} \end{aligned} \tag{0.0.11}$$

Using (0.0.10) in (0.0.11) we get

$$E(\theta|X) = \frac{S+\frac{1}{2}}{n+1} \tag{0.0.12}$$

For $E(\theta|X)$ to be greater than MLE,

$$\frac{S+\frac{1}{2}}{n+1} > \frac{S}{n}$$
$$\therefore n > 2S \tag{0.0.13}$$

1) This option is incorrect since $E(\theta|X) = \frac{S+\frac{1}{2}}{n+1}$ (0.0.12) and $n > 1$ which means that $E(\theta|X)$ exists.
2) This option is correct since $E(\theta|X) = \frac{S+\frac{1}{2}}{n+1}$ (0.0.12) and $n > 1$ which means that $E(\theta|X)$ exists.
3) This option is incorrect as from (0.0.13) we see that $E(\theta|X) \ngtr$ MLE for all values of S.
4) This option is correct as from (0.0.13) we see that $E(\theta|X) >$ MLE for some values of S.

$\therefore$ Option 2 and 4 are correct.