NCERT DISCRETE

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GATE 2023 BM.48: The function $f(z) = \frac{1}{z-1}$ of a complex variable z on a closed contour in an anti-clockwise direction. For which of the following contours, does this integral have a non-zero value?

(A) $|z - 2| = 0.01$
(B) $|z - 1| = 0.1$
(C) $|z - 3| = 5$
(D) $|z| = 2$

Solution:

Cauchy's Integral Formula and Residue Theorem.

$$\oint_c f(z) = 2\pi j Res[f(z), z_0] \tag{1}$$

$$Res[f(z), z_0] = \lim_{z \to z_0} [(z - z_0) f(z)] \tag{2}$$
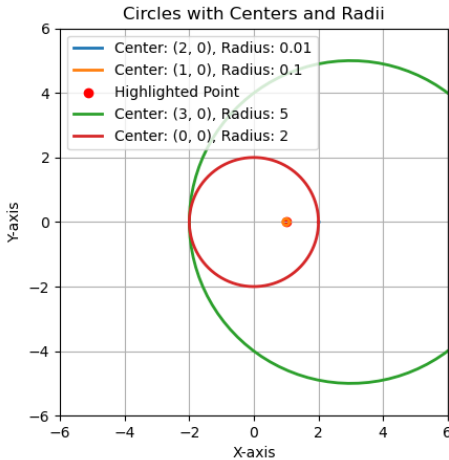
Here z_0 is pole of the f(z)

Using (1)

Fig. 0. graphs of all the given contours

$$\oint_c \frac{1}{z-1} dz = 2\pi j Res\left[\frac{1}{z-1}, 1\right] \tag{3}$$

For option A the pole is outside the contour, then Residue is zero.

$$\oint_c \frac{1}{z-1} dz = 2\pi j (0) \implies = 0 \tag{4}$$

For option B the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z \to 1} (z - 1) \frac{1}{z-1} \tag{5}$$

$$= 1 \tag{6}$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (1) \tag{7}$$

$$= 2\pi j \implies \neq 0 \tag{8}$$

For option C the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z \to 1} (z - 1) \frac{1}{z-1} \tag{9}$$

$$= 1 \tag{10}$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (1) \tag{11}$$

$$= 2\pi j \implies \neq 0 \tag{12}$$

For option D the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z \to 1} (z - 1) \frac{1}{z-1} \tag{13}$$

$$= 1 \tag{14}$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (1) \tag{15}$$

$$= 2\pi j \implies \neq 0 \tag{16}$$

We can conclude that for options B,C,D contours have the non-zero value for this integral.