Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.
    com/gadepall/
EE1310/master/filter/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')
#print(a)
#print(b)
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)
```

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```
#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/
    EE1310/raw/master/filter/codes/
    xnyn.py
```

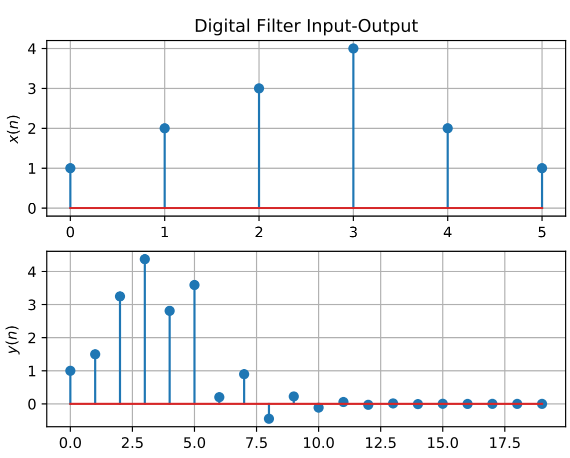

Fig. 3.2

3.3 Repeat the above exercise using a C code.

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution: From (4.1) we get,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= x(0)\,z^{-0} + x(1)\,z^{-1} + \cdots + x(5)\,z^{-5} \tag{4.8}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.20}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.21}$$

$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{4.22}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.23}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: From (4.12) we get

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right) \tag{4.24}$$

$$= \frac{1 + \left(e^{-2\jmath\omega}\right)}{1 + \frac{1}{2}\left(e^{-1\jmath\omega}\right)} \tag{4.25}$$

$$= \frac{\left(e^{\jmath\omega}\right) + \left(e^{-\jmath\omega}\right)}{\left(e^{\jmath\omega}\right) + \frac{1}{2}} \tag{4.26}$$

$$= \frac{2\cos(\omega)}{\left(e^{\jmath\omega}\right) + \frac{1}{2}} \tag{4.27}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{2\cos(\omega)}{\sqrt{\left(\cos(\omega) + \frac{1}{2}\right)^2 + \left(\sin^2(\omega)\right)}} \tag{4.28}$$

$$= \frac{2\cos(\omega)}{\sqrt{\cos^2(\omega) + \sin^2(\omega) + \cos(\omega) + \frac{1}{4}}} \tag{4.29}$$

