An Introduction to Geometric Algebra

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History

- Geometric algebra is the Clifford algebra of a finite dimensional vector space over real scalars cast in a form most appropriate for physics and engineering. This was done by David Hestenes (Arizona State University) in the 1960's. From this start he developed the geometric calculus whose fundamental theorem includes the generalized Stokes theorem, the residue theorem, and new integral theorems not realized before. Hestenes likes to say he was motivated by the fact that physicists and engineers did not know how to multiply vectors.

- Researchers at Arizona State and Cambridge have applied these developments to classical mechanics,quantum mechanics, general relativity (gauge theory of gravity), projective geometry, conformal geometry, etc.

Axioms of Geometric Algebra

Let $\mathcal{V}(p,q)$ be a finite dimensional vector space of signature (p,q)[1] over $\Re$. Then $\forall a,b,c \in \mathcal{V}$ there exists a geometric product with the properties -

$$
\begin{aligned}
(ab)c &= a(bc) \\
a(b+c) &= ab+ac \\
(a+b)c &= ac+bc \\
aa &\in \Re
\end{aligned}
$$

If $a^2 \neq 0$ then $a^{-1} = \dfrac{1}{a^2}a$.

[1] To be completely general we would have to consider $\mathcal{V}(p,q,r)$ where the dimension of the vector space is $n = p+q+r$ and p, q, and r are the number of basis vectors respectively with positive, negative and zero squares.

Why Learn This Stuff?

- The geometric product of two (or more) vectors produces something "new" like the $\sqrt{-1}$ with respect to real numbers or vectors with respect to scalars. It must be studied in terms of its effect on vectors and in terms of its symmetries.

- Geometric algebra greatly simplifies understanding rotations and reflections in a N dimensional space of mixed signature.

- The geometric calculus unifies many diverse areas in mathematics and many areas of physics and engineering are greatly simplified (Maxwell's equations are $\nabla F = J$).

- Geometric algebra allows physicists to make a geometric interpetation of a **spinor**. Anything that can make spinors simple is worthwhile!

- Quantum mechanics does not require complex numbers anymore!

Inner, $\cdot$, and outer, $\wedge$, product of two vectors and their basic properties

$$a \cdot b \equiv \frac{1}{2}(ab + ba) \tag{1}$$

$$a \wedge b \equiv \frac{1}{2}(ab - ba) \tag{2}$$

$$ab = a \cdot b + a \wedge b \tag{3}$$

$$a \wedge b = -b \wedge a \tag{4}$$

$$\begin{gathered} c = a + b \\ c^2 = (a + b)^2 \\ c^2 = a^2 + ab + ba + b^2 \\ 2a \cdot b = c^2 - a^2 - b^2 \\ a \cdot b \in \Re \end{gathered} \tag{5}$$

$$a \cdot b = |a|\,|b| \cos(\theta) \ \text{ if } a^2, b^2 > 0 \tag{6}$$

Orthogonal vectors are defined by $a \cdot b = 0$.
For orthogonal vectors $a \wedge b = ab$.

Now compute $(a \wedge b)^2$

$$\begin{aligned}
(a \wedge b)^2 &= -(a \wedge b)(b \wedge a) && (7)\\
&= -(ab - a \cdot b)(ba - a \cdot b) && (8)\\
&= -\left(abba - (a \cdot b)(ab + ba) + (a \cdot b)^2\right) && (9)\\
&= -\left(a^2 b^2 - (a \cdot b)^2\right) && (10)\\
&= -a^2 b^2 \left(1 - \cos^2(\theta)\right) && (11)\\
&= -a^2 b^2 \sin^2(\theta) && (12)
\end{aligned}$$

Thus in a Euclidean space, $a^2, b^2 > 0$, $(a \wedge b)^2 \leq 0$ and $a \wedge b$ is proportional to $\sin(\theta)$. If $e_{\|}$ and $e_{\perp}$ are any two orthonormal unit vectors in a Euclidean space then $\left(e_{\|} e_{\perp}\right)^2 = -1$. Who needs the $\sqrt{-1}$?

