Poisson Process.

$N(t) = \#$ of arrivals in $(0, t]$ $\rightarrow$ Poisson$(\lambda t)$

$X_i$ = inter-arrival times $\overset{iid}{\sim}$ Exponential$(\lambda)$

$S_n = \sum X_i$ : Sum of inter-arrival times $\sim$ Erlang$(n, \lambda)$

$$f_{x_1, x_2}(x_1, x_2) = \prod_{i=1}^{2} f_{x_i}(x_i) = f_{x_1}(x_1) \cdot f_{x_2}(x_2)$$

$$f_{x_1 \cdots x_n}(x_1 \cdots x_n) = \prod_{i=1}^{n} f_{x_i}(x_i)$$

$S_i$'s are not independent

$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$

$$f_{S_1, S_2}(S_1, S_2) = f_{S_2|S_1}(S_2|S_1) \cdot f_{S_1}(S_1)$$

$P(X \cap Y) = P(X|Y)P(Y)$

$$F_{S_2|S_1}(s_2/s_1) = P(S_2 \leq s_2 \mid S_1 = s_1)$$

$f_{S_1 \cdots S_n}(s_1 \cdots s_n)$

Let $X_i \overset{iid}{\sim}$ Exp$(\lambda)$ with density $f^n$ $f_{X_i}(x_i) = \lambda e^{-\lambda x_i}$, $x_i \geq 0$, $\lambda > 0$

$S_n = \sum_{i=1}^{n} X_i$ for each $n \geq 1$. Then for each $n \geq 2$, $f_{S_1 \cdots S_n}(s_1 \cdots s_n) = \lambda^n e^{-\lambda s_n}$

ie $0 \leq s_1 \leq s_2 \leq s_3 \cdots s_n$

Proof by induction.

for $n = 2$

$$f_{S_1 S_2}(s_1, s_2) = \lambda^2 e^{-\lambda s_2}$$

Let's assume the result holds true for n,

$$f_{S_1 \dots S_n}(s_1 \dots s_n) = \lambda^n e^{-\lambda s_n}$$

for k = n+1

$$f_{S_1 \dots S_n, S_{n+1}}(s_1 \dots s_n, s_{n+1}) = f_{S_{n+1}|S_1 \dots S_n} \cdot f_{S_1 \dots S_n}$$

$$= f_{S_{n+1}|S_1 \dots S_n}(s_{n+1}/s_1 \dots s_n) \cdot f_{S_1 \dots S_n}(s_1 \dots s_n)$$

{ to find conditional, we use CDF }

$$F_{S_{n+1}|S_1 \dots S_n}(s_{n+1}/s_1 \dots s_{n+1}) = P(S_{n+1} \le s_{n+1} \mid S_1 = s_1, \dots S_n = s_n)$$

$$= P(X_{n+1} + S_n \le s_{n+1} \mid S_1 = s_1 \dots S_n = s_n)$$

$$= P(X_{n+1} \le s_{n+1} - s_n \mid S_1 = s_1 \dots S_n = s_n)$$

$$= \frac{P[(X_{n+1} \le s_{n+1} - s_n) \cap (S_1 = s_1, S_2 = s_2 \dots S_n = s_n)]}{P(S_1 = s_1 \dots S_n = s_n)}$$

$$= P(X_{n+1} \le s_{n+1} - s_n) = F_{X_{n+1}}(s_{n+1} - s_n)$$

$$= 1 - e^{-\lambda(s_{n+1} - s_n)}$$ diff. wrt $s_{n+1}$

$$f_{S_{n+1}|S_1 \dots S_n}(s_{n+1}|s_1 \dots s_n) = \lambda e^{-\lambda(s_{n+1} - s_n)}$$

$$f_{S_1 \dots S_{n+1}}(s_1 \dots s_{n+1}) = f_{S_{n+1}|S_1 \dots S_n}(s_{n+1}|s_1 \dots s_n) \cdot f_{S_1 \dots S_n}(s_1 \dots s_n)$$

$$= \lambda e^{-\lambda(s_{n+1} - s_n)} \cdot \lambda^n e^{-\lambda s_n}$$

$$= \lambda^{n+1} e^{-\lambda s_{n+1}}$$

* Try solving for $\lambda_i$'s → non iid $X_i$'s. | * have join, work with margins

$$M_{S_n}(s) = E[e^{\sum X_i t}] = \left(\frac{\lambda}{\lambda - s}\right)^n$$

Result : $\{S_n \le t\} = \{N(t) \ge n\}$ → trivial.

