The Normal-Gamma conjugate family

Dr. Merlise Clyde

Duke University

prior for mean & variance

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$$s_n^2 = \frac{1}{v_n}\left[s_0^2 v_0 + s^2(n-1) + \frac{n_0 n}{n_n}(\bar{Y} - m_0)^2\right]$$

inference about μ

- joint distribution

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$$\mu \mid \text{data} \sim \textsf{t}(v_n, m_n, s_n^2/n_n)$$

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$$\mu \mid \text{data} \sim \mathsf{t}(v_n, m_n, s_n^2/n_n) \Leftrightarrow t = \frac{\mu - m_n}{s_n/\sqrt{n_n}} \sim \mathsf{t}(v_n, 0, 1)$$

total trihalomethane in tapwater (TTHM)

prior $\mathsf{NormalGamma}(35, 25, 156.25, 24)$

https://pixabay.com/en/wine-glass-water-clear-glass-739540/

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data $\bar{Y} = 55.5,\ s^2 = 540.7,\ n = 28$

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data $\bar{Y} = 55.5,\ s^2 = 540.7,\ n = 28$

$$n_n = 25 + 28 = 53$$

$$m_n = \frac{28 \times 55.5 + 25 \times 35}{53} = 45.8$$

$$v_n = 24 + 28 = 52$$

$$s_n^2 = \frac{(n-1)s^2 + v_0 s_0^2 + n_0 n (m_0 - \bar{Y})^2 / n_n}{52}$$

$$= \frac{1}{52}\left[27 \times 540.7 + 24 \times 156.25 + \frac{25 \times 28}{53} \times (35 - 55.5)^2\right] = 459.6$$

posterior NormalGamma$(45.8, 53, 459.6, 52)$

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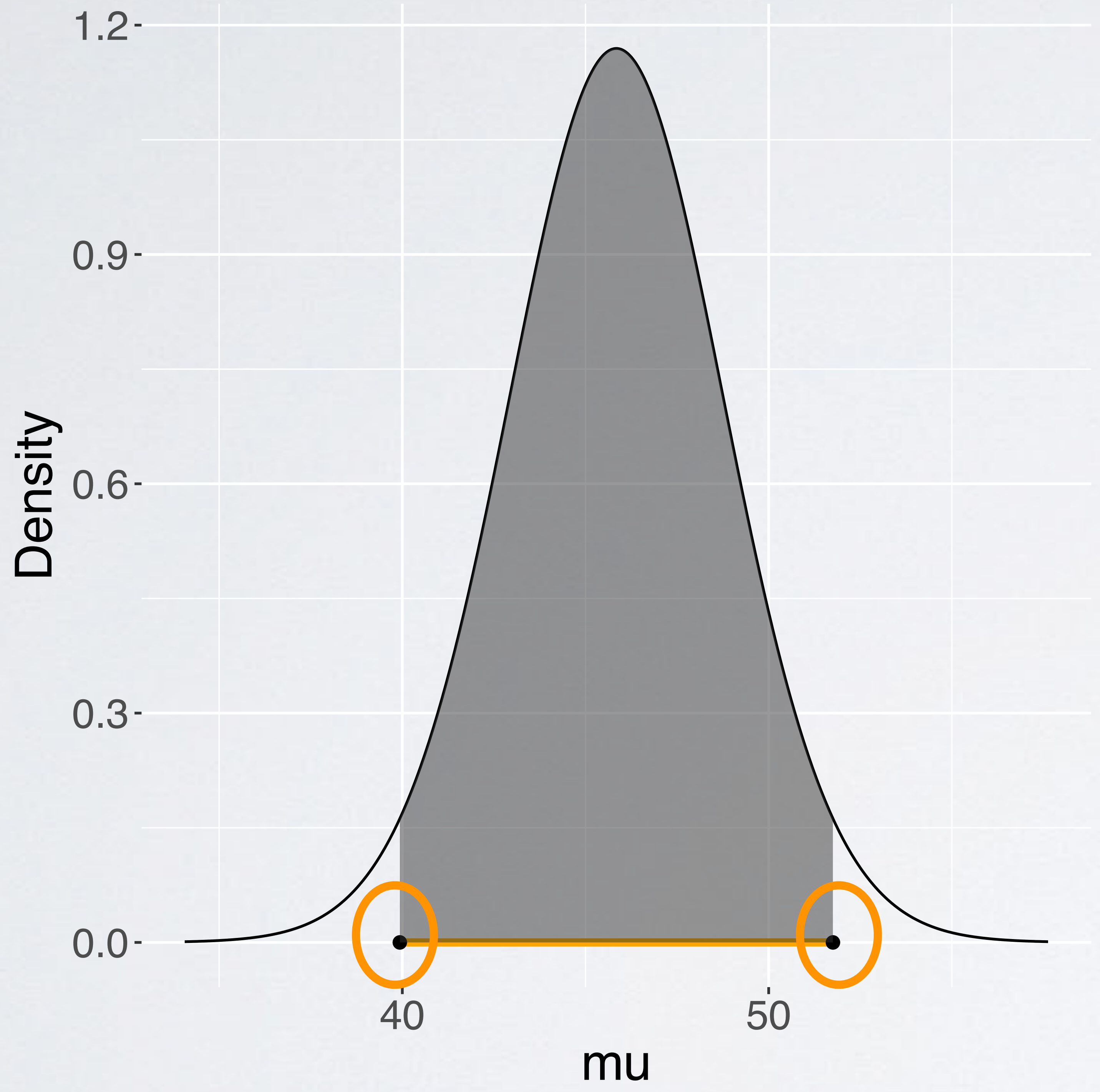
credible interval for μ

