CS 577: Introduction to Algorithms Fall 2015

Network Flow

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DRAFT

(Last updated: November 19, 2015)

When we covered greedy algorithms, we thought of them as making a sequence of irrevocable decisions. By making the right decisions, we get a simple, correct algorithm. However, we don't always want to forced to make our decisions irrevocable; we want to be able to change our mind after considering new information about the input instance. In general, this can lead to a tremendous blow-up in the running time, since most algorithms along these lines end up considering an exponential number of possible solutions. When we covered dynamic programming, we saw how to mitigate this: the idea is to give an algorithm that organizes the solutions concisely, so that the algorithm ultimately only considers a small number of possible solutions.

In these notes, we present a different approach to extending greedy algorithms, that of *network flow*. The basic framework here will be a bit different than in previous sections, because network flow is itself a single computational problem. The way we will apply "network flow" to other computational problems will be by *reducing* them to network flow.

For instance, in Section 2, we'll define the maximum bipartite matching problem. The way we'll solve this problem is by turning instances of it into instances of the network flow problem, solving the network flow problem, and then turning the result of the network flow problem into a solution to the original matching problem. The main idea here is that it's much easier to reduce matching to network flow than it is to produce from scratch an algorithm for matching. We present a few more examples demonstrating the power and flexibility of this idea of reducing to network flow, as well as to give an intuitive idea of when a problem can nicely reduce to network flow.

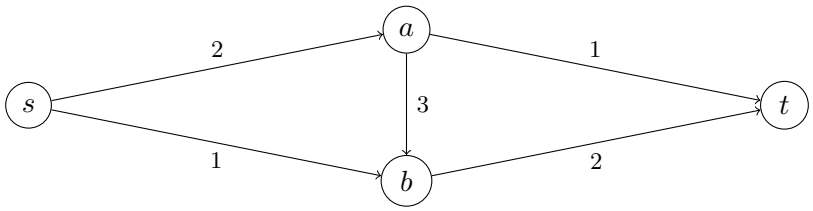
Another motivation for studying network flow is its duality theory. As we'll see, network flow is a maximization problem, and it has as a dual problem, "minimum cut", which is a minimization problem. The duality theory for network flow has as its main theorem that these two problems are very closely related. Other than being a neat result in its own right, this duality itself has a number of mathematical consequences—the well-known Hall's Marriage Theorem and Menger's Theorem are two we will specifically cover. It will also expand the range of problems which reduce simply to network flow, such as the project selection problem in Section 7, where the most natural reduction is to the minimum cut problem.

1 Network Flow

A *network* is a directed graph $G = (V, E)$ with two distinct, distinguished vertices, $s, t \in V$, called the *source* and *sink* respectively. These vertices have the property that the in-degree of s is zero, and that the out-degree of t is zero. Each network additionally has a *capacity function* on the edges, $c : E \to [0, +\infty)$. Figure 1 sketches an example of a network.

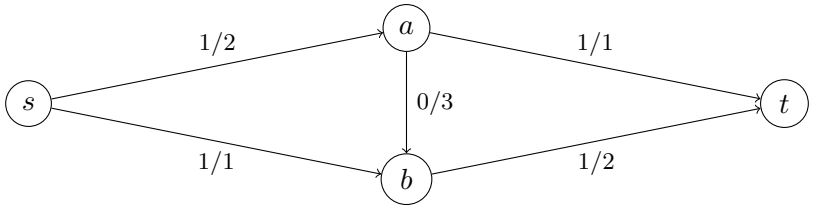
A *flow* through a network G is a function on the edges, $f : E \to [0, +\infty)$ that satisfies two properties:

Figure 1: A network



A depiction of a network on four vertices. The capacities of each edge are written along the corresponding edge.

Figure 2: A flow through a network



A depiction of a flow through the network from Figure 1. The flows and capacities are written as fractions—the expression a/b next to an edge e means $f(e) = a$ and $c(e) = b$. Note that this satisfies the conservation and capacity constraints.

- **(Conservation)** For every vertex besides the source and sink, the total flow *entering* the vertex is equal to the total flow *leaving* the vertex. Symbolically,

$$f_{\text{in}}(v) = f_{\text{out}}(v)$$

- **(Capacity)** For every edge $e \in E$, the flow *across* the edge e is at most the capacity of e. Symbolically,

$$f(e) \leq c(e)$$

The notation $f_{\text{in}}(v)$ denotes the sum over all edges *entering into* v of the flow across that edge. The notation $f_{\text{out}}(v)$ denotes the sum over all edges *exiting out of* v of the flow across that edge. Symbolically:

$$f_{\text{in}}(v) \doteq \sum_{e \in E \,:\, e=(u,v)} f(e)$$

and

$$f_{\text{out}}(v) \doteq \sum_{e \in E \,:\, e=(v,w)} f(e)$$

Figure 2 sketches a representation of a flow through a network.

At an intuitive level, a network can be thought of as a series of tubes through which something, say oil, might flow. Each tube has a capacity, indicating the maximum amount of oil that can flow through that tube. The source represents an oil supplier, who is trying to transport oil to the sink. The source can only do so by pushing the oil through the tubes. The ways of pushing oil through the tubes correspond to flows. The conservation constraint simply says that the intermediate nodes neither add nor remove oil, and the capacity constraints ensure that no more oil is being sent through a single edge than is possible.

