Chandrashekar Mass Limit

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January 11, 2023

Abstract

This is the Winter project report 2022[1], where we derived the limit on the maximum mass that a star can have to form a white dwarf at the end of its life. This limit on maximum mass is called Chandrasekhar limit. The derivation uses concepts from statistical mechanics, thermodynamics, relativity, and gravitation.

1 Introduction

The Chandrasekhar limit is derived by equating the degeneracy pressure due to electron gas and gravitational pull by protons and other heavy nuclei present in the star.

2 Theory

Inside a normal star, the gravitational attraction is balanced by the thermal pressure due to the thermonuclear reactions in the interior of the star. Once the nuclear fuel of the star comes to an end, thermal pressures (that balances the gravity) vanishes. The electrons of the stellar matter contribute to Fermi gas. Once the density of contracting star become sufficiently high, these electrons eventually form 'degenerate' gas and exert pressure to further balance the gravitational attraction.

3 Derivation

The pressure P of the gas as per kinetic theory is given by [1]

$$P = \frac{1}{3}\int vpf(p)4\pi p^2 dp \tag{1}$$

Electrons being **Fermi particles**, they follow Pauli exclusion principle and no more than two electrons can occupy same state. When a gas of Fermi particles is compressed to very high density, many of the particles are forced to remain in non-zero momentum states even at

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zero K; this gives rise to the degeneracy pressure. As mass of electrons being much smaller than protons and other heavy nuclei present in the star, they have smaller momentum[2]. Thus, in a momentum-space, electrons occupy a much smaller volume and have a larger number density than other heavy particles in a given volume. This density is sufficient for making electrons degenerate and hence only electrons exert degenerate pressure.[3]
Assuming there is no finite-temperature effects at 0 K and that all states below the Fermi momentum p_F are occupied (whereas all states above p_F are unoccupied), the number density n_e of electrons is given by [4]

$$n_e = \int_0^{p_f} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_F^3 \tag{2}$$

Now, the number of particles having momentum between p and $p + dp$ will be equal to the number of states per unit volume within the shell in the momentum space. Thus,

$$4\pi f(p) p^2 dp = \frac{8\pi p_f^3}{3h^3} \tag{3}$$

Hence, considering Fermi-Dirac statistics:

$$f(p) = \begin{cases} \frac{2}{h^3}, & \text{if } p < p_f \\ 0, & \text{if } p > p_f \end{cases} \tag{4}$$

Putting (4) in (1), we get

$$P = \frac{8\pi}{3h^3} \int_0^{p_f} v p^3 dp \tag{5}$$

Using relativistic expression of momentum $p = mv\gamma$, where γ is Lorentz factor, and substituting the value of v [5] in (5):

$$P = \frac{8\pi}{3h^3} \int_0^{p_f} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}} dp \tag{6}$$

In order to establish the relation between density and pressure, we first find relation between density ρ and number density n_e. As protons and other heavier nuclei are non-degenerate, they do contribute to the density but not pressure.

Consider ionized H atoms, with mass fraction X. The electron number density is given by [6]

$$n_e = \frac{X\rho}{m_H} + \frac{(1-X)\rho}{2m_H} = \frac{\rho(1+X)}{2m_H} = \frac{\rho}{\mu_e m_H} \tag{7}$$

where, $\mu_e = 2/(1+X)$. Equating equations (2) and (7) we get

$$p_F = \left(\frac{3h^3 \rho}{8\pi \mu_e m_H} \right)^{1/3}$$

For the sake of simplicity, we will consider two extreme cases -non-relativistic and fully relativistic- instead of solving the integral over all possible momentum (eq 6).

Non relativistic:

$$\sqrt{p^2c^2 + m_e^2c^4} \approx m_ec^2 \tag{8}$$

Fully relativistic:

$$\sqrt{p^2c^2 + m_e^2c^4} \approx pc \tag{9}$$

Substituting values (8), (9) and p_F in (6), we get following two cases (eq in SI unit):

Non - relativistic:

$$P = k_1\rho^{5/3}; \quad k_1 = \frac{10^7}{\mu_e^{5/3}} \tag{10}$$

Fully - relativistic:

$$P = k_2\rho^{4/3}; \quad k_2 = \frac{1.24 * 10^{10}}{\mu_e^{4/3}} \tag{11}$$

We can write combined form of above two pressure equations as (12), which is also known as **Polytropic relation** between density and pressure.

Thus generalizing one of the four pressure relations [7]:

$$P = K\rho^{(1+\frac{1}{n})} \tag{12}$$

where, $n = \frac{3}{2}$ corresponds to non-relativistic case and $n = 3$ corresponds to fully relativistic case. We can plot $log(T)$ against $log(\rho)$ and divide the graph into four regions corresponding to four equation of state - Radiation, Ideal gas, relativistic and non-relativistic, as shown in Fig:1 [2].