$$= \frac{2\cos(\omega)}{\sqrt{\frac{5}{4} + \cos(\omega)}} \tag{4.30}$$

$$\tag{4.31}$$

For a periodic function of period T,

$$f(x) = f(x + T), T \neq 0 \tag{4.32}$$

Both the numerator and denominator have a period of 2π since $\cos(\omega)$ has a period of 2π. Which means that $H(e^{\jmath(\omega)})$ has a period of 2π $\therefore$ Period is 2π.
The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/dtft.py
```

From Fig. 4.6 we can see that $\left|H\left(e^{\jmath\omega}\right)\right|$ is periodic with period 2π.

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.
Solution: Since $H\left(e^{\jmath\omega}\right)$ is the DTFT of $h(n)$,

$$H\left(e^{\jmath\omega}\right) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-\jmath\omega n} \tag{4.33}$$

Multiplying both sides with $e^{\jmath\omega n}$ and integrating we get

$$\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-\jmath\omega n} e^{\jmath\omega k} \tag{4.34}$$

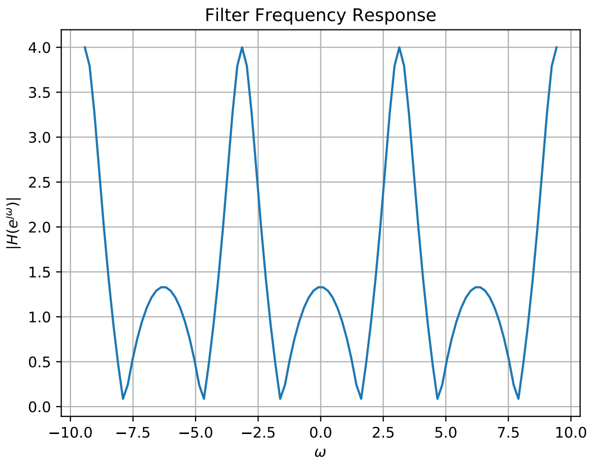

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

Case 1: $n \neq k$

$$\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-\jmath\omega n} e^{\jmath\omega k} d\omega \tag{4.35}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \frac{e^{\jmath\pi(k-n)} - e^{-\jmath\pi(k-n)}}{\jmath(k-n)} \tag{4.36}$$

$$= 0 \tag{4.37}$$

Case 2: $n = k$

$$\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-\jmath\omega n} e^{\jmath\omega k} d\omega \tag{4.38}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} d\omega \tag{4.39}$$

$$= 2\pi \tag{4.40}$$

$$\implies \int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega k} d\omega = 2\pi h(n) \tag{4.41}$$

$$\implies h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega n} d\omega \tag{4.42}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12).

Solution: From (4.12) we get,

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$\implies H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.3}$$

$$\text{ROC}: \tag{5.4}$$

$$\left|\frac{z^{-1}}{2}\right| \leq 1 \tag{5.5}$$

$$\implies \frac{1}{2} \leq |z| \tag{5.6}$$

$$H(z) = \sum_{-\infty}^{\infty} h(n) z^{-n} \tag{5.7}$$

$$\frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \sum_{-\infty}^{\infty} h(n) z^{-n} \tag{5.8}$$

$$\begin{array}{r l}
 & 1 - \frac{z^{-1}}{2} + \frac{5z^{-2}}{4} - \frac{5z^{-3}}{8} + \frac{5z^{-4}}{16} + \dots \\
1 + \frac{z^{-1}}{2} \big) & 1 + 0z^{-1} + z^{-2} + 0z^{-3} + 0z^{-4} + \dots \\
 & 1 + \frac{z^{-1}}{2} \\ \hline
 & \frac{-z^{-1}}{2} + z^{-2} \\
 & \frac{-z^{-1}}{2} - \frac{z^{-1}}{4} \\ \hline
 & \frac{5z^{-1}}{4} + 0z^{-3} \\
 & \frac{5z^{-1}}{4} + \frac{5z^{-3}}{8} \\ \hline
 & \frac{-5z^{-3}}{8} + 0z^{-4} \\
 & \frac{-5z^{-3}}{8} - \frac{5z^{-4}}{16} \\ \hline
 & \frac{5z^{-4}}{16} + 0z^{-5} \\
 & \vdots
\end{array}$$

Comparing coefficients of z^{-n} we get,

$$h(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ \frac{-1}{2} & \text{if } n = 1 \\ \frac{5}{4} & \text{if } n = 2 \\ \frac{-5}{8} & \text{if } n = 3 \\ \frac{5}{16} & \text{if } n = 4 \end{cases} \tag{5.9}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.10}$$

and there is a one to one relationship between

$h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.11}$$

$$\therefore h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.12}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution:

$$|u(n)| \leq 1 \tag{5.13}$$

$$\left|\left(\frac{-1}{2}\right)^n\right| \leq 1 \tag{5.14}$$

$$\implies \left|\left(\frac{-1}{2}\right)^n u(n)\right| \leq 1 \tag{5.15}$$

$$\text{Similarly,} \tag{5.16}$$

$$\implies \left|\left(\frac{-1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.17}$$

$$\implies \left|\left(\frac{-1}{2}\right)^n u(n)\right| + \left|\left(\frac{-1}{2}\right)^{n-2} u(n-2)\right| \leq 2 \tag{5.18}$$

$$\implies \left|\left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)\right| \leq 2 \tag{5.19}$$

The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/hn.py
```