A natural question to ask in this setting is, how much oil can the source push to the sink? Or more abstractly, how much flow can be sent from the source node to the sink node? To measure this, we need a way to quantify the total amount of flow being sent from the source to the sink. We call this the *value* of the flow, and denote the value of a flow f by $\mu(f)$.

One way to define the value of a flow is to just add up the total amount of flow leaving the source vertex, *i.e.*, $\mu(f) \doteq f_{\text{out}}(s)$. But what's so special about s? Why not define the value of a flow to be the total flow coming into the sink vertex? Intuitively, these quantities should actually be the same. More generally, we should expect that, whenever we cut the network into two pieces, S and T, so that the source is in S and the sink is in T, then the value of the flow should just be the total amount of flow crossing from S to T, and that this really should not depend on the choice of cut.

Indeed, this does turn out to be the case. Formally, we define an s-t cut to be a cut (S, T) of G (the graph underlying the network) which has the property that s is in S and t is in T. The value of the flow f *with respect to the* s-t *cut* (S, T) is the *net* flow crossing the cut from S to T. *i.e.*, , the value of the flow is the total flow crossing the cut *from* S *to* T minus the total flow crossing the cut *from* T *to* S. We denote this quantity by $\mu(f, (S, T))$. Symbolically,

$$\mu(f, (S, T)) \doteq \sum_{e \in E(S,T)} f(e) - \sum_{e \in E(T,S)} f(e) \tag{1}$$

(The notation $E(S, T)$ means the edges $(u, v) \in E$ for which u is in S and v is in T.) Our first step toward understanding network flow is in proving that $\mu(f, (S, T))$ does not depend on the s-t cut chosen. Formally, we have Proposition 1.

Proposition 1. *For every* s-t *cut* (S, T)*, we have* $\mu(f) = \mu(f, (S, T))$.

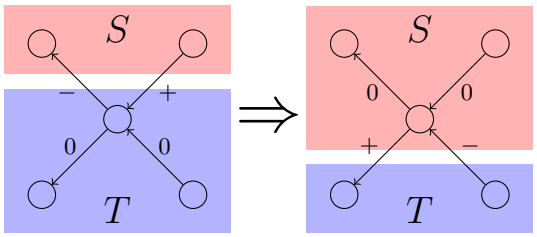
Proof. The proof essentially boils down to the conservation property of flows and the fact that any s-t cut has s on one side and t on the other. We give two arguments. Figure 3 gives a visual aide to understanding.

The first argument is inductive; we sketch the main ideas. The claim is trivial when $S = \{s\}$: $\mu(f)$ is exactly the first sum in Equation (1), and since the in-degree of s is zero, the second sum is zero. For cuts (S, T) with larger S, we can think of starting with the cut $(\{s\}, V \setminus \{s\})$ and then moving vertices v from the 't-side' of the cut to the 's-side' of the cut until we have (S, T). When we move a single vertex across, the sums in (1) change. In particular, the whole expression decreases by $f_{\text{in}}(v)$, as terms either disappear from the left sum in (1) or appear in the right sum in (1). Similarly, it increases by $f_{\text{out}}(v)$, as terms either disappear from the left sum or appear in the right sum. However, when v isn't the source or sink, we know that $f_{\text{in}}(v) = f_{\text{out}}(v)$, which means the overall change to the sum is zero.

The second argument is a single-step, purely algebraic proof, we can add the following set of equations:

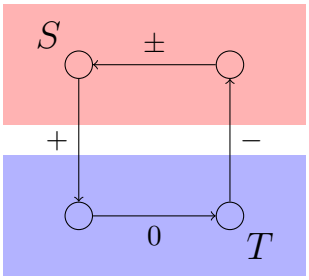
$$\begin{array}{rccccc} & \mu(v) & \doteq & f_{\text{out}}(s) & - & f_{\text{in}}(s) \\ + & 0 & = & f_{\text{out}}(v) & - & f_{\text{in}}(v) \qquad \forall v \in S \setminus \{s\} \\ \hline & \mu(v) & = & \sum_{v \in S} f_{\text{out}}(v) & - & \sum_{v \in S} f_{\text{in}}(v) \end{array}$$

Figure 3: Proof of Proposition 1



(a) The inductive step in the first proof

The + and − signs indicate the sign by which the edge is counted; the symbol 0 means the edge is not counted. Note that as the center node crosses from the blue region to the red region, the signs of edges leaving the center node increase by one, and the signs of edges entering the center node decrease by one.



(b) The cancellation in the second proof

The + and − signs indicate the sign by which the edge is counted. The symbol 0 indicates the edge is not counted in the sum. The symbols ± indicates that the edge is counted with both signs, and hence cancels.

and observe that we can rewrite the sums on the right-hand-side as

$$\sum_{e \in E(S,V)} f(e) - \sum_{e \in E(V,S)} f(e)$$

By canceling the common terms in the sums (corresponding to edges in $E(S,S)$) to get

$$\mu(f) = \sum_{e \in E(S,T)} f(e) - \sum_{e \in E(T,S)} f(e) \doteq \mu(f,(S,T))$$

□

Proposition 1 tells us that there is a meaningful way to value a flow, and that this meaning matches our intuition. Our objective in the network flow problem will be to find a flow through a network with *maximum* value. Formally the network flow problem meets the following description:

Input: A network $G = (V, E)$ with capacity function $c : E \to [0, +\infty)$.