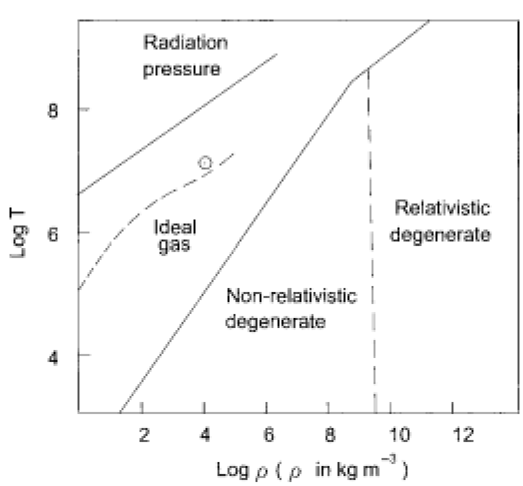

Figure 1: log(T) vs log(ρ)

[2] Adapted from Chap 5 *Astrophysics for Physicist by Arnab Rai Choudhuri*

The dashed line separates relativistic and non-relativistic regions.[3]
Now, let's consider a model of the star with degenerate matter. Combining two of the stellar structure equations[8] and eliminating M_r, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{13}$$

Simplifying (12), ρ and r, equation by introducing some variables, we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{14}$$

Where

1. $r = a\xi$ and

$$a = \sqrt{\frac{(n+1)K\rho_c^{\left(\frac{1-n}{n}\right)}}{4\pi G}}$$

2. $\rho = \rho_c(\theta^n)$. [4]

Equation (14) is known as *Lane- Emden equation*.
Now, lets apply boundary conditions, to solve second-order differential equation (14):

- $\theta(\xi = 0) = 1$ and
- $\left(\frac{d\theta}{d\xi}\right)_{\xi=0} = 0$ [5]

Considering a general star of radius R and made up of degenerate matter, mass M and density ρ_c. If $n < 5$, then θ tends to 0, for some finite value of $\xi = \xi_1$, which we interpret as a surface where density and pressure go to 0 (2 and 12). Therefore, $R = a\xi_1$ be the physical radius of the star, where is ξ_1 is constant.
From (1) we see that,

$$R \propto \rho_c^{\frac{1-n}{2n}} \tag{15}$$

We can write mass M in terms of density ρ and eventually our new variables ξ and θ as

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \theta^n d\xi \tag{16}$$

The integral on RHS is a constant [9]. The variable a has ρ_c dependence and thus, mass will depend on ρ_c as

$$M \propto \rho_c^{\left(\frac{3-n}{2n}\right)} \tag{17}$$

[3]The dashed curve in ideal gas region corresponds to the curve of temperature and density inside the Sun, indicating that the ideal gas equation of state is sufficient in dealing with stars like the Sun.
[4]ρ_c is density at center of star and θ is dimensionless variable ($\theta = 1$ at center)
[5]Considering that we do not want a cusp in the density at the centre of the star.

For $n = 3/2$ i.e non-relativistic case: $R \propto \rho_c^{-1/6}$ and $M \propto \rho_c^{1/2}$.Hence,

$$R \propto M^{-1/3} \tag{18}$$

This non-relativistic mass-radius relation is shown as dashed curve in Fig:2
For $n = 3$ i.e fully relativistic case: Mass M is independent of ρ_c. Hence, simplifying (16) in terms of M_o [6] we get a constant value of mass $M(= M_{ch})$ for the relativistic equation of state

$$M_{Ch} = 1.46 M_o \left(\frac{2}{\mu_e}\right)^{1/2} \tag{19}$$

White dwarf is the end state of a star, where all of it's H is used up in nuclear fusion reaction, to form He. Hence, we can make a fair assumption that $X \approx 0$, thus $\mu_e = 2$ (from 3). Therefore, we get

$$M_{Ch} = 1.4 M_o, \tag{20}$$

4 Conclusions

The solid curve in fig:2 [7] shows full equation of state from eq (6), instead of considering the non-relativistic and fully relativistic limits. As shown in the figure, for white dwarfs of lower masses, the curve coincides with non-relativistic curve, as the interior density is not so high and non-relativistic state of equation hold. As mass and interior density increases, Fermi

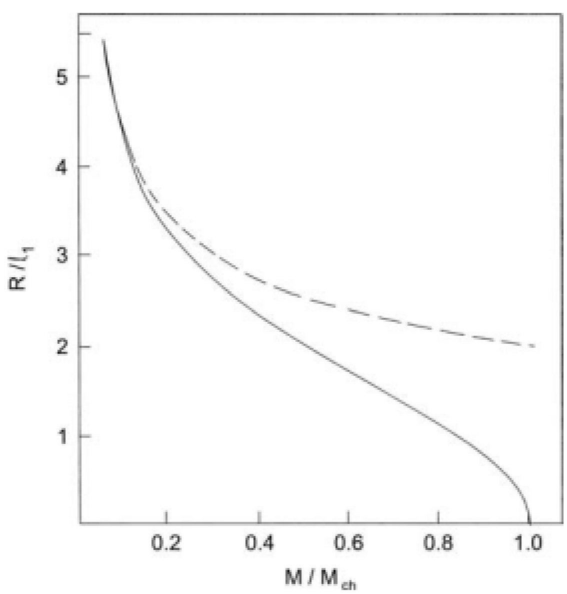

Figure 2: R vs M (scaled)

momentum p_F becomes large as per (3).

