Bernoulli :- single arrival, $\{{1 \atop 0}\}$, discrete time.

| | 0 | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|---|
| $z_t$ | 0 | 1 | 0 | 0 | 1 | | ... |
| | q | p | q | q | p | | |

$N_t$ = # of arrivals till time t. $N_1 = Z_1 = 1$

$N_2 = Z_1 + Z_2 = 1$

$$N_t = \sum_{i=0}^{t} Z_i \sim Binomial(t, p)$$

$E(x) = \sum_{x=0}^{n} x \, {}^nC_x p^x (1-p)^x =$

$E((X_1+X_2)^2) - (E(X_1+X_2))^2$

$E(X_1^2 + \cancel{X_2}^2 + 2x_1x_2) - (E(x_1)+E(x_2))^2$

$E(x_1^2) + E(x_2^2) + 2E(x_1x_2) - E(x_1)^2 + E(x_1)^2$
$+2E(x_1)E(x_2)$

$var(x_1) + var(x_2)$

$+2E(x_1x_2)$
$+2E(x_1)(x_2)$

$Z_t \overset{iid}{\sim}$ Bernoulli

$E(Z_t) = p$

$Var(Z_t) = pq$

$N_t \overset{iid}{\sim}$ Binomial

$E(N_t) = E(\sum Z_i) = \sum E(Z_i) = t(p)$

$Var(N_t) = E((N_t)^2) - (E(N_t))^2$

$\{ var(X_1 + X_2) = V(X_1) + V(X_2) + 2\,Cov(X_1, X_2) \}$

$= \sum V(Z_i) + 2 \sum_i \sum_j (Cov(Z_i Z_j))$ } Cov is 0 for independent vars.

$= \sum V(Z_i)$

$= t\,pq$

Mgf → uniquely identifies a distribution.

$\frac{d}{ds} M_y(s) \Big|_{s=0} = E(Y)$ $\qquad \frac{d^2}{ds^2} M_y(s) \Big|_{s=0} = E(Y^2)$

mgf of Bernoulli

$$M_X(s) = E[e^{Xs}] = \sum_X e^{xs} P(X=x) = (q+pe^s)$$

$$Z_t \sim \text{Bernoulli}(p)$$

$$M_{Z_t}(s) = (q+pe^s)$$

$$N_t \sim \text{Binomial}(t,p)$$

$$M_{N_t}(s) \overset{(i)}{=} E[e^{N_t s}] = \sum_k e^{N_t s} P(N_t = k)$$

$$\overset{(ii)}{=} E[e^{N_t s}]$$

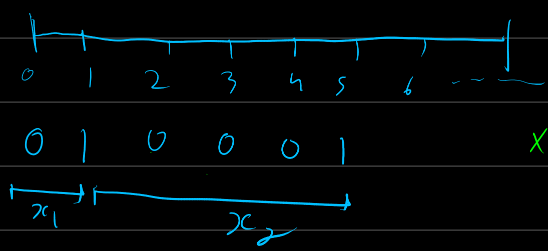
$$= E[e^{(\sum_i Z_i)s}]$$

$$= E[e^{Z_0 s} + e^{Z_1 s} + e^{Z_2 s} \cdots e^{Z_t s}]$$

$\{\because Z_i\text{'s are independent}\}$

$$= \prod_{i=0}^{t} E[e^{Z_i s}]$$

$$= \prod_{i=0}^{t} M_{Z_i}(s) = (p+qe^s)^t$$



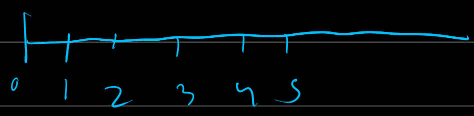

$X_1$: first inter arrival time.

$X_2$: $2^{nd}$ inter arrival time.

⋮

$X_k$: $k^{th}$ interarrival time.

Sum of inter arrival times
$$\begin{cases} S_1 = X_1 \\ S_2 = X_1 + X_2 \\ S_3 = X_1 + X_2 + X_3 \\ \vdots \\ S_k = \sum_{j=1}^{k} X_i \end{cases}$$




$\{X_1 = 1\} = \{Z_1 = 1\} = p$

$X_1 \quad \underbrace{0 \ 1}$

$\{X_2 = 1\} = \{Z_1 = 0, Z_2 = 1\} = pq$

$E(X_1) = 1/p$

$X_2 \quad \underbrace{0 \ 0 \ 1}$

$\{X_k = 1\} = pq^{k-1}$

$Var(X_1) = \dfrac{q}{p^2}$

$M_{X_1}(s) = \dfrac{pe^s}{1 - qe^s}$