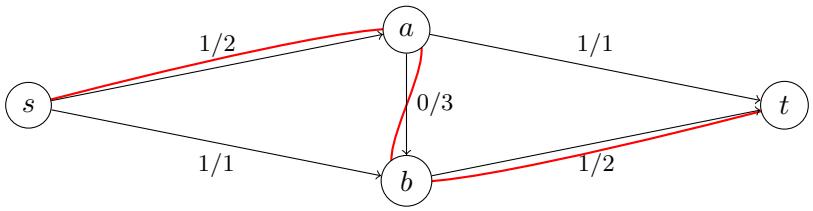
Output: A flow f through G whose value $\mu(f)$ is maximum.

1.1 Weak Duality

Consider the flow given in Figure 4. It has value two. There is a flow with larger value: simply 'push' one unit of flow from s to a to b to t (as indicated by the red path). This yields a new flow with value three. Can we do better than three? The answer in this case is no: every flow has to satisfy the capacity constraints, so every flow f in the example network must have $f_{\text{out}}(s) \leq 3$, and thus $\mu(f) \leq 3$.

In general, every flow must have value at most the total capacity of edges leaving s. Similarly, every flow must have value at most the total capacity of edges entering t. Even more generally, we can consider arbitrary s-t cuts and define a notion of capacity for the whole cut, with the property that the value of the flow is at most the capacity of the cut.

Figure 4: A non-maximal flow



The flow given by the fractions has value two, and is not maximal. Pushing one additional unit of flow along the red curve yields a flow with value three. The new flow is maximal, because the s-t cut with $S = \{s\}, T = \{a, b, t\}$ has capacity three.

Fix an s-t cut (S, T) arbitrarily. We want to define the *capacity of the cut* to be a quantity which easily upper bounds the value of any flow. An easy way to do this is to simply bound the value of the flow relative to the cut (S, T), and then appeal to Proposition 1. We can do this by using two facts:

- For flow crossing from S to T, the total flow has to be at most the sum of the capacities of the edges from S to T. This is because, by the capacity constraints, the flow across any given edge crossing from S to T has to be at most the capacity of that edge.
- For flow crossing from T to S, the total flow has to be nonnegative; this simply follows from the fact that the flow along any given edge must be nonnegative.

Thus the *net flow* from S to T has to be at most the total capacity of the edges from S to T (minus zero, for the edges from T to S). We denote it by $c(S, T)$, and define it formulaically as

$$c(S, T) = \sum_{e \in E(S,T)} c(e)$$

It's easy to see then that the value of every flow has to be at most the capacity of every cut: for any flow f and s-t cut (S, T), we have

$$\begin{aligned} \mu(f) &= \mu(f, (S, T)) \\ &= \sum_{e \in E(S,T)} f(e) - \sum_{e \in E(T,S)} f(e) \\ &\leq \sum_{e \in E(S,T)} c(e) - \sum_{e \in E(T,S)} 0 \\ &\doteq c(S, T) \end{aligned}$$

This fact is referred to as *weak duality*. We can formulate it more traditionally as follows:

$$\max_{\text{flows } f} \mu(f) \leq \min_{s\text{-}t \text{ cuts } (S,T)} c(S, T)$$

or even more succinctly as

$$\textbf{Max-Flow} \leq \textbf{Min-Cut}$$

1.2 Strong Duality

A natural question is to ask for the conditions under which equality holds in the weak duality statement. The answer, perhaps surprisingly, is *always*.

We will prove this by showing partial correctness of an algorithm which computes the maximum flow in a network. But to introduce the algorithm, we need a couple more concepts.

The first is that of an *augmenting path*. This is simply a formalization of the red path in Figure 4. In the example, we could 'push' an extra unit of flow along the red path, and this increased the value of the flow. In general, let G be a network with capacity function c, and let f be a flow through G. Let P be a path from s to t through G with the property that, for every edge e in P, the flow across e is strictly less than the capacity of e, *i.e.*, $f(e) < c(e)$. Let $\alpha > 0$ be such that $f(e) + \alpha \leq c(e)$ for all edges e in the path P. Then we can *augment* the flow f along the path P by adding α units of flow to each edge in P. In other words, we get a new flow f' defined to be

$$f'(e) = \begin{cases} f(e) + \alpha & : \quad e \in P \\ f(e) & : \quad e \notin P \end{cases}$$

It's easy to see that f' satisfies the conservation and capacity constraints, so it is also a flow. Furthermore, the value of the flow has increased by α.

Augmenting paths will be the essential tool for computing a maximum flow. To demonstrate that, let's fix a flow f. Consider the set S of vertices to which s can send positive flow. Either t is in this set of vertices or it is not. We have the following alternative:

- If t is in S, then there is an augmenting path from s to t, and so we can improve our flow f.
- If t is not in S, then we get an s-t cut, namely $(S, V \setminus S)$.

