Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is
$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$
2. The Laplace transform of $g(t)$ is defined as
$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

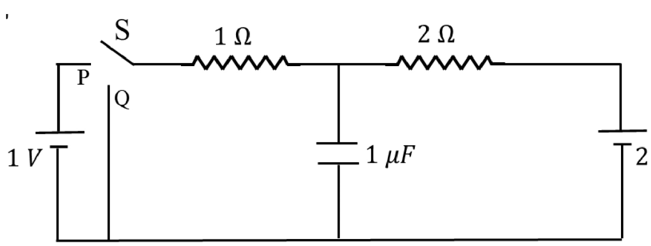

Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:

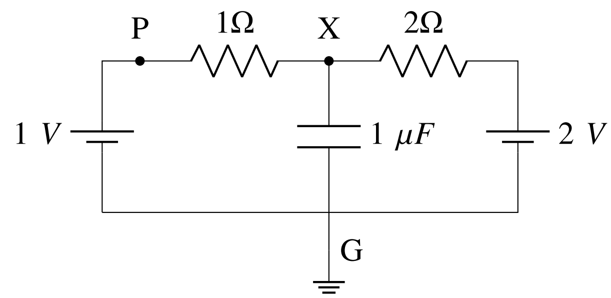

Fig. 2.2

3. Find q_1.
Solution: After a long time, the capacitor starts to behave like an open switch, which means that no current will flow through the capacitor. Assume that the circuit is grounded at the negative terminals of the battery and current in the circuit is i. Applying KVL in the loop:
$$1 + i + 2i - 2 = 0 \tag{2.1}$$
$$\implies i = \frac{1}{3}A \tag{2.2}$$
$$\frac{q_1\mu}{C} = 1 + \frac{1}{3} \tag{2.3}$$
$$\implies q_1 = \frac{4}{3} \tag{2.4}$$
4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:
$$\mathcal{U}(s) = \int_0^{\infty} u(t)e^{-st}dt \tag{2.5}$$
$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^{\infty} e^{-st}dt = \frac{1}{s} \tag{2.6}$$
$$\text{R.O.C: } Re(s) > 0 \tag{2.7}$$
5. Show that
$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \tag{2.8}$$
and find the ROC.

Solution:

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty u(t)e^{-(s+a)t}dt \tag{2.9}$$

$$= \frac{1}{s+a} \tag{2.10}$$

$$\text{R.O.C: } Re(s) > -a \tag{2.11}$$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

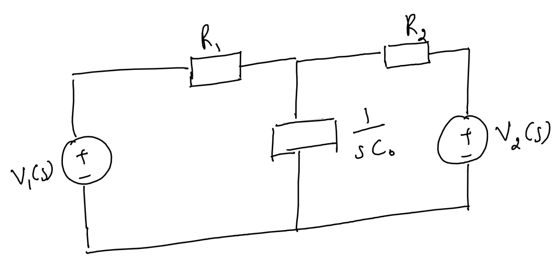

Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.12}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.13}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: Applying KCL at X:

$$\frac{V_x - \frac{1}{s}}{R_1} + s(VC_0) = \frac{\frac{2}{s} - V_x}{R_2} \tag{2.14}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.15}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.16}$$

$$= \frac{4}{3}\left(\frac{1}{s} - \frac{1}{\frac{3}{2C_0} + s}\right) \tag{2.17}$$

7. Find $v_{C_0}(t)$. Plot using python.

$$v_{C_0}(t) = \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.18}$$

$$v_{C_0}(t) = \frac{4}{3}\left(1 - e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{2.19}$$