Outer, $\wedge$, product for r Vectors in terms of the geometric product

We define the outer product of r vectors to be

$$a_1 \wedge \ldots \wedge a_r \equiv \frac{1}{r!} \sum_{i_1,\ldots,i_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} a_{i_1} \ldots a_{i_r} \tag{13}$$

where $\varepsilon^{i_1 \ldots i_r}_{1 \ldots r}$ is the Levi-Civita permutation symbol.

If the a_j's are orthogonal $(a_i \cdot a_j = 0)$

$$a_1 \wedge \ldots \wedge a_r = a_1 \ldots a_r \tag{14}$$

The outer product of r vectors is called a blade of grade r.

Projection Operator

- A multivector, the basic element of the geometric algebra, is made of of a linear combination of scalars, vectors, blades.

- A multivector is homogeneous (pure) if all the blades in it are of the same grade.

- The grade of a scalar is 0 and the grade of a vector is 1.

- The general multivector A is decomposed with the grade projection operator $\langle A \rangle_r$ as (N is dimension of the vector space):

$$A = \sum_{r=0}^{N} \langle A \rangle_r \tag{15}$$

As an example consider ab, the geometric product of two vectors. Then

$$ab = a \cdot b + a \wedge b = \langle ab \rangle_0 + \langle ab \rangle_2 \tag{16}$$

Basis Blades

The geometric algebra of a vector space, $\mathcal{V}(p,q)$, is denoted $\mathcal{G}(p,q)$ or $\mathcal{G}(\mathcal{V})$ where (p,q) is the signature of the vector space (first p unit vectors square to $+1$ and next q unit vectors square to -1, dimension of the space is $p+q$). Examples are:

p	q	Type of Space
3	0	3D Euclidean
1	3	Relativistic Space Time
4	1	3D Conformal Geometry

- If the orthogonal basis set of the vector space is $e_1, \ldots, e_N$, the basis of the geometric algebra (multivector space) is formed from the geometric products (since we have chosen an orthogonal basis) of the basis vectors.

- For grade r multivectors the basis blades are all the combinations of basis vectors products taken r at a time from the set of N vectors.

- The number of basis blades of grade r are $\binom{N}{r}$, the binomial expansion coefficient and the total dimension of the multivector space is the sum of $\binom{N}{r}$ over r which is 2^N.

- The highest grade blade (grade N) is the pseudoscalar defined by

$$I = e_1 \ldots e_N \tag{17}$$

$\mathcal{G}(3,0)$ Euclidian 3-Space

The basis blades for $\mathcal{G}(3,0)$ are:

Grade			
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

Note that e_1e_2, e_1e_3, and e_2e_3 are proportional to the quaternions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Where

$$\mathbf{i} = e_3e_2 \quad \mathbf{j} = e_1e_3 \quad \mathbf{k} = e_2e_1 \tag{18}$$

and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \tag{19}$$

Reduction of Geometric Product

The geometric product of orthogonal bases is reduced using the relationships:

$$e_i^2 = \pm 1 \quad \text{(Depends upon signature)} \tag{20}$$

$$e_i e_j = -e_j e_i \tag{21}$$

Consider the geometric product $(e_1 e_2 e_3)(e_1 e_2)$

$$\begin{aligned}
(e_1 e_2 e_3)(e_1 e_2) &= e_1 e_2 e_3 e_1 e_2 && (22)\\
&= -e_1 e_2 e_1 e_3 e_2 && (23)\\
&= +e_1^2 e_2 e_3 e_2 && (24)\\
&= -e_1^2 e_2^2 e_3 && (25)\\
&= -e_3 && (26)
\end{aligned}$$

$\mathcal{G}(3,0)$ **Euclidian 3-Space**

The multiplication table for the $\mathcal{G}(3,0)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1. The highest rank basis blade (in this case $e_1e_2e_3$) is ususally denoted by I and is called the pseudoscalar.

$\mathcal{G}(1,3)$ Space Time

$$\gamma_0^2 = -\gamma_1^2 = -\gamma_2^2 = -\gamma_3^2 = 1$$

The basis blades for $\mathcal{G}(1,3)$ are:

Grade				
0	1	2	3	4
1	γ_0	$\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$
	γ_1	$\gamma_0\gamma_2$	$\gamma_0\gamma_1\gamma_3$	
	γ_2	$\gamma_1\gamma_1$	$\gamma_0\gamma_2\gamma_3$	
	γ_3	$\gamma_0\gamma_3$	$\gamma_1\gamma_2\gamma_3$	
		$\gamma_1\gamma_3$		
		$\gamma_2\gamma_3$		

Note that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \tag{27}$$

Which are the anticommutation relations for the Dirac γ matrices.

