From Exact Computation to Geometric Interpretation: ①

$$\dot{x} = \sin(x) \quad \text{where} \quad \dot{x} = \frac{dx}{dt}$$
$$x(0) = x_0$$

So the ODE is seperable, where

$$\frac{dx}{\sin(x)} = dt \quad \text{or} \quad \int_{x_0}^{x} \frac{du}{\sin(u)} = \int_0^t ds = t$$

↳ So if $\displaystyle\int \frac{du}{\sin(u)} = C - \ln|\csc(x) + \cot(x)|$

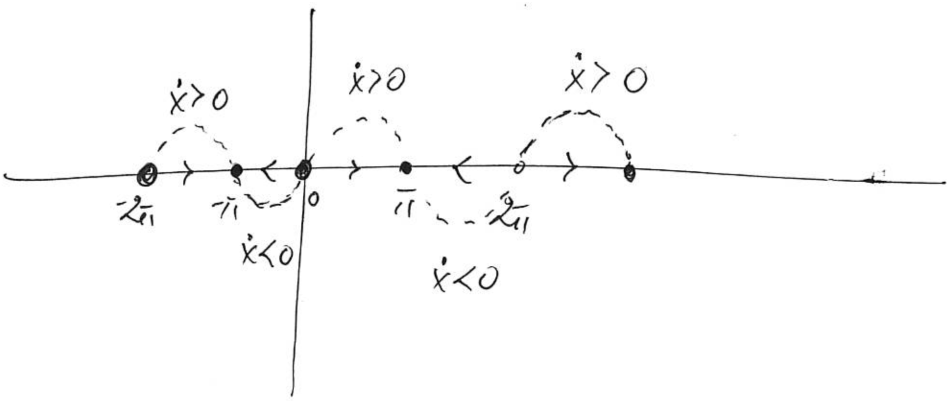
↳ $\displaystyle\int_{x_0}^{x} \frac{du}{\sin(u)} = \ln\left|\frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)}\right| = t$

So okay, you've "solved" the problem...

but so what?

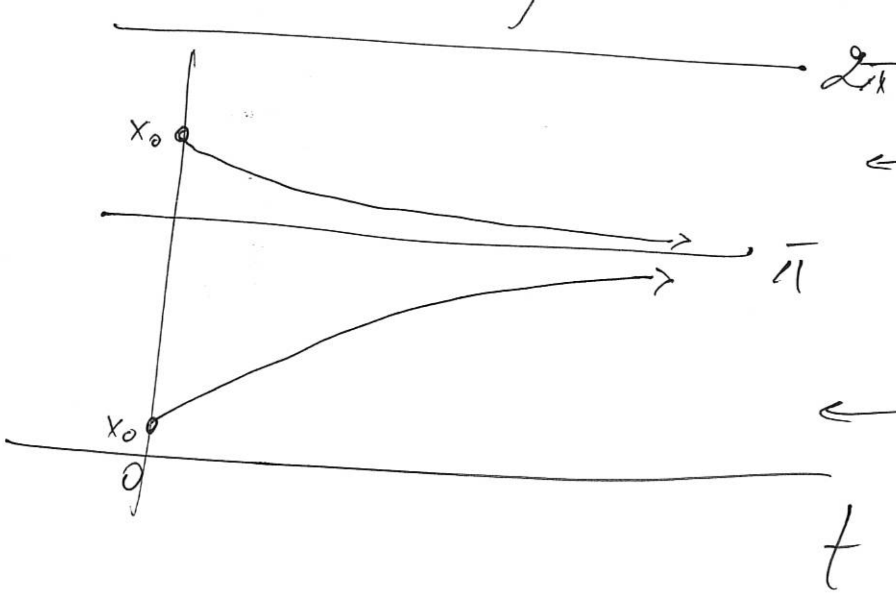
Instead:

$\dot{x} = \sin(x)$



So we see, while we do not know the exact details we see that as $t \to \infty$, we move away from the points $x_* = \pm 2n\pi$ towards the points $x_* = \pm(2n+1)\pi$

So without really doing much of anything, we get



$\leftarrow$ if $x_0 \in (\pi, 2\pi)$

$\hookrightarrow x(t) \to \pi$

$\leftarrow$ if $x_0 \in (0, \pi)$

$\hookrightarrow x(t) \to \pi$

We note that since $\dot{x} = \sin(x)$

$$\dot{x}\Big|_{x = \pm 2n\pi,\ \pm(2n+1)\pi} = 0$$

we call these points at which $\dot{x} = 0$

fixed points.