8. Verify your result using ngspice.
Solution:

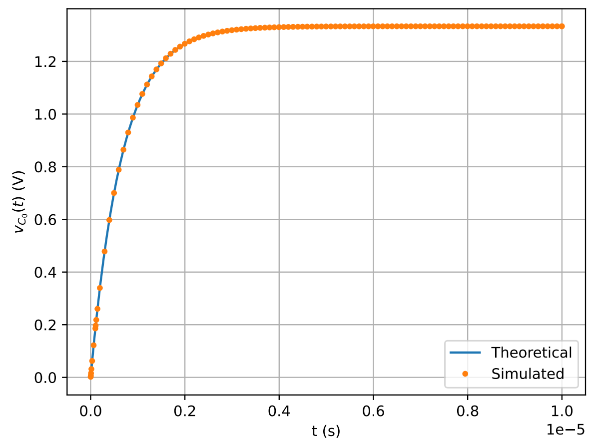

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

9. Obtain Fig. 2.3 using the equivalent differential equation.
Solution: Using KVL in the two separate loops and assuming currents i_1, i_2 in the loops such that $dq/dt = i_1 + i_2$

$$1 - i_1 = \frac{q}{C} = 2 - 2i_2 \tag{2.20}$$

$$\implies 2 - 2i_2 + 2 - 2i_1 = q_1/\mu + 2q_1/\mu \tag{2.21}$$

$$\implies 4 - 2\frac{dq_1}{dt} = 3q_1/\mu \tag{2.22}$$

$$\implies 2 - 1.5q_1/\mu = \frac{dq_1}{dt} \tag{2.23}$$

$$\implies \int_0^{q_1} \frac{dq_1}{2 - 1.5q_1/\mu} = \int_0^t dt \tag{2.24}$$

$$\implies ln(\frac{2 - 1.5q_1/\mu}{2}) = 1.5t \times 10^6 \tag{2.25}$$

$$\implies q_1 = \frac{4}{3}(1 - e^{1.5t\times10^6}) \tag{2.26}$$

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The circuit at steady state when the switch is at Q:

At steady state: Capacitor is charged
Applying KCL at X.

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \tag{3.1}$$

$$\implies V = \frac{2}{3}\text{V} \quad q_2 = \frac{2}{3}\,\mu\text{C} \tag{3.2}$$

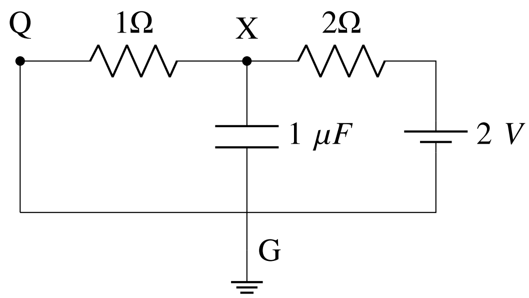

Fig. 3.1

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:

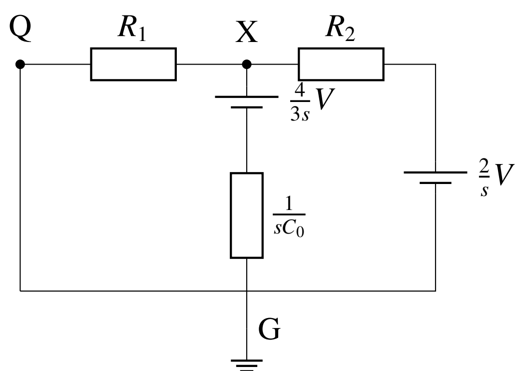

Fig. 3.2

3. $V_{C_0}(s) = ?$ **Solution:**
Applying KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.3}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.4}$$

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.5}$$

4. $v_{C_0}(t) = ?$ Plot using python. Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.6}$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.7}$$

