4 March

Chebyshev's Inequality

If X is a R.V with mean (μ) & variance (σ^2) then for any positive no. k

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$ – upper bound

$$P\{|X-\mu| < k\sigma\} \geq 1-\frac{1}{k^2}$$ – Lower Bound

Note:- Let $k\sigma = c > 0$

$$P\{|X-\mu| \geq k\sigma = c\} \leq \frac{1}{k^2} = \frac{\sigma^2}{c^2}$$

$$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$$

Also; $$P\{|X-\mu| < c\} \geq 1-\frac{\sigma^2}{c^2}$$

Q Two unbiased dice are thrown. If the sum of the numbers showing up then Prove that

$$P(|X-7| \geq 3) \leq \frac{35}{54}$$

Sol use; $P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$

$SS = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$P(X=2) = \frac{1}{36}$ $P(X=3) = \frac{2}{36}$ $P(X=4) = \frac{3}{36}$

$P(X=5) = \frac{4}{36}$ $P(X=6) = \frac{5}{36}$ $P(X=7) = \frac{6}{36}$

$P(X=8) = 5/36$ $P(X=9) = 4/36$ $P(X=10) = 3/36$

$P(X=11) = 2/36$ $P(X=12) = 1/36$

$$E(X) = 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} +$$

$$7\times\frac{6}{36} + 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + \frac{12}{36}$$

$$= \frac{252}{36} = \boxed{7} = \text{Mean}$$

Variance $= E(X^2) - (E(X))^2$

$$= 4\times\frac{1}{36} + 9\times\frac{2}{36} + 16\times\frac{3}{36} + 25\times\frac{4}{36} + 36\times\frac{5}{36} + 49\times\frac{6}{36} + 64\times\frac{5}{36} + 81\times\frac{4}{36}$$

$$100\times\frac{3}{36} + 121\times\frac{2}{36} + 144\times\frac{1}{36} = 54.83$$

Variance $(\sigma^2) = 54.83 - 49$

$= 5.833 \Rightarrow \frac{1974}{36} - 49$

$= \frac{210}{36} = \frac{35}{6}$

$\frac{\sigma^2}{c^2} = \frac{210}{36} = \frac{35}{54}$

By Chebyshev's Inequality we have

$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$

$P(|X-7| \geq 3) \leq \frac{\sigma^2}{c^2} = \frac{35}{6 \times 9} = \frac{35}{54}$

$P(|X-7| \geq 3) + P(|X-7| < 3) = 1$

$= 1 - P(|X-7| < 3)$

$1 - P(-3 < X-7 < 3)$

$1 - P(-3+7 < X < 3+7)$

$1 - P(-4 < X < 10)$

$= \frac{1}{3}$

lower bound → Chebyshev inequality

Q2 A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 times

$P(80 \leq X \leq 120)$

$\mu = np$ Binomial distr

$\sigma^2 = npq$

$\boxed{\mu = 600 \times \frac{1}{6} = 100}$

$\sigma^2 = 600 \times \frac{1}{6} \times \frac{5}{6}$

$\boxed{\sigma^2 = \frac{500}{6}}$

$P\{|X-100| < c\} \geq 1 - \frac{500}{6 \times c^2}$

$P\{-c \leq X-100 \leq c\} \geq 1 - \frac{500}{6 \times c^2}$

$P\{100 - c \leq X \leq 100 + c\} \geq 1 - \frac{500}{6 \times c^2}$

$100 - c = 80$, $100 + c = 120$

$c = 20$ $c = 20.$

Put $c = 20$;

$P\{80 \leq X \leq 120\} > \frac{19}{24}$

5 March

Q. For Geometric Distribution $P(X=x) = 2^{-x}$, $x = 1, 2, 3 \ldots$
Prove that Chebyshev's Inequality gives

$$P\{|X-2| \le 2\} > \frac{1}{2}$$

Sol. $E(X) = \sum x\, p(x)$

$= 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

$= 2^{-1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right)$ $\qquad 1 + 2A + 3A^2 + 4A^3 \ldots = \frac{1}{(1-A)^2}$

$2^{-1}\left(\frac{1}{(1-\frac{1}{2})^2}\right)$

$2^{-1} \times 4 = \boxed{2} =$ Mean.

Variance $= 2$ $\qquad \boxed{\mu = 2,\ \sigma^2 = 2}$

$P\ 2-2 < 2 \quad > 1 - \frac{2}{4}$

$P\{0 \le 2\} > \frac{2}{4}\ \frac{1}{2}$ H.P Chebyshev's inequality

If said least upper bound → Cheb
upper bound ↳ Markov's

MARKOV'S INEQUALITY

Markov's inequality is a probabilistic inequality. It provides an upper bound to the probability that the realization of a Random Variable exceeds a given threshold

Markov's Inequality is a quick way of estimating probabilities based on the mean of a R.V

Let X be a positive R.V & $a > 0$ be a real number then

$$\boxed{P(X \ge a) \le \frac{E(X)}{a}}$$

Corollary.