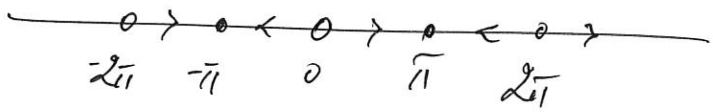
Fixed Point:

For $\dot{x} = f(x)$, any point x_*:

$$f(x_*) = 0$$

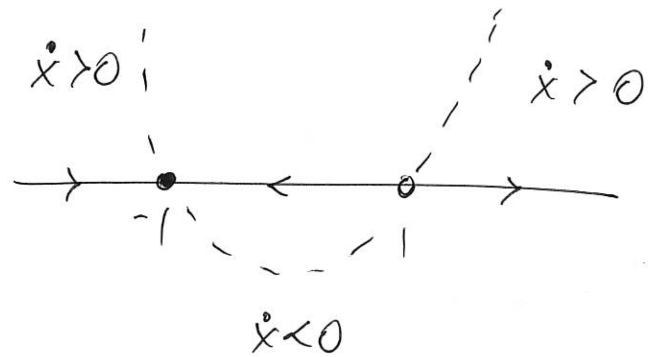
So we see everything is organized around those fixed points which <u>repel</u> ($x_* = \pm 2n\pi$) and those which <u>attract</u> ($x_* = \pm(2n+1)\pi$)

We call the plot



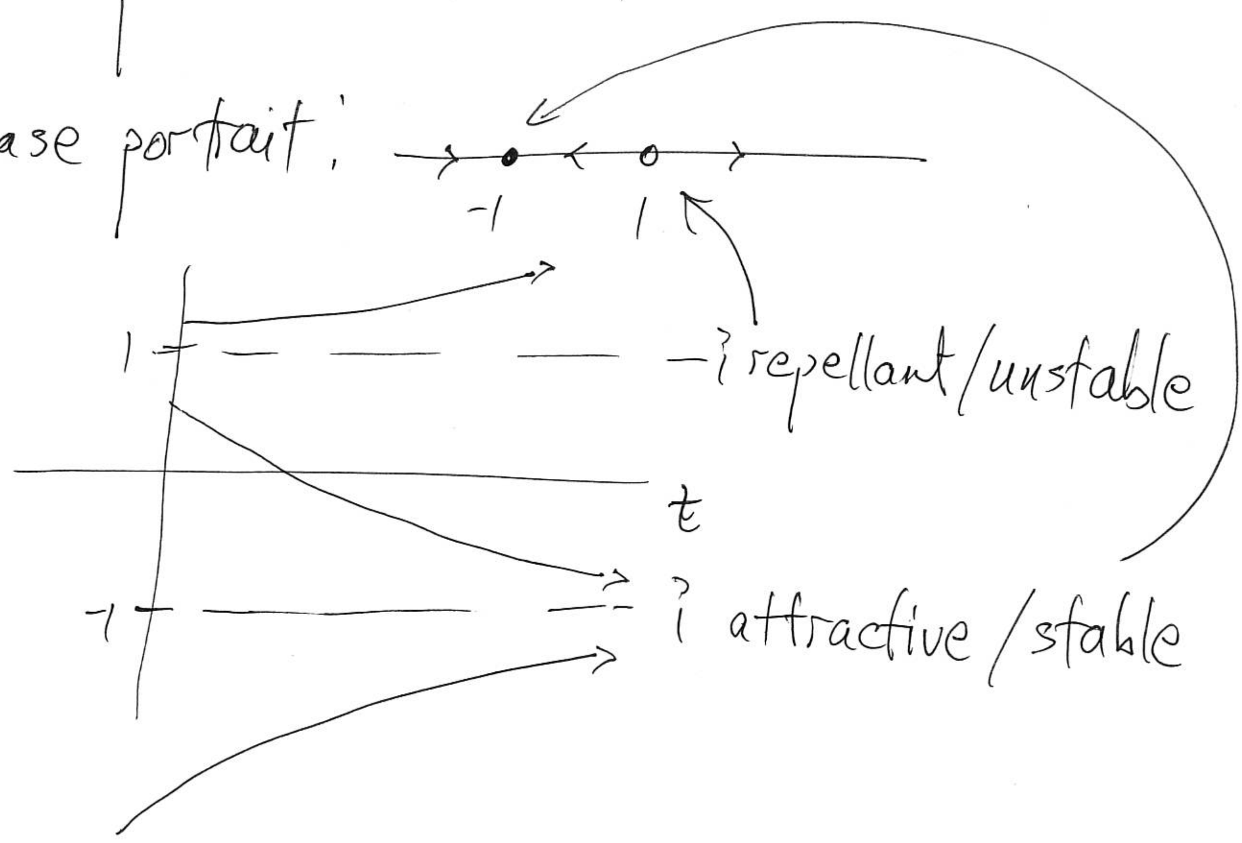
a "phase portrait".