It would be really nice if, in the second case, the s-t cut has capacity exactly equal to value of f. In particular, this would imply that f is a maximum flow by weak duality. The first case of the alternative even tells us how to compute it: simply repeatedly find augmenting paths until none exist.

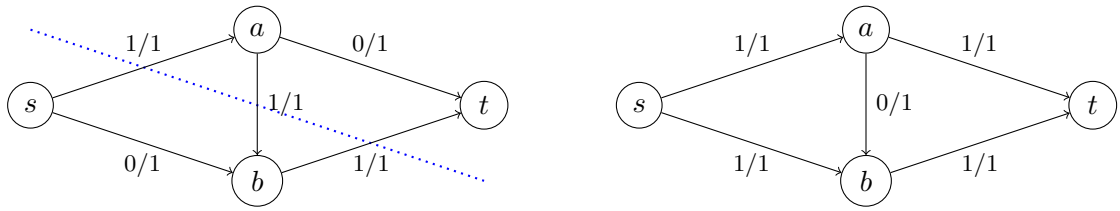
This plan almost works: suppose we have a flow f in the second case above, and consider any edge (u, v) in $E(S, V \setminus S)$. Since u is in S, there is a way to send positive flow from s to u, and because v is not in S, there is no way to send positive flow from s to v. This implies that $c((u, v)) = f((u, v))$, since otherwise we could send positive flow from s to v by first sending it from s to u and then along the edge (u, v). In other words, the cut we find will be *full in the forward direction*. Thus, in order for our plan to work, we just need for there to be no flow crossing the cut in the backward direction.

Strictly speaking, however, this does not always happen. Figure 5 gives an example.

What we need to make this work is the following observation: if f is a flow through a network, and e is an edge in this network for which $f(e) > 0$, then we can view this flow as the *capacity to send flow in the reverse direction*. Thus what we want are augmenting paths in a different network, called the *residual* network with respect to the flow f.

Formally, let G be a network with capacity function c, and let f be a flow through G. We define the *residual network* of G with respect to f to be the network $G_f = (V_f, E_f)$ with capacity

Figure 5: The necessity of residual networks



The left figure features a simple network with a flow with value one, formed by augmenting one unit of flow along the path $s \to a \to b \to t$. There is no path from s to t along which the flow values are strictly less than the capacity; in particular, the only vertices reachable from s in this manner are s and b. But the cut $(\{s, b\}, \{a, t\})$ has capacity two, which is larger than one. And indeed, there is a flow with value two, as shown in the figure on the right.

function c_f defined as follows:[1][2]

- **(Vertices)** G_f is a network on the same vertices as G; *i.e.*, $V_f = V$.
- **(Forward edges)** Let e be an edge in E. We say that e is a *forward edge*. If the flow across e is equal to the capacity (*i.e.*, $f(e) = c(e)$), then we do *not* include e in E_f. Otherwise, we do include e in E_f. The capacity of a forward edge in the residual network is simply the *left-over capacity*,
$$c_f(e) \doteq c(e) - f(e)$$
- **(Backward edges)** Let e be an edge in E, and let $\overleftarrow{e}$ denote the *reverse* of e. *i.e.*, if $e = (u, v)$, then $\overleftarrow{e} = (v, u)$. We say that $\overleftarrow{e}$ is a *back edge*. If the flow across e is zero (*i.e.*, $f(e) = 0$), then we do not include $\overleftarrow{e}$ in E_f. Otherwise, we do include $\overleftarrow{e}$ in E_f. The capacity of a backward edge in the residual network is simply the *flow in the forward direction*,
$$c_f(\overleftarrow{e}) \doteq f(e)$$

Figure 6 sketches a flow and associated residual network in a simple network.

We also need to adapt our notion of an augmenting path to fit the new idea. One way to do this is to say that an augmenting path through the residual network G_f is a path P through G_f, from s to t, so that, for every edge e in P, the residual capacity of e is strictly positive. More succinctly, for every edge e in P, $c_f(e) > 0$. The old notion of augmenting path then corresponds to augmenting paths in G_f that only use forward edges.

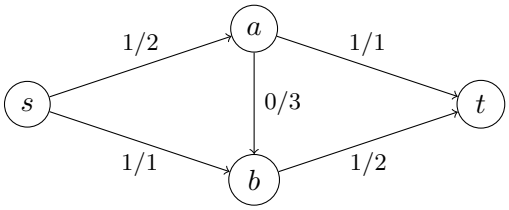
The important property of residual networks is that augmenting paths in the residual network can always be used to extend a flow.

The augmentation itself is straightforward; we simply realize the intuition that capacity on backward edges represents the potential to subtract flow. Specifically, let P be an augmenting path

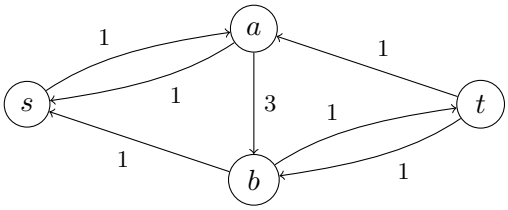
[1] If G has a pair of vertices u and v for which the edges (u, v) and (v, u) exist and have positive capacity, this scheme could introduce multiple edges of the form (u, v) or (v, u), which is technically not allowed in our definition of a directed graph. For the rest of this discussion, however, simply extend the definition of a network to allow for multiple such edges, and with different capacities on each. We then think of "the forward edge (u, v)" and "the backward edge $\overleftarrow{(v, u)}$" (which are both directed edges from u to v) as distinct edges in the residual network.

