

Assignment 6

Raghav Juyal - EP20BTECH11018

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<https://github.com/RaghavJuyal/AI1103/tree/main/Assignment6/Assignment6.tex>

STATISTICS 2015 PAPER I, Q.2 (A)

Let C be a circle of unit area with centre at origin and let S be a square of unit area with $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ as the four vertices. If X and Y be two independent standard variates, show that

$$\iint_C \phi(x) \phi(y) dx dy \geq \iint_S \phi(x) \phi(y) dx dy$$

where $\phi(\cdot)$ is the pdf of $N(0, 1)$ distribution.

SOLUTION

Definition 1. PDF of normal distribution is given as

$$\phi_X(x) = N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (0.0.1)$$

Corollary 0.1.

$$\phi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (0.0.2)$$

$$\phi_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad (0.0.3)$$

Proof. Since $\phi(\cdot)$ is the pdf of $N(0, 1)$ distribution (given in question), $\mu = 0$ and $\sigma^2 = 1$.

$$\Rightarrow \phi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ and } \phi_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

□

Lemma 0.1. For circle C ,

$$\iint_C \phi(x) \phi(y) dx dy \approx 0.147136 \quad (0.0.4)$$

Proof. C has unit area with centre at origin.

$$\Rightarrow \pi \times r^2 = 1 \quad (0.0.5)$$

$$\Rightarrow |r| = \frac{1}{\sqrt{\pi}} \quad (0.0.6)$$

For the area inside circle C we get,

$$x^2 + y^2 \leq \frac{1}{\pi} \Rightarrow |y| \leq \sqrt{\frac{1}{\pi} - x^2} \quad (0.0.7)$$

From (0.0.7) we get,

$$\begin{aligned} \iint_C \phi(x) \phi(y) dx dy &= \\ \int_{x=-\frac{1}{\sqrt{\pi}}}^{x=\frac{1}{\sqrt{\pi}}} \int_{y=-\sqrt{\frac{1}{\pi}-x^2}}^{y=\sqrt{\frac{1}{\pi}-x^2}} \phi(x) \phi(y) dy dx & \quad (0.0.8) \end{aligned}$$

Using (0.0.2) and (0.0.3) in (0.0.8) we get,

$$\int_{x=-\frac{1}{\sqrt{\pi}}}^{x=\frac{1}{\sqrt{\pi}}} \int_{y=-\sqrt{\frac{1}{\pi}-x^2}}^{y=\sqrt{\frac{1}{\pi}-x^2}} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} dy dx \quad (0.0.9)$$

Converting it to polar coordinates we get,

$$\int_{r=0}^{r=\frac{1}{\sqrt{\pi}}} \int_{\theta=0}^{\theta=2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r d\theta dr \quad (0.0.10)$$

$$= \int_0^{\frac{1}{\sqrt{\pi}}} e^{-\frac{r^2}{2}} r dr \quad (0.0.11)$$

$$= 1 - e^{-\frac{1}{2\pi}} \quad (0.0.12)$$

$$\approx 0.147136 \quad (0.0.13)$$

□

Definition 2. The Q function is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \quad (0.0.14)$$

Lemma 0.2.

$$\int_{-a}^a e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} (1 - 2Q(a)) \quad (0.0.15)$$

Proof. Since $e^{-\frac{x^2}{2}}$ is an even function,

$$\int_{-a}^a e^{-\frac{x^2}{2}} dx = 2 \int_0^a e^{-\frac{x^2}{2}} dx \quad (0.0.16)$$

$$\begin{aligned} \implies 2 \int_0^a e^{-\frac{x^2}{2}} dx &= 2 \left(\int_0^\infty e^{-\frac{x^2}{2}} dx - \int_a^\infty e^{-\frac{x^2}{2}} dx \right) \\ &= 2 \sqrt{2\pi} \times \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty e^{-\frac{x^2}{2}} dx - \int_a^\infty e^{-\frac{x^2}{2}} dx \right) \end{aligned} \quad (0.0.17)$$

Comparing (0.0.17) with (0.0.14) we get,

$$\int_{-a}^a e^{-\frac{x^2}{2}} dx = 2 \sqrt{2\pi} (Q(0) - Q(a)) \quad (0.0.18)$$

$$= 2 \sqrt{2\pi} \left(\frac{1}{2} - Q(a) \right) \quad (0.0.19)$$

$$= \sqrt{2\pi} (1 - 2Q(a)) \quad (0.0.20)$$

□

Lemma 0.3. For square S ,

$$\iint_S \phi(x) \phi(y) dx dy \approx 0.146631 \quad (0.0.21)$$

Proof. For square S (given in question),

$$\frac{-1}{2} \leq x \leq \frac{1}{2} \quad (0.0.22)$$

$$\frac{-1}{2} \leq y \leq \frac{1}{2} \quad (0.0.23)$$

From this we get,

$$\begin{aligned} \iint_S \phi(x) \phi(y) dx dy &= \\ \int_{x=-\frac{1}{2}}^{x=\frac{1}{2}} \int_{y=-\frac{1}{2}}^{y=\frac{1}{2}} \phi(x) \phi(y) dy dx \end{aligned} \quad (0.0.24)$$

Using (0.0.2) and (0.0.3) in (0.0.24) we get,

$$\int_{x=-\frac{1}{2}}^{x=\frac{1}{2}} \int_{y=-\frac{1}{2}}^{y=\frac{1}{2}} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} dy dx \quad (0.0.25)$$

Using (0.0.15) twice we get,

$$\int_{x=-\frac{1}{2}}^{x=\frac{1}{2}} \int_{y=-\frac{1}{2}}^{y=\frac{1}{2}} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} dy dx = \left(1 - 2Q\left(\frac{1}{2}\right) \right)^2 \quad (0.0.26)$$

$$\approx 0.146631 \quad (0.0.27)$$

□

Solution: Since from (0.0.4) and (0.0.21),

$$0.147136 > 0.146631 \quad (0.0.28)$$

This proves that

$$\iint_C \phi(x) \phi(y) dx dy \geq \iint_S \phi(x) \phi(y) dx dy \quad (0.0.29)$$