The python code used to plot Fig. 3.3

https://github.com/RaghavJuyal/EE3900/blob/main/cktsig/codes/e3.4.py

5. Verify your result using ngspice.
Solution: The following ngspice script simulates the given circuit

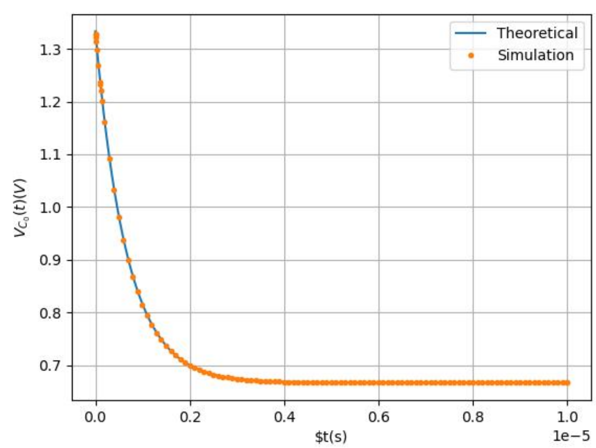

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

https://github.com/RaghavJuyal/EE3900/blob/main/cktsig/codes/e3.cir

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution:

$$v_{C_0}(0-) = \lim_{t\to0-} v_{C_0}(t) = \frac{q_1}{C} = \frac{4}{3}\text{ V} \tag{3.8}$$

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}\text{ V} \tag{3.9}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}\text{ V} \tag{3.10}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution: The equivalent circuit in the t-domain is shown below.

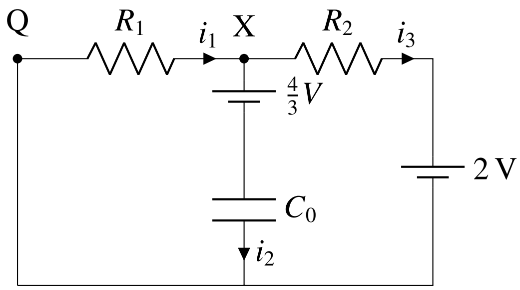

Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.11}$$
$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt = 0 \tag{3.12}$$
$$\frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt - i_3 R_2 - 2 = 0 \tag{3.13}$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$\text{Let } i(t) \overset{\mathcal{L}}{\longleftrightarrow} I(s) \tag{3.14}$$
$$\implies I_1 = I_2 + I_3 \tag{3.15}$$
$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \tag{3.16}$$
$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \tag{3.17}$$

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

Solution: The equivalent circuit in the t-domain is shown below.
Applying KCL and KVL,

$$i_1 = i_2 + i_3 \tag{4.1}$$
$$i_1 R_1 + \frac{1}{C_0}\int_0^t i_2\, dt = 0 \tag{4.2}$$
$$i_3 R_2 + 2 - \frac{1}{C_0}\int_0^t i_2\, dt = 0 \tag{4.3}$$

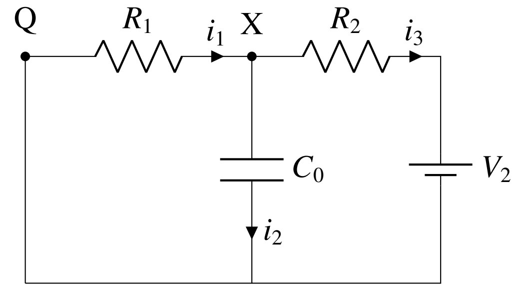

Fig. 4.1

Differentiating the above equations,

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{\mathrm{d}i_3}{\mathrm{d}t} \tag{4.4}$$
$$R_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + \frac{i_2}{C_0} = 0 \tag{4.5}$$
$$R_2\frac{\mathrm{d}i_3}{\mathrm{d}t} - \frac{i_2}{C_0} = 0 \tag{4.6}$$

Using (4.4) and (4.6) in (4.5),

$$R_1\left(\frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{\mathrm{d}i_3}{\mathrm{d}t}\right) + \frac{i_2}{C_0} = 0 \tag{4.7}$$
$$R_1\frac{\mathrm{d}i_2}{\mathrm{d}t} + \left(1 + \frac{R_1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.8}$$
$$\frac{\mathrm{d}i_2}{\mathrm{d}t} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.9}$$
$$\frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{i_2}{\tau} = 0 \tag{4.10}$$

where $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ is the RC time constant
$i_2(0) = \frac{V_2}{R_2}$ and $i_2 = C_0\frac{\mathrm{d}V}{\mathrm{d}t}$, where $V = V_{C_0}$ is the voltage of the capacitor.