Let $\dot{x} = x^2 - 1 \longrightarrow$

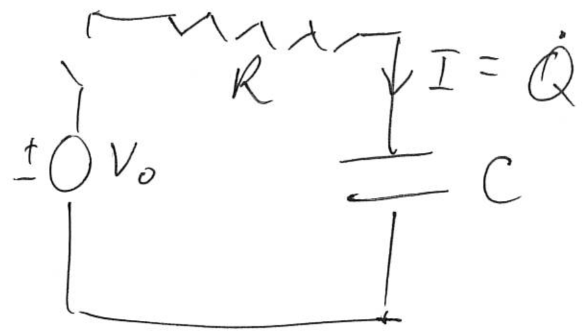


So fixed points: $x_* = \pm 1$

phase portrait:



R-C Circuit



Charge: $Q = Q(t)$

Current: $I = \dot{Q}$

Resistance: $V = IR$

Capacitance: $C = Q/V$

Kirchoff's Law: Potential Drop around loop is zero.
(Conservation of Energy)

$$V_0 - IR - Q/C = 0$$

$$\to V_0 - \dot{Q}R - Q/C = 0$$

or $$\dot{Q} = -\frac{Q}{RC} + \frac{V_0}{R}$$

Method of Integrating Factors

$$\dot{x} + p(t)x = f(t)$$

Let $G(t) = e^{\int_{t_0}^{t} p(s)\,ds} \rightarrow \dot{G} = pG$

↳ $G\dot{x} + pxG = f(t)G(t)$

↳ $\frac{d}{dt}(xG) = f(t)G(t)$

↳ $x(t)G(t) - x(t_0) = \int_{t_0}^{t} f(u)G(u)\,du$

or: $x(t) = x_0 e^{-\int_{t_0}^{t} p(s)\,ds} + e^{-\int_{t_0}^{t} p(s)\,ds} \int_{t_0}^{t} f(u)\, e^{\int_{t_0}^{u} p(s)\,ds}\,du$

So

$$\dot{Q} + \frac{1}{RC} Q = \frac{V_0}{R}$$

$\to$ $p = \frac{1}{RC}$; $f = \frac{V_0}{R}$

$$e^{\int_0^t p(s)\,ds} = e^{t/RC}$$

$$\to Q(t) = Q_0 e^{-t/RC} + e^{-t/RC} \int_0^t \frac{V_0}{R} e^{u/RC} du$$

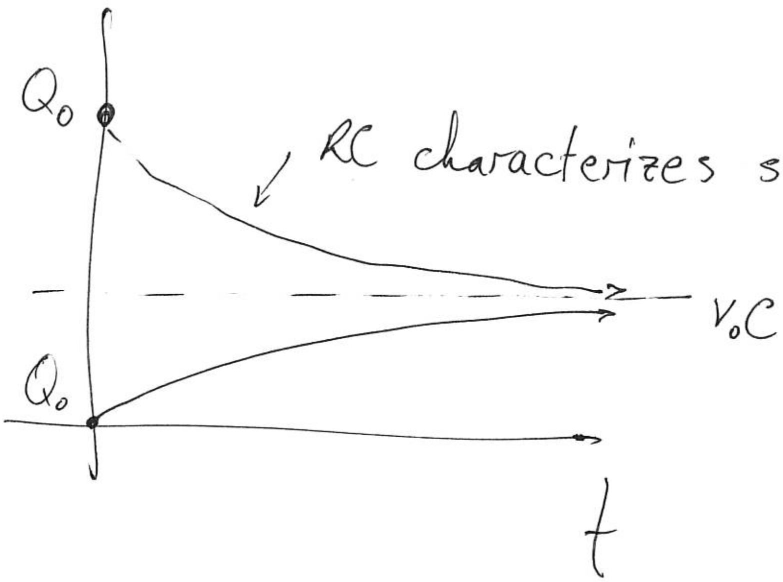
$$= Q_0 e^{-t/RC} + \frac{V_0}{R}(RC) e^{-t/RC} \left\{ e^{t/RC} - 1 \right\}$$

