
Proof 1.3 Escape Condition Proof Using Triangle Inequality

We start with the Mandelbrot iteration:

$$z_{n+1} = z_n^2 + c$$

Step 1: Apply Triangle Inequality

For any complex numbers a, b:

$|a + b| \le |a| + |b|$

$|a + b| \ge ||a| - |b||$

Now substitute $a= z_n^2$, $b = c$:

$$|z_{n+1}|= | z_n^2 + c | \ge ||z_n^2| - |c||$$

Since $|a^2|=|a|^2$

This becomes $|z_{n+1}| \ge ||z_n|^2 - |c||$

Step 2 — Removal of the outer modulus (two cases, solved separately)

From Step 1 we have the basic triangle-inequality bound

$|z_{n+1}| \ge ||z_n|^2 - |c||$

Let $u := |z_n|^2$ and $v := |c|$

The bound is

$$|z_{n+1}| \ge |u - v|$$

Expand absolute value

By the definition of absolute value, we split into two cases:

- **Case 1:** If $u \ge v$, then

  $$|z_{n+1}| \ge u-v$$

- **Case 2:** If $u< v$, then

  $$|z_{n+1}| \ge v-u$$

Return to original notation

- **Case 1:**

  $|z_{n+1}| \geq |z_n|^2 - |c|$ **if** $|z_n|^2 >= |c|$

- **Case 2:**

  $|z_{n+1}| \geq |c| - |z_n|^2$ **if** $|z_n|^2 < |c|$

Step 3a: Solving case 1 where u ≥v (i.e. $|z_n|^2 >= |c|$**)**

We have: $|z_{n+1}| \geq |z_n|^2 - |c|$

From *Proof 1.1* we already established that if $|c|>2$ then c is not in the Mandelbrot set. Therefore, when testing points that *may* belong to the Mandelbrot set, we may restrict to $|c| \leq 2$.

To obtain a simple, uniform lower bound (the worst-case subtraction) put $|c|=2$:

$|z_{n+1}| \geq |z_n|^2 - 2$

Step 3a.1: Solve the quadratic equation

From Case 3a (the branch where the outer modulus was removable) we derived the uniform lower bound

$$|z_{n+1}| \geq |z_n|^2 - 2$$

A sufficient condition for one-step growth is

$$|z_{n+1}| > |z_n|$$

Using the bound above it suffices that

$$|z_n|^2 - 2 > |z_n|$$

Let $r := |z_n|$ This becomes the real quadratic inequality

$$r^2 - r - 2 > 0$$

Solve the equality $r^2 - r - 2 = 0$. Factorize:

$r^2 - r - 2 = (r-2)(r+1)$

Hence roots of the equation are $r^2 - r - 2$ are $r=-1$ and $r=2$.

Because $r=|z_n|$ **is modulus** then left interval is impossible, so the only relevant condition is $r>2$.

Therefore:

If $|c| \le 2$ and $|z_n| > 2 \Longrightarrow |z_{n+1}| > |z_n|$ always.

Thus, point will grow in next iteration.

Step 3b: Solving case 1 where u <v (i.e. $|z_n|^2 < |c|$)

We have: $|z_{n+1}| \ge |c| - |z_n|^2$

From *Proof 1.1* we already established that if **$|c|>2$** then **c** is not in the Mandelbrot set. Therefore, when testing points that *may* belong to the Mandelbrot set, we may restrict to **$|c| \le 2$**.

To obtain a simple, uniform lower bound (the worst-case subtraction) put $|c|=2$:

$|zn+1| \ge 2-|z_n|^2$

Step 3b.1— Solve the quadratic equation

From Case 3b (the branch where the outer modulus was removable) we derived the uniform lower bound

$$|z_{n+1}| \ge 2-|z_n|^2$$

A sufficient condition for one-step growth is

$$|z_{n+1}| > |z_n|$$

Using the bound above it suffices that

$$2-|z_n|^2 > |z_n|$$

Let **$r := |z_n|$** This becomes the real quadratic inequality

$$2-r^2-r<0$$

$$r^2+r-2<0$$

Solve the equality $r^2+r-2=0$. Factorize:

$r^2+r-2=(r+2)(r-1)$

Hence roots of the equation are **r^2+r-2** are **$r=-2$** and **$r=1$.**

Because **$r=|z_n|$ is modulus** then left interval is impossible, so the only relevant condition is **$r<1$**.

Therefore:

If $|c| \leq 2$ and $|z_n| < 1 \Longrightarrow |z_{n+1}| > |z_n|$ always.

Thus, point will grow in next iteration.

Step 4: Growth Regions (One-Step Growth)

From Steps **3a** and **3b**, one-step growth occurs when **$|z_n| < 1$** or **$|z_n| > 2$**.

Number-line visualization:

$|z_n| = 0$ $|z_n| = 1$ $|z_n| = 2$ $|z_n| > 2$
|---------------------------|-------------------------|--------------------|------------------->

Small growth Uncertain growth Critical point Growth occurs

Explanation:

- **$|z_n| < 1$ → next iterate increases (Case 3b)**
- **$1 \leq |z_n| \leq 2$ → growth not guaranteed in one step**
- **$|z_n| = 2$ → critical threshold**
- **$|z_n| > 2$ → next iterate increases (Case 3a)**

Conclusion:

Based on the triangle inequality, one-step growth occurs when **$|z_n| < 1$** or **$|z_n| > 2$**.
In the case **$|z_n| > 2$**, the next step will always increase and the point will eventually reach infinity.
For **$|z_n| < 1$**, growth occurs only for the next step and is not guaranteed beyond that as it may or may not grow between 1 and 2.
Therefore, $|z_n| > 2$ can be used as the escape threshold.