[2] This definition also can introduce edges which make the in-degree of s and the out-degree of t greater than zero. Strictly speaking, this doesn't fit our definition of a network; however, as long as we stick to the convention that the value of a flow is the *net* flow across a cut, then we can allow for edges to enter s or exit t without any problems.

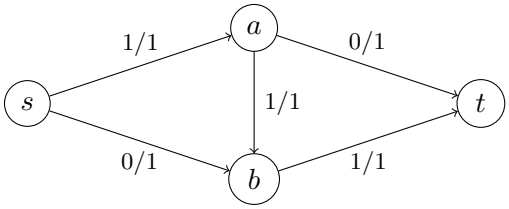
Figure 6: Residual networks



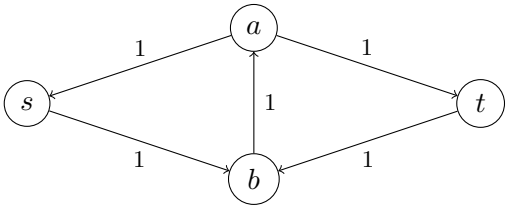
(a) A flow f through a network G



(b) The residual network G_f



(c) The flow from Figure 5.



(d) The residual network for the flow in Figure 5.

The residual network G_f is a network for which flows can be added to f without violating the capacity constraints in G. We view existing flow in f as having the *capacity to be undone*. Adding flow along backward edges really means subtracting flow from the forward edges.

in the residual network G_f, and let $\alpha > 0$ be so that $\alpha \leq c_f(e)$ for every edge e in P. We will define a function $f' : E \to \mathbb{R}$ which will be the flow we are interested in.

- For every forward edge e in P, we set $f'(e) = f(e) + \alpha$.
- For every backward edge $\overleftarrow{e}$ in P (where e is the corresponding forward edge in G), we set $f'(e) = f(e) - \alpha$.
- For every edge e for which both e and $\overleftarrow{e}$ are not in P, we just set $f'(e) = f(e)$.

An example of this augmentation (and a hint of what is to come) is given in Figure 7.

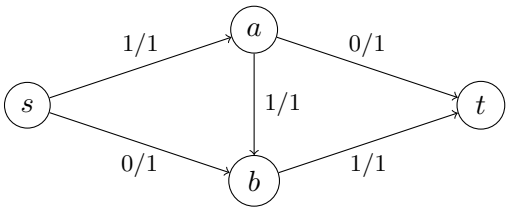
What remains to show is that the result of this augmentation is in fact a flow, *i.e.*, that it is nonnegative and satisfies the capacity and conservation constraints. We state and prove this formally in Proposition 2.

Proposition 2. *The function f' defined above is non-negative, and thus can be regarded as a function $f' : E \to [0, +\infty)$. Furthermore, f' is a flow.*

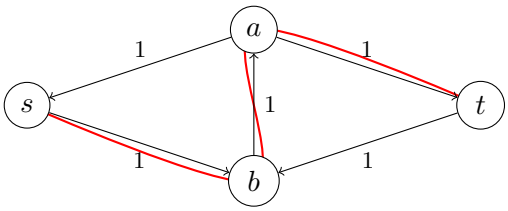
Proof. The function f' is nonnegative. Since $\alpha \geq 0$ and $f(e) \geq 0$ for all edges e, we have $f(e) \geq 0$ and $f(e) + \alpha \geq 0$ for all edges e. Thus we only have to worry about $f'(e)$ which was updated by a backward edge. But in this case, we know that $\alpha \leq c_f(\overleftarrow{e}) = f(e)$ for all backward edges $\overleftarrow{e}$, so $f'(e) = f(e) - \alpha \geq 0$.

The function f' additionally meets the capacity constraints. Since $f(e) \leq c(e)$ for all edges e, and $\alpha \geq 0$, it follows easily that $f(e) \leq c(e)$ and $f(e) - \alpha \leq c(e)$ for all edges e. Thus we only have to worry about the capacity constraints when f' was updated by a forward edge. But in this case, because $\alpha \leq c_f(e) = c(e) - f(e)$ for all forward edges, it follows that $f'(e) = f(e) + \alpha \leq c(e)$.

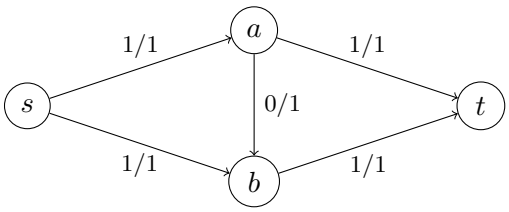
Figure 7: Augmenting paths in residual networks



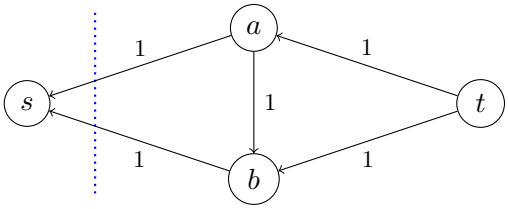
(a) The flow f from Figure 5.



