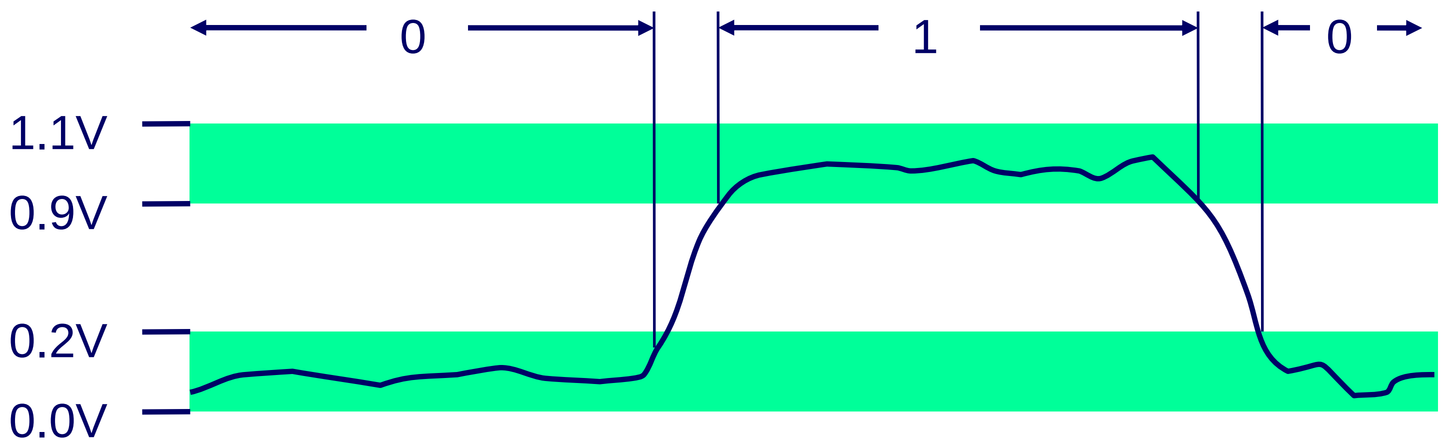
Computer Systems Organization

Topic 2

Based on chapter 2 from Computer Systems by Randal E. Bryant and David R. O'Hallaron

Everything is bits

- Each bit is 0 or 1 (binary digits)
- Form basis of digital revolution
- Why bits? Electronic Implementation
  - Storing/performing computations is simple/reliable
  - Easy to store with bistable elements
  - Reliably transmitted on noisy and inaccurate wires



Everything is bits

- Single bit may not be useful but bit patterns do (groups of bits)
- 3 important representations of numbers
  - Unsigned encodings based on binary notation
  - Two's complement encoding to represent signed integers
  - Floating point encoding are base-2 version for representing real numbers
- Limited number of bits to encode numbers can have surprising effects e.g., 200 * 300 * 400 * 500 yields -884901888 (32 bit representation)

Binary representation

- Base 2 Number Representation
  - Represent 15213_{10} as 11101101101101_2
  - Represent 1.20_{10} as $1.0011001100110011[0011]..._2$
  - Represent $1.5213 \text{ X } 10^4$ as $1.1101101101101_2 \text{ X } 2^{13}$

How to convert

- $11 = (1011)_2 = 2^3*1 + 2^2*0 + 2^1*1 + 2^0*1$
- 11/2 = 5 (1)
- 5/2 = 2 (1)
- 2/2 = 1 (0)
- 1/2 = 0 (1)

- $12 = (1100)_2 = 2^3*1 + 2^2*1 + 2^1*0 + 2^0*0$
- 12/2 = 6(0)
- 6/2 = 3(0)
- 3/2 = 1(1)
- 1/2 = 0(1)

How to convert (fraction)

- Convert 0.8125
- .8125 * 2 = 1.6250 (1)
- .6250 * 2 = 1.250 (1)
- .250 * 2 = 0.5 (0)
- .5 * 2 = 1.0 (1)
- 0 * 2 = 0 (0)
- Soln: 0.11010
- Converting back: 1*2^-1 + 1*2^-2 + 0 + 1*2^-3 + 0 = .5 + .25 + 0 + .0625 + 0 = 0.8125

Encoding Byte Values

- Byte = 8 bits
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters '0' to '9' and 'A' to 'F'
    - Write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Hex Notation

- 314156 = 19634.16 + 12
- 19634 = 1227.16 + 2
- 1227 = 76.16 + 11
- 76 = 4.16 + 12
- 4 = 0.16 + 4
- Hence 0x4CB2C
- What is binary and hex notation for 158 ?
- What is 0x605c + 0x5 ?
- Please practice…

Data sizes

- Each computer has word size indicating nominal size of pointer data
- Virtual address is encoded by such a word, hence word size determines size of virtual address space
- For machine with w-bit word size, virtual address can range from 0 to (2^w)-1, giving program access to at most 2^w bytes
- 32 bit word size limits virtual address space to 4GB [4*10^9 bytes] while 64 bit leads to 16 exabytes i.e., 1.84 * 10^19 bytes
- Most 64-bit machines can run programs compiled to use for 32-bit machines i.e., backward compatible
- 32 bit vs. 64 bit programs [rather than machine distinction lies in how the program is compiled]

Typical Data Representations in C

C Data Type	Typical 32-bit	Typical 64-bit
`char`	1	1
`short`	2	2
`int`	4	4
`long`	4	8
`float`	4	4
`double`	8	8
`long double`	−	−
pointer	4	8

Most data types encode signed values unless prefixed by unsigned

Exception for char – need to declare signed char

Addressing and Byte Ordering

- In almost all machines, a multi-byte object stored as a contiguous sequence of bytes
  – E.g., variable x of type int has address 0x100 i.e., address expression &x is 0x100. Assuming int is 4 bytes, x would be stored in location 0x100, 0x101, 0x102 and 0x103.
- Two notations – big endian vs. little endian
- Number 0x01234567 stored as {01}{23}{45}{67} in big endian notation while stored as {67}{45}{23}{01} in little endian notation for the addresses 0x100, 0x101, 0x102, 0x103.
- Most intel compatible machines are little endian while most IBM/Oracle machines are big endian