Let X be a discrete R.V & $h : R \to R$ be a non negative function then

$$\boxed{P(h(x) \ge a) \le \frac{E\, h(x)}{a}}$$

* Benefits
1) Only assumption is that R.V is Non Negative
2) Only have to find expected value (mean)

* Demerits

1. Applies to only Non-Negative variables
2. Ignores distribution shape - It considers only mean & does not use finer details
3. It provides an upper bound that is often very weak
4. Not useful for Tight prob bounds

Q1 Suppose that the average grade on the upcoming statistics exam ~~was~~ is 70%. Given upper bound on the proportion of students who score atleast 90%

Sol.

$$E(X) = 70\% = 0.7$$

$$X \geq 90\%$$

$$P(X \geq .90) \leq \frac{70}{0.9}$$

$$\boxed{P(X \geq 0.9) \leq \frac{7}{9}} \rightarrow \text{upper bound}$$

Ans.

bound / upper bound → Markov's

Q2 A coin is weighted so that its probability of landing on heads is 20%. Suppose the coin is flipped 20 times. Find a bound for the prob. if it lands on heads atleast 16 times

$$\text{Mean} = np = 2\cancel{0} \times \frac{2\cancel{0}}{10\cancel{0}} = \boxed{4}$$

$$P(X \geq 16) < \frac{\cancel{4}^{1}}{\cancel{16}_{4}}$$

$$\boxed{P(X \geq 16) < \frac{1}{4}}$$

Ans.

Q3 Consider a R.V X that takes the value 0 with probability $\rightarrow \frac{24}{25}$ & value 1 with prob $\rightarrow \frac{1}{25}$. Find a bound on prob that X is atleast 5

$$E(X) = 0 \times \frac{24}{25} + 1 \times \frac{1}{25} = \frac{1}{25}$$

$$P(X \geq 5) \leq \frac{1}{25 \times 5}$$

$$\boxed{P(X \geq 5) \leq \frac{1}{125}}$$

Ans.

Q4. A biased coin which lands heads with prob $\frac{1}{10}$ each time it is flipped, is flipped 200 times consectively. Give an upper bound on the prob that it lands heads atleast 120 times?

Sol. Mean $E(X) = np$

$$200 \times \frac{1}{10} = 20$$

$$P(X \geq 120) \leq \frac{20}{120} \; \frac{1}{6}$$

$$\boxed{P(X \geq 120) \leq \frac{1}{6}}$$

Q5. Let us flip a coin N times Let X_i be the indicator R.V for the events that the i^{th} coin flip is head. Find probability to obtain 80% or more heads in such a sequence of coin flips

$$E(X) = \frac{n}{2}$$

Sol.

$$P\left(X \geq \frac{80n}{100}\right) = \frac{\frac{n}{2}}{\frac{80n}{100}} = \frac{1}{2} \times \frac{100}{80} \; 50$$

$$P(X > 0.8n) \leq \frac{5}{8}$$

6 March

LAW OF LARGE NUMBERS

$\frac{1}{n} \to 0$ Convergence value? ... $\to 0$

Convergence reduces computation w/ big Data

* Convergence in Probability

A sequence of R.V $X_1, X_2 \dots X_n$ is said to converge in probability to (α) alpha if for any $\epsilon > 0$

$$\lim_{n \to \infty} P(|X_n - \alpha| < \epsilon) = 1$$

Probabilistic convergence will be 1

OR

$$\lim_{n \to \infty} P(|X_n - \alpha| \geq \epsilon) = 0$$

OR $X_n \xrightarrow{P} \alpha$ as $n \to \infty$

* Chebyshev's Theorem

If mean can be determined & Var tends to 0 then r.v will converge to the mean itself

If $X_1, X_2 \dots X_n$ is a sequence of R.V and if mean (μ_n) & standard deviation (σ_n) of X_n exists $\forall n$ and if $\sigma_n \to 0$ as $n \to \infty$ then $X_n - \mu_n \xrightarrow{P} 0$ as $n \to \infty$ i.e

$$X_n \xrightarrow{P} \mu_n$$

OR

$$\lim_{n \to \infty} P(|X_n - \mu_n| < \varepsilon) = 1$$

X_n will prob. converge to its mean if $P = 1$

* Weak Law of Large Numbers (WLLN)

If $X_1, X_2 \ldots X_n$ is a sequence of R.V and $\mu_1, \mu_2 \ldots \mu_n$ be their respective Means & let

$$B_n = Var(X_1 + X_2 + \ldots X_n) < \infty$$

then,

$$P\left\{\left|\frac{X_1 + X_2 \ldots + X_n}{n} - \frac{\mu_1 + \mu_2 - + \mu_n}{n}\right| < \varepsilon\right\} \geq 1 - \eta$$

$\forall n \geq n_0$ where ε, η are arbitrary small positive numbers provided

$$\lim_{n \to \infty} \frac{B_n}{n^2} \to 0$$

10 March

For the existance of Weak Law of Large NO. The following conditions are needed

1. $E(X_i)$ exists $\forall i$ Mean of all r.v must exist
2. $B_n = Var(X_1 + X_2 + \ldots + X_n)$ is finite Variance of sum of r.v is finite
3. $\lim_{n \to \infty} \frac{B_n}{n^2} \to 0$

* Avg of R.V will converge to avg of means
→ probabilistic convergence

Weak Law of Large No. for i.i.d (Independently Identically Distributed R)V

R.V with same mean & var

If the variables $x_1, x_2 \ldots x_n$ are independent & identically distributed i.e $E(x_i) = \mu$, $Var(x_i) = \sigma^2 \; \forall i$

Then it satisfies WLLN.

To Check if WLLN holds for i.i.d RV

1. $E(x_i) = \mu$ exists $\forall i$
2. $B_n = var(x_1 + x_2 \ldots x_n)$
   $var(x_1) + V(x_2) \ldots V(x_n)$
   $\sigma^2 + \sigma^2 \ldots + \sigma^2$
   $\boxed{B_n = n\sigma^2}$
3. $\lim_{n \to \infty} \frac{n\sigma^2}{n^2} \to 0$

How to check WLLN Holds or Not ?