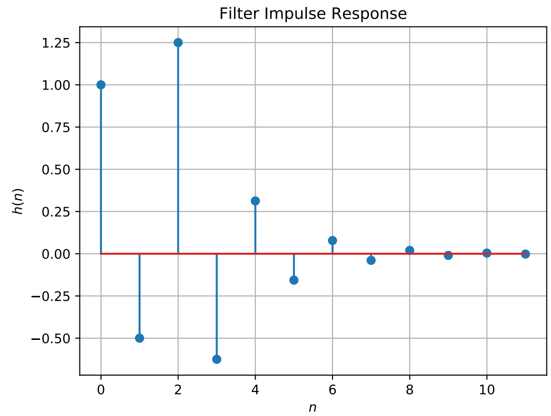

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 Convergent? Justify using the ratio test.
Solution:

$$h(n) = \begin{cases} 0 \quad \text{if} \quad n < 0 \\ 1, \quad \text{if} \quad n = 0 \\ -\frac{1}{2}, \quad \text{if} \quad n = 1 \\ 5\left(-\frac{1}{2}\right)^n, \quad \text{if} \quad n \geq 2 \end{cases} \tag{5.20}$$

Using ratio test:

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \left|\frac{5\left(-\frac{1}{2}\right)^{n+1}}{5\left(-\frac{1}{2}\right)^n}\right| = \frac{1}{2} < \infty \tag{5.21}$$

$$\implies h(n) \text{ is convergent}$$

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.22}$$

Is the system defined by (3.2) stable for the impulse response in (5.10)
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = 0 + 1 - \frac{1}{2} + 5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n \tag{5.23}$$

$$= \frac{1}{2} + 5\left(1 - \frac{1}{2} - \left(\frac{1}{1+\frac{1}{2}}\right)\right) \tag{5.24}$$

$$= \frac{1}{2} + \frac{5}{6} = \frac{8}{6} = 1.333 < \infty \tag{5.25}$$

$\therefore h(n)$ is Stable

5.6 Verify the above result using a python code.
Solution: The Following code computes and proves the aboves result

```
wget https://github.com/DarkWake9/
    EE3900/blob/main/Assignment
    %201/e5-6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \qquad (5.26)$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/hndef.py
```

