Injecting a signal into a $\mathcal{M}$-bit quantised data stream

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Assuming that the background noise is represented by a Gaussian distribution, the probability density function (pdf) of the noise $\mathcal{N}$ is:

$$P(\mathcal{N}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\mathcal{N}-\mu)^2}{2\sigma^2}}$$

where the noise level σ is given by the **radiometer equation** for a single antenna:

$$\sigma = \frac{T_{sys}}{G\sqrt{N_{\mathrm{pol}}\delta t \Delta\nu}}$$

The cumulative distribution function of noise is given by:

$$\phi(\mathcal{N}) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mathcal{N}-\mu}{\sigma\sqrt{2}}\right)\right]$$

For a given signal $\mathcal{S}$ injected to data samples, we get digitized values ranging from 0 to $2^{\mathcal{M}} - 1$ for the $\mathcal{M}$ bit case. The probability matrix for digit changes upon injecting a signal would be:

$$P_{ij} = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0(2^{\mathcal{M}}-1)} \\ 0 & P_{11} & \cdots & P_{1(2^{\mathcal{M}}-1)} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & P_{(2^{\mathcal{M}}-2)(2^{\mathcal{M}}-1)} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where P_{nm} is the probability to change the bit from state n to the state m and is given by:

$$P_{nm} = \frac{\text{Probability to go from state } n \text{ to state } m}{\text{Probability to remain in original state } n}$$

Say we have $\mathcal{M}$ bit data; thus there are $2^{\mathcal{M}}$ levels. Consider state n lying between $\mu + il\sigma$ and $\mu + (i+1)l\sigma$, where μ is mean of Gaussian noise and l is level setting number for digitization. Here, i runs from $-(\frac{2^{\mathcal{M}}}{2} - 1)$ to $+(\frac{2^{\mathcal{M}}}{2} - 1)$ and n varies from 0 to $2^{\mathcal{M}} - 1$. As shown in the figure (1), the probability of bit shift can be calculated as,

$$P_{n\to n+m} = \frac{\text{Area under pdf between intersection of states } n \text{ and } n+m}{\text{Area under pdf up to state } n}$$

$$\boxed{P_{n\to n+m} = \frac{\phi(\min(\mu + (i+m)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+m-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)}} \tag{1}$$

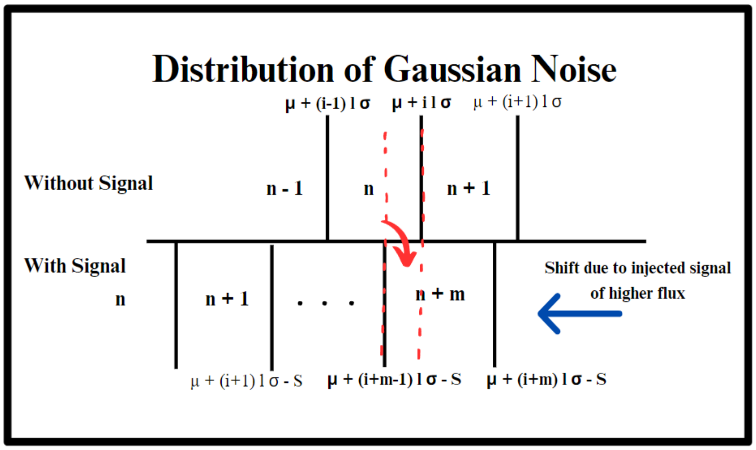

Figure 1: Distribution of Gaussian noise

where n and i are related as $n - i = \frac{2^{\mathcal{M}}}{2} - 1$.

