Pingala Series

Raghav Juyal*

Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download the python code for the above problem using.

```
$ wget https://raw.githubusercontent.
    com/prajwal-3-14159/
    EE3900_ma20btech11013/main/
    Pinagala_Series/pingala/codes/
    problem_1.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: The python code for stem plot:

```
$ wget https://raw.githubusercontent.
    com/prajwal-3-14159/
    EE3900_ma20btech11013/main/
    Pinagala_Series/pingala/codes/
    problem_2-2.py
```

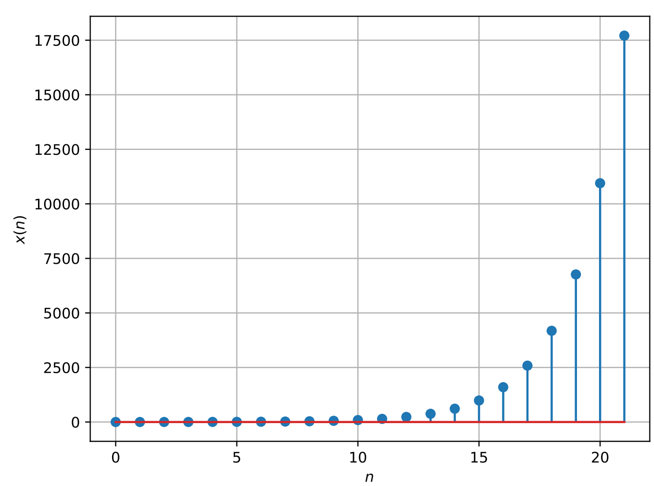

Fig. 2.2: The stem plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: Take the one-sided Z-transform on

both sides of the equation (2.2),

$$\mathcal{Z}^+ [x(n+2)] = \mathcal{Z}^+ [x(n+1)] + \mathcal{Z}^+ [x(n)] \tag{2.3}$$

$$z^2X^+(z) - z^2x(0) - zx(1) = zX^+(z) - zx(0) + zX^+(z \tag{2.4}$$

Now, as $x(o) = x(1) = 1$

$$\left(z^2 - z - 1\right) X^+(z) = z^2 \tag{2.5}$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

Let, α and β be solutions to eqn $x^2 - x - 1 = 0$, then we get,

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > \alpha \tag{2.7}$$

2.4 Find $x(n)$.
Solution: On expanding the $X^+(z)$ as given in eqn (2.7) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta) z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right] \tag{2.8}$$

$$= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1} \tag{2.9}$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \tag{2.10}$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} z^{-n} \tag{2.11}$$

As $k := n + 1$. hence,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n) \tag{2.12}$$

therefore, $x(n) = a_{n+1}u(n)$.

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.13}$$

Solution: Download the python code for plot of $y(n)$ using:

```
$ wget https://raw.githubusercontent.
    com/prajwal-3-14159/
    EE3900_ma20btech11013/main/
    Pinagala_Series/pingala/codes/
    problem_2-5.py
```

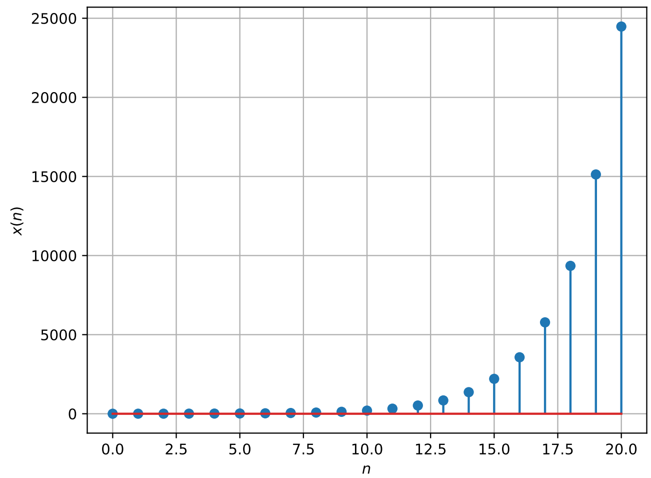

Fig. 2.5: The stem plot of $y(n)$

2.6 Find $Y^+(z)$.
Solution: Take the one-sided Z-transform on the both sides of the eqn (2.13),

$$\mathcal{Z}^+ [y(n)] = \mathcal{Z}^+ [x(n+1)] + \mathcal{Z}^+ [x(n-1)] \tag{2.14}$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \tag{2.15}$$

On substituting the values of $X^+(z)$, $x(0)$, $x(1)$ and $x(-1)$. [here, $x(-1) = 0$, as $x(n) = 0\ \forall n < 0$]

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.16}$$

$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha \tag{2.17}$$

2.7 Find $y(n)$.
Solution: As $Y^+(z) = (1 + 2z^{-1})X^+(z)$, hence using the (2.1),

$$Y^+(z) = (1 + 2z^{-1}) \sum_{n=0}^{\infty} x(n) z^{-n} \tag{2.18}$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n} + \sum_{n=1}^{\infty} 2x(n-1) z^{-n} \tag{2.19}$$

$$= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1)) z^{-n} \tag{2.20}$$

Now, as $y(0) = x(0) = 1$ and using the fact that

α and β are the roots of the eqn $z^2 - z - 1 = 0$,

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \tag{2.21}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \tag{2.22}$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta\left(\alpha^n + \beta^n\right)}{\alpha - \beta} \tag{2.23}$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \tag{2.24}$$

$$= \alpha^{n+1} + \beta^{n+1} \tag{2.25}$$

Hence, $y(n) = \alpha^{n+1} + \beta^{n+1}$; $\forall n \geq 0$. as $\alpha + \beta = 1$. Comparing (2.22) with the definition of b_n, we get $y(n) = b_{n+1} = \alpha^{n+1} + \beta^{n+1}$ ($\therefore$ $b_n = \alpha^n + \beta^n$).

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution: From the eqn (2.12), we can see that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$\left[x\left(n+1\right) - 1\right] u\left(n\right) \tag{3.7}$$

Solution: From the eqn (2.12),

$$a_{n+2} - 1 = \left[x(n+1) - 1\right], \quad n \geq 0 \tag{3.8}$$

Now, using the definition of $u(n)$,

$$a_{n+2} - 1 = \left[x(n+1) - 1\right] u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x\left(k\right)}{10^k} = \frac{1}{10} X^+\left(10\right) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.12}$$

$$= \frac{1}{10} X^+(z) \tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.16}$$

and find $W(z)$.

Solution: Let, $n = k + 1$ in (3.15) and using the definition of $u(n)$, we get:

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{3.17}$$

Now, eqn (3.15) can be expressed as,

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \tag{3.18}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.19}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y\left(k\right)}{10^k} = \frac{1}{10} Y^+\left(10\right) \tag{3.20}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.21}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.22}$$

$$= \frac{1}{10} Y^+(z) \tag{3.23}$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.24}$$

3.6 Solve the JEE 2019 problem.
Solution: As we know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.25}$$

also as

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \tag{3.26}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{3.27}$$

$$= z\left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}}\right] \tag{3.28}$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} \left(x(n) - 1\right) z^{-n} \tag{3.29}$$

$$= \sum_{n=0}^{\infty} \left(x(n) - 1\right) z^{-n+1} \tag{3.30}$$

$$= \sum_{n=0}^{\infty} \left(x(n+1) - 1\right) z^{-n} \tag{3.31}$$

And from (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.32}$$

We have already established the remaining options in the problems (3.3), (2.7), (3.5).
$\therefore$ the correct options are: a, b, and c
and, incorrect option is: d.