Representing code

- Consider the following C function:

```
int sum(int x, int y) {
    Return x + y;
}
```

- Following machine code generated when compiled
- Linux 32:   55 89 e5 8b 45 0c 03 45 08 c9 c3
- Windows: 55 89 e5 8b 45 0c 03 45 08 5d c3
- Instruction codings can be different – binary code is seldom portable across combinations of machines + OS
- From machine perspective – program is simply a sequence of bytes and has no/minimal information of original source program

Boolean Algebra

- Developed by George Boole in 19th Century
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- **A&B = 1 when both A=1 and B=1**

&	0	1
0	0	0
1	0	1

Or

- **A|B = 1 when either A=1 or B=1**

|	0	1
0	0	1
1	1	1

Not

- **~A = 1 when A=0**

~	
0	1
1	0

Exclusive-Or (Xor)

- **A^B = 1 when either A=1 or B=1, but not both**

^	0	1
0	0	1
1	1	0

General Boolean Algebras

- Operate on Bit Vectors (strings of 0's and 1's of fixed length w) - operations applied bitwise

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  01000001     01111101     00111100     10101010
```

- All of the Properties of Boolean Algebra Apply
- Let a and b denote bit vectors [a_w-1, a_w-2, ..., a_0] and [b_w-1, b_w-2, ..., b_0].
- a&b would be a bit vector of length w, where the ith element would be a_i&b_i

Representing & Manipulating Sets

- Representation
  - Can encode any subset {0, …, w–1} with a bit vector [a_w-1, a_w-2, …, a_0]
  - represents subsets of Width w
  - $a_j = 1$ if $j \in A$

    - 01101001 { 0, 3, 5, 6 }
    - *76543210*

    - 01010101 { 0, 2, 4, 6 }
    - *76543210*
- Operations
  - & Intersection 01000001 { 0, 6 }
  - | Union 01111101 { 0, 2, 3, 4, 5, 6 }
  - ^ Symmetric difference 00111100 { 2, 3, 4, 5 }
  - ~ Complement 10101010 { 1, 3, 5, 7 }

Bit-Level Operations in C

- Operations & (AND), | (OR), ~ (NOT), ^ (Exclusive-OR) Available in C
  - Apply to any "integral" data type
    - long, int, short, char, unsigned
  - View arguments as bit vectors
- Examples (Char data type)
  - ~0x41 is 0xBE
    - $\sim 01000001_2$ is 10111110_2
  - ~0x00 is 0xFF
    - $\sim 00000000_2$ is 11111111_2
  - 0x69 & 0x55 is 0x41
    - 01101001_2 & 01010101_2 is 01000001_2
  - 0x69 | 0x55 is 0x7D
    - 01101001_2 | 01010101_2 is 01111101_2

Contrast: Logic Operations in C

- Contrast to Logical Operators
  - && (AND), || (OR), ! (NOT)
    - View 0 as "False"
    - Anything nonzero as "
    - Always return
    - Early termina
- Examples (ch
  - !0x41 is 0x00
  - !0x00 is 0x01
  - !!0x41 is 0x01
  - 0x69 && 0x55 is
  - 0x69 || 0x55 is 0x01
  - p && *p - avoids null pointer access since logical operators do not evaluate second argument if result can be determined with first
  - Similarly a && 5/a will never cause a division by 0

Watch out for && vs. & (and || vs. |)… logical vs. bit level operators one of the more common oopsies in C programming

Shift Operations