$\mathcal{G}(1,3)$ **Space Time**

The multiplication table for the $\mathcal{G}(1,3)$ basis blades is (Part I)

	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

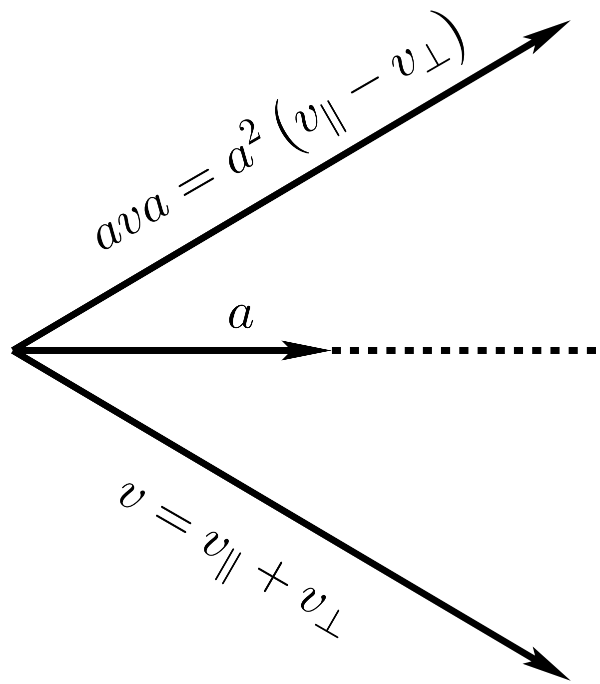
The multiplication table for the $\mathcal{G}(1,3)$ basis blades is (Part II)

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

Note that $(\gamma_0\gamma_i)^2 = 1$ and $(\gamma_i\gamma_j)^2 = -1$

Reflections

We wish to show that $a, v \in \mathcal{V} \to ava \in \mathcal{V}$ and v is reflected about a if $a^2 = 1$.



1. Decompose $v = v_\| + v_\perp$ where $v_\|$ is the part of v parallel to a and $v_\perp$ is the part perpendicular to a.

2. $av = av_{\|} + av_{\perp} = v_{\|}a - v_{\perp}a$ since a and $v_{\perp}$ are orthogonal.

3. $ava = \left(v_{\|} - v_{\perp}\right) a^2$ is a vector since a^2 is a scalar.

4. ava is the reflection of v about the direction of a if $a^2 = 1$.

5. Thus $a_1 \ldots a_r v a_r \ldots a_1 \in \mathcal{V}$ and produces a composition of reflections of v if $a_1^2 = \cdots = a_r^2 = 1$.

Rotations, Part 1

First define the reverse of a product of vectors. If $R = a_1 \ldots a_r$ then the reverse is $R^\dagger = (a_1 \ldots a_r)^\dagger = a_r \ldots a_1$, the order of multiplication is reversed. Then let $R = ab$ so that

$$RR^\dagger = (ab)(ba) = ab^2a = a^2b^2 = R^\dagger R \tag{28}$$

Let $RR^\dagger = 1$ and calculate $\left(RvR^\dagger\right)^2$, where v is an arbitrary vector.

$$\left(RvR^\dagger\right)^2 = RvR^\dagger RvR^\dagger = Rv^2R^\dagger = v^2RR^\dagger = v^2 \tag{29}$$

Thus $RvR^\dagger$ leaves the length of v unchanged.