Conditions for WLLN.

→ PTO

Conditions for WLLN

I (i) $E(X_i)$ exist $\forall i$

(ii) $B_n = Var(X_1 + X_2 - - X_n)$ is finite

(iii) $\lim_{n \to \infty} \frac{B_n}{n^2} \to 0$

when we have sequence x_n.

II When $\{x_n\}$ is i.i.d

Then $E(X_i)$ enough for the existance of WLLN

This result is called **Khinchin's Theorem**

If atleast one of the Conditions is not met then we apply the further test Markov's Theorem

The WLLN holds if for some delta (δ) exists for some bounded $E\left(|X_i|^{1+\delta}\right)$ exists bounded

17 March

Q1. Examine whether WLLN hold for the sequence $\{X_i\}$ of independent r.v $P(X_k = \pm 2^k) = 2^{-(2k+1)}$

$P(X_k = 0) \quad 1 - 2^{-2k}$

Sol. $E(X_i) = \sum x_i \, p(x_i)$ independent

X_i	2^k	-2^k	0
$P(X_i)$	$2^{-(2k+1)}$	$2^{-(2k+1)}$	$1 - 2^{-2k}$

$$E(X) = 2^k \cdot 2^{-(2k+1)} - 2^k \cdot 2^{-(2k+1)} + 0$$

$$2^k \left(\cancel{2^{-2k+1}} - \cancel{2^{-2k+1}}\right)$$

$$= \boxed{0}$$

$$B_n = Var(X_1 + X_2 + \cdots X_n)$$
$$= Var(X_1) + Var X_2 + \cdots Var(X_n)$$

$$Var(X_i) = E(X_i)^2 - [E(X_i)]^2$$

$(X_i)^2$	2^{2k}	2^{2k}	0
	$2^{-(2k+1)}$	$2^{-(2k+1)}$	$1-2^{-2k}$

AMBA SINGH

$$2^{2k} \times 2^{(2k+1)} + 2^{2k} \times 2^{-(2k+1)} \neq 0$$

$$= 2^{2k}\left(2 \cdot 2^{-(2k+1)}\right)$$

$$= 2^{2k+1-2k-1} = 2^0 = 1$$

$V(x_1) = 1$

$V(x_1) = 1 \quad V(x_2) = 1 \quad V(x_3) = 1 \ \ldots\ n$

(II) $B_n = V(S_n) = n$

(III) $\lim_{n\to\infty} \frac{B_n}{n^2} = \frac{n}{n^2} = \frac{1}{n} \to 0$

As all conditions are true Hence
∴ WLLN holds True.

$\boxed{Q2}$ Examine if WLLN holds for the sequence $\{X_n\}$ of i.i.d r.v with

$$P\left(X_i = (-1)^{k-1} k\right) = \frac{6}{\pi^2 k^2} \ ; \ k = 1, 2, 3 \ldots \quad i = 1, 2, 3$$

Ans-

X_i	$(-1)^{k-1} k$
$P(X_i)$	$\frac{6}{\pi^2 k^2}$

$$E(X) = \left((-1)^{k-1} \cdot k\right)\left(\frac{6}{\pi^2 k^2}\right)$$

$$(-1)^k \cdot (-1)^{-1} \frac{6}{\pi^2 k} = \frac{-6}{\pi^2 k}(-1)^k$$

$$\Sigma k = \frac{-6}{\pi^2}\left(\frac{(-1)^k}{k}\right)$$

$$\frac{-6}{\pi^2}\left(\frac{-1}{1} - \frac{1^2}{2} - \frac{-1^3}{3} - \frac{1^4}{4}\right)$$

$$= \frac{-6}{\pi^2}\left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \ldots\right)$$

$\log_e 2$

Expectation $\boxed{\frac{-6}{\pi^2} \log_e 2}$

As it is the case of iid only expectation is enough to say that WLLN holds True.

X_i	i	$-i$
P_i	$1/2$	$1/2$

20 March

Q. Let x_i assume that values i & $-i$ with equal probabilities. Show that WLLN cannot be applied to the independent variables $x_1, x_2, \ldots$

Exi)

$$\text{Prob} = P$$

$$P(i) + P(-i) = 0$$

$$P(x_i = 1) = \frac{1}{2}, \quad P(x_i = -i) = 1/2$$

$$E(x) = i \cdot 1/2 + (-i)(1/2) = 0$$

$$E(x_i) = 0$$

$$Var(x_i) = \cancel{Var}\,E(x_i)^2 - E(x_i)^2$$

$$= i^2(1/2) + (-i)^2(1/2)$$

$$= \boxed{i^2}$$

$$var(x_i) = i^2 - 0 = \boxed{i^2}$$

$$B_n = Var(x_1 + \cdots x_n)$$

$$= 1^2 + 2^2 + \cdots n^2$$

$$B_n = \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n}$$

$$\lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right) \cdot n\left(2+\frac{1}{n}\right)}{6n} = \boxed{\infty}$$

Now we apply Markov's $E(|x_i|^{1+\delta})$ exists & bounded

$$\frac{i^{1+\delta}}{2} + \frac{i^{1+\delta}}{2} = i^{1+\delta}$$

∴ It is Unbounded.