- Left Shift: `x << y`
  - Shift bit-vector **x** left **y** positions
    - Throw away extra bits on left
    - Fill with 0's on right
- Right Shift: `x >> y`
  - Shift bit-vector **x** right **y** positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- Undefined Behavior
  - Shift amount < 0 or ≥ word size

Argument **x**	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument **x**	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

Integer Representations: Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- C `short` 2 bytes long

	Decimal	Hex	Binary
x	15213	**3B 6D**	**00111011 01101101**
y	-15213	**C4 93**	**11000100 10010011**

- Sign Bit
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Two’s-complement Encoding Example

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

- Unsigned Values
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1

- Two's Complement Values
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1
- Other Values
  - Minus 1
    111…1

Values for w = 16 bits

	Decimal	Hex	Binary
UMax	**65535**	**FF FF**	11111111 11111111
TMax	**32767**	**7F FF**	01111111 11111111
TMin	**-32768**	80 00	10000000 00000000
-1	**-1**	**FF FF**	11111111 11111111
0	**0**	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- Observations
  - $|TMin| = TMax + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$

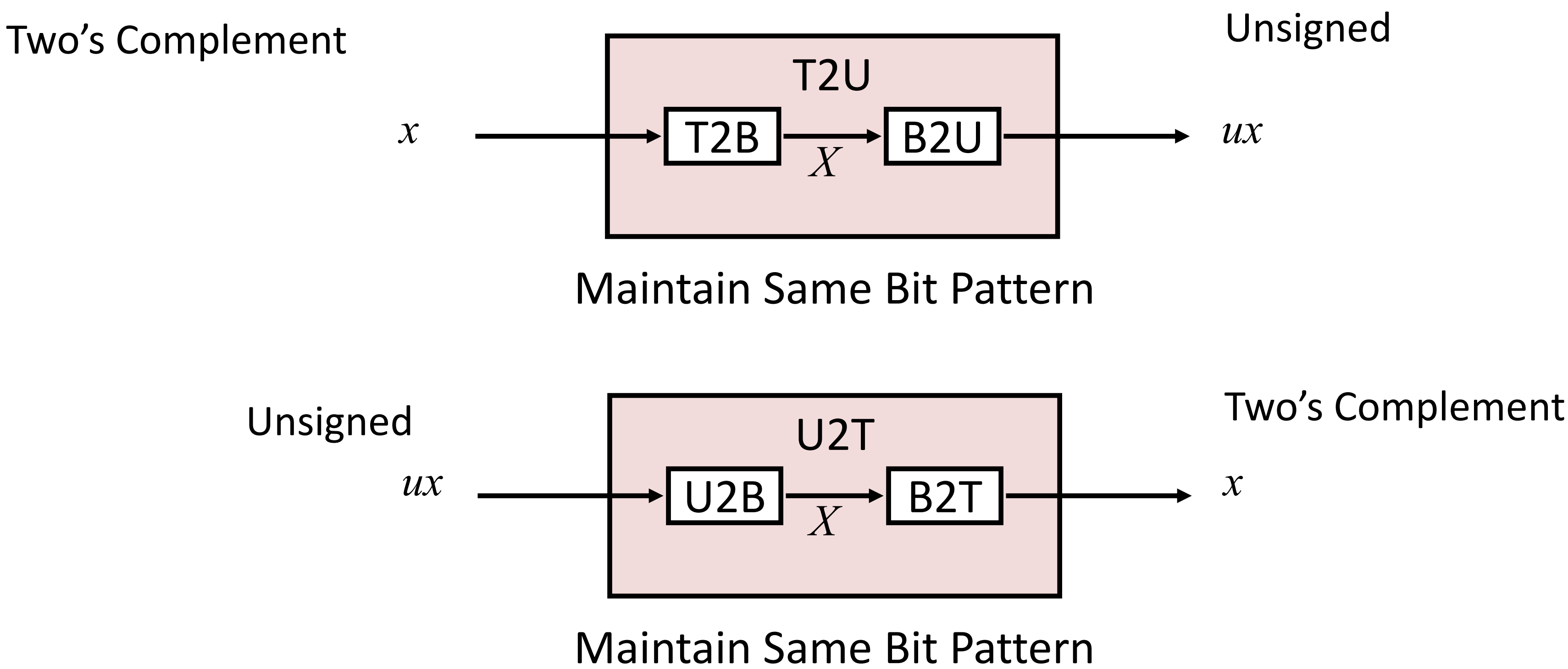
- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values platform specific

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- Equivalence
  - Same encodings for nonnegative values
- Uniqueness
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- $\Rightarrow$ Can Invert Mappings
  - $\mathrm{U2B}(x) = \mathrm{B2U}^{-1}(x)$
    - Bit pattern for unsigned integer
  - $\mathrm{T2B}(x) = \mathrm{B2T}^{-1}(x)$
    - Bit pattern for two's comp integer

Mapping Between Signed & Unsigned



- Mappings between unsigned and two's complement numbers:
  Keep bit representations and reinterpret

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

$\rightarrow$ T2U $\rightarrow$

$\leftarrow$ U2T $\leftarrow$

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

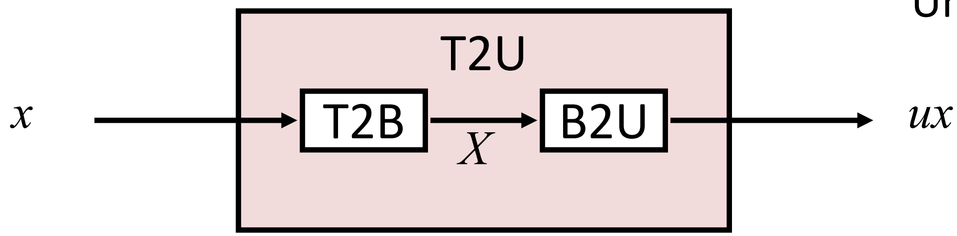
=

+/- 16

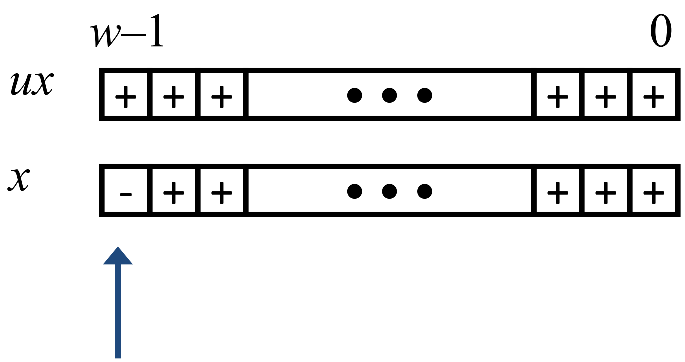
Relation between Signed & Unsigned

Two's Complement

Unsigned



Maintain Same Bit Pattern

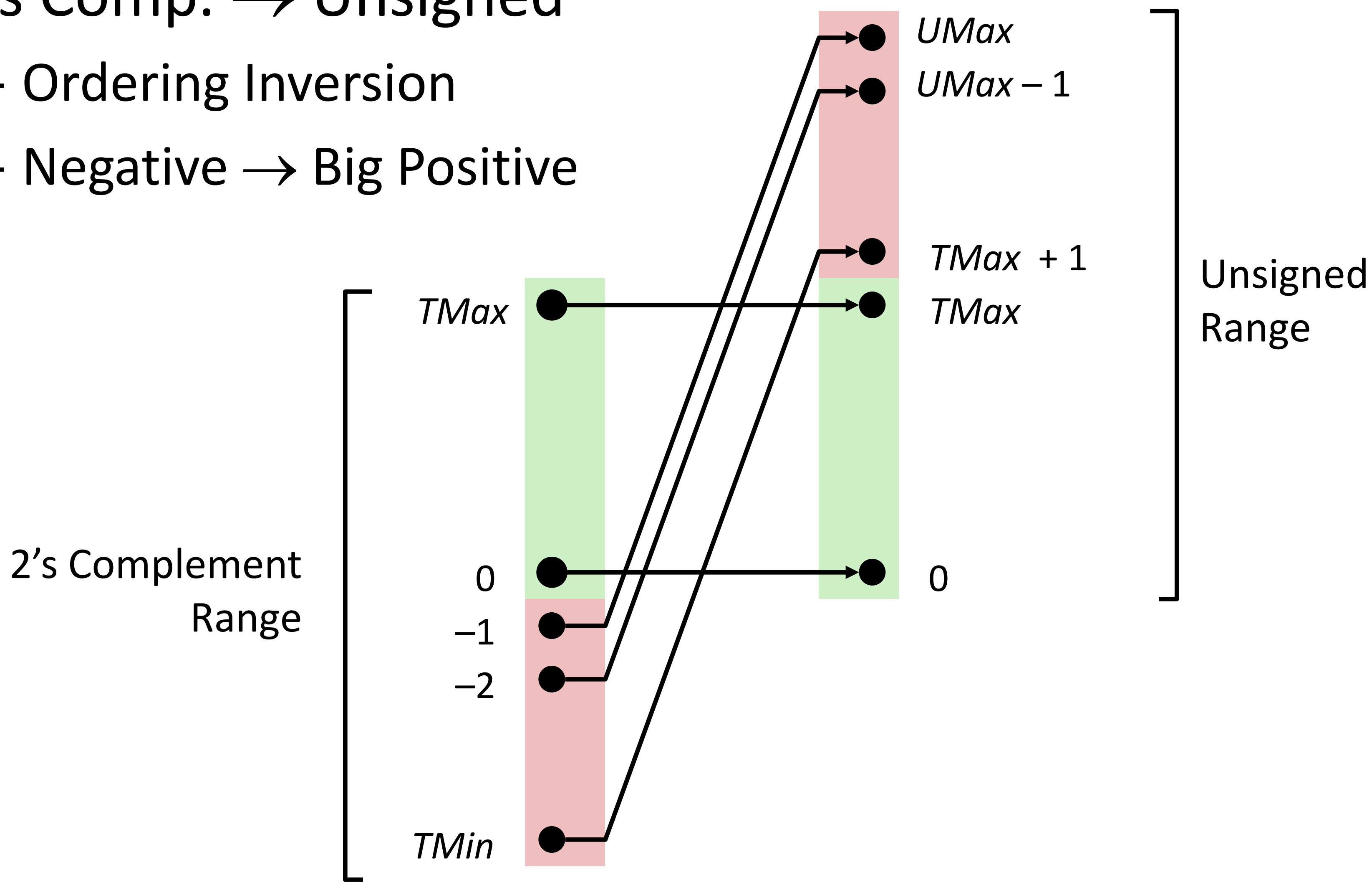


Large negative weight
becomes
Large positive weight

Add 2^w if x < 0,
otherwise remains same

Conversion Visualized

- 2's Comp. → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Signed vs. Unsigned in C

- Constants
  - By default are considered to be signed integers
  - Unsigned if have "U" as suffix

    0U, 4294967259U
- Casting
  - Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

  - Implicit casting also occurs via assignments and procedure calls

