**CS 577: Introduction to Algorithms** 9/24/2015

# Homework 3

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This assignment covers divide & conquer and graph primitives. Good luck!

## Review problems

1. Suppose you have a divide-and-conquer algorithm that reduces a problem instance of size $n$ to 4 instances of size $n/3$, spending $O(n^2)$ time in constructing the subproblems and building the solution out of the solutions to the subproblems. What is the resulting running time? What if you manage to improve the local running time from $O(n^2)$ to $O(n)$?

2. You are given an array $A$ of length $n$ and want to color each entry such that every subarray of the form $A[i \dots j]$ for $1 \leq i \leq j \leq n$ contains an entry whose color differs from all other entries in the subarray. For example, for $n = 3$ the sequence of colors (green,red,green) works.

   Express the minimum number $k(n)$ of distinct colors you need as a function of $n$, for all $n$ of the form $n = 2^m - 1$ where $m$ is a positive integer. For example, $k(3) = 2$ as the above example shows that 2 colors suffice, and there is no solution with only 1 color.

## Graded written problem

3. [10 points] You are given a sequence of $n$ real numbers $a_1, a_2, \dots, a_n$ and a corresponding sequence of weights $w_1, w_2, \dots, w_n$. The weights are nonnegative reals that add up to 1, i.e., $\sum_{i=1}^{n} w_i = 1$. The weighted median of the sequence is the number $a_k$ such that

   $$\sum_{a_i < a_k} w_i < \frac{1}{2} \quad \text{and} \quad \sum_{a_i \leq a_k} w_i \geq \frac{1}{2}.$$

   For example, the weighted median of the following instance is 2.5:

   | $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
   |---|---|---|---|---|---|---|---|
   | $a_i$ | 40 | -5 | 4 | 0 | 2.5 | 6 | -2 |
   | $w_i$ | .25 | .1 | .05 | .18 | .15 | .2 | .07 |

   Give an algorithm that finds the weighted median using $O(n)$ elementary operations. An addition or a multiplication of two real numbers counts as one elementary operation.

## Additional written problem

4. You are given a topographical map that provides the maximum altitude along the direct road between any two neighboring cities, and two cities $s$ and $t$. Develop a linear-time algorithm that finds a route from $s$ to $t$ that minimizes the maximum altitude. All roads can be traversed in both directions.

## Optional programming problem

5. [2.5 points] Solve SPOJ problem Minimum Knight Moves (problem code NAKANJ).

## Challenge problem

6. Let $P$ be a set of $n$ points in the plane. A point $(x, y)$ in $P$ is called undominated if for every other point $(x', y')$ in $P$, either $x' < x$ or $y' < y$ (or both).

   Develop an algorithm for finding all of the undominated points in a given set $P$ that runs in time$O(n \log u)$, where $u$ denotes the number of undominated points.