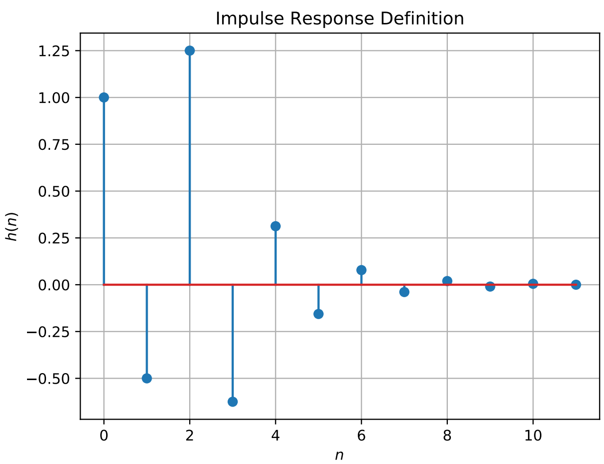

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \qquad (5.27)$$

Comment. The operation in (5.27) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/ynconv.py
```

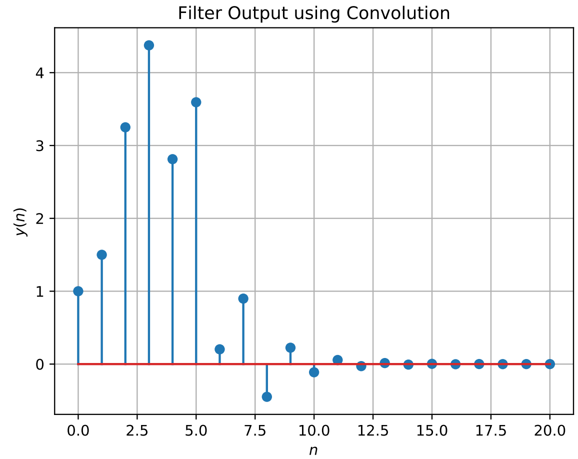

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution: From (3.1) $x(n) = \{1, 2, 3, 4, 2, 1\}$

From (5.27) $y(n) = x(n) * h(n)$

$$\begin{pmatrix}
x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
h(1) \\ h(2) \\ h(3) \\ h(4) \\ h(5) \\ h(6) \\ h(7) \\ h(8) \\ h(9) \\ h(10) \\ h(11) \\ h(12)
\end{pmatrix} \qquad (5.28)$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 2&1&0&0&0&0&0&0&0&0&0&0\\ 3&2&1&0&0&0&0&0&0&0&0&0\\ 4&3&2&1&0&0&0&0&0&0&0&0\\ 2&4&3&2&1&0&0&0&0&0&0&0\\ 1&2&4&3&2&1&0&0&0&0&0&0\\ 0&1&2&4&3&2&1&0&0&0&0&0\\ 0&0&1&2&4&3&2&1&0&0&0&0\\ 0&0&0&1&2&4&3&2&1&0&0&0\\ 0&0&0&0&1&2&4&3&2&1&0&0\\ 0&0&0&0&0&1&2&4&3&2&1&0\\ 0&0&0&0&0&0&1&2&4&3&2&1\\ 0&0&0&0&0&0&0&1&2&4&3&2\\ 0&0&0&0&0&0&0&0&1&2&4&3\\ 0&0&0&0&0&0&0&0&0&1&2&4\\ 0&0&0&0&0&0&0&0&0&0&1&2\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 1.\\ -0.5\\ 1.25\\ -0.625\\ 0.3125\\ -0.15625\\ 0.078125\\ -0.0390625\\ 0.01953125\\ -0.00976562\\ 0.00390625\\ -0.00195312 \end{pmatrix} \tag{5.29}$$

$$= \begin{pmatrix} 1\\ 1.5\\ 3.25\\ 4.375\\ 2.8125\\ 3.59375\\ 0.203125\\ 0.8984375\\ -0.44921875\\ 0.224609375\\ -0.112304688\\ 0.0561523438\\ -0.0280761719\\ 0.0140380859\\ -7.01904297\times10^{-3}\\ 3.50952148\times10^{-3}\\ -1.75476074\times10^{-3}\\ 8.77380371\times10^{-4}\\ -4.38690186\times10^{-4}\\ 0 \end{pmatrix} \tag{5.30}$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.31}$$

Solution:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{5.32}$$

$$H(z) = \mathcal{Z}\{h(m)\} = \sum_{m=-\infty}^{\infty} h(m)z^{-m} \tag{5.33}$$

$$Y(z) = \mathcal{Z}\{h(m)\} = \sum_{k=-\infty}^{\infty} y(m)z^{-k} \tag{5.34}$$

$$X(z)H(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \sum_{m=-\infty}^{\infty} h(m)z^{-m} \tag{5.35}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n]h[m]z^{-(n+m)} \tag{5.36}$$

Let $m = k - n$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x[n]h[n-k] \right) z^{-k} \tag{5.37}$$