∴ WLLN does not Hold True

STRONG LAW OF LARGE NUMBERS

Let $X_1, X_2, \ldots X_n$ be i.i.d r.v with a finite expected value

$$E(x_i) = \mu < \infty$$

almost surely

Then $\bar{X}_n \xrightarrow{as} \mu$ where $\bar{X}_n = \frac{X_1 + X_2 + \cdots X_n}{n}$

Q. Consider the following random Experiment. A fair coin tossed once. Then the sample space consists of elements $S = \{H, T\}$

We define a sequence of r.v $X_1, X_2 \ldots$ on this SS as follows

$$X_n(S) = \begin{cases} \frac{n}{n+1}; & \text{if } S = H \\ (-1)^n; & \text{if } S = T \end{cases}$$

Determine whether the resulting seq for each outcome H or T of real no. converge or not ?

When the outcome is Head i.e $S = H$ then we obtain the following sequence

$X_n(H) = \frac{n}{n+1}$ where

$\lim_{n\to\infty} X_n(H) = \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} = \boxed{1}$

which is CONVERGING TO 1

If the outcome is Tail we obtain seq. as

$X_n(T) = (-1)^n$ i.e ~~S=T~~ $-1, 1, -1, 1, \ldots\ldots$

which is oscillating b/w -1 & 1

HENCE DOES NOT CONVERGE

Therefore; X_n will only converge if Head appears & does not converge when Tail appears.

ALMOST SURE CONVERGENCE

If the probability that the seq. X_n converge to X is equal to 1 then we say that X_n converges to X almost surely. i.e

$$X_n \xrightarrow{a.s} X$$

i.e $P\left(\lim_{n\to\infty} X_n \to X\right) = 1$

24 March, Mon

CENTRAL LIMIT THEOREM (CLT)

CLT is an imp result in statistics which states that normal distribution is the limit distribution to the sum of independent r.v with finite variance as the no. of r.v get indefinately large

State C.L.T

If x_i $i = 1, 2 \ldots n$ be independently distributed r.v such that $E(x_i) = \mu_i$, $var(x_i) = \sigma_i^2$ then As $n \to \infty$, the distribution of the sum of these r.v namely $S_n = x_1 + x_2 + x_3 + \ldots$ tends to normal distribution with mean μ & variance σ^2 where $\mu = \sum_{i=1}^{n} \mu_i$, $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$

Q. A coin is tossed 200 times. Find approximate prob. that no. of heads obtained between 80 & 120

Sol. $n = 200 \quad p = \frac{1}{2} \quad q = \frac{1}{2}$

$\mu = 200 \times \frac{1}{2} = 100$

$\sigma^2 = npq = 100 \times \frac{1}{2} = 50$

$\sigma = 5\sqrt{2}$

$= P(80 \le x \le 120)$

$= P\left(\frac{800-100}{5\sqrt{2}} < z < \frac{120-100}{5\sqrt{2}}\right)$

$P\left(\frac{-4}{\sqrt{2}} < z < \frac{4}{\sqrt{2}}\right)$

$P(-2\sqrt{2} < z < 2\sqrt{2})$

$= 2P(0 < z < 2\sqrt{2}) = 0.9952$

Q A Random Sample of size 100 is taken from population where mean is 60, variance is 400 using central limit theorem with what prob can we assess that mean of sample will not differ from $\mu = 60$ by more than 4

Sol. $n = 100 \quad \mu = 60 \quad \sigma^2 = 400$

$|\bar{x} - 60| \le 4$

$P(|\bar{x} - 60| \le 4)$

$P(-4 \le \bar{x} - 60 \le 4)$

$P(56 \le \bar{x} \le 64)$

$n = 100 \quad \mu = 60 \quad \sigma^2 = 400 \quad z = \frac{\bar{x} - \mu}{\sigma}$

$E(ax) = a\,E x$

$V(ax) \quad a^2 V(x)$

$E(\bar{x}) = \frac{E(x_1 + x_2 + \dots x_{100})}{100}$

$\frac{1}{100}\left(E(x_1) + E(x_2) + E(x_3) \dots E(x_{100})\right)$

$= \frac{1}{100} \times 100 \times 60$

$V(\bar{x}) = \frac{V(x_1 + x_2 + \dots x_n)}{100}$

$= \frac{1}{(100)^2}\left(V(x_1) + V(x_2) + \dots V(x_n)\right)$

$P\left(\frac{56-\mu}{\sigma} \le \frac{\bar{x}-\mu}{\sigma} \le \frac{64-\mu}{\sigma}\right)$

$P\left(\frac{56-60}{2} \le z \le \frac{64-60}{2}\right)$

$P(-2 \le z \le 2)$

$2P(0 \le z \le 2)$

$= \underline{0.9544}$

*** BOREL - CANTELLI LEMMA (Zero One Law)**

Let $A_1, A_2 \ldots$ be a sequence of events on probability space & let $A = \overline{\lim}\, A_n$ then $P(A) = 0$ if $\sum_{n \geq 1}^{\infty} P(A_n) < \infty$

CONVERSE → Let A_1, A_2 & so on be independent events on probability space & $A = \lim A_n$ If

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \text{ then } P(A) = 1$$

25 March

R.V. = real value is assign
R.P. – real value assign at t
→ assign function of time to sample space

RANDOM PROCESS

The random process X assigns a real function of time t to each outcome s of the experiment. This fxn is denoted by $X(t,s)$

For example $x(t,s) = x_1(t) = t$; if $s = H$
$x(t,s) = x_2(t) = \sin(2t)$; if $s = T$

* If this x is independent of time then it becomes Random Variable

Application - Stock Market

In other words, Random Process is a collection of random variables usually indexed by time

Consider a Random Exp with sample space S if a time function $x(t,s)$ is assigned to each outcome $s \in S$ where $t \in T$ then the family of all such functions denoted by $X(t,s)$ is called Random Process i.e

$$\{X(t),\ t \in [0, \infty)\}$$

For a fixed t, say $t = 2$
$X(t,s) = X(2,s)$ is a random Variable as s varies over the sample space S. On the other hand

For fixed s say; $s = s_i$

$X(t, s_i) = X_i(t)$ is a single function of time t called as sample function or ensemble member or a realization of a Random Process.