The first state (0) extends from $-\infty$ to $\mu - (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma$ and the last state $(2^{\mathcal{M}} - 1)$ extends from $\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma$ to $+\infty$. Therefore, we need to accommodate these variation at the boundary states. This can be done by modifying above formula as,

$$P_{0\to 0} = \frac{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{2}$$

$$P_{0\to m} = \frac{\phi(\min(\mu + (m + 1 - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S}, \mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)) - \phi(\mu + (m - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{3}$$

$$P_{n\to 2^{\mathcal{M}}-1} = \frac{\phi(\mu + (n + 1 - \frac{2^{\mathcal{M}}}{2})l\sigma) - \phi(\max(\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma - \mathcal{S}, \mu + (n - \frac{2^{\mathcal{M}}}{2})l\sigma))}{\phi(\mu + (n + 1 - \frac{2^{\mathcal{M}}}{2})l\sigma) - \phi(\mu + (n - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{4}$$

$$P_{0\to 2^{\mathcal{M}}-1} = \frac{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma) - \phi(\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} = 1 - \frac{\phi(\mu + (\frac{2^{\mathcal{M}}}{2} - 1)l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} \tag{5}$$

and,

$$P_{2^{\mathcal{M}}-1\to 2^{\mathcal{M}}-1} = 1 \tag{6}$$

Equation (1) to (6) are general formulas to calculate probability of bit change from one state to other when a signal $\mathcal{S}$ is injected into the quantized data stream.

Now, **for the 8-bit case**, we have $n - i = \frac{2^8}{2} - 1 = 127$. Thus,

$$P_{n\to n+m} = \frac{\phi(\min(\mu + (n + m - 127)l\sigma - \mathcal{S}, \mu + (n - 127)l\sigma)) - \phi(\max(\mu + (n + m - 128)l\sigma - \mathcal{S}, \mu + (n - 128)l\sigma))}{\phi(\mu + (n - 127)l\sigma) - \phi(\mu + (n - 128)l\sigma)}$$

The formula is appropriate given we are away from boundary states (the first and the last state). Probability of changing bit from 0 to 0 and 255 to 255, is

$$P_{0\to 0} = \frac{\phi(\mu - 127l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)} \text{ and } P_{255\to 255} = 1, \text{ respectively.} \tag{7}$$

For bit change from state 0 to any higher state m, the formula modifies as

$$P_{0\to m} = \frac{\phi(\min(\mu + (m-127)l\sigma - \mathcal{S}, \mu - 127l\sigma)) - \phi(\mu + (m-128)l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)}$$

Similarly, for bit change from any non-zero state n to the highest state 255, the probability is

$$P_{n\to 255} = \frac{\phi(\mu + (n-127)l\sigma) - \phi(\max(\mu + 127l\sigma - \mathcal{S}, \mu + (n-128)l\sigma))}{\phi(\mu + (n-127)l\sigma) - \phi(\mu + (n-128)l\sigma)}$$

Lastly, the probability for bit change from the lowest state 0 to the highest state 255 is

$$P_{0\to 255} = \frac{\phi(\mu - 127l\sigma) - \phi(\mu + 127l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)} = 1 - \frac{\phi(\mu + 127l\sigma - \mathcal{S})}{\phi(\mu - 127l\sigma)}$$

For a 1-bit case, $\mathcal{M}$ is 1 and thus there are only 2 levels 0 and 1. So, n and i are equal for 1-bit data. Only possible transition between these two states is 0 to 1, and the resulting probability from equation 5 is

$$P_{0\to 1} = \frac{\phi(\mu + (1 - \frac{2^1}{2})l\sigma) - \phi(\mu + (\frac{2^1}{2} - 1)l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^1}{2})l\sigma)} = 1 - \frac{\phi(\mu - \mathcal{S})}{\phi(\mu)} = 1 - \frac{\phi(\mu - \mathcal{S})}{0.5} \tag{8}$$

$$P_{1\to 0} = 1 - P_{0\to 1} = \frac{\phi(\mu - \mathcal{S})}{0.5} \tag{9}$$

For a 2-bit case, $\mathcal{M}$ is 2 and thus there are 4 levels namely 0, 1, 2, and 3. In this case, n and i are related as $n - i = \frac{2^2}{2} - 1 = 1$.