```
tx = ux; /* cast to signed */
uy = ty; /* cast to unsigned */
```

Casting Surprises

- Expression Evaluation
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations **<, >, ==, <=, >=**
  - Examples for $W = 32$: **TMIN = -2,147,483,648 , TMAX = 2,147,483,647**

Casting Surprises

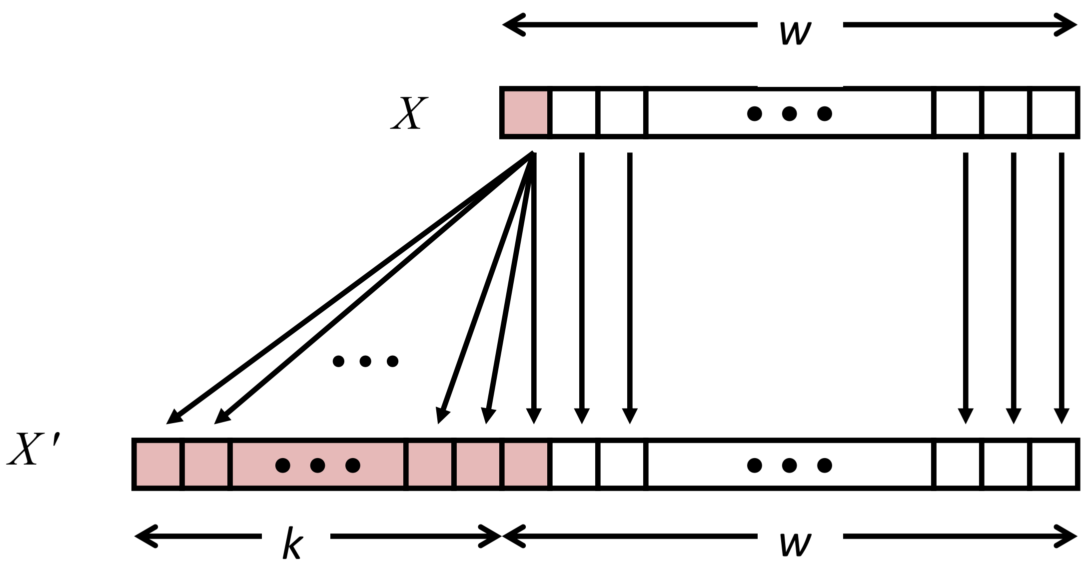
Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	==	Unsigned
-1	0	<	Signed
-1	0U	>	Unsigned
2147483647	-2147483647-1	>	Signed
2147483647U	-2147483647-1	<	Unsigned
-1	-2	>	Signed
(unsigned)-1	-2	>	Unsigned
2147483647	2147483648U	<	Unsigned
2147483647	(int) 2147483648U	>	signed

Summary: Casting Signed ↔ Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- Expression containing signed and unsigned int
  - `int` is cast to `unsigned`!!

Sign Extension

- Task:
  - Given *w*-bit signed integer *x*
  - Convert it to *w*+*k*-bit integer with same value
- Rule:
  - Make *k* copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Truncation of number

- When truncating a w bit number to a k-bit number, we drop the high order w-k bits
- Result: x’ = x mod 2^k
- While similar property holds for twos-complement, it converts the most significant bit into a sign bit