V.Imp.

CLASSIFICATION OF A RANDOM PROCESS

$E \rightarrow$ state space/ set of outcomes of random exp.

$T \rightarrow$ indexset (dependending on time factor)

Let E be the state space of the Random Process
T is the indexset or Parameter Set

Consider a Random Process $\{X(t) : t \in T\}$
If the index set T is discrete then the Process is called a Discrete Parameter or Discrete time Process. It is also called a Random Sequence & denoted by

$$\{X_n, n = 1, 2, 3 \ldots \ldots T\}$$

1. If T and E are Both discrete then Random Process is called as Discrete Random Sequence

$E \rightarrow D$
$T \rightarrow D$

Ex- If X_n represent the no. of heads obtained in the n^{th} toss of 2 fair coins then $\{X_n, n \geq 1\}$ is a discrete Random sequence.

$T = \{1, 2, 3 \ldots \ldots\}$
$E = \{0, 1, 2\}$

2. If T is Discrete and E is Continous then Random Process is Called Continous Random Sequence

$E \rightarrow C$
$T \rightarrow D$

Example - If X_n represent the Temperature at the end of n^{th} hour of the day then the States $\{X_n, n = 1, 2 \ldots 24\}$ can take any value in an Interval & hence are Continous

$T = \{1, 2 \ldots \ldots 24\}$

$E = [0, 100]$

3. If the ~~Safe~~ State space of R.P is discrete then R.P is called Discrete Random Process. or Discrete State Process

$E \rightarrow D$
$T \rightarrow C$

$\rightarrow$ A discrete Random Process also called as Chain

For Eg - If X_t represents the no. of SMS Messages receive in a Cellphone in the interval $[0, t]$ then $\{X(t)\}$ is a Discrete Random Process.

(4) If the state space E of a Random Process is continous then the R.P is called as <u>Continous Random Process</u> or Continous State Space.

For Eg:- If X_t is the min temp recorded in a City in the interval $(0,t)$ then $\{X(t)\}$ is a <u>Continous Random Process</u>

E-c
T-c

So We Have 4 Types of Random Process

- TIME
  - Discrete-Time
    - Discrete State → Discrete Random Sequence
    - Cont. State → Cont. Random Sequence
  - Continous-Time
    - Discrete State → Discrete Random Process
    - Cont State → Cont. Random Process

Deterministic - find new values from old values

27 March

Deterministic & Non Deterministic Random Process

A random process is called a deterministic r.p if future values of any sample function can be predicted exactly from past observations or past values.

For eg:-
$$X(t) = A\cos(\omega)t \text{ is deterministic}$$

If future values of any sample function cannot be predicted exactly from past observations or past values. Then it is called <u>Non Deterministic R.P</u>

DESCRIPTION OF RANDOM PROCESSES

* Consider a r.p $\{X(t): t \in T\}$ for a fixed time t_1 $X(t_1) = X_1$ is a random variable & its cumulative distribution function (C.D.F) denoted by $F_X(x_1, t_1)$ & defined as

$$F_X(x_1,t_1) = P(X(t_1) \leq x_1)$$

and called as first order distribution of r.p $X(t)$

$$F_X(x) = P(X \leq x)$$

* For given t_1, t_2 $X(t_1) = X_1$, $X(t_2) = X_2$ are 2 r.v then the joint CDF is defined as

$$F(x_1, x_2, t_1, t_2) = P(X(t_1) < x_1, X(t_2) \leq x_2)$$

& called as second order distribution of r.p $X(t)$

STATIONARY RANDOM PROCESS

with change in time no change in prob.

A random process $x(t)$ such that $\{X(t) : t \in T\}$ is stationary if its statistical properties do not change by time i.e for stationary process $X(t)$ and $X(t+\Delta)$ have the same probability distribution.

$$\text{CDF} \rightarrow F_{X(t)}(x) = F_{X(t+\Delta)}(x) \quad \forall\ t, t+\Delta \in T$$

OR

$$\text{P.DF} \rightarrow f_X(x, t) = f_X(x, t+\Delta)$$

In ~~some~~ summary, A random process is stationary if a time shift does not change its statistical properties.

STRICT SENSE STATIONARY PROCESS.