Using equation (1),

$$\begin{aligned} P_{1\to 1} &= \frac{\phi(\min(\mu + (i+1)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+1-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)} && (10)\\ &= \frac{\phi(\mu - \mathcal{S}) - \phi(\mu - l\sigma)}{\phi(\mu) - \phi(\mu - l\sigma)} && (11)\\ &= \frac{\phi(\mu - \mathcal{S}) - \phi(\mu - l\sigma)}{0.5 - \phi(\mu - l\sigma)} && (12) \end{aligned}$$

$$\begin{aligned} P_{1\to 2} &= \frac{\phi(\min(\mu + (i+1)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+1-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)} && (13)\\ &= \frac{\phi(\min(\mu + l\sigma - \mathcal{S}, \mu)) - \phi(\max(\mu - \mathcal{S}, \mu - l\sigma))}{\phi(\mu) - \phi(\mu - l\sigma)} && (14)\\ &= \frac{\phi(\min(\mu + l\sigma - \mathcal{S}, \mu)) - \phi(\max(\mu - \mathcal{S}, \mu - l\sigma))}{0.5 - \phi(\mu - l\sigma)} && (15) \end{aligned}$$

and,

$$P_{2\to 2} = \frac{\phi(\min(\mu + (i+1)l\sigma - \mathcal{S}, \mu + il\sigma)) - \phi(\max(\mu + (i+1-1)l\sigma - \mathcal{S}, \mu + (i-1)l\sigma))}{\phi(\mu + il\sigma) - \phi(\mu + (i-1)l\sigma)} \tag{16}$$

$$= \frac{\phi(\mu + l\sigma - \mathcal{S}) - \phi(\mu)}{\phi(\mu + l\sigma) - \phi(\mu)} \tag{17}$$

$$= \frac{\phi(\mu + l\sigma - \mathcal{S}) - 0.5}{\phi(\mu + l\sigma) - 0.5} \tag{18}$$

Using equation (2),

$$P_{0\to 0} = \frac{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma - \mathcal{S})}{\phi(\mu + (1 - \frac{2^{\mathcal{M}}}{2})l\sigma)} = \frac{\phi(\mu - l\sigma - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{19}$$

From equation (3),

$$P_{0\to m} = \frac{\phi(\min(\mu + (m-1)l\sigma - \mathcal{S}, \mu - 1l\sigma)) - \phi(\mu + (m-2)l\sigma - \mathcal{S})}{\phi(\mu - 1l\sigma)}$$

$$P_{0\to 1} = \frac{\phi(\min(\mu - \mathcal{S}, \mu - l\sigma)) - \phi(\mu - l\sigma - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{20}$$

$$P_{0\to 2} = \frac{\phi(\min(\mu + l\sigma - \mathcal{S}, \mu - l\sigma)) - \phi(\mu - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{21}$$

from equation (4).

$$P_{n\to 3} = \frac{\phi(\mu + (n-1)l\sigma) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu + (n-2)l\sigma))}{\phi(\mu + (n-1)l\sigma) - \phi(\mu + (n-2)l\sigma)}$$

$$P_{1\to 3} = \frac{\phi(\mu) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu - l\sigma))}{\phi(\mu) - \phi(\mu - l\sigma)} = \frac{0.5 - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu - l\sigma))}{0.5 - \phi(\mu - l\sigma)} \tag{22}$$

$$P_{2\to 3} = \frac{\phi(\mu + l\sigma) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu))}{\phi(\mu + l\sigma) - \phi(\mu)} = \frac{\phi(\mu + l\sigma) - \phi(\max(\mu + l\sigma - \mathcal{S}, \mu))}{\phi(\mu + l\sigma) - 0.5} \tag{23}$$

from equation (5)

$$P_{0\to 3} = \frac{\phi(\mu - l\sigma) - \phi(\mu + l\sigma - \mathcal{S})}{\phi(\mu - l\sigma)} \tag{24}$$

and lastly from equation (6)

$$P_{3\to 3} = 1 \tag{25}$$

Equations (8) to (25) match with the probability formulas given in the paper Luo et al. [2022], Appendix B.

References

R. Luo, G. Hobbs, S. Y. Yong, A. Zic, L. Toomey, S. Dai, A. Dunning, D. Li, T. Marshman, C. Wang, P. Wang, S. Wang, and S. Zhang. Simulating high-time resolution radio-telescope observations. *Monthly Notices of the Royal Astronomical Society*, 513(4):5881–5891, 04 2022. ISSN 0035-8711. doi: 10.1093/mnras/stac1168. URL https://doi.org/10.1093/mnras/stac1168.