Assignment 5

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Download latex-tikz codes from

https://github.com/RaghavJuyal/AI1103/tree/main/Assignment5/Assignment5.tex

Question 113, CSIR UGC NET EXAM (Dec 2014)

Let $X_1, X_2, ..., X_n$ be independent and identically distributed Bernoulli(θ), where $0 < \theta < 1$ and $n > 1$. Let the prior density of θ be proportional to $\frac{1}{\sqrt{\theta(1-\theta)}}$, $0 < \theta < 1$. Define $S = \sum_{i=1}^{n} Xi$.
Then valid statements among the following are:

1. The posterior mean of θ does not exist;
2. The posterior mean of θ exists;
3. The posterior mean of θ exists and it is larger than the maximum likelihood estimator for all values of S.
4. The posterior mean of θ exists and it is larger than the maximum likelihood estimator for some values of S.

Solution

Let $f_\Theta(\theta)$ be the prior density and $f_{X|\Theta}(x|\theta)$ be the likelihood function.

$$f_\Theta(\theta) \propto \frac{1}{\sqrt{\theta(1-\theta)}} \tag{0.0.1}$$

$$\begin{aligned} f_{X|\Theta}(x|\theta) &= \prod_{i=1}^{n} \theta^{X_i}(1-\theta)^{1-X_i} \\ &= \theta^S (1-\theta)^{n-S} \end{aligned} \tag{0.0.2}$$

Let MLE be the maximum likelihood estimator.

$$\ln f_{X|\Theta}(x|\theta) = S \ln\theta + (n-S)\ln(1-\theta) \tag{0.0.3}$$

$$\frac{\partial \ln f_{X|\Theta}(x|\theta)}{\partial\theta} = \frac{S}{\theta} + \frac{S-n}{1-\theta} = 0$$

$$\therefore \text{MLE} = \frac{S}{n} \tag{0.0.4}$$

$$\begin{aligned} f_{\Theta|X}(\theta|x) &\propto f_{X|\Theta}(x|\theta)\, f_\Theta(\theta) \\ &\propto \theta^{S-\frac{1}{2}}(1-\theta)^{n-S-\frac{1}{2}} \end{aligned} \tag{0.0.5}$$

where $f_{\Theta|X}(\theta|x)$ is the posterior density.

$$\int_0^1 f_{\Theta|X}(\theta|x)\, d\theta = 1$$

$$\therefore f_{\Theta|X}(\theta|x) = \frac{\theta^{S-\frac{1}{2}}(1-\theta)^{n-S-\frac{1}{2}}}{B(S+\frac{1}{2}, n-S+\frac{1}{2})} \tag{0.0.6}$$

where $B(x, y)$ is the beta function. From definition of beta function we get

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \\ &= \frac{x+y}{xy} \times \frac{1}{{}^{x+y}C_x} \end{aligned} \tag{0.0.7}$$

Let posterior mean be $E(\Theta)$

$$\begin{aligned} E(\Theta) &= \int_0^1 \theta\, f_{\Theta|X}(\theta|x)\, d\theta \\ &= \int_0^1 \frac{\theta^{S+\frac{1}{2}}(1-\theta)^{n-S-\frac{1}{2}}}{B(S+\frac{1}{2}, n-S+\frac{1}{2})} \\ &= \frac{B(S+\frac{3}{2}, n-S+\frac{1}{2})}{B(S+\frac{1}{2}, n-S+\frac{1}{2})} \end{aligned} \tag{0.0.8}$$

Using (0.0.7) in (0.0.8) we get

$$E(\Theta) = \frac{S+\frac{1}{2}}{n+1} \tag{0.0.9}$$

Since $n > 1$, $E(\Theta)$ exists.
For $E(\Theta)$ to be greater than MLE,

$$\frac{S+\frac{1}{2}}{n+1} > \frac{S}{n}$$

$$n > 2S\,(S > 0) \text{or } n < 2S\,(S < 0) \tag{0.0.10}$$

$$\tag{0.0.11}$$

Since $n > 1$, for $E(\Theta) >$ MLE we get $n > 2S$.
$\therefore$ Option 2. and 4. are correct.