```
int x = 53191
short sx = (short) x  /* -12345 */
int y = sx;           /* -12345 */
```

Truncation of number

- Summary:
- $B2U_k$([xk-1, xk-1, …, x0]) = $B2U_k$([xw-1, xw-2, …, x0]) mod 2^k
- $B2T_k$([xk-1, xk-1, …, x0]) = $U2T_k$(B2U([xw-1, xw-2, …, x0]) mod 2^k)

Summary: Expanding, Truncating: Basic Rules

- Expanding (e.g., short int to int)
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result

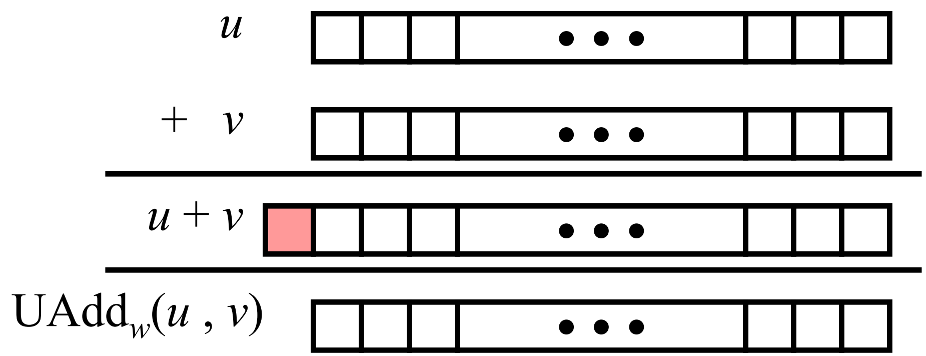
- Truncating (e.g., unsigned to unsigned short)
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behavior

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



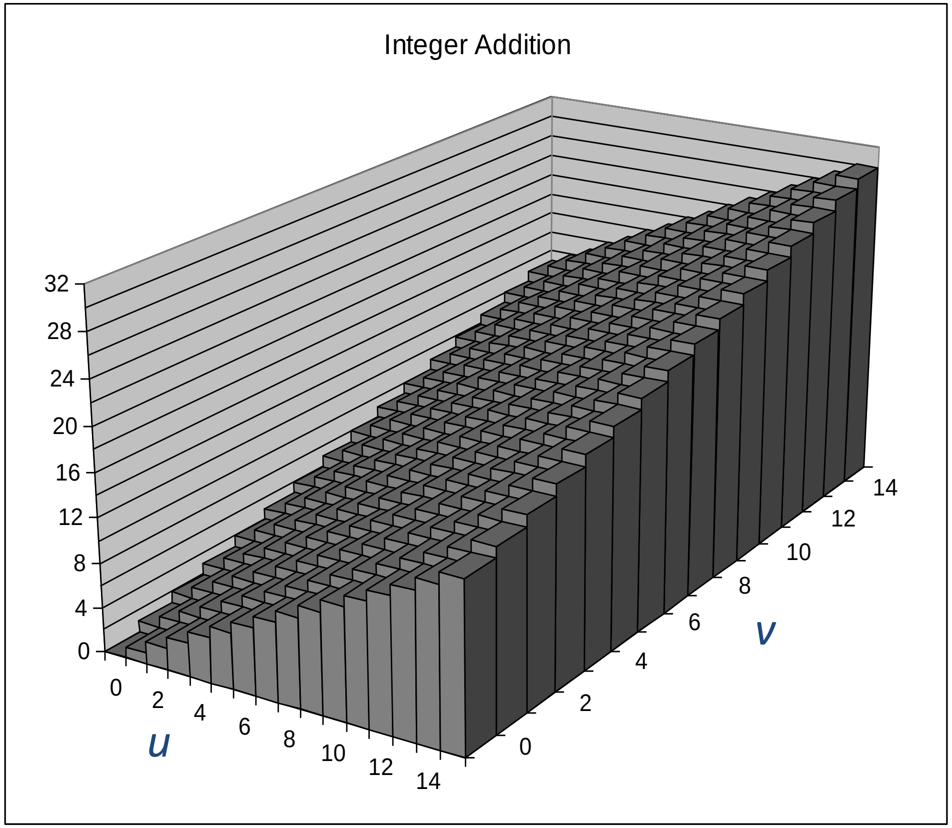
- Standard Addition Function
  - Ignores carry output
- Implements Modular Arithmetic

$$s \;=\; \mathrm{UAdd}_w(u\,,\,v) \;=\; u + v \mod 2^w$$

Visualizing (Mathematical) Integer Addition

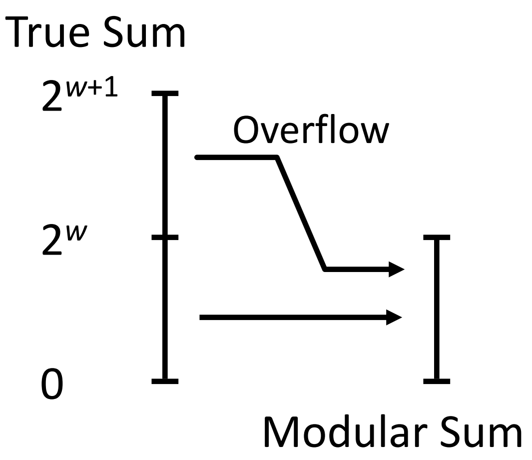
- Integer Addition
  - 4-bit integers *u*, *v*
  - Compute true sum $\text{Add}_4(u\ ,\ v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

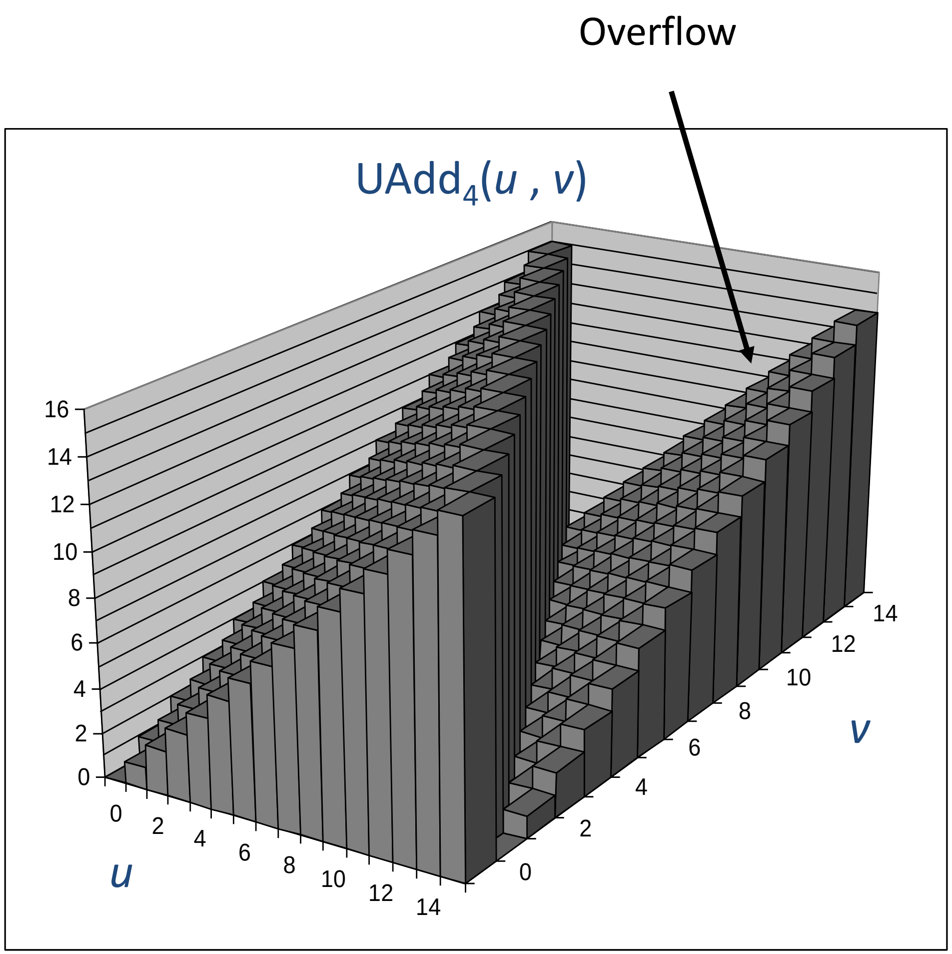