Now we must also prove $Rv_1R^\dagger \cdot Rv_2R^\dagger = v_1 \cdot v_2$. Since $Rv_1R^\dagger$ and $Rv_2R^\dagger$ are both vectors we can use the definition of the dot product for two vectors

$$
\begin{aligned}
Rv_1R^\dagger \cdot Rv_2R^\dagger &= \frac{1}{2}\left(Rv_1R^\dagger Rv_2R^\dagger + Rv_2R^\dagger Rv_1R^\dagger\right) \\
&= \frac{1}{2}\left(Rv_1v_2R^\dagger + Rv_2v_1R^\dagger\right) \\
&= \frac{1}{2}R\left(v_1v_2 + v_2v_1\right)R^\dagger \\
&= R\left(v_1 \cdot v_2\right)R^\dagger \\
&= v_1 \cdot v_2 RR^\dagger \\
&= v_1 \cdot v_2
\end{aligned}
$$

Thus the transformation $RvR^\dagger$ preserves both length and angle and must be a rotation. The normal designation for R is a rotor.

- If we have a series of successive rotations $R_1, R_2, \ldots, R_k$ to be applied to a vector v then the result of the k rotations will be

$$R_k R_{k-1} \ldots R_1 v R_1^\dagger R_2^\dagger \ldots R_k^\dagger$$

- Each individual rotation can be written as the geometric product of two vectors, the composition of k rotations can be written as the geometric product of $2k$ vectors.

- The multivector that results from the geometric product of r vectors is called a **versor** of order r.

- A composition of rotations is always a versor of even order.

Rotations, Part 2

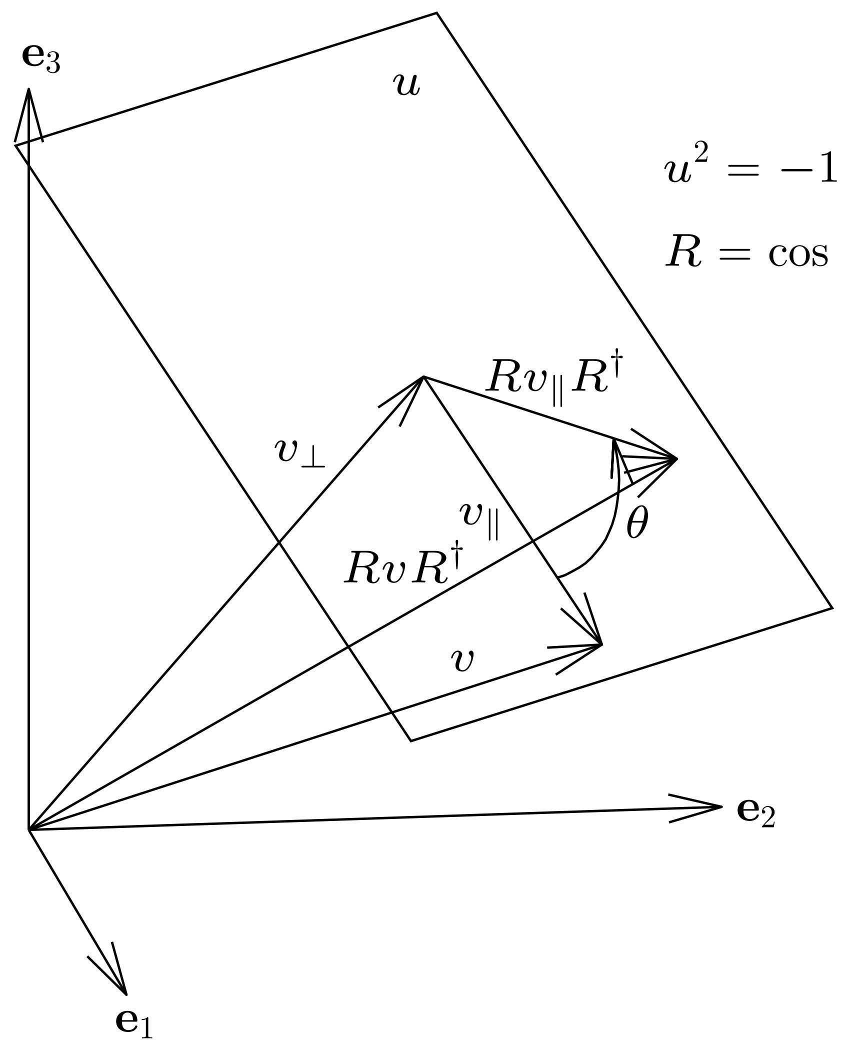
The general rotation can be represented by $R = e^{\frac{\theta}{2}u}$ where u is a unit bivector in the plane of the rotation and θ is the rotation angle in the plane.[2] The two possible non-degenerate cases are $u^2 = \pm 1$

