posterior probabilities of hypotheses and Bayes factors

Duke University

Dr. Mine Çetinkaya-Rundel

prior odds

ratio of the prior probabilities of hypotheses

$$O[H_1 : H_2] = \frac{P(H_1)}{P(H_2)}$$

posterior odds

ratio of the posterior probabilities of hypotheses

$$PO[H_1 : H_2] = \frac{P(H_1 \mid data)}{P(H_2 \mid data)}$$

posterior odds - expanded

$$PO[H_1 : H_2] = \frac{P(H_1 \mid data)}{P(H_2 \mid data)}$$

$$= \frac{\big(P(data|H1) \times P(H1)\big)/P(data)}{\big(P(data|H2) \times P(H2)\big)/P(data)}$$

$$= \frac{P(data \mid H_1) \times P(H_1)}{P(data \mid H_2) \times P(H_2)}$$

bayes factor

prior odds

$$= \frac{P(data \mid H_1)}{P(data \mid H_2)} \times \frac{P(H_1)}{P(H_2)}$$

$$PO[H_1 : H_2] = BF[H_1 : H_2] \times O[H_1 : H_2]$$

Bayes factor

$$BF[H_1 : H_2] = \frac{P(data \mid H_1)}{P(data \mid H_2)}$$

- quantifies the evidence of data arising from H_1 vs. H_2
- discrete case: ratio of the likelihoods of the observed data under the two hypotheses
- continuous case: ratio of the marginal likelihoods

$$BF[H1 : H2] = \frac{\int P(data \mid \theta, H_1)\ d\theta}{\int P(data \mid \theta, H_2)\ d\theta}$$

example: HIV testing with ELISA

hypotheses

H_1: Patient does not have HIV

H_2: Patient has HIV

priors

$P(H_1) = 0.99852$

$P(H_2) = 0.00148$

$$O[H_1 : H_2] = \frac{P(H_1)}{P(H_2)} = \frac{0.99852}{0.00148} = 674.6757$$

posteriors

$P(H_1 \mid +) = 0.8788551$

$P(H_2 \mid +) = 0.1211449$

$$PO[H_1 : H_2] = \frac{P(H_1 \mid +)}{P(H_2 \mid +)} = \frac{0.8788551}{0.1211449} = 7.254578$$

Bayes factor

$$BF[H_1 : H_2] = \frac{PO[H_1 : H_2]}{O[H_1 : H_2]} = \frac{7.254578}{674.6757} \approx 0.0108$$

$$= \frac{P(+ \mid H_1)}{P(+ \mid H_2)} = \frac{0.01}{0.93} \approx 0.0108$$

interpreting the Bayes factor

Jeffreys (1961)

BF[H_1 : H_2]	Evidence against H_2
1 to 3	Not worth a bare mention
3 to 20	Positive
20 to 150	Strong
> 150	Very strong

changing the order of hypotheses

$$BF[H_2 : H_1] = \frac{1}{BF[H_1 : H_2]}$$

$$= \frac{1}{0.0108} = 92.59259$$

BF[H_2 : H_1]	Evidence against H_1
1 to 3	Not worth a bare mention
3 to 20	Positive
20 to 150	Strong
> 150	Very strong

interpreting the Bayes factor - another look

Kass & Raftery (1995)

$2\ log(BF[H_2 : H_1])$	Evidence against H_1
0 to 2	Not worth a bare mention
2 to 6	Positive
6 to 10	Strong
> 10	Very strong

$$2 \times log(92.59259) = 9.056418$$

recap

- prior odds, posterior odds, Bayes factor
- interpreting Bayes factors by Jeffreys or Kass & Raftery scales
- order of hypotheses doesn't matter