reference priors

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objective prior distributions

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$$s_n^2 = \frac{1}{v_n}\left[s_0^2 v_0 + s^2(n-1) + \frac{n_0 n}{n_n}(\bar{Y} - m_0)^2\right] \to s^2$$

reference priors

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- Non-Generative or "improper prior distribution"
- "Formal" posterior distribution, proper distribution if $n \geq 2$

$$\mathsf{NormalGamma}(\bar{Y}, n, s^2, n-1)$$

$$\mu \mid \sigma^2, \text{data} \sim \mathsf{N}(\bar{Y}, \sigma^2/n)$$

$$1/\sigma^2 \mid \text{data} \sim \mathsf{Gamma}((n-1)/2, s^2(n-1)/2)$$

reference analysis

- under the reference prior $p(\mu, \sigma^2) \propto 1/\sigma^2$

$$\frac{\mu - \bar{Y}}{\sqrt{s^2/n}} \mid \text{data} \sim \mathrm{t}(n-1, 0, 1)$$

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- intervals estimates $(\bar{Y} - t_{1-\alpha/2} s/\sqrt{n}, \bar{Y} + t_{1-\alpha/2} s/\sqrt{n})$

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- intervals estimates $(\bar{Y} - t_{1-\alpha/2} s/\sqrt{n}, \bar{Y} + t_{1-\alpha/2} s/\sqrt{n})$
- Bayesian allows probability statements after seeing the data

$$P(\bar{Y} - t_{1-\alpha/2} s/\sqrt{n} < \mu < \bar{Y} + t_{1-\alpha/2} s/\sqrt{n}) = 1 - \alpha$$