$$e^{\frac{\theta}{2}u} = \left\{ \begin{array}{lrl} \text{(Euclidean plane)} & u^2 = -1: & \cos\left(\frac{\theta}{2}\right) + u\sin\left(\frac{\theta}{2}\right) \\ \text{(Minkowski plane)} & u^2 = 1: & \cosh\left(\frac{\theta}{2}\right) + u\sinh\left(\frac{\theta}{2}\right) \end{array} \right\} \tag{30}$$

Decompose $v = v_{\|} + \left(v - v_{\|}\right)$ where $v_{\|}$ is the projection of v into the plane defined by u. Note the $v - v_{\|}$ is orthogonal to all vectors in the u plane. Now let $u = e_{\perp}e_{\|}$ where $e_{\|}$ is parallel to $v_{\|}$ and of course $e_{\perp}$ is in the plane u and orthogonal to $e_{\|}$. $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ and $v_{\|}$ anticommutes with $e_{\perp}$ (it is left to the viewer to show $RR^{\dagger} = 1$).

[2] e^A is defined as the Taylor series expansion $e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$ where A is any multivector.

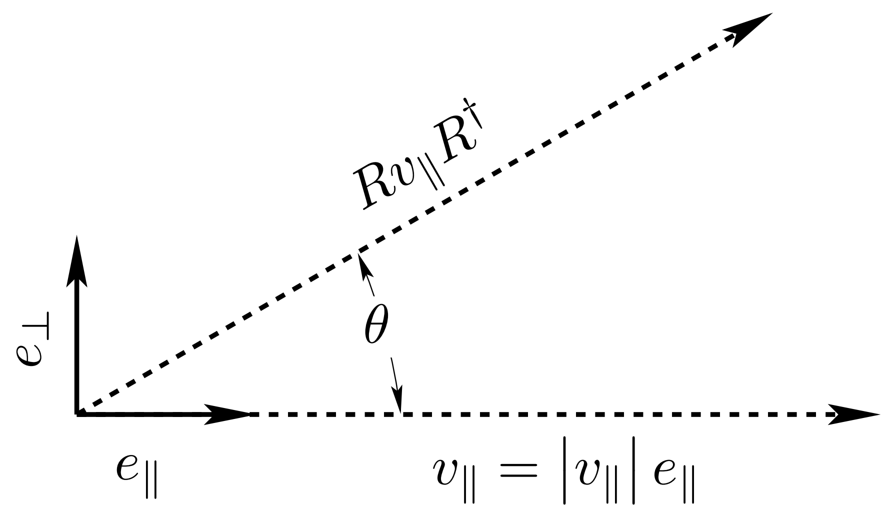
Euclidean Case



$$u^2 = -1$$

$$R = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) u$$

For the case of $u^2 = -1$



$$RvR^\dagger = \left(\cos\left(\frac{\theta}{2}\right) + e_\perp e_\parallel \sin\left(\frac{\theta}{2}\right)\right)\left(v_\parallel + (v - v_\parallel)\right)\left(\cos\left(\frac{\theta}{2}\right) + e_\parallel e_\perp \sin\left(\frac{\theta}{2}\right)\right)$$

Since $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ it commutes with R and

$$RvR^{\dagger} = Rv_{\|}R^{\dagger} + \left(v - v_{\|}\right) \tag{31}$$

So that we only have to evaluate

$$Rv_{\|}R^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sin\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cos\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right) \tag{32}$$

Since $v_{\|} = \left|v_{\|}\right| e_{\|}$

$$Rv_{\|}R^{\dagger} = \left|v_{\|}\right| \left(\cos\left(\theta\right) e_{\|} + \sin\left(\theta\right) e_{\perp}\right) \tag{33}$$

and the component of v in the u plane is rotated correctly.