(b) The residual network G_f with augmenting path.



(c) The flow f' after augmentation.



(d) The residual network $G_{f'}$ after augmentation, with cut.

The red path gives an augmenting path through the residual network G_f. We think of augmenting f along this path as pushing one unit of flow from s to t along the red path. In the case of the edge $a \to b$, this means *reducing* the flow. The result is a flow f' satisfying the conservation and capacity constraints. In this case, the vertices reachable from s in the residual network $G_{f'}$ define an s-t cut in G, the capacity of which is equal to the value of the flow f'.

Finally, the function f' satisfies the conservation constraints. For vertices v which are not visited in the path P, there are no changes to the total flow entering and exiting v; therefore, since f satisfies the conservation constraints, so too does f'.

For each vertex v along P besides s and t, P enters v along some edge a and exits v along some edge b. There are a few cases:

- If both a and b are forward edges, then the flow coming into v by a and the flow leaving v by b both increase by α.
- If both a and b are backward edges, then the flow coming into v by $\overrightarrow{b}$ and the flow leaving v by $\overrightarrow{a}$ both decrease by α. (Here $\overrightarrow{b}$ for a backward edge b just means its corresponding forward edge.)
- If a is a forward edge and b is a backward edge, then the flow coming into v by a increases by α and the flow coming into v by $\overrightarrow{b}$ decreases by α, resulting in no net change of flow entering v. Additionally, there is no change to the flow leaving v.
- If a is a backward edge and b is a forward edge, then the flow coming into v does not change. The flow leaving v by $\overrightarrow{a}$ decreases by α, and the flow leaving v by b increases by α, resulting in no net change of flow leaving v.

In any case, we have conservation of flow at v (even if v is visited multiple times by P).

Finally, the value of f' is α more than the value of f. This follows from applying a similar case analysis as above to the first edge of P, *i.e.*, the edge leaving s. □

We are now ready to prove strong duality. We'll start by recalling the alternative we stated above, except adapted to the setting of residual networks. Let f be a flow through a network G, and G_f be the associated residual network. Let S denote the set of vertices reachable from s in the residual network along edges e with $c_f(e) > 0$. The alternative states that one of the following is true:

- The sink t is in S. In this case there is an augmenting path in G_f.
- The sink t is not in S. In this case the cut $(S, V \setminus S)$ is an s-t cut.

This suggests the following algorithm for computing the maximum flow in a network G with capacity function c.

1. Initialize f to be the zero flow, and compute G_f (which is just G).
2. While there is a path P from s to t in G_f, do the following: Compute the maximum amount of flow that can be pushed along P; this is just the minimum residual capacity of the edges in P. Update f and G_f according to pushing this amount of flow along the path P.
3. Output the flow f.

Note that the process of finding a path from s to t is not completely determined here; there may be many paths to choose from, and choices of different paths may lead to different answers. In fact, this algorithm is better thought of as a *family* of algorithms, where different subroutines for finding the augmenting path lead to different behaviors. It is referred to as the "Ford–Fulkerson scheme" (or Ford–Fulkerson algorithm), after its discoverers.

In general, however, these different choices will lead only to different performance guarantees. The *partial correctness* of the algorithm will not depend on the choice of augmenting path. We will focus on that now, since this is where strong duality comes in, and return to the termination and performance guarantees later.

Thus we want to prove the partial correctness of the above algorithm. So let's suppose that the algorithm was given a network G with capacity function c, and it computed a flow f. We know by the termination condition of the while loop in step 2 that there is no augmenting path in the residual network G_f. Thus we are in the second case of our alternative, and so we have an s-t cut (S, T), where S is defined by the vertices reachable from s in G_f. We want to show that this implies that f is a maximum flow. The following theorem does this:

Theorem 1. *The following are equivalent:*

1. *f is a maximum flow.*
2. *There is no augmenting path from s to t in G_f.*
3. *The cut (S, T), where S is the set of vertices reachable from s in G_f, is an s-t cut. It furthermore satisfies the property that, for every edge e in $E(S, T)$, $f(e) = c(e)$, and for every edge e in $E(T, S)$, $f(e) = 0$.*

Proof. We prove that (1) implies (2), (2) implies (3), and (3) implies (1), showing that all three statements are in fact equivalent.

(1) $\implies$ **(2)** We prove this by contraposition. Suppose that there is an augmenting path in G_f. Then by augmenting f along this path, f becomes a flow with strictly larger value, implying f was not originally a maximum flow.

(3) $\implies$ **(1)** The capacity of the cut (S, T) can be written as

$$\sum_{e \in E(S,T)} c(e) + \sum_{e \in E(T,S)} 0$$

which is term-by-term equal to $\mu(f, (S, T))$ by (3). By Proposition 1, we have $\mu(f, (S, T)) = \mu(f)$. Thus, by weak duality, every flow has value at most $\mu(f)$, which means that f is a maximum flow.