A continous r.p $\{x(t) : t \in T\}$ is strict sense stationary process or simply stationary if $\forall\ t_1, t_2 \cdots t_r \in T$ all $\Delta \in R$

$$F_{X(t_1) X(t_2) \cdots X(t_r)}(x_1, x_2 \cdots x_r) = F_{X(t_1+\Delta) X(t_2+\Delta) \cdots X(t_r+\Delta)}(x_1, x_2 \cdots x_r)$$

OR

$$f_X(x_1, x_2 \cdots x_r, t_1, t_2 \cdots t_r) = f_X(x_1, x_2 \cdots x_r, t_1+\Delta \cdots t_r+\Delta)$$

A Random Process which is not stationary in any sense is called EVOLUTIONARY

1Q. Consider the discrete time r.p $\{X(n), n \in Z\}$ in which $X(n)$'s are i.i.d with CDF $F_{X(n)}(x) = F(x)$ Show that this is a Strict Sense Stationary Process

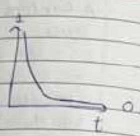
$$F_{X(t_1) X(t_2) \cdots X(t_r)}(x_1, x_2 \cdots x_n) = F_X(t_1)\ F_X(t_2) \cdots F_X(t_r)$$
$$= F(x_1)\ F(x_2) \cdots F(x_r)$$

$$F_{X(t_1+\Delta) X(t_2+\Delta) \cdots X(t_r+\Delta)}(x_1, x_2 \cdots x_r) = F_X(t_1+\Delta)\ F_X(t_2+\Delta) \cdots F_X(t_r+\Delta)$$
$$= F(x_1)\ F(x_2) \cdots F(x_r)$$

31 March

RELIABILITY

★ Reliability is the probability that a system, device or process performs its function (intended) without failure for a given time when operated correctly under stated conditions



$R(0) = 1$
$R(\infty) \to 0$

$$R(t) = P[T>t] = 1 - P[T \le t] \; ; \; t > 0$$

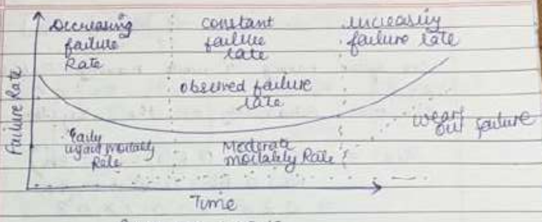
$$\therefore P(T>t) + P(T \le t) = 1$$

★ Terms [Factors of Rel.]

1. Adequate Performance
2. Time
3. operating conditions
4. Probability

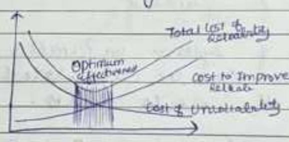
★ **Importance**

1) Gain competitive advantage
2) controlling the cost of unreliability
3) Compare the reliability & profit of other similar industries
4) Fix the price of product manufactured
5) upper limits of payments of replacement / repair



BATH TUB CURVE

★ Reliability & Cost



★ Types of Reliability

1. Series Configuration
2. Parallel Configuration
3. Series & Parallel Config^n

1 April

① Series config $\boxed{R_1} - \boxed{R_2} - \boxed{R_3} - \boxed{R_4}$ (0.3, 0.5, 0.7, 0.8)

One system having 4 units having Reliability $R_1 = 0.3$, $R_2 = 0.5$, $R_3 = 0.7$, $R_4 = 0.8$ then total Reliability of the system ?

$$R(t) = P(E_1 \cap E_2 \cap E_3 \cap \dots E_n)$$
$$P(E_1) \cap P(E_2) \cap P(E_3) \cap (E_4)$$
$$= 0.3 \times 0.5 \times 0.7 \times 0.8$$
$$= 0.084$$

If one system stops working, all systems will stop work.

2) In case of Parallel.

Reliability of system in Parallel combination can be calculated by either of the units work.

$$R(t) = P(E_1 \cup E_2 \cup E_3 \cup E_4)$$
$$P(E_1) \cup P(E_2) \cup P(E_3) \cup P(E_4)$$
$$= \cancel{0.3 + 0.5 + 0.7 + 0.8}$$
$$= 1 - P(\overline{E_1 \cup E_2 \cup E_3 \cup E_4})$$
$$1 - P(\bar{E_1} \cap \bar{E_2} \cap \dots \cap \bar{E_4})$$

$\because P A + P(\bar{A}) = 1$

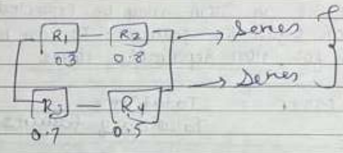
$\cancel{1 - 0.084}$ $1 - (0.7)(0.5)(0.3)(0.2)$
$\cancel{0.916}$

$\boxed{0.979}$ ✓

$f(t) \rightarrow$ Prob density func
$F(T) \rightarrow$ Cumulative distribution func.

Reliability $1 - \prod_{i=1}^{4} (1 - P(E_i))$

Q



$= R_1 + R_2 = 0.24$ $R_3\ R_4 = 0.35$

Parallel $= 1 - (0.76 \times 0.65)$

$1 - 0.494$

$\boxed{R(t) = 0.506}$

$R(t) = P(T > t) = 1 - P(T \le t)$

$f(t) = \lambda e^{-\lambda t}$ $F(t) = \int_0^t f(t)\,dt$

$\boxed{R(t) = e^{-\lambda t}}$ exponentially

Reliability in terms of Hazard Rate

$$z(t) = \lim_{\Delta t \to 0} = \frac{N_S(t) - N_S(t + \Delta t)}{N_S\, t(\Delta t)}$$

7 April

★ MTTF [Mean Time To Failure]

Average time an item may be expected to function before the failure. It can be used only for non-repairable items.

$$\boxed{MTTF = \frac{\text{Total Time}}{\text{Total No. of failures}}}$$

For eg:- We have 4 bulbs each has been working for 3000, 4000, 5000, 4000 hours before failure. From this we take an average which will be MTTF. Therefore $MTTF = \frac{16000}{4} = 4000$

★ Mean Time Between Failure [MTBF]

Time between two failures is called MTBF. It can be used for both repairable as well as non-repairable items.