Minkowski Case

For the case of $u^2 = 1$ there are two possibilities, $v_{\|}^2 > 0$ or $v_{\|}^2 < 0$. In the first case $e_{\|}^2 = 1$ and $e_{\perp}^2 = -1$. In the second case $e_{\|}^2 = -1$ and $e_{\perp}^2 = 1$. Again $v - v_{\|}$ is not affected by the rotation so that we need only evaluate

$$Rv_{\|}R^{\dagger} = \left(\cosh\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sinh\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cosh\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sinh\left(\frac{\theta}{2}\right)\right)$$

Note that in this case $\left|v_{\|}\right| = \sqrt{\left|v_{\|}^2\right|}$ and

$$Rv_{\|}R^{\dagger} = \left\{ \begin{array}{l} v_{\|}^2 > 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} + \sinh\left(\theta\right) e_{\perp}\right) \\ v_{\|}^2 < 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} - \sinh\left(\theta\right) e_{\perp}\right) \end{array} \right\} \tag{34}$$

Lorentz Transformation

We now have all the tools needed to derive the Lorentz transformation with Geometric Algebra. Consider a two dimensional time-like plane with with coordinates t[3] and x and basis vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_x$ $\left(\boldsymbol{e}_t^2 = -\boldsymbol{e}_x^2 = 1\right)$. Then a general space-time vector in the plane is given by

$$x = t\boldsymbol{e}_t + x\boldsymbol{e}_x = t'\boldsymbol{e}'_t + x'\boldsymbol{e}'_x \tag{35}$$

where the basis vectors of the two coordinate systems are related by

$$\boldsymbol{e}'_{[t,x]} = R\boldsymbol{e}_{[t,x]}R^\dagger \tag{36}$$

and R is a Minkowski plane rotor since $\left(\boldsymbol{e}_t\boldsymbol{e}_x\right)^2 = 1$

$$R = e^{\dfrac{\alpha}{2}\boldsymbol{e}_t\boldsymbol{e}_x} = \sinh\left(\frac{\alpha}{2}\right) + \cosh\left(\frac{\alpha}{2}\right)\boldsymbol{e}_t\boldsymbol{e}_x \tag{37}$$

[3]We let the speed of light $c = 1$.

so that

$$R\boldsymbol{e}_t R^\dagger = \cosh(\alpha)\,\boldsymbol{e}_t + \sinh(\alpha)\,\boldsymbol{e}_x \tag{38}$$

and

$$R\boldsymbol{e}_x R^\dagger = \cosh(\alpha)\,\boldsymbol{e}_x + \sinh(\alpha)\,\boldsymbol{e}_t \tag{39}$$

Now consider the special case that the primed coordinate system is moving with velocity β in the direction of $\boldsymbol{e}_x$ and the two coordinate systems were coincident at time $t = 0$. Then $x = \beta t$ and $x' = 0$ so we may write

$$t\boldsymbol{e}_t + \beta t\boldsymbol{e}_x = t' R\boldsymbol{e}_t R^\dagger \tag{40}$$

$$\frac{t}{t'}\left(\boldsymbol{e}_t + \beta\boldsymbol{e}_x\right) = \cosh(\alpha)\,\boldsymbol{e}_t + \sinh(\alpha)\,\boldsymbol{e}_x \tag{41}$$

Equating components gives

$$\cosh(\alpha) = \frac{t}{t'} \tag{42}$$

$$\sinh(\alpha) = \frac{t}{t'}\beta \tag{43}$$

Solving for α and $\dfrac{t}{t'}$ in equations 42 and 43 gives

$$\tanh(\alpha) = \beta \tag{44}$$

$$\frac{t}{t'} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{45}$$

Now consider the general case of x, t and x', t' giving

$$\begin{aligned} t\boldsymbol{e}_t + x\boldsymbol{e}_x &= t'R\boldsymbol{e}_tR^\dagger + x'R\boldsymbol{e}_xR^\dagger && (46) \\ &= t'\gamma\left(\boldsymbol{e}_t + \beta\boldsymbol{e}_x\right) + x'\gamma\left(\boldsymbol{e}_x + \beta\boldsymbol{e}_t\right) && (47) \end{aligned}$$

Equating basis vector coefficients recovers the Lorentz transformation

$$\begin{aligned} t &= \gamma\left(t' + \beta x'\right) \\ x &= \gamma\left(x' + \beta t'\right) \end{aligned} \tag{48}$$