(2) $\implies$ **(3)** Let S be the set of vertices reachable from s in G_f, and T be its complement in V. By assumption, (2) tells us that t is not in S. Therefore (S, T) is an s-t cut.

For every edge (u, v) in $E(S, T)$, we know that u is reachable from s in G_f, and that v is not reachable from s in G_f. Thus the forward edge (u, v) cannot be present in $E(S, T)$. Since it is a forward edge, this means that $f((u, v)) = c((u, v))$.

For every edge (v, u) in $E(T, S)$, we know that v is not reachable from s in G_f, and that u is reachable from s in G_f. Thus the backward edge $\overleftarrow{(v, u)}$ cannot be present in G_f. Since it is a backward edge, this means that $f((v, u)) = 0$. □

Phrased in more general terms, Theorem 1 tells us that the maximum value of any flow in any network is exactly equal to the minimum capacity of any s-t cut in this network. More succinctly, we have the following strengthening of weak duality:

Max-Flow = Min-Cut

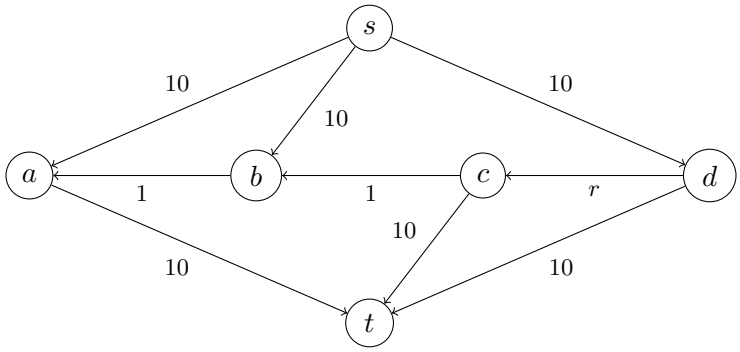
Finally, Theorem 1 tells us that the Ford–Fulkerson scheme, when it terminates, will always yield a maximum flow through its input network.

1.3 Termination and performance of the Ford–Fulkerson scheme

The final point of discussion for the Ford–Fulkerson scheme is its termination. It's easy to see that, with every augmentation, the value of the flow strictly increases: an augmenting path by definition has positive residual capacity on every edge, so increasing the flow by the smallest of these capacities is still a strictly positive increase. Furthermore, there is an upper bound on the total value of flow, witnessed by any s-t cut in the network. At an intuitive level then, it seems as though the Ford–Fulkerson algorithm will always terminate: it always makes progress, and it only has to make a finite amount of progress before it terminates.

Unfortunately, this is not always the case. In some specially-crafted networks and with poor choices of augmenting paths, the Ford–Fulkerson scheme can be made to run indefinitely. It always finds an augmenting path, which strictly increases the value of the flow, but this increase may shrink over time. For instance, one might imagine the first augmentation increases the flow value by 1, the second by 1/2, the third by 1/4, and so on, but the maximum flow has value 2 or larger. More concretely, the network given in Figure 8 gives an example for which this actually happens.

Figure 8: An example on which the Ford–Fulkerson scheme might not terminate



The number r is the (unique) positive real number satisfying $1 - r = r^2$ This value r additionally satisfies $1 - 2r = r^3$, $r^k(1 - r) = r^{k+2}$, and $(1 - r) < r < 1$. It is easy (if tedious) to use these facts to check that one can repeatedly augment using the sequence of paths P_1, P_2, P_1, P_3, defined by $P_1 = s \to d \to c \to b \to a \to t$, $P_2 = s \to b \to c \to d \to t$, and $P_3 = s \to a \to b \to c \to t$.

On the other hand, there are settings in which we can realize our intuition. The missing piece to our approach is that we need for there to be *finitely* many values between 0 and the value of the maximum flow. This suffices since, if we start with a flow of value 0, and each augmentation strictly increases the value of the flow, then the value of the maximum flow must eventually be reached. The example in Figure 8 takes advantage of the fact that there are infinitely many real values between 0 and any positive number.

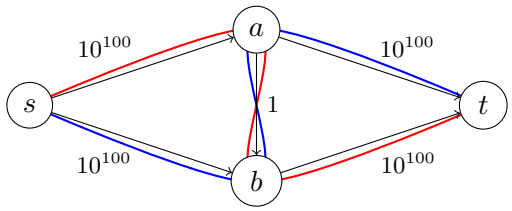
However, the counterexample has to do so in a rather careful way. The reason is that there is a relationship between the capacities in the original network and the values of the flows seen during the Ford–Fulkerson algorithm. To demonstrate this, let's focus on the case where all of the capacities in the input network are integers. When this is the case, it's easy to see that the Ford–Fulkerson algorithm will push an integral amount of flow along the first augmenting path. When this happens, the residual capacities are either increased or decreased by this integral value. Since the set of integers is closed under addition and subtraction, this means that all the residual capacities in the network will again be integers. Thus the Ford–Fulkerson scheme will again push an integral amount of flow along its second augmenting path, and so on. In other words, the value of the flow is *always an integer*. Since the number of integers between 0 and a fixed positive integer is finite, this means that the Ford–Fulkerson algorithm will always terminate on networks with integral capacities. More formally, we have Claim 1.

