comparing two paired means using Bayes factors

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zinc in drinking water

location	Surface	Bottom	Difference
1	0.43	0.415	0.015
2	0.266	0.238	0.028
3	0.567	0.39	0.177
4	0.531	0.41	0.121
5	0.707	0.605	0.102
⋮	⋮	⋮	⋮
10	0.723	0.612	0.111

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$$BF[H_1 : H_2] = \frac{\int p(\text{data}|\mu{=}0,\sigma^2)p(\sigma^2|H_1)\ d\sigma^2}{\int\int p(\text{data}|\mu,\sigma^2)p(\mu|\sigma^2,H_2)p(\sigma^2|H_2)\,d\mu\,d\sigma^2}$$

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priors:

- $\mu \mid \sigma^2, H_2 \sim \mathsf{N}(0, \sigma^2/n_0)$
- $p(\sigma^2) \propto 1/\sigma^2$ for both H_1 and H_2

Bayes factor with reference prior σ^2

$$BF[H_1 : H_2] = \left(\frac{n+n_0}{n_0}\right)^{1/2} \left(\frac{t^2 \frac{n_0}{n+n_0} + \nu}{t^2 + \nu}\right)^{\frac{\nu+1}{2}}$$

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  $n_0 \to 0, \; BF[H1 : H_2] \to \infty$
- no improper or vague prior on μ

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$$BF[H_1 : H_2] = \left(\frac{n+n_0}{n_0}\right)^{1/2} \left(\frac{t^2 \frac{n_0}{n+n_0} + \nu}{t^2+\nu}\right)^{\frac{\nu+1}{2}}$$

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- Jeffrey-Zellner-Siow or 'JZS' prior

zinc concentration example

```
bayes_inference(difference, data=zinc, statistic="mean", type="ht",
                prior="JZS", mu_0=0, method="theo", alt="twosided")

## Single numerical variable
## n = 10, y-bar = 0.0804, s = 0.0523
## (Using Zellner-Siow Cauchy prior:  mu ~ C(0, 1*sigma)
## (Using Jeffreys prior: p(sigma^2) = 1/sigma^2
##
## Hypotheses:
## H1: mu = 0 versus H2: mu != 0
## Priors:
## P(H1) = 0.5 , P(H2) = 0.5
## Results:
## BF[H2:H1] = 50.7757
## P(H1|data) = 0.0193  P(H2|data) = 0.9807
##
```

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next: comparing two means