$\text{Add}_4(u\ ,\ v)$



Visualizing Unsigned Addition

- Wraps Around
  - If true sum ≥ 2^w
  - At most once
  - Decrements by 2^w



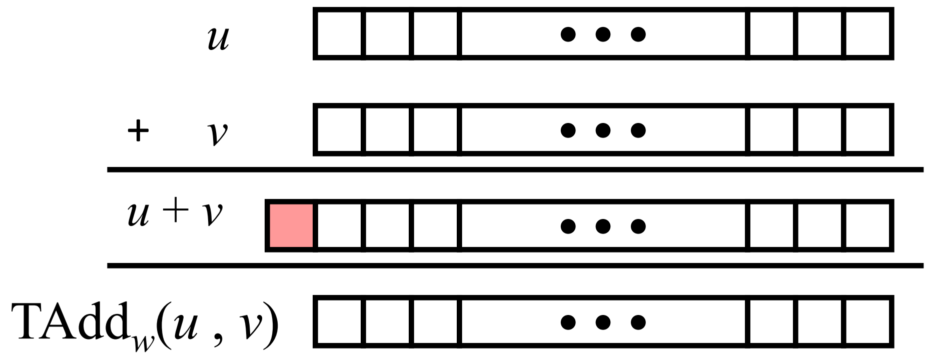


Two’s Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



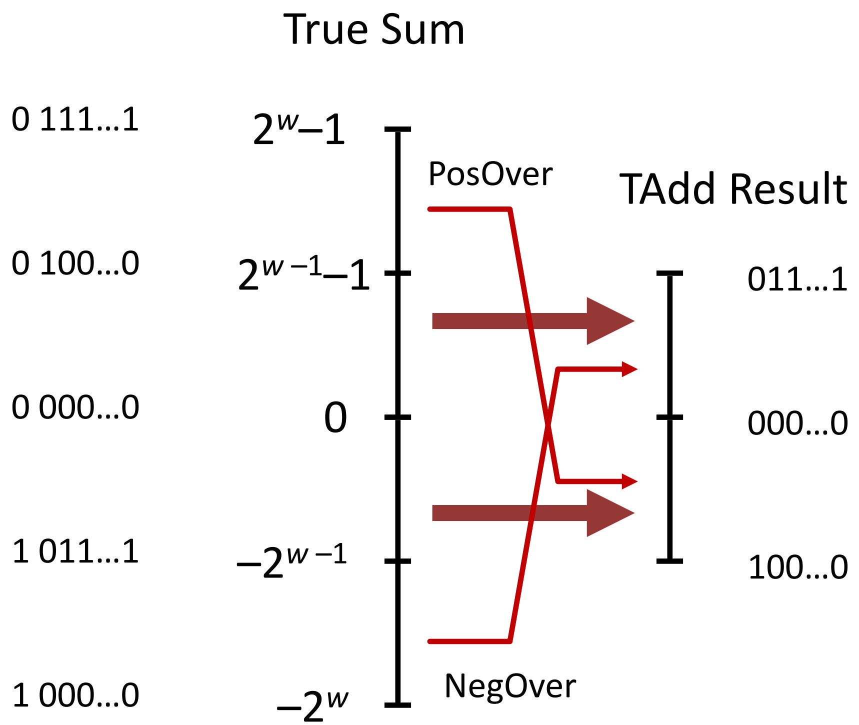
- TAdd and UAdd have Identical Bit-Level Behavior
  - Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

TAdd Overflow

- Functionality
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer

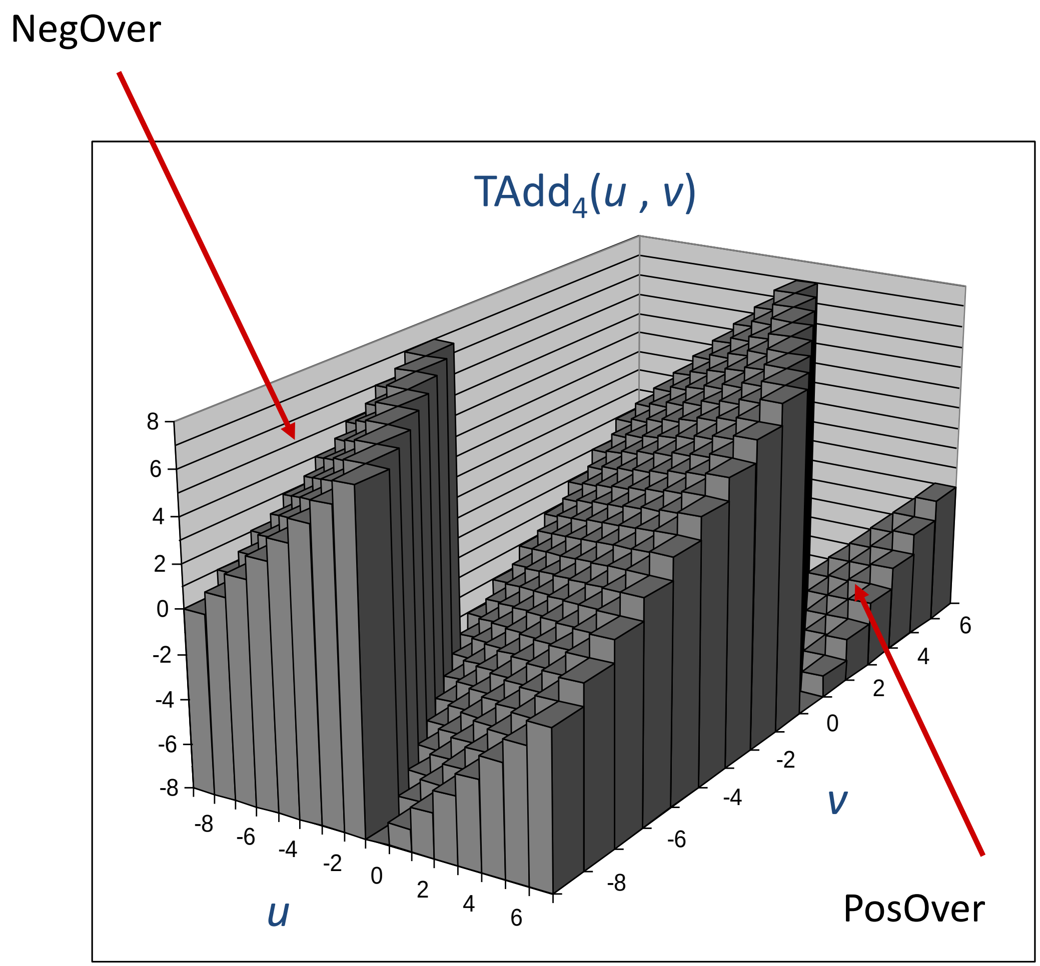


Two’s Complement Addition

- In summary, subtract 2^w if positive overflow
- Add 2^w if negative overflow
- No changes if 2^(w-1) <= sum < 2^(w-1)
- For w = 4 bits,
- -8 [1000] + -5 [1011] = -13 [10011] = 3 [0011]
- 5 [0101] + 5 [0101] = 10 [01010] = -6 [1010]

Visualizing 2's Complement Addition

- Values
  - 4-bit two's comp.
  - Range from -8 to +7
- Wraps Around
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Two's Complement Negation

- Complement the bits, increment the result by 1
- 0101 [5] → 1010 [-6] → 1011 [-5]
- 1000 [-8] → 0111 [7] → 1000 [-8]
