

NCERT DISCRETE

EE23BTECH11020 - Raghava Ganji*

GATE 2023 BM.48: The function $f(z) = \frac{1}{z-1}$ of a complex variable z on a closed contour in an anti-clockwise direction. For which of the following contours, does this integral have a non-zero value?

(A) $|z - 2| = 0.01$

(B) $|z - 1| = 0.1$

(C) $|z - 3| = 5$

(D) $|z| = 2$

Solution:

Cauchy's Integral Formula and Residue Theorem.

$$\oint_c f(z) = 2\pi j \text{Res}[f(z), z_0] \quad (1)$$

$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} [(z - z_0) f(z)] \quad (2)$$

Here z_0 is pole of the $f(z)$

Using (1)

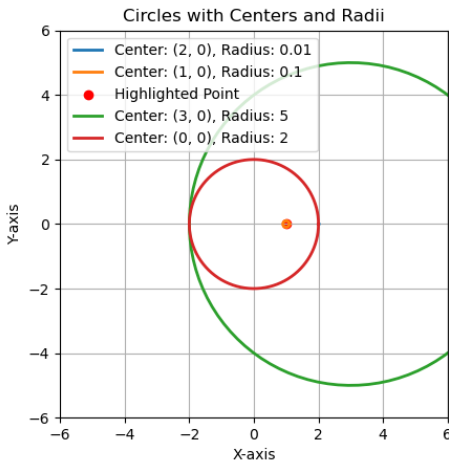


Fig. 0. graphs of all the given contours

$$\oint_c \frac{1}{z-1} dz = 2\pi j \text{Res}\left[\frac{1}{z-1}, 1\right] \quad (3)$$

For option A the pole is outside the contour, then Residue is zero.

$$\Rightarrow \oint_c \frac{1}{z-1} dz = 2\pi j(0) = 0 \quad (4)$$

For option B the pole is inside the contour. Then, using (2)

$$\text{Res}\left[\frac{1}{z-1}, 1\right] = \lim_{z \rightarrow 1} (z-1) \frac{1}{z-1} \quad (5)$$

$$= 1 \quad (6)$$

$$\Rightarrow \oint_c \frac{1}{z-1} dz = 2\pi j(1) \quad (7)$$

$$\Rightarrow 2\pi j \neq 0 \quad (8)$$

For option C the pole is inside the contour. Then, using (2)

$$\text{Res}\left[\frac{1}{z-1}, 1\right] = \lim_{z \rightarrow 1} (z-1) \frac{1}{z-1} \quad (9)$$

$$= 1 \quad (10)$$

$$\Rightarrow \oint_c \frac{1}{z-1} dz = 2\pi j(1) \quad (11)$$

$$\Rightarrow 2\pi j \neq 0 \quad (12)$$

For option D the pole is inside the contour. Then, using (2)

$$\text{Res}\left[\frac{1}{z-1}, 1\right] = \lim_{z \rightarrow 1} (z-1) \frac{1}{z-1} \quad (13)$$

$$= 1 \quad (14)$$

$$\Rightarrow \oint_c \frac{1}{z-1} dz = 2\pi j(1) \quad (15)$$

$$\Rightarrow 2\pi j \neq 0 \quad (16)$$

We can conclude that for options B,C,D contours have the non-zero value for this integral.