$$C_0\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \tag{4.11}$$
$$\implies \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \tag{4.12}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
Solution:

$$H(s) = \frac{V_{C_0}(s)}{V_2(s)} \tag{4.13}$$

In the s domain:

$$\frac{V_{C_0}}{R_1} + \frac{V_{C_0}}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2}{R_2} = 0 \tag{4.14}$$

$$H(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{R_2} \tag{4.15}$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} = \frac{1}{3 + 2sC_0} \tag{4.16}$$

3. Plot $H(s)$. What kind of filter is it?

https://github.com/RaghavJuyal/EE3900/blob/main/cktsig/codes/e4.3.py

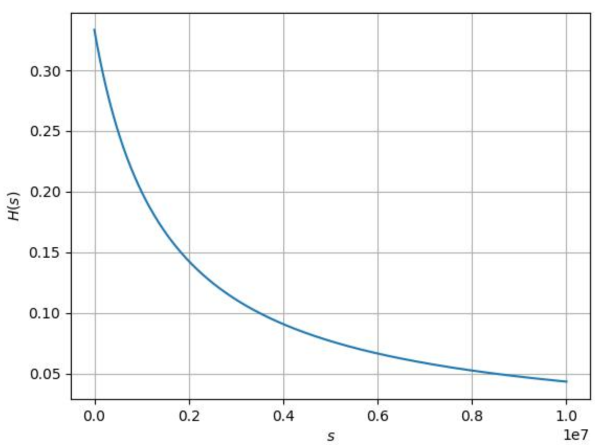

Fig. 4.2: Plot of $H(s)$.

It is a Low-Pass filter

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.17}$$

Solution:
$v_{in} = v_2(n) = 2u(n)$ and $v_{out} = v(n)$
Integrating (4.12) from n to n+1 and using (2.13):

$$v(n+1) - v(n) + \frac{v(n+1) + v(n)}{2\tau} = \frac{2}{C_0R_2}\frac{(u(n+1) + u(n))}{2} \tag{4.18}$$

$$v(n+1)(2\tau + 1) + v(n)(1 - 2\tau) = \frac{2}{C_0R_2}\tau(u(n+1) + u(n)) \tag{4.19}$$

$$v(n)(2\tau + 1) + v(n-1)(1 - 2\tau) = \frac{2}{C_0R_2}\tau(u(n) + u(n-1)) \tag{4.20}$$

5. Find $H(z)$.
Solution:

$$H(z) = \frac{V(z)}{V_2(z)} \tag{4.21}$$

We know that

$$v_2(n) = 2u(n) \implies V_2(z) = \frac{2}{1 - z^{-1}} \tag{4.22}$$

Applying $\mathcal{Z}$-transform on (4.20)

$$V(z)(2\tau + 1) + z^{-1}V(z)(1 - 2\tau) = \frac{2\tau(1 + z^{-1})}{C_0R_2(1 - z^{-1})} \tag{4.23}$$

$$V(z)\left(2\tau + 1 - z^{-1}(2\tau - 1)\right) = \frac{\tau V_2(z)(1 + z^{-1})}{C_0R_2} \tag{4.24}$$

$$H(z) = \frac{V(z)}{V_2(z)} = \frac{\tau(1 + z^{-1})}{C_0R_2\left(2\tau + 1 - z^{-1}(2\tau - 1)\right)} \tag{4.25}$$

$$H(z) = \frac{\tau(1 + z^{-1})}{R_2C_0\left(2\tau + 1 - z^{-1}(2\tau - 1)\right)} \tag{4.26}$$

$$= \frac{\tau(z + 1)}{R_2C_0(2\tau(z - 1) + z + 1)} \tag{4.27}$$

$$= \frac{1}{R_2C_0\left(2\frac{z-1}{z+1} + \frac{1}{\tau}\right)} \tag{4.28}$$

But $\tau = \frac{C_0R_1R_2}{R_1 + R_2} = \frac{2C_0}{3}$ and $R_2 = 2$

$$\implies H(z) = \frac{1}{2C_0\left(2\frac{z-1}{z+1} + \frac{3}{2C_0}\right)} \tag{4.29}$$