$$= \sum_{k=-\infty}^{\infty} y[n]z^{-k} = Y(z) \tag{5.38}$$

$$\implies Y(z) = X(z) \cdot H(z) \tag{5.39}$$

now put $n + m = k \quad n = -\infty$

$$\Rightarrow Y(z) = \sum_{k=-\infty}^{\infty} x(m-k) \sum_{m=-\infty}^{\infty} h(m)z^{-k} \tag{5.40}$$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m-k]h[k] \right) z^{-k} \tag{5.41}$$

$$\text{but } Y(z) = \sum_{k=-\infty}^{\infty} y(m)z^{-k} \tag{5.42}$$

$$\Rightarrow y(m) = \sum_{m=-\infty}^{\infty} x[m-k]h(k) \tag{5.43}$$

$$\implies y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.44}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y\,(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y\,(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. (6.3) and computes $X(k)$ and $Y(k)$. Note that this is the same as $y(n)$ in Fig. (3.2).

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e6_3.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT. **Solution:** Download the code from

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e6_4.py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

6.5 Wherever possible, express all the above equations as matrix equations. **Solution:** We use the

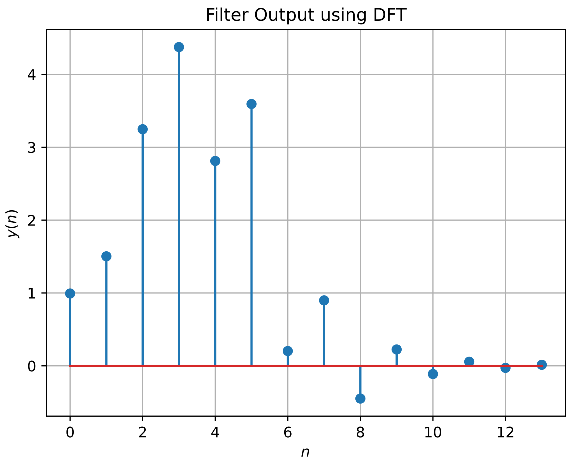

Fig. 6.3: $y(n)$ from the DFT

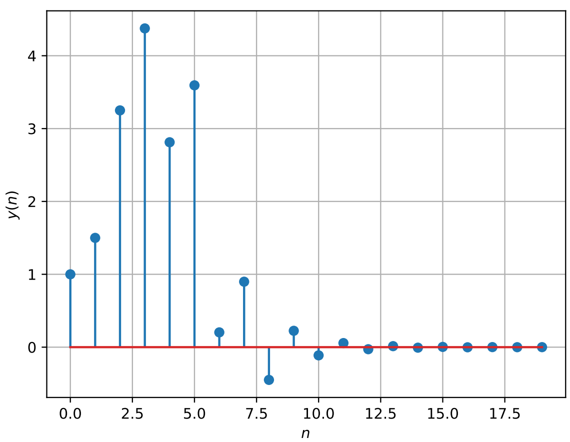

Fig. 6.4: $y(n)$ using FFT and IFFT

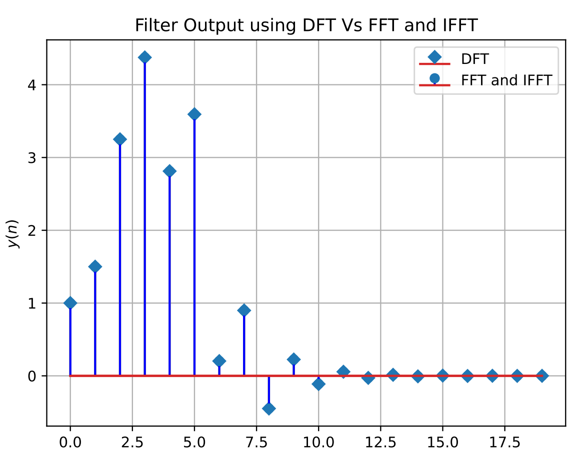

Fig. 6.4: $y(n)$ by DFT vs FTT and IFFT

DFT Matrix, where $\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.4}$$

i.e. $W_{jk} = \omega^{jk}$, $0 \leq j, k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \tag{6.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{6.6}$$

Using (6.3), the inverse Fourier Transform is given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \tag{6.7}$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \tag{6.8}$$

where H denotes hermitian operator. We can rewrite (6.2) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{W}\mathbf{h}) \cdot (\mathbf{W}\mathbf{x}) \tag{6.9}$$

The plot of $y(n)$ using the DFT matrix in Fig. (5) is the same as $y(n)$ in Fig. (3.2). Download the code using

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e6_5.py
```