$$= Q_0 e^{-t/RC} + V_0 C \left(1 - e^{-t/RC}\right)$$

So, decay why?

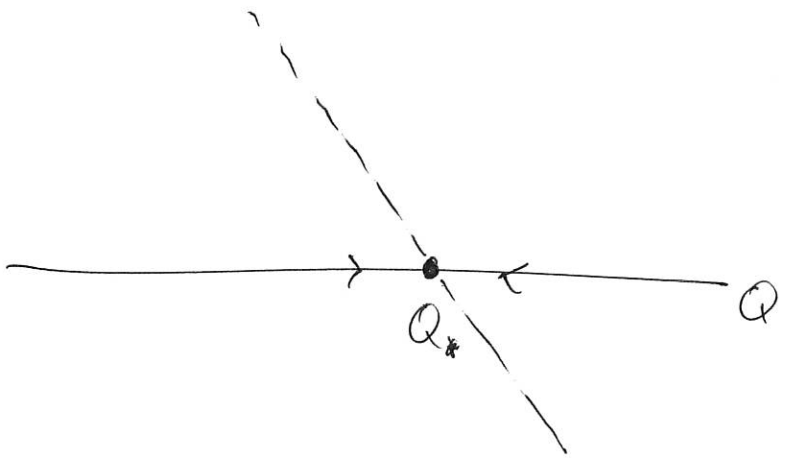
Now we know something physical!

time scale for decay: RC



Now: $\dot{Q} = \dfrac{V_0}{R} - \dfrac{Q}{RC}$, $\quad \dfrac{V_0}{R} - \dfrac{Q_*}{RC} = 0$

$\hookrightarrow Q_* = V_0 C$



So phase plane certainly tells us about the attractive fixed point, but <u>not RC</u>.

Scaling:

So how else can we see RC?

$$Q(t) = Q_s \tilde{Q}(t)$$

$\tilde{Q}$: no units, i.e. non-dimensionalized

Q_s: units of charge

$$\tau = t/t_s$$

t_s: units of time

τ: no units

For t, we have the rule:

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \frac{1}{t_s}\frac{d}{d\tau}$$

ODE Becomes:

$$\frac{Q_s}{t_s}\frac{d\tilde{Q}}{d\tau} = \frac{V_0}{R} - \frac{Q_s}{RC}\tilde{Q}$$

$$\Rightarrow \frac{d\tilde{Q}}{d\tau} = \frac{t_s V_0}{Q_s R} - \frac{t_s}{RC}\tilde{Q}$$

Let $t_s = RC$, $Q_s = CV_0$

↳ $$\frac{d\tilde{Q}}{d\tau} = 1 - \tilde{Q}$$ ← no parameters in non-dimensional form!

← though not always the case

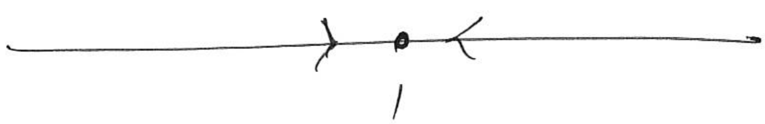
↳ $\tilde{Q}(\tau) = \tilde{Q}_0 e^{-\tau} + 1 - e^{-\tau}$

↳ $Q(t) = Q_s \tilde{Q}(t/t_s)$

$$= Q_0 e^{-t/RC} + CV_0(1 - e^{-t/RC})$$

So nothing has changed

But if we plot the phase plane for $\tilde{Q}(\tau)$



← know $t_s = RC$, $Q_s = CV_0$ } know physical dimensions