- …

Multiplication

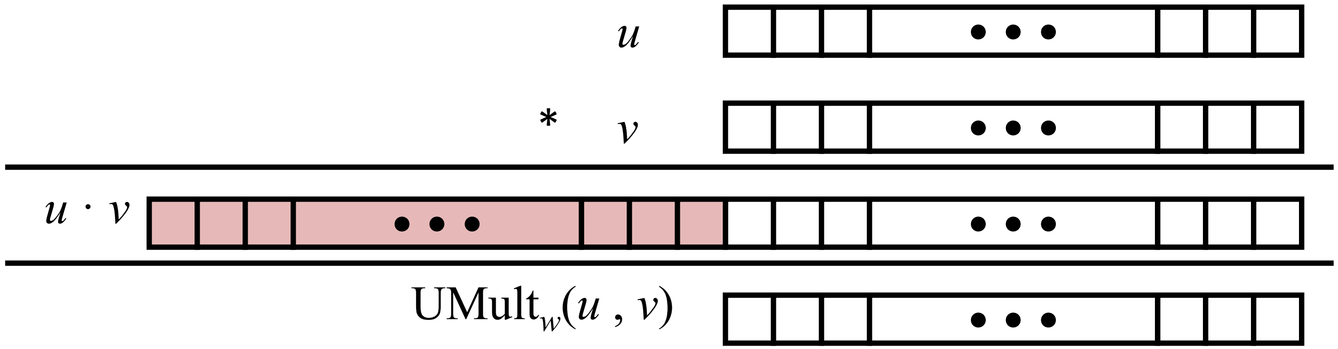
- Goal: Computing Product of w-bit numbers x, y
  - Either signed or unsigned
- But, exact results can be bigger than w bits
  - Unsigned: up to $2w$ bits
    - Result range: $0 \le x * y \le (2^{w} - 1)^{2} = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to $2w$-1 bits
    - Result range: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to $2w$ bits, but only for $(TMin_{w})^{2}$
    - Result range: $x * y \le (-2^{w-1})^{2} = 2^{2w-2}$
- So, maintaining exact results…
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2*w bits

Discard *w* bits: *w* bits



- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic

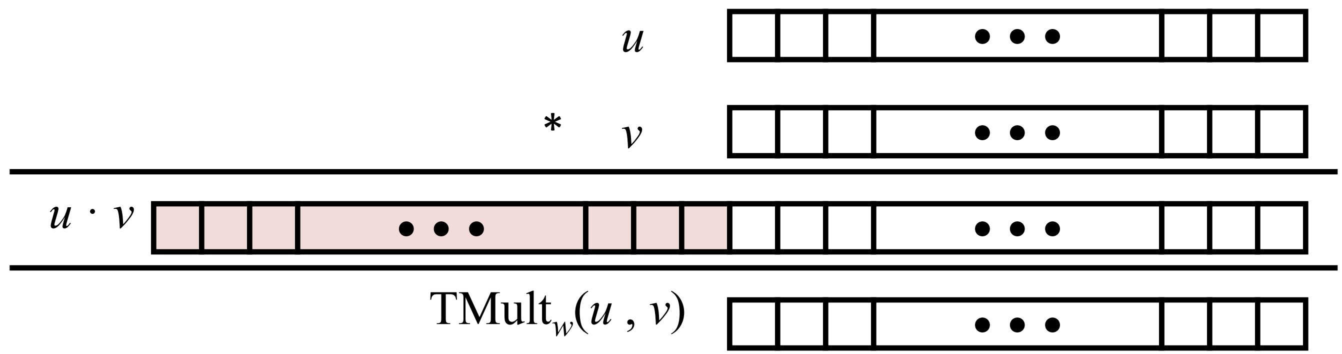
  $\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$

Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- Standard Multiplication Function
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

Example

- Unsigned: 5 [101] * 3 [011] = 15 [01111] → 7 [111] Truncated
- 101
- 011
- 101
- 101
- 000

- 01111

Example

