NCERT DISCRETE

EE23BTECH11020 - Raghava Ganji*

GATE 2023 BM.48: The function $f(z) = \frac{1}{z-1}$ of a complex variable z on a closed contour in an anti-clockwise direction.For which of the following contours, does this integral have a non-zero value?
$(A)|z-2| = 0.01$
$(B)|z-1| = 0.1$
$(C)|z-3| = 5$
$(D)|z| = 2$

Solution:
Using (**??**)

$$\oint_c \frac{1}{z-1} dz = 2\pi j Res\left[\frac{1}{z-1}, 1\right] \tag{1}$$

For option A the pole is outside the contour, then

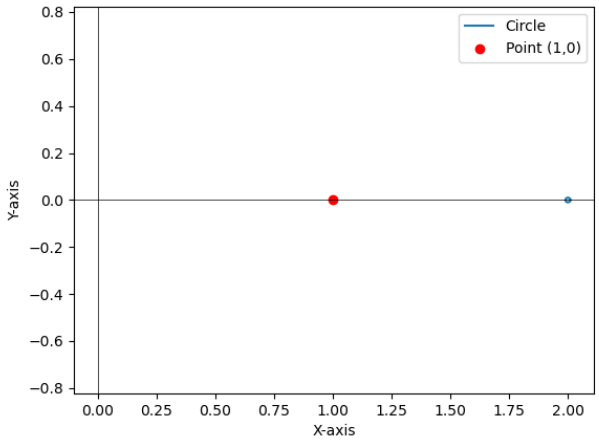

Fig. 0. graph of option A

$$z = 2 + 0.01e^{jt}$$

$$\implies \oint_c \frac{1}{z-1} dz = \int_0^{2\pi} \frac{0.01je^{jt}}{1+0.01e^{jt}} dt \tag{2}$$

$$= \int_0^{2\pi} \frac{je^{jt}}{100+e^{jt}} dt \tag{3}$$

$$\alpha = e^{jt}$$

$$= \int_1^1 \frac{1}{100+\alpha} d\alpha = 0 \tag{4}$$

For option B the pole is inside the contour. Then, using (**??**)

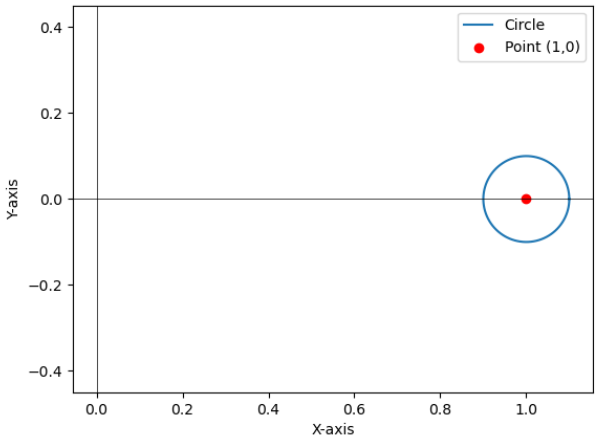

Fig. 0. graph of option B

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{5}$$

$$= 1$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j(1) \tag{6}$$

$$= 2\pi j \neq 0$$

For option C the pole is inside the contour. Then, using (**??**)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{7}$$

$$= 1$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j(1) \tag{8}$$

$$= 2\pi j \neq 0$$

For option D the pole is inside the contour.

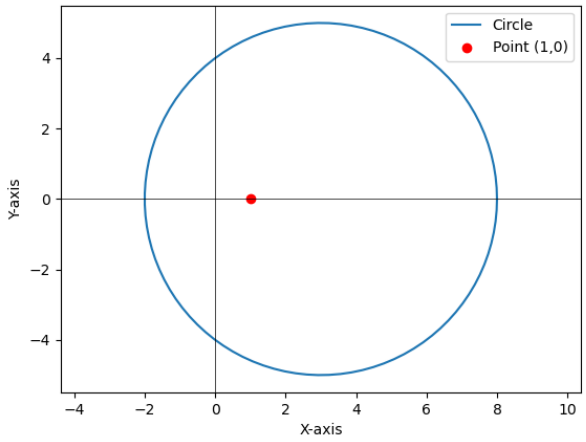

Fig. 0. graph of option C

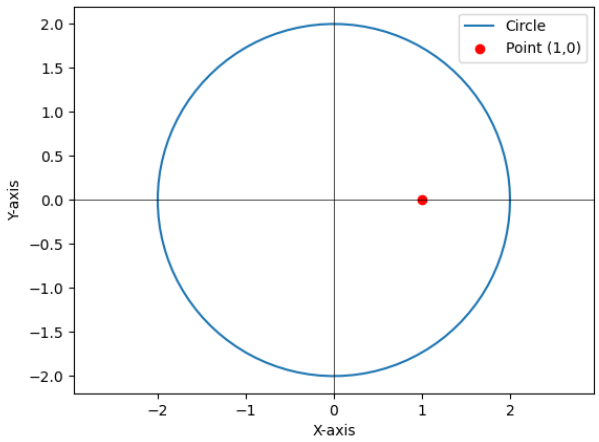

Fig. 0. graph of option D

Then, using (**??**)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{9}$$
$$= 1$$
$$\implies \oint_c \frac{1}{z-1}dz = 2\pi j\,(1) \tag{10}$$
$$= 2\pi j \neq 0$$

We can conclude that for options B,C,D contours have the non-zero value for this integral.