NCERT DISCRETE

EE23BTECH11020 - Raghava Ganji*

GATE 2023 BM.48: The function $f(z) = \frac{1}{z-1}$ of a complex variable z on a closed contour in an anti-clockwise direction.For which of the following contours, does this integral have a non-zero value?
$(A)|z-2| = 0.01$
$(B)|z-1| = 0.1$
$(C)|z-3| = 5$
$(D)|z| = 2$
Solution:
Cauchy's Integral Formula and Residue Theorem.

$$\oint_c f(z) = 2\pi j Res\left[f(z), z_0\right] \tag{1}$$

$$Res\left[f(z), z_0\right] = \lim_{z\to z_0}\left[(z-z_0) f(z)\right] \tag{2}$$

Here z_0 is pole of the f(z)
Using (1)

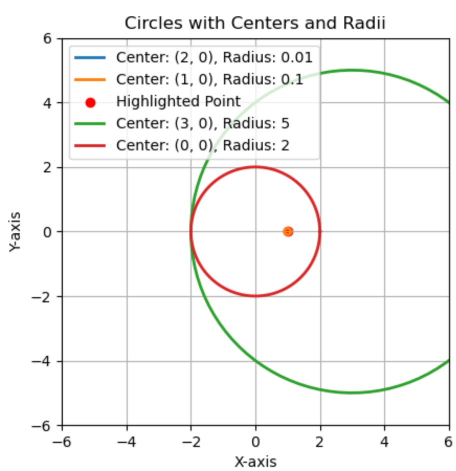

Fig. 0. graphs of all the given contours

$$\oint_c \frac{1}{z-1} dz = 2\pi j Res\left[\frac{1}{z-1}, 1\right] \tag{3}$$

For option A the pole is outside the contour, then Residue is zero.

$$\oint_c \frac{1}{z-1} dz = 2\pi j(0) \tag{4}$$
$$= 0$$

For option B the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{5}$$
$$= 1$$
$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j(1) \tag{6}$$
$$= 2\pi j \neq 0$$

For option C the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{7}$$
$$= 1$$
$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j(1) \tag{8}$$
$$= 2\pi j \neq 0$$

For option D the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{9}$$
$$= 1$$
$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j(1) \tag{10}$$
$$= 2\pi j \neq 0$$

We can conclude that for options B,C,D contours have the non-zero value for this integral.