[6] M_o is solar mass = 1.989 x 10^{30} kg.
[7] Adapted from Chap 5 *Astrophysics for Physicist by Arnab Rai Choudhuri*

Moving towards right in the figure, the radius continues to decrease and eventually becomes zero. At this point, the interior density is so high that the relativistic limit of the equation of state is approached. The mass M_{Ch} corresponding to the relativistic limit of the equation of state is the limiting mass for which the radius goes to zero. This limit is the well-known **Chandrasekhar mass limit**, derived by Subrahmanyan Chandrasekhar, formulated in 1930.

Thus, there exist a limit on maximum mass of a stable white dwarf star. But, beyond this mass limit, protons and neutrons also start forming degenerate gas(once the interior of star reaches certain density) and then apply degeneracy pressure to oppose further contraction due to gravity. Such a star later becomes a **neutron star**. The upper limit on the mass of neutron star is not known precisely, due to uncertainties in the knowledge of the equation of the state of matter. But, the limit is not more than $2M_o$.

5 Appendix

1. This is the pressure of gas with $4\pi f(p)p^2dp$ number of particles having momentum between p and $p+dp$ and v as particle velocity. Here, we have assumed the distribution function $f(p)$ to be isotropic i.e $f(p)$ does not vary in magnitude according to the direction of measurement.

2. Considering equal partition of energy, as per expression of kinetic energy $p^2/2m$, electrons should have smaller momentum (due to smaller mass) compared to other heavy nuclei.

3. Particles except electrons remain in non-degenerate state, as they require much higher density to become degenerate than electrons. In neutron star, however, protons and neutrons become degenerate.

4. In short, the number density is the product of number of state of given energy and average number of particles in that state. Electrons occupy $2Vd^3p/h^3$ states (twice due to exclusion principle) and number of states per unit volume is $\frac{8\pi}{h^3}p^2dp$. Hence, number density is integral over all momentum (momentum is zero beyond Fermi-momentum p_F).

5. Here we used the following relation from relativity:
$$E = mc^2\gamma = \sqrt{p^2c^2 + m^2c^4}$$
where γ is a Lorentz factor:
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

6. Number density of *ionized* H atoms is $X\rho/m_H$ and each H atom contributes the same number of electrons $X\rho/m_H$ per unit volume. But, for heavier atoms (He and beyond), each atom can contribute only *around* 0.5 electron per atomic mass unit. In

other words, the number of electrons is half the number of nucleons, for other heavier atoms(including He). Therefore, in the unit volume of stellar matter, these heavy atoms provide $(1-X)\rho/m_H$ nucleons and $(1-X)\rho/2m_H$ number of corresponding electrons.

7. Four pressure relations, corresponding to four regions in Fig1:
   - Ideal gas: $P = \frac{K_b}{\mu m_H}\rho T$, where $\mu = (2X + 3Y/4 + Z/2)^{-1}$
   - Black Body Radiation: $P = \frac{1}{3}a_B T^4$
   - Non-relativistic: 10
   - Fully relativistic: 11

8. Consider a thin shell element of star with density ρ, and of radius r from center of star. Hence it's mass will be $dM_r = 4\pi r^2\rho$. Thus

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \tag{21}$$

   Now, consider force due to pressure on this shell; $(P + dP)dA$ on outer surface and PdA on inner surface of the shell. Thus net force would be $-dPdA$, which will be balanced by the force of gravitational field $(-\frac{GM_r}{r^2})$. Thus

$$-dPdA - \frac{GM_r}{r^2}\rho dr dA = 0 \qquad\qquad \frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \tag{22}$$

   Above two equations(21 and 22) forms first two **stellar structure equations.** Remaining two stellar structure equations do not come into play here.

9. Integrating this integral from $\xi = 0$ to $\xi = 1$, we get $|\xi^2\theta'(\xi_1)| = 2.018$, which is a constant.

References

[1] Arnab Rai Choudhuri *Astrophysics for physicist*, (Cambridge University Press, 2010).

[2] T. Padmanabhan *Theoretical Astrophysics, Vol 2 Stars and stellar systems* (Cambridge University Press, 2000)