$$H(z) = \frac{1}{3 + 4C_0 \frac{z-1}{z+1}} \tag{4.30}$$

R.O.C: $|z| < 1$

6. How can you obtain $H(z)$ from $H(s)$?
Solution:
Apply a Bilinear Transform

$$s \rightarrow \frac{2}{T}\frac{z-1}{z+1} \tag{4.31}$$

$$H(z) = \frac{1}{3 + 2C_0 \frac{2}{T}\frac{z-1}{z+1}} \tag{4.32}$$

Putting T = 1 will give (4.30)

7. Find $v(n)$. Verify using ngspice and differential equation
Solution:

$$V(z) = H(z)V_2(z) \tag{4.33}$$

$$= \frac{1}{3 + \frac{4}{T}C_0 \frac{1-z^{-1}}{1+z^{-1}}} \frac{2}{1-z^{-1}} \tag{4.34}$$

$$= \frac{2}{3}\left(\frac{1}{1 + \frac{2\tau}{T}\frac{1-z^{-1}}{1+z^{-1}}} \frac{1}{1-z^{-1}}\right) \tag{4.35}$$

$$= \frac{2}{3}\left(\frac{1+z^{-1}}{1 + z^{-1} + \frac{2\tau}{T}\left(1-z^{-1}\right)} \frac{1}{1-z^{-1}}\right) \tag{4.36}$$

$$= \frac{2}{3}\left(\frac{1}{1-z^{-1}} - \frac{\frac{2\tau}{T}}{1 + z^{-1} + \frac{2\tau}{T}\left(1-z^{-1}\right)}\right) \tag{4.37}$$

$$= \frac{2}{3}\left(\frac{1}{1-z^{-1}} - \frac{\frac{2\tau}{T}}{1 + \frac{2\tau}{T} + z^{-1}\left(1-\frac{2\tau}{T}\right)}\right) \tag{4.38}$$

$$= \frac{2}{3}\left(\frac{1}{1-z^{-1}} - \frac{\frac{2\tau}{T}}{1+\frac{2\tau}{T}} \frac{1}{1 - \left(\frac{\frac{2\tau}{T}-1}{\frac{2\tau}{T}+1}\right)z^{-1}}\right) \tag{4.39}$$

Taking Inverse $\mathcal{Z}$ transform on both sides

$$v(n) = \frac{2}{3}u(n)\left[1 - \frac{\frac{2\tau}{T}}{\frac{2\tau}{T}+1}\left(\frac{\frac{2\tau}{T}-1}{\frac{2\tau}{T}+1}\right)^n\right] \tag{4.40}$$

Python code used to plot theoretical and simulated values of $v(n)$ can be found at:

https://github.com/RaghavJuyal/EE3900/blob/main/cktsig/codes/e4.7.py

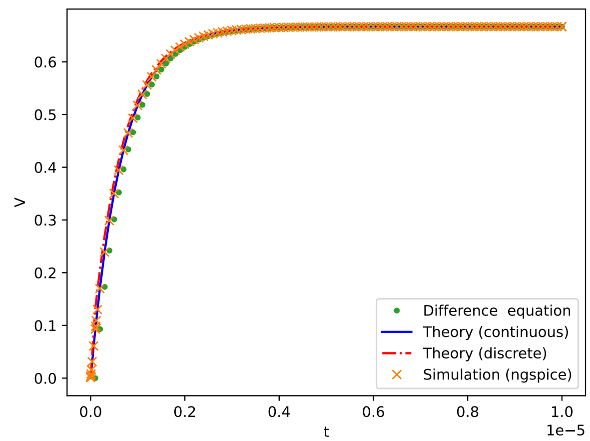

Fig. 4.3