95% credible interval: (L, U) where $P(L \leq \mu \leq U \mid \text{data}) = 0.95$



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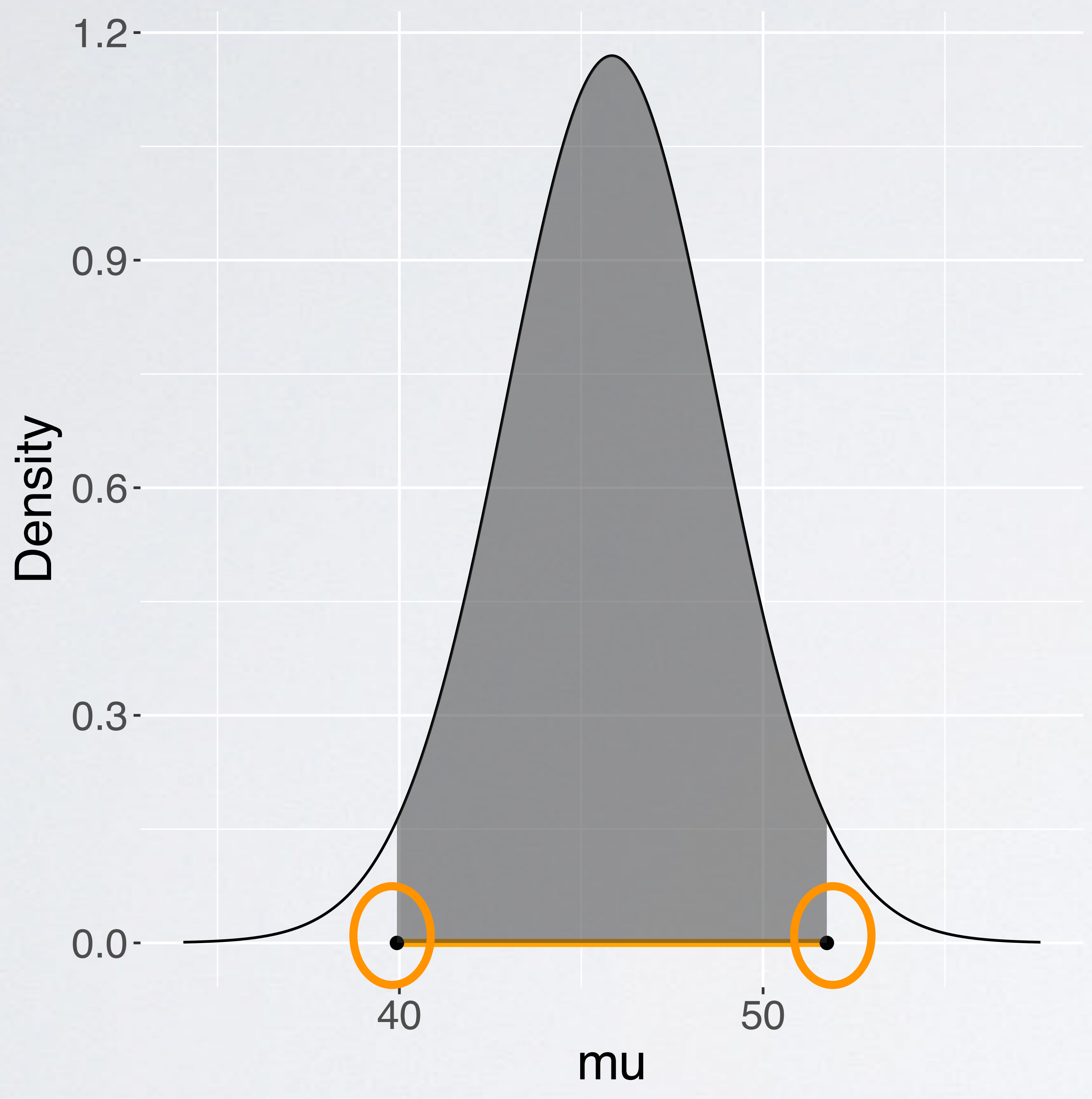

$$L = t_{0.025}\sqrt{s_n^2/n_n} + m_n$$

$$U = t_{0.975}\sqrt{s_n^2/n_n} + m_n$$

using R

```
m_0 = 35; n_0 = 25; s2_0 = 156.25; v_0 = n_0 - 1
data(tapwater); Y = tapwater$tthm
ybar = mean(Y); s2 = var(Y); n = length(Y)
n_n = n_0 + n
m_n = (n*ybar + n_0*m_0)/n_n
v_n = v_0 + n
s2_n = ((n-1)*s2 + v_0*s2_0 + n_0*n*(m_0 - ybar)^2/n_n)/v_n
# CI
L = qt(.025, v_n)*sqrt(s2_n/n_n) + m_n
U = qt(.975, v_n)*sqrt(s2_n/n_n) + m_n
c(L, U)

## [1] 39.93192 51.75374
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n_n = n_0 + n
m_n = (n*ybar + n_0*m_0)/n_n
v_n = v_0 + n
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L = qt(.025, v_n)*sqrt(s2_n/n_n) + m_n
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there is a 95% chance that the mean TTHMs are between 39.9 ppb and 51.8 ppb

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next: using Monte Carlo simulation to explore prior and posterior distributions