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

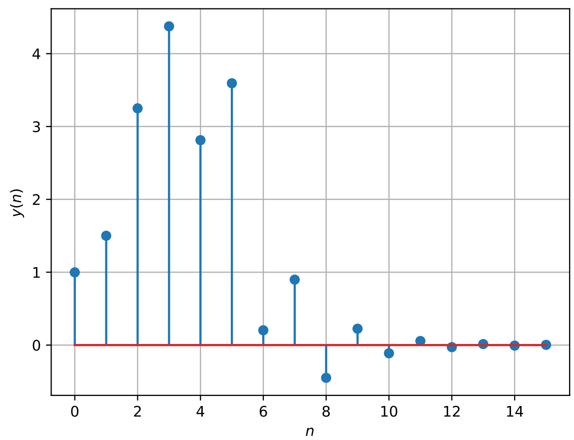

Fig. 5: $y(n)$ using the DFT matrix

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_4^0 & W_4^1 & W_4^2 & W_4^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$\implies W_{N/2} = e^{-j2\pi/(N/2)} \tag{7.9}$$
$$\therefore W_N^2 = e^{2(-j2\pi/N)} = e^{-j2\pi/(N/2)} = W_{N/2} \tag{7.10}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.11}$$

Solution:

$$\mathbf{F}_2 = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.12}$$
$$\mathbf{D}_{4/2} = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \tag{7.13}$$
$$\text{R.H.S} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_{4/2}\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_{4/2}\mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.14}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{bmatrix} \mathbf{P}_4 \tag{7.15}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \mathbf{F}_4 \tag{7.17}$$

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.18}$$

Solution:
If we perform odd-even permutation on the columns of $\mathbf{F}_N$ and compare the terms with that of $\mathbf{F}_{N/2}$ and $\mathbf{D}_{N/2}$ we get

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.19}$$
$$= \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.20}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.21}$$

Solution:

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.22}$$
$$\mathbf{x}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \tag{7.23}$$
$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix} 1 & 3 & 2 & 4 \end{pmatrix} \tag{7.24}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.25}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution:

$$(\mathbf{F}_N\mathbf{x})_k = \sum_{m=0}^{N-1} W_N^{mk} x(m) \tag{7.26}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N} = X(k) = \mathbf{X}_k \tag{7.27}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.28}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.29}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.30}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.31}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.32}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.33}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.34}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.35}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.36}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.40}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.41}$$

compte the DFT using (7.25)
Solution:

$$\mathbf{X} = \mathbf{F}_6\mathbf{x} \tag{7.42}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right) & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^5 \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^9 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{16} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^5 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} & \left(e^{\frac{-j2\pi}{6}}\right)^{25} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.43}$$

$$= \begin{pmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{pmatrix} \tag{7.44}$$

12. Repeat the above exercise using the FFT after zero padding **x**.
Solution:

```
wget https://github.com/RaghavJuyal/EE3900/
    blob/main/Sound/codes/e7_12.py
```

From the above code we get this output:

$$\begin{bmatrix} 13 \\ -3.1213-6.5355j \\ j \\ 1.1213-0.5355j \\ -1 \\ 1.1213+0.5355j \\ -j \\ -3.1213+6.5355j \end{bmatrix}$$

13. Write a C program to compute the 8-point FFT.
Solution:

```
wget https://github.com/RaghavJuyal/EE3900/
    blob/main/Sound/codes/e7_13.c
```

From the above code we get this output:

$$\begin{bmatrix} 13 \\ -3.1327-j6.5545 \\ j \\ 1.1327-j0.5545 \\ -1 \\ 1.1327+j0.5545 \\ -j \\ -3.1327+j6.5545 \end{bmatrix}$$

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.
    lfilter(b, a, input_signal
    )
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: The implementation is at

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e8_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b. **Solution:** For the given values, the difference equation is

$$\begin{aligned} &y(n)-(4.44)\,y(n-1)+(8.78)\,y(n-2) \\ &-(9.93)\,y(n-3)+(6.90)\,y(n-4) \\ &-(2.93)\,y(n-5)+(0.70)\,y(n-6) \\ &-(0.07)\,y(n-7)=\left(5.02\times10^{-5}\right)x(n) \\ &+\left(3.52\times10^{-4}\right)x(n-1)+\left(1.05\times10^{-3}\right)x(n-2) \\ &+\left(1.76\times10^{-3}\right)x(n-3)+\left(1.76\times10^{-3}\right)x(n-4) \\ &+\left(1.05\times10^{-3}\right)x(n-5)+\left(3.52\times10^{-4}\right)x(n-6) \\ &+\left(5.02\times10^{-5}\right)x(n-7) \end{aligned} \quad (8.2)$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{\sum_{k=0}^{M} a(k)z^{-k}} \quad (8.3)$$

$$= \sum_i \frac{r(i)}{1-p(i)z^{-1}} + \sum_j k(j)z^{-j} \quad (8.4)$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.19),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \quad (8.5)$$

