Injecting a signal into a $\mathcal{M}$-bit quantised data stream

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Assuming that the background noise is represented by a Gaussian distribution, the probability density function (pdf) of the noise $\mathcal{N}$ is:

$$P(\mathcal{N}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\mathcal{N}-\mu)^2}{2\sigma^2}}$$

where the noise level σ is given by the **radiometer equation** for a single antenna:

$$\sigma = \frac{T_{sys}}{G\sqrt{N_{\mathrm{pol}}\delta t \Delta\nu}}$$

The cumulative distribution function of noise is given by:

$$\phi(\mathcal{N}) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mathcal{N}-\mu}{\sigma\sqrt{2}}\right)\right]$$

For a given signal $\mathcal{S}$ injected to data samples, we get digitized values ranging from 0 to $2^{\mathcal{M}} - 1$ for the $\mathcal{M}$ bit case. The probability matrix for digit changes upon injecting a signal would be:

$$P_{ij} = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0(2^{\mathcal{M}}-1)} \\ 0 & P_{11} & \cdots & P_{1(2^{\mathcal{M}}-1)} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & P_{(2^{\mathcal{M}}-2)(2^{\mathcal{M}}-1)} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where P_{nm} is the probability to change the bit from state n to the state m and is given by:

$$P_{nm} = \frac{\text{Probability to go from state } n \text{ to state } m}{\text{Probability to remain in original state } n}$$

Say we have $\mathcal{M}$ bit data; thus there are $2^{\mathcal{M}}$ levels. Consider state n lying between $\mu + il\sigma$ and $\mu + (i+1)l\sigma$, where μ is mean of Gaussian noise and l is level setting number for digitization. Here, i runs from $-(\frac{2^{\mathcal{M}}}{2} - 1)$ to $+(\frac{2^{\mathcal{M}}}{2} - 1)$ and n varies from 0 to $2^{\mathcal{M}} - 1$. As shown in the figure (1), the probability of bit shift can be calculated as,

$$P_{n\to n+m} = \frac{\text{Area under pdf between intersection of states } n \text{ and } n+m}{\text{Area under pdf up to state } n}$$

$$\boxed{P_{n\to n+m} = \frac{\phi(\min(\mu + (i+m)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+m-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)}} \quad (1)$$

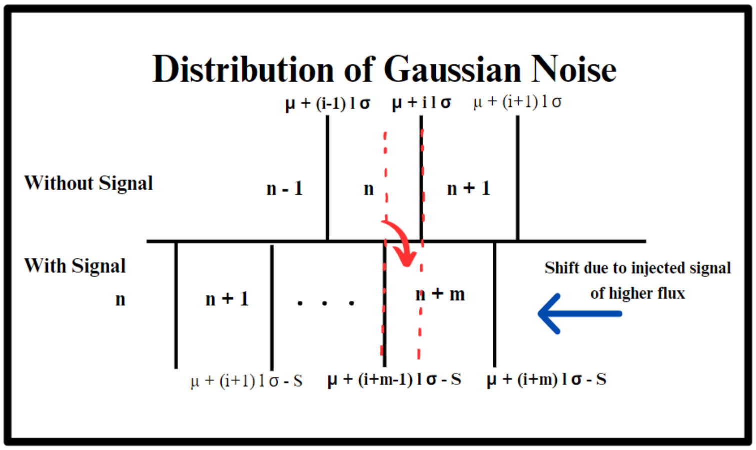

Figure 1: Distribution of Gaussian noise

where n and i are related as $n - i = \frac{2^{\mathcal{M}}}{2} - 1$.

The first state (0) extends from $-\infty$ to $\mu - (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma$ and the last state $(2^{\mathcal{M}} - 1)$ extends from $\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma$ to $+\infty$. Therefore, we need to accommodate these variation at the boundary states. This can be done by modifying above formula as,

$$P_{0\to 0} = \frac{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{2}$$

$$P_{0\to m} = \frac{\phi(\min(\mu + (m + 1 - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S}, \mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)) - \phi(\mu + (m - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{3}$$

$$P_{n\to 2^{\mathcal{M}}-1} = \frac{\phi(\mu + (n + 1 - \frac{2^{\mathcal{M}}}{2})l\sigma) - \phi(\max(\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma - \mathcal{S}, \mu + (n - \frac{2^{\mathcal{M}}}{2})l\sigma))}{\phi(\mu + (n + 1 - \frac{2^{\mathcal{M}}}{2})l\sigma) - \phi(\mu + (n - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{4}$$

$$P_{0\to 2^{\mathcal{M}}-1} = \frac{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma) - \phi(\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} = 1 - \frac{\phi(\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{5}$$

and,

$$P_{2^{\mathcal{M}}-1\to 2^{\mathcal{M}}-1} = 1 \tag{6}$$

Equation (1) to (6) are general formulas to calculate probability of bit change from one state to other when a signal $\mathcal{S}$ is injected into the quantized data stream.