Spinors

The general definition of a spinor is a multivector, $\psi \in \mathcal{G}(p,q)$, such that $\psi v \psi^\dagger \in \mathcal{V}(p,q)\ \ \forall v \in \mathcal{V}(p,q)$. Practically speaking a spinor is the composition of a rotation and a dilation (stretching or shrinking) of a vector by a factor ρ. Thus we can write

$$\psi v \psi^\dagger = \rho R v R^\dagger \tag{49}$$

where R is a rotor $\left(RR^\dagger = 1\right)$. Letting $U = R^\dagger \psi$ we must solve

$$U v U^\dagger = \rho v \tag{50}$$

U must generate a pure dilation. The most general form for U based on the fact that the l.h.s of equation 50 must be a vector is

$$U = \alpha + \beta I \tag{51}$$

so that

$$UvU^\dagger = \alpha^2 v + \alpha\beta\left(Iv + vI^\dagger\right) + \beta^2 IvI^\dagger = \rho v \tag{52}$$

Using $vI^\dagger = (-1)^{\frac{(n-1)(n-2)}{2}} Iv$, $vI^\dagger = (-1)^{n-1} I^\dagger v$, and $II^\dagger = (-1)^q$ we get

$$\alpha^2 v + \alpha\beta\left(1 + (-1)^{\frac{(n-1)(n-2)}{2}}\right) Iv + (-1)^{n+q-1}\beta^2 v = \rho v \tag{53}$$

If $\dfrac{(n-1)(n-2)}{2}$ is even $\beta = 0$ and $\alpha \neq 0$, otherwise $\alpha, \beta \neq 0$. For the odd case

$$\psi = R(\alpha + \beta I) \tag{54}$$

where $\rho = \alpha^2 + (-1)^{n+q-1}\beta^2$. In the case of $\mathcal{G}(1,3)$ (relativistic space time) we have $\rho = \alpha^2 + \beta^2$, $\rho > 0$.

Spinors cannot be represented as tensors since they are the sum of even grade blades!

Expansion of geometric product and generalization of $\cdot$ and $\wedge$

If A_r and B_s are respectively grade r and s pure grade multivectors then

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{\min(r+s, 2N-(r+s))} \tag{55}$$

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|} \tag{56}$$

$$A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s} \tag{57}$$

Thus if $r + s > N$ then $A_r \wedge B_s = 0$, also note that these formulas are the most efficient way of calculating $A_r \cdot B_s$ and $A_r \wedge B_s$.

One can prove that for a vector a and a grade r multivector B_r

$$a \cdot B_r = \frac{1}{2}\left(aB_r - (-1)^r B_r a\right) \tag{58}$$

$$a \wedge B_r = \frac{1}{2}\left(aB_r + (-1)^r B_r a\right) \tag{59}$$

If equations 58 and 59 are true for a grade r blade they are also true for a grade r multivector (superposition of grade r blades).

Duality and the Pseudoscalar

If $e_1, \ldots, e_n$ is an orthonormal basis for the vector space the the pseudoscalar I is defined by

$$I = e_1 \ldots e_n \tag{60}$$

Since one can transform one orthonormal basis to another by an orthogonal transformation the I's for all orthonormal bases are equal to within a ± 1 scale factor with depends on the ordering of the basis vectors.

If A_r is a pure r grade multivector $(A_r = \langle A_r \rangle_r)$ then

$$A_r I = \langle A_r I \rangle_{n-r} \tag{61}$$

or $A_r I$ is a pure $n - r$ grade multivector.