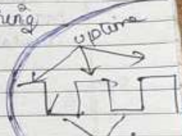
For eg: If total device hours 20,000 hours & no of failures = 5 then

$$\boxed{MTBF = \frac{\text{Total Device Hours}}{\text{No of failure}}}$$

$$MTBF = \frac{20,000}{5} = 4000 \text{ hrs}$$

★ Mean Time To Repair [MTTR]

It is average time to repair something



$$MTBF = \frac{\text{total uptime}}{\text{No of Breakdowns}}$$

$$MTTR = \frac{\text{total downtime}}{\text{no of breakdown}}$$

$$\text{Availability} = \frac{\text{Total uptime}}{\text{Total uptime} + \text{Total Downtime}}$$

$$\text{Availability} = \frac{MTBF}{MTBF + MTTR}$$

FAILURE RATE (λ) USING MTTF, MTBF, MTTR

$\lambda = \frac{1}{MTTF}$ for Non Repairable Products

$\lambda = \frac{1}{MTBF}$ for Repairable Products

$$MTBF = MTTF + MTTR$$

8 April

Q1. A product has MTBF of 200 hours & a MTTR of 10 hours. What is system availability

Sol. ∴ Avail. $= \dfrac{MTBF}{MTBF + MTTR}$

$= \dfrac{200}{210} = \boxed{0.95}$ Ans

Q2. If MTTF is 3 years & MTTR is 1 day Calculate system Availability ?

Sol. $MTBF = MTTF + MTTR$
$1095 + 1 = 1096$

$\text{Availability} = \dfrac{1096}{1079} = \boxed{0.9990}$ Ans

Q3. The mean time b/w failure MTBF of a machine is 400 hours. If Availability of Machine is 80% find the MTTR in hours

$\dfrac{\cancel{0.80}\ 80}{100} = \dfrac{400}{400 + x}$

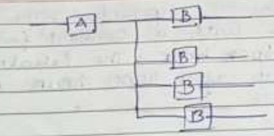
$80(400 + x) = 40000$

$32000 + 80x = 40000$

$80x = 8000$

$\boxed{x = 100}$ hours

Q4.

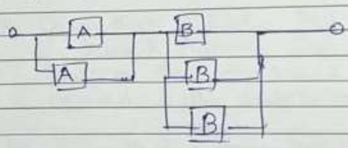


$P(A) \times \left[1 - P(B)_i\right]$

$= P(A) \times \left[1 - (1 - P_B)(1 - P_B)(1 - P_B)(1 - P_B)\right]$

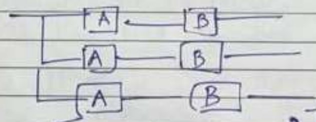
$= P(A)\left[1 - (1 - P_B)^4\right]$ Ans

Q5.



Sol. $\left[1 - (1 - P_A)^2\right]\left[1 - (1 - P_B)^3\right]$

Q6.



Sol → $\left[1 - (P(A)\ P(B))^3\right] \rightarrow 1 - (1 - PA \cdot PB)^3$

1 − P

6. A product has an exponential time to failure distribution with a constant failure rate of 0.00006 per hour. The Reliability of the product after 4000 hours of operation is ________.

$$\text{Reliability} = e^{-\lambda t}$$

$\lambda \to 0.00006$, $t \to 4000$

$$= e^{-0.24} \quad \text{Ans}$$

A process that changes with time | we have to measure Eg stock market

STOCHASTIC PROCESS

System that changes over time in an <u>uncertain manner</u> is called a stochastic process. For eg. In a certain market there are 3 brands of Lipsticks A, B & C

⇒ Given that a lady last purchased Lipstick of brand A. Assume that there are 70% chances that she would continue with Brand A. 20% & 10% chances that she would shift to Brand B & C respectively

⇒ Given that a lady last purched Lipstick of Brand B Assume that there is 50% chances that she would shift to Brand A & 10% Chance to Brand C

⇒ Given that a lady last purchased lipstick of Brand C there is 60%, 20% chance that she would Shift to Brand A & B respectively.

What are the Market Shares of 3 brands at the end of the year?

In all these cases the most imp thing is the "CHANGE", the behaviour of the factor / states over the time

STATE - snapshot of the system at some fixed point of time is called state

TRANSITION - movement from one state to another

Hence we are interested in How a Random Variable changes over-time.

Note:- In prob theory A Markov Model is a Statistical Model used model randomly changing system over time

9 April

Let $\{X_i ; n = 0, 1, 2 \ldots\}$ be a stochastic process that takes on a finite or countable no. of possible values

$$\boxed{X_n = i} \leftarrow \text{state} \rightarrow \text{Markov's Representation}$$

time

It means that process is said to be in state i at time n

$X_3 = B \quad X_4 = A$

A lady will purchase lipstick Brand B after the time period 2

In Markov Model, 2 terms play an imp role

1. State
2. Transition probability

For eg:-

Present → Next Day.