- Two’s C: -3 [101] * 3 [011] = -9 [110111] → -1 [111] Truncated
- Need to sign extend and then multiply
- 111101
- 000011
- 111101
- 111101
- 000000
- 000000
- 000000
- 000000
- ----------------
- 000101**110111**

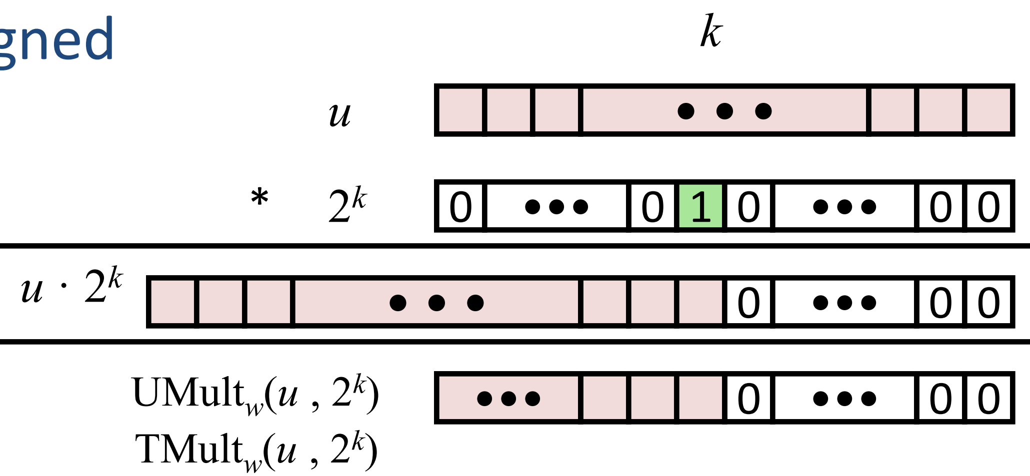
Power-of-2 Multiply with Shift

- Operation
  - **`u << k`** gives **`u * 2`**^{k}
  - Both signed and unsigned

Operands: *w* bits

True Product: *w*+*k* bits

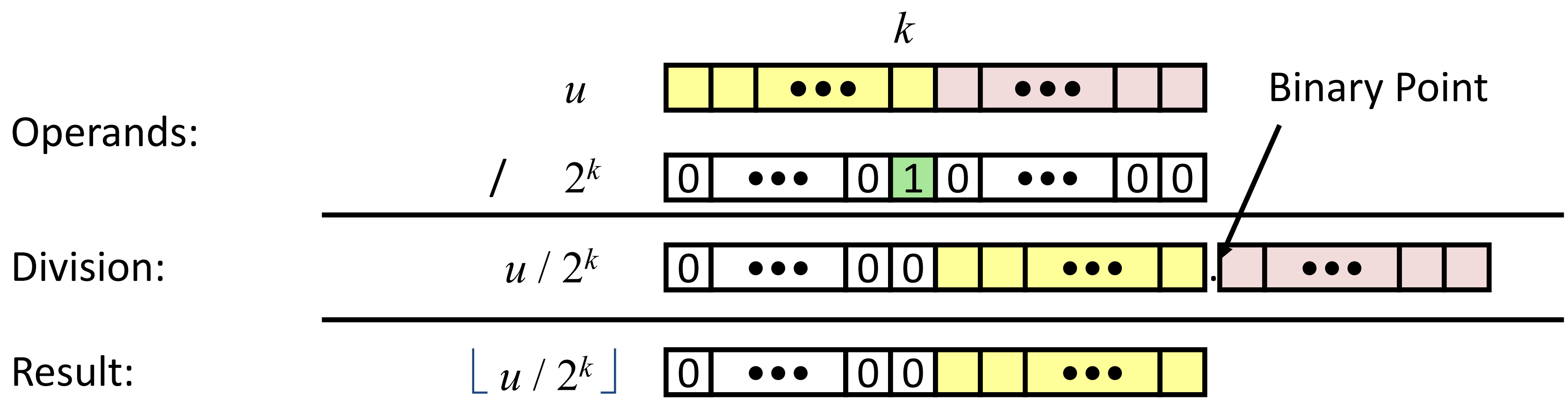
Discard *k* bits: *w* bits



- Examples
  - **`u << 3 == u * 8`**
  - **`(u << 5) - (u << 3) == u * 24`**
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

- Quotient of Unsigned by Power of 2
  - `u >> k` gives $\lfloor$ `u / 2`^{k} $\rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Arithmetic: Basic Rules

- Addition:
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2^w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2^w

- Multiplication:
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Using Unsigned

- *Don't* use without understanding implications
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```