Substituting the values,

$$\begin{aligned} h(n) = &\,[(2.76)\,(0.55)^n \\ &+(-1.05-1.84\jmath)\,(0.57+0.16\jmath)^n \\ &+(-1.05+1.84\jmath)\,(0.57-0.16\jmath)^n \\ &+(-0.53+0.08\jmath)\,(0.63+0.32\jmath)^n \\ &+(-0.53-0.08\jmath)\,(0.63-0.32\jmath)^n \\ &+(0.20+0.004\jmath)\,(0.75+0.47\jmath)^n \\ &+(0.20-0.004\jmath)\,(0.75-0.47\jmath)^n]u(n) \\ &+\left(-6.81\times10^{-4}\right)\delta(n) \end{aligned} \quad (8.6)$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e8_2_1.py
```

The filter frequency response is plotted at

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e8_2_1.py
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We note that observe that $|p(i)| < 1$ and so, as $h(n)$ is the sum of convergent series, we see that $h(n)$ converges. From Fig. (8.2), it is clear that $h(n)$ is bounded. From (4.1),

$$\sum_{n=0}^{\infty} h(n) = H(1) = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (8.2), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2).

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e8_2_1.py
```

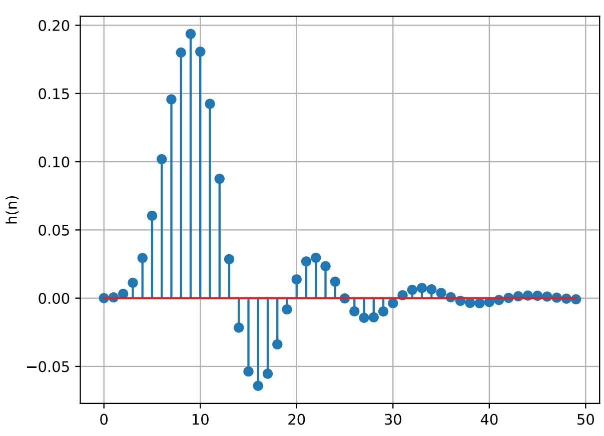

Fig. 8.2: Plot of $h(n)$

8.3 What is the sampling frequency of the input signal? **Solution:** Sampling frequency f_s = 44.1 kHZ.

8.4 What is type, order and cutoff frequency of the above Butterworth filter? **Solution:** The given Butterworth filter is low pass with order 4 and cutoff frequency 4 kHz.

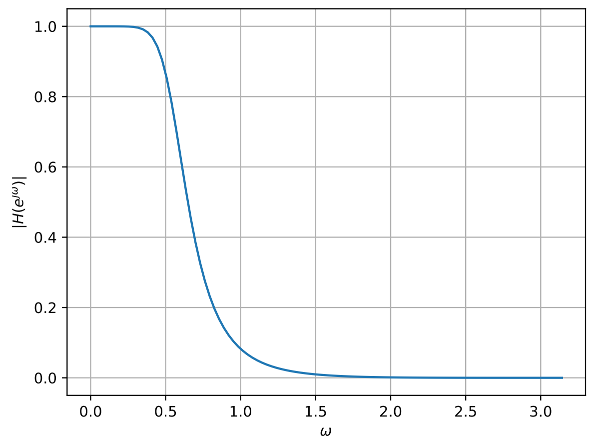

Fig. 8.2: Filter frequency response

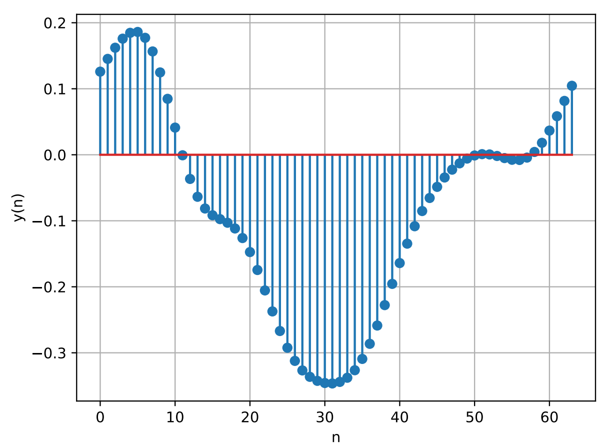

Fig. 8.2: Plot of $y(n)$

8.5 Modifying the code with different input parameters and to get the best possible output. **Solution:** A better filtering was found on setting the order of the filter to be 7.