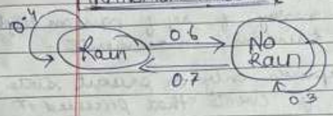
Rain → Rain 40%

Rain → No Rain 60%

No Rain → Rain 70%

No Rain → 30%

TRANSITION MODEL



TRANSITION MATRIX

$$\begin{array}{c|cc} & \text{Rain} & \text{No Rain} \\ \hline \text{Rain} & 0.4 & 0.6 \\ \text{No Rain} & 0.7 & 0.3 \end{array}$$

MARKOV CHAIN

It is the process $X_1, X_2 \ldots x_n$ where system states are observable & fully autonomous

Markov Property or Memory Less Property

The basic property of the Markov Chain is that X_{t+1} depends upon X_t but it does not depends upon $X_{t-1}, X_{t-2}, \ldots, X_1, X_2$. Mathematically the Markov property is stated as

Markov Property is stated as :

$$P(X_{t+1} = j / X_t = i, X_{t-1} = i_1, \ldots X_0 = i_0)$$

$$= P(X_{t+1} = j / X_t = i) \ \forall \ t = 1, 2, 3 \ldots$$ for all states

For eg- Random Walk

H.W. What is Random Walk & How it is Related to Markov Property ?

Ans- It is a path consisting of seq of random steps. Sequence of Random movements.

Future state depends only on present state not on sequence of events that preceded it.

Random Walk satisfies this property bcoz next step is only determined by current position.

Transition Probabilities

Prob. from state i to state j after one step time period denoted by $P_{ij} = P\{X_{n+1} = j / X_n = i\}$

Eg. $P_{23} = \{X_{n+1} = 3 / X_n = 2\}$

$P_{13} \ \{X_3 = 3 / X_2 = 1\}$

N Step Probability →

Probabilities from state i to state j after n step time period denoted by $P_{ij}(n)$ or $P_{ij}^{(n)}$ is defined as

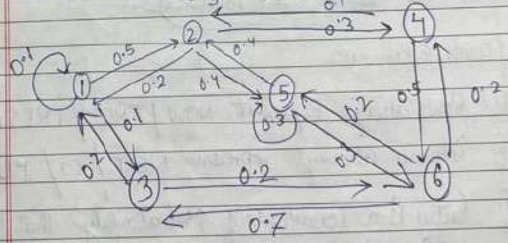
$$P_{ij}(n) = \{X_{n+1} = j / X_1 = i\}$$

Eg. $\{X_3 = 4 / X_0 = 2\} \longrightarrow P_{24}(3)$ → difference of time Ans

Eg $\{X_2 = 2 / X_1 = 0\} \rightarrow P_{02}(1)$ Ans.

Eg $\{X_6 = 1 / X_2 = 0\} \longrightarrow P_{01}(4)$ Ans

Q. The Transition Model of a system is given below. Find $P(X_3 = 6, X_2 = 4, X_1 = 2, X_0 = 5)$



10 April

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0 & 0.1 & 0.3 & 0 \\ 0.1 & 0 & 0.2 & 0 & 0 & 0.7 \\ 0 & 0.3 & 0 & 0.2 & 0 & \boxed{0.5} \\ 0 & 0.7 & 0 & 0 & 0.3 & 0.3 \\ 0 & 0 & 0.2 & 0.3 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

$P(X_3=6 / X_2=4, X_1=2, X_0=5) = P(X_3=6 / X_2 ...$

$= \boxed{0.5}$ Ans.

Note → In other Words Transition Matrix [TPM] is the matrix describing Markov Chain. The matrix is said to be TPM if

* It is a SQUARE MATRIX
* $P_{ij} \geq 0 \ \forall i, j$
* $\Sigma P_{ij} = 1$ (rowwise)

Points To Remember

1. Rows always Represent NOW / FROM / PRESENT
2. Column always represent NEXT / TO / FUTURE
3. Entry is a conditional probability that the next state j given that now = i

Q A Market Survey is made on 2 Brands of Breakfast food A & B. Everytime a Customer purchases He/she buy the same brand or switch to another Brand. If the probability of Buying Brand B after Brand A is 0.2 & prob of buying Brand A after Brand B is 0.6. Find the TPM?

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Q Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will be Coke. If a person's last cola purchase was Pepsi there is 80% chance that his next cola purchase will also be pepsi. Construct TPM

$$\begin{array}{c} \\ C \\ P \end{array}\begin{array}{c} \begin{array}{cc} C & P \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

Q. Three Boys A, B, C are throwing a ball to each other A always throws ball to B & B always throws ball to C But C is as likely to throw the ball to B as to A. Find the TPM?

Sol.

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

NOTE $P(X_3=6, X_2=4, X_1=2)$

$= P(X_3=6 / X_2=4, X_1=2)\ P(X_2=4, X_1=2)$

$P(X_3=6 / X_2=4, X_1=2)\ P(X_2=4 / X_1=2)\ P(X_1=2)$

$P(X_3=6 / X_2=4)\ P(X_2=4 / X_1=2)\ P(X_1=2)$

$P_{46} \times P_{24}\ P(X_1=2)$

Formula $\boxed{P(A,B) = P(A/B)\ P(B)}$

Q. For the given TPM

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

with

$P[X_0=i] = \frac{1}{3}$, $i = 0, 1, 2$

find $P[X_2=2, X_1=1, X_0=2]$

Sol. $P(X_2=2 / X_1=1, X_0=2)\ P(X_1=1, X_0=2)$

$P(X_2=2 / X_1=1, X_0=2)\ P(X_1=1 / X_0=2)\ P(X_0=2)$

$P_{12}^{(1)} \cdot P_{21}^{(1)}\ P(X_0=2)$

$\frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{3} = \boxed{\frac{1}{16}}$ Ans.