$$\{N(t) \ge n\} = \{S_n \le t\}$$

$\{P(S_n \le t) = P(N(t) \ge n)\}$ → either can be easy

$$\{S_n > t\} = \{N(t) < n\}$$

eg $\lambda = 1$ per hour.

(i) P( atmost 1 acciden in $[0,5]$ & atleast 2 accidents in $[5,10]$ & ~~ext~~ exactly 1 acc in $(10,13)$ )

$$P(N(5) \le 1 \cap \tilde{N}(5,10) \ge 2 \cap \tilde{N}(10,13) = 1)$$

$$P(N(5) \le 1) \cdot P(\tilde{N}(5,10) \ge 2) \cdot P(\tilde{N}(10,13) = 1)$$

Stationary increment.

$$P(N(5) \le 1) \cdot P(N(5) \ge 2) \cdot P(N(3) = 1)$$

$$\left(\sum_{i=0}^{1} \frac{e^{-\lambda 5}(\lambda 5)^i}{i!}\right) \cdot \left(\sum_{i=2}^{\infty} \frac{e^{-\lambda 5}(5\lambda)^i}{i!}\right) \cdot \left(\frac{e^{-3\lambda} \cdot 3\lambda}{3!}\right)$$

ii $P($2 arr in $[0,5)$ & atmost 3 in $[0,7])$

= 2 in $[0,5]$ & $\le 1$ in $[5,7]$

$$(N(5) = 2) \cap (N(2) \le 1)$$

(iii) P( third arrival occurs after 8)

$P(S_3 > 8)$ or $P(N(8) < 3)$

2 Quests → 1 week. deadline = 17. GPT

$S_1 = X_1$

$S_2 = X_1 + X_2 = S_1 + X_2$

$S_3 = X_1 + X_2 + X_3 = S_2 + X_3$

$\vdots$

$S_n = X_1 + \dots + X_n = S_{n-1} + X_n$

$S_1 < S_2 < S_3 < \dots < S_n$

$S_i$'s ~~are~~ not indep

$f_{S_1,S_2}(s_1, s_2) = ?$

$F_{S_1,S_2}(s_1, s_2) = ??$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$P(S_2|S_1) = \dfrac{P(S_2 \cap S_1)}{P(S_1)}$

$f_{S_2|s_1} = \dfrac{f_{S_2,s_1}}{f_{s_1}}$

$f_{S_1,S_2}(s_1, s_2) = f_{S_2|s_1}(s_2|s_1)\, f_{S_1}(s_1) = \lambda e^{-\lambda(s_2 - x_1)} \lambda e^{-\lambda x_1} = \lambda^2 e^{-\lambda s_2}$

$F_{S_2|S_1}(s_2|s_1) = P(S_2 \le s_2 \mid S_1 = s_1)$

$= P(X_1 + X_2 \le s_2 \mid X_1 = x_1)$

$= P(X_2 \le s_2 - x_1 \mid X_1 = x_1)$

$= \dfrac{P(X_2 \le s_2 - x_1 \cap X_1 = x_1)}{P(X_1 = x_1)}$

$= \dfrac{P(X_2 \le s_2 - x_1)\, P(X_1 = x_1)}{P(X_1 = x_1)} = P(X_2 \le s_2 - x_1) = F_{X_2}(s_2 - x_1)$

$= 1 - e^{-\lambda(s_2 - x_1)}$

$f_{S_2|s_1}(s_2|s_1) = \lambda e^{-\lambda(s_2 - x_1)}$