Now, **for the 8-bit case**, we have $n - i = \frac{2^8}{2} - 1 = 127$. Thus,

$$P_{n\to n+m} = \frac{\phi(\min(\mu + (n + m - 127)l\sigma - \mathcal{S}, \mu + (n - 127)l\sigma)) - \phi(\max(\mu + (n + m - 128)l\sigma - \mathcal{S}, \mu + (n - 128)l\sigma))}{\phi(\mu + (n - 127)l\sigma) - \phi(\mu + (n - 128)l\sigma)}$$

The formula is appropriate given we are away from boundary states (the first and the last state). Probability of changing bit from 0 to 0 and 255 to 255, is

$$P_{0\to 0} = \frac{\phi(\mu - 127l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)} \text{ and } P_{255\to 255} = 1\text{, respectively.} \tag{7}$$

For bit change from state 0 to any higher state m, the formula modifies as

$$P_{0\to m} = \frac{\phi(\min(\mu + (m-127)l\sigma - \mathcal{S}, \mu - 127l\sigma)) - \phi(\mu + (m-128)l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)}$$

Similarly, for bit change from any non-zero state n to the highest state 255, the probability is

$$P_{n\to 255} = \frac{\phi(\mu + (n-127)l\sigma) - \phi(\max(\mu + 127l\sigma - \mathcal{S}, \mu + (n-128)l\sigma))}{\phi(\mu + (n-127)l\sigma) - \phi(\mu + (n-128)l\sigma)}$$

Lastly, the probability for bit change from the lowest state 0 to the highest state 255 is

$$P_{0\to 255} = \frac{\phi(\mu - 127l\sigma) - \phi(\mu + 127l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)} = 1 - \frac{\phi(\mu + 127l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)}$$

For a 1-bit case, $\mathcal{M}$ is 1 and thus there are only 2 levels 0 and 1. So, n and i are equal for 1-bit data. Only possible transition between these two states is 0 to 1, and the resulting probability from equation 5 is

$$P_{0\to 1} = \frac{\phi(\mu + (1 - \frac{2^1}{2})l\sigma) - \phi(\mu + (\frac{2^1}{2} - 1)l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^1}{2})l\sigma)} = 1 - \frac{\phi(\mu - \mathcal{S})}{\phi(\mu)} = 1 - \frac{\phi(\mu - \mathcal{S})}{0.5} \tag{8}$$

$$P_{0\to 0} = 1 - P_{0\to 1} = \frac{\phi(\mu - \mathcal{S})}{0.5} \tag{9}$$

$$P_{1\to 0} = 0; P_{1\to 1} = 1 \tag{10}$$

For a 2-bit case, $\mathcal{M}$ is 2 and thus there are 4 levels namely 0, 1, 2, and 3. In this case, n and i are related as $n - i = \frac{2^2}{2} - 1 = 1$.

Using equation (1),

$$\begin{aligned} P_{1\to 1} &= \frac{\phi(\min(\mu + (i+1)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+1-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)} \qquad (11) \\ &= \frac{\phi(\mu - \mathcal{S}) - \phi(\mu - l\sigma)}{\phi(\mu) - \phi(\mu - l\sigma)} \qquad (12) \\ &= \frac{\phi(\mu - \mathcal{S}) - \phi(\mu - l\sigma)}{0.5 - \phi(\mu - l\sigma)} \qquad (13) \end{aligned}$$

$$\begin{aligned} P_{1\to 2} &= \frac{\phi(\min(\mu + (i+1)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+1-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)} \qquad (14) \\ &= \frac{\phi(\min(\mu + l\sigma - \mathcal{S}, \mu)) - \phi(\max(\mu - \mathcal{S}, \mu - l\sigma))}{\phi(\mu) - \phi(\mu - l\sigma)} \qquad (15) \\ &= \frac{\phi(\min(\mu + l\sigma - \mathcal{S}, \mu)) - \phi(\max(\mu - \mathcal{S}, \mu - l\sigma))}{0.5 - \phi(\mu - l\sigma)} \qquad (16) \end{aligned}$$

and,

$$P_{2\to 2} = \frac{\phi(\min(\mu + (i+1)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+1-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)} \tag{17}$$

$$= \frac{\phi(\mu + l\sigma - \mathcal{S}) - \phi(\mu)}{\phi(\mu + l\sigma) - \phi(\mu)} \tag{18}$$

$$= \frac{\phi(\mu + l\sigma - \mathcal{S}) - 0.5}{\phi(\mu + l\sigma) - 0.5} \tag{19}$$

Using equation (2),

$$P_{0\to 0} = \frac{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} = \frac{\phi(\mu - l\sigma - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{20}$$

From equation (3),

$$P_{0\to m} = \frac{\phi(\min(\mu + (m-1)l\sigma - \mathcal{S}, \mu - 1l\sigma)) - \phi(\mu + (m-2)l\sigma - \mathcal{S})}{\phi(\mu - 1l\sigma)}$$

$$P_{0\to 1} = \frac{\phi(\min(\mu - \mathcal{S}, \mu - l\sigma)) - \phi(\mu - l\sigma - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{21}$$

$$P_{0\to 2} = \frac{\phi(\min(\mu + l\sigma - \mathcal{S}, \mu - l\sigma)) - \phi(\mu - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{22}$$

from equation (4).

$$P_{n\to 3} = \frac{\phi(\mu + (n-1)l\sigma) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu + (n-2)l\sigma))}{\phi(\mu + (n-1)l\sigma) - \phi(\mu + (n-2)l\sigma)}$$

$$P_{1\to 3} = \frac{\phi(\mu) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu - l\sigma))}{\phi(\mu) - \phi(\mu - l\sigma)} = \frac{0.5 - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu - l\sigma))}{0.5 - \phi(\mu - l\sigma)} \tag{23}$$

$$P_{2\to 3} = \frac{\phi(\mu + l\sigma) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu))}{\phi(\mu + l\sigma) - \phi(\mu)} = \frac{\phi(\mu + l\sigma) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu))}{\phi(\mu + l\sigma) - 0.5} \tag{24}$$

from equation (5)

$$P_{0\to 3} = \frac{\phi(\mu - l\sigma) - \phi(\mu + l\sigma - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{25}$$

and lastly from equation (6)

$$P_{3\to 3} = 1 \tag{26}$$

Equations (8) to (26) match with the probability formulas for injection in 1-bit and 2-bit data given in the paper Luo et al. [2022], Appendix B.

References

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