Further by the symmetry properties of I we have

$$I A_r = (-1)^{(n-1)r} A_r I \tag{62}$$

I can also be used to exchange the $\cdot$ and $\wedge$ products as follows

$$a \cdot (A_r I) = (a \wedge A_r)\, I \tag{63}$$

More generally if A_r and B_s are pure grade multivectors with $r + s \leq n$ we have

$$A_r \cdot (B_s I) = (A_r \wedge B_s)\, I \tag{64}$$

Finally we can relate I to $I^\dagger$ by

$$I^\dagger = (-1)^{\frac{n(n-1)}{2}}\, I \tag{65}$$

Reciprocal Frames

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a set of linearly independent vectors that span the vector space that are not necessarily orthogonal. These vectors define the frame (frame vectors are shown in bold face since they are almost always associated with a particular coordinate system) with volume element

$$E_n \equiv \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \tag{66}$$

So that $E_n \propto I$. The reciprocal frame is the set of vectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$ that satisfy the relation

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \quad \forall i, j = 1, \ldots, n \tag{67}$$

The $\mathbf{e}^i$ are constructed as follows

$$\mathbf{e}^j = (-1)^{j-1}\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1} \tag{68}$$

So that the dot product is (using equation 63 since $E_n^{-1} \propto I$)

$$\begin{aligned}
\mathbf{e}_i \cdot \mathbf{e}^j &= (-1)^{j-1}\, \mathbf{e}_i \cdot \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (69)\\
&= (-1)^{j-1} \left(\mathbf{e}_i \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (70)\\
&= 0, \quad \forall i \neq j && (71)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}^1 &= \mathbf{e}_1 \cdot \left(\mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (72)\\
&= \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (73)\\
&= 1 && (74)
\end{aligned}$$

Coordinates

The reciprocal frame can be used to develop a coordinate representation for multivectors in an arbitrary frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with reciprocal frame $\mathbf{e}^1, \ldots, \mathbf{e}^n$.

Since both the frame and it's reciprocal span the base vector space we can write any vector a in the vector space as

$$a = a^i \mathbf{e}_i = a_i \mathbf{e}^i \tag{75}$$

where if an index such as i is repeated it is assumes that the terms with the repeated index will be summed from 1 to n. Using that $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$ we have

$$\begin{aligned} a_i &= a \cdot \mathbf{e}_i && (76) \\ a^i &= a \cdot \mathbf{e}^i && (77) \end{aligned}$$

In tensor notation a_i would be the covarient representation and a^i the contravarient representation of the vector a.

It can be proved that

$$\left(\mathbf{e}^{k_r} \wedge \ldots \wedge \mathbf{e}^{k_1}\right) \cdot \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_r}\right) = \delta^{j_1}_{k_1}\delta^{j_2}_{k_2}\ldots\delta^{j_r}_{k_r} \tag{78}$$

so that the general multivector A can be expanded in terms of the blades of the frame and reciprocal frame as

$$A = \sum_{i<j<\cdots<k} A_{ij\cdots k}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \cdots \wedge \mathbf{e}^k \tag{79}$$

where

$$A_{ij\cdots k} = \left(\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_j \wedge \mathbf{e}_i\right) \cdot A \tag{80}$$

The components $A_{ij\cdots k}$ are totally antisymmetric on all indices and are usually referred to as the components of an *antisymmetric tensor*.

Blade Orientation Theorem

A blade only depends on the relative orientation of the vectors in the plane defined by the blade. Since any blade can be defined by the geometric product of two orthogonal vectors let them be $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$. Then any two vectors in the plane can be define by:

$$a = a_x \boldsymbol{e}_x + a_y \boldsymbol{e}_y \tag{81}$$

$$b = b_x \boldsymbol{e}_x + b_y \boldsymbol{e}_y \tag{82}$$

and any rotor in the plane by

$$R = ab = (a \cdot b) + (a_x b_y - a_y b_x)\, \boldsymbol{e}_x \boldsymbol{e}_y = (a \cdot b) + (a \times b)\, \boldsymbol{e}_x \boldsymbol{e}_y \tag{83}$$

as long as

$$RR^\dagger = 1 \tag{84}$$

but

$$R\boldsymbol{e}_x\boldsymbol{e}_y = \boldsymbol{e}_x\boldsymbol{e}_y R \tag{85}$$

and

$$R\boldsymbol{e}_x R^\dagger R\boldsymbol{e}_y R^\dagger = R\boldsymbol{e}_x\boldsymbol{e}_y R^\dagger = \boldsymbol{e}_x\boldsymbol{e}_y R R^\dagger = \boldsymbol{e}_x\boldsymbol{e}_y \tag{86}$$

and absolute orientations of $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ does not matter for $\boldsymbol{e}_x\boldsymbol{e}_y$.