# Assignment 5

Raghav Juyal - EP20BTECH11018

Download latex-tikz codes from

https://github.com/RaghavJuyal/AI1103/tree/main/Assignment5/Assignment5.tex

## Question 113, CSIR UGC NET EXAM (Dec 2014)

Let $X_1, X_2, ..., X_n$ be independent and identically distributed Bernoulli($\theta$), where $0 < \theta < 1$ and $n > 1$. Let the prior density of $\theta$ be proportional to $\frac{1}{\sqrt{\theta(1-\theta)}}$, $0 < \theta < 1$. Define $S = \sum_{i=1}^{n} X_i$.
Then valid statements among the following are:

1. The posterior mean of $\theta$ does not exist;
2. The posterior mean of $\theta$ exists;
3. The posterior mean of $\theta$ exists and it is larger than the maximum likelihood estimator for all values of S.
4. The posterior mean of $\theta$ exists and it is larger than the maximum likelihood estimator for some values of S.

## Solution

**Definition 1.** *Posterior mean is the mean of the posterior distribution of $\theta$, i.e.,*

$$E(\theta|X) = \int \theta\, f(\theta|X)\, d\theta \tag{0.0.1}$$

**Definition 2.** *The beta function, $B(x, y)$, is defined by the integral*

$$\begin{aligned} B(x, y) &= \int_0^1 t^{x-1}\, (1-t)^{y-1}\, dt \\ &= \frac{x+y}{xy} \times \frac{1}{{}^{x+y}C_x} \end{aligned} \tag{0.0.2}$$

*where $Re(x) > 0$ and $Re(y) > 0$.*

Let $f(\theta)$ be the prior density of $\theta$.

$$\begin{aligned} f(\theta) &\propto \frac{1}{\sqrt{\theta(1-\theta)}} \\ \implies f(\theta) &= \frac{K}{\sqrt{\theta(1-\theta)}} \end{aligned} \tag{0.0.3}$$

where $K$ is the proportionality constant.

$$\begin{aligned} \int_0^1 f(\theta)\, d\theta &= 1 \\ \implies K \int_0^1 \frac{1}{\sqrt{\theta(1-\theta)}}\, d\theta &= 1 \end{aligned} \tag{0.0.4}$$

From (0.0.2) we get,

$$K \times B\left(\frac{1}{2}, \frac{1}{2}\right) = 1$$

$$\implies K = \frac{1}{B\left(\frac{1}{2}, \frac{1}{2}\right)} \tag{0.0.5}$$

$$\therefore f(\theta) = \frac{\theta^{-\frac{1}{2}}\, (1-\theta)^{-\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{1}{2}\right)} \tag{0.0.6}$$

Let $f(X|\theta)$ be the likelihood function.

$$\begin{aligned} f(X|\theta) &= \prod_{i=1}^{n} \theta^{X_i}\, (1-\theta)^{1-X_i} \\ &= \theta^{S}\, (1-\theta)^{n-S} \end{aligned} \tag{0.0.7}$$

**Definition 3.** *The maximum likelihood estimator is the value which maximizes the likelihood function, i.e.,*

$$MLE = arg\ max\,(f(X|\theta)) \tag{0.0.8}$$

Using log of likelihood function and differentiating we get,

$$\ln(f(X|\theta)) = S\ln(\theta) + (n-S)\ln(1-\theta) \tag{0.0.9}$$

$$\frac{\partial \ln(f(X|\theta))}{\partial\theta} = \frac{S}{\theta} + \frac{S-n}{1-\theta} = 0$$

$$\therefore \text{MLE} = \frac{S}{n} \tag{0.0.10}$$

From (0.0.6) and (0.0.7) we get,

$$\begin{aligned} f(\theta|X) &\propto f(X|\theta)\, f(\theta) \\ \implies f(\theta|X) &\propto \theta^{S-\frac{1}{2}}\, (1-\theta)^{n-S-\frac{1}{2}} \\ \implies f(\theta|X) &= C \times \theta^{S-\frac{1}{2}}\, (1-\theta)^{n-S-\frac{1}{2}} \end{aligned} \tag{0.0.11}$$

where $f(\theta|X)$ is the posterior density of $\theta$ and $C$ is the proportionality constant.

$$\int_0^1 f(\theta|X)\, d\theta = 1$$
$$\implies C \int_0^1 \theta^{S-\frac{1}{2}} (1-\theta)^{n-S-\frac{1}{2}}\, d\theta = 1 \tag{0.0.12}$$

From (0.0.2) we get,

$$C \times B\left(S+\frac{1}{2}, n-S+\frac{1}{2}\right) = 1$$
$$\implies C = \frac{1}{B\left(S+\frac{1}{2}, n-S+\frac{1}{2}\right)} \tag{0.0.13}$$
$$\therefore f(\theta|X) = \frac{\theta^{S-\frac{1}{2}} (1-\theta)^{n-S-\frac{1}{2}}}{B\left(S+\frac{1}{2}, n-S+\frac{1}{2}\right)} \tag{0.0.14}$$

From (0.0.1) we get,

$$E(\theta|X) = \int_0^1 \theta f(\theta|X)\, d\theta$$
$$= \int_0^1 \frac{\theta^{S+\frac{1}{2}} (1-\theta)^{n-S-\frac{1}{2}}}{B\left(S+\frac{1}{2}, n-S+\frac{1}{2}\right)}\, d\theta$$
$$= \frac{B\left(S+\frac{3}{2}, n-S+\frac{1}{2}\right)}{B\left(S+\frac{1}{2}, n-S+\frac{1}{2}\right)} \tag{0.0.15}$$

Using (0.0.2) in (0.0.15) we get

$$E(\theta|X) = \frac{S+\frac{1}{2}}{n+1} \tag{0.0.16}$$

For $E(\theta|X)$ to be greater than MLE,

$$\frac{S+\frac{1}{2}}{n+1} > \frac{S}{n}$$
$$\therefore n > 2S \tag{0.0.17}$$

1) This option is incorrect since $E(\theta|X) = \frac{S+\frac{1}{2}}{n+1}$ (0.0.16) and $n > 1$ which means that $E(\theta|X)$ exists.
2) This option is correct since $E(\theta|X) = \frac{S+\frac{1}{2}}{n+1}$ (0.0.16) and $n > 1$ which means that $E(\theta|X)$ exists.
3) This option is incorrect as from (0.0.17) we see that $E(\theta|X) \ngtr$ MLE for all values of S.
4) This option is correct as from (0.0.17) we see that $E(\theta|X) >$ MLE for some values of S.

$\therefore$ Option 2 and 4 are correct.