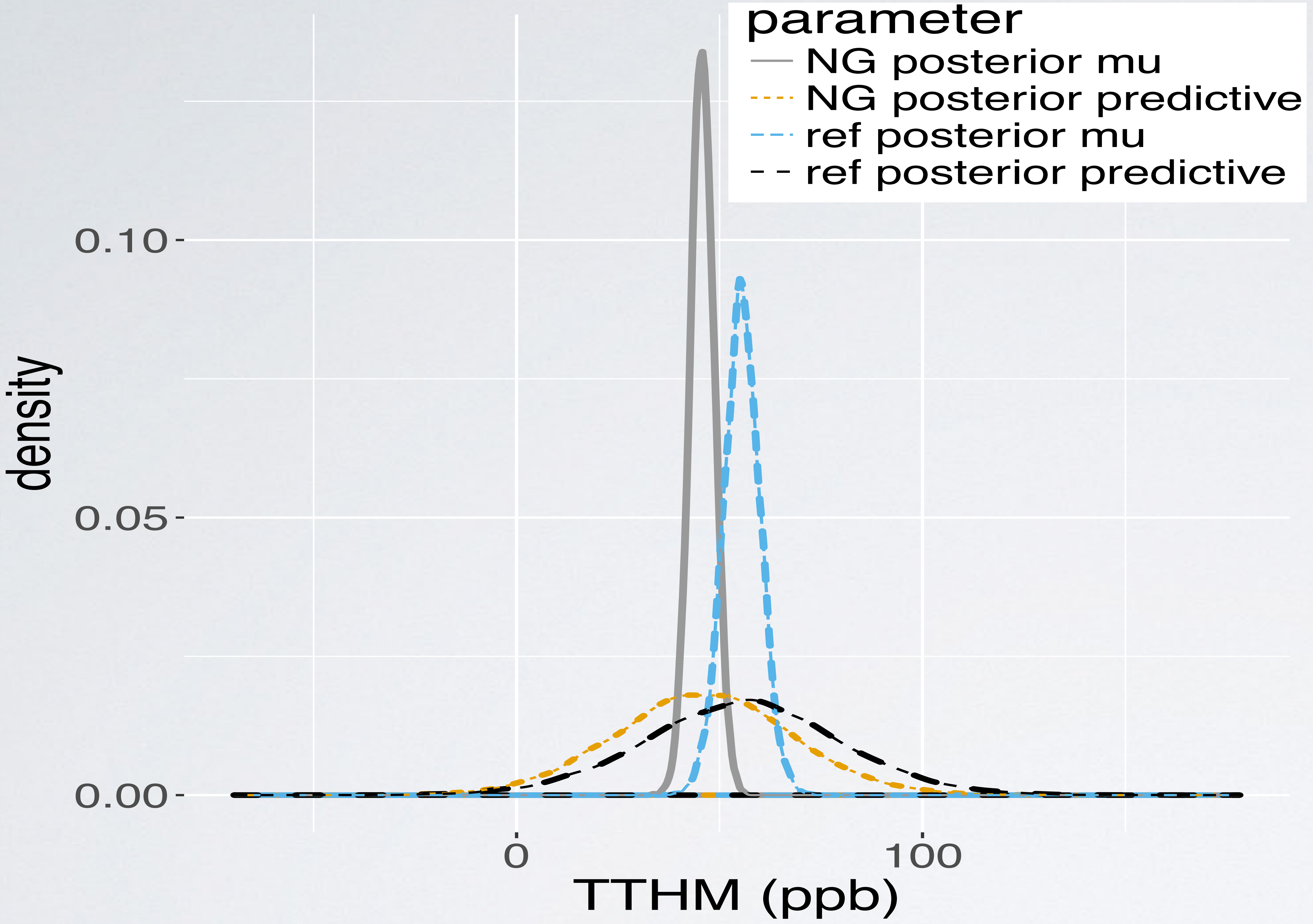
tap water example: reference analysis

R Code

```
phi = rgamma(10000, (n-1)/2, s2*(n-1)/2)
sigma = 1/sqrt(phi)
post_mu = rnorm(10000, mean=ybar, sd=sigma/(sqrt(n)))
pred_y =  rnorm(10000,post_mu, sigma)
quantile(pred_y, c(.025, .975))

##       2.5%      97.5%
##   6.692877 104.225954
```

comparison of posterior densities



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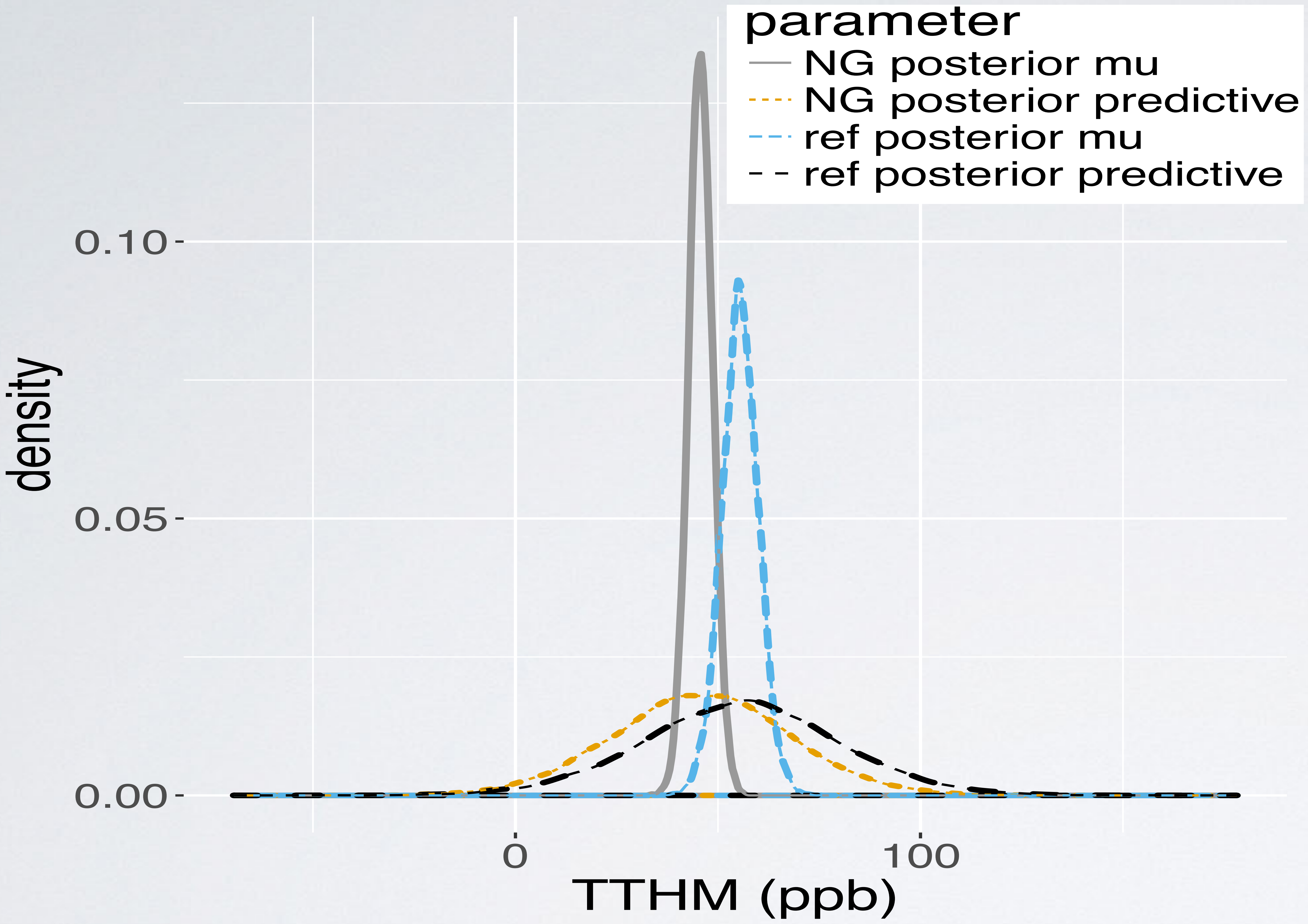


```r
sum(pred_y > 80)/length(pred_y)   # P(Y > 80 | data)

## [1] 0.1534
```

summary

- introduced reference priors

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- sensitivity analysis

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- check assumptions of model and prior

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next: mixtures of conjugate priors,
robustness and MCMC