Claim 1. *If the input network's capacities are integers, then the Ford–Fulkerson scheme will always terminate. In particular, if the maximum flow has value F, then the Ford–Fulkerson scheme terminates after at most F augmentations. Furthermore, the flow computed by the Ford–Fulkerson scheme will have integral values.*

Proof. As argued above, if the input network's capacities are integers, it follows that every augmentation increases the value of the flow by an integer. Since each augmentation strictly increases the value of the flow, this means that the value of the flow always increases by at least 1.

This means that after at most F augmentations, the Ford–Fulkerson scheme will have found a flow with value at least F. Since the maximum flow has value at most F, it follows that after at most F augmentations, the Ford–Fulkerson scheme will find a maximum flow.

Figure 9: An instance of network flow for which the Ford–Fulkerson algorithm is slow



The Ford–Fulkerson algorithm may choose to augment along the red path, then the blue path, then the red path, etc. The effect is that the intermediate flow values only increase by one at each augmentation. The result is that the Ford–Fulkerson algorithm needs $2 \cdot 10^{100}$ iterations to find the maximum flow in this network.

The fact that the flow returned by Ford–Fulkerson is integral is a consequence of the fact that every augmentation increases the values of the flow by integral amounts. □

Since the augmenting path can be found using *e.g.*, breadth-first search, Claim 1 implies that the Ford–Fulkerson scheme can be implemented to run in time $O(F \cdot (n+m))$ on any network with at most n vertices and m edges and maximum flow value F.

This is not a particularly nice upper bound. In particular, it is only a pseudo-polynomial time bound. However, without specifying any information on how the augmenting paths are chosen, this is essentially the best bound achievable by the Ford–Fulkerson scheme. Consider for instance Figure 9, where $2 \cdot 10^{100}$ augmentations are needed when the augmenting path is chosen as described in the figure. Thus a natural question is where there are smart ways to choose the augmenting paths, so that the running time does not depend so poorly on the value of the flow.

A better bound It turns out that there are a few approaches that succeed at this. One approach is to simply always choose the augmenting path to be one for which the *most* flow can be pushed. Finding this path can still be done in linear time (using the routine for median-finding as a subroutine!) for each iteration, and the end result is an algorithm with running time $O(\log(F)(m)(n+m))$. (We leave out the analysis.)

Another approach completely removes the dependence on F from the running time. This approach is deceptively simple—the augmenting path is found by simply doing a breadth-first search! The difficulty lies in the analysis, however, which we do not cover here. The end result, however, is a bound of $O(n \cdot m)$ on the number of iterations, and thus a bound of $O(n \cdot m \cdot (n+m)) = O(nm^2 + n^2 m)$ on the running time. This approach is generally referred to as the Edmonds–Karp algorithm, after the authors who proved this bound on the running time.

There are yet more approaches to the network flow problem. An algorithm known as "Dinic's Algorithm" is a variation on the Ford–Fulkerson scheme, and can be implemented straightforwardly to run in time $O(n^2 m)$. Using more sophisticated data structures, this can be reduced to $O(nm \log(n))$, though this asymptotic performance is rarely worth the extra effort.

A rather different approach, known both as the "push–relabel algorithm" and the "preflow–push algorithm", achieves a running time of $O(n^2\sqrt{m})$ when implemented simply. Sophisticated data structures can be employed here as in Dinic's Algorithm to bring the running time down to $O(nm \log(n^2/m))$.

Finally, there exists an $O(n \cdot m)$ algorithm for maximum flow whose discovery was only completed (and published) in 2012.

2 Matching

Our first application of network flow is to the bipartite maximal matching problem. In a bipartite maximal matching problem, the input is a bipartite graph $G = (L \cup R, E)$. A matching in this graph is a subset $M \subseteq E$ of the edges with the property that each vertex is incident to at most one edge of M; if $(u, v) \in M$, then one thinks of v as being matched to u, and vice-versa. A maximal matching is a matching of maximum size. Formally, we have the following problem description:

Input: A bipartite graph $G = (L \cup R, E)$

Output: A matching M of maximum size

As promised, the bipartite maximum matching problem can be solved via reduction to network flow. The reduction is fairly straightforward: the main intuition is to think of a match (u, v) as being represented by a unit of flow going from u to v through G. If we can accomplish this, then flows will correspond to matchings, and maximal flows will correspond to maximal matchings.

Since we want flows to go "through" G, it makes sense to have the network be based on G itself. Toward this end, we'll just add a source s and a sink t to the vertices, add edges from the source to the vertices in L, edges from the vertices in R to the sink, and finally direct all the edges of G from L to R. The end result is that every path from s to t has to travel into a vertex in L, through an edge of G, into a vertex in R, and finally on to t. We refer to the network that results from this as G'. Pictorially, we have Figure 10.

To enforce that the flow represents a matching, we need to make sure there is at most one unit of flow through each vertex of L and R. We can accomplish this by imposing a capacity constraint of 1 on all the edges of the form (s, u) and (v, t), where u is in L and v is in R. This will suffice for all of our arguments (as long as the edges from E have capacity at least one), but we still have to choose capacities for the edges in E. It will simplify our arguments in Section **??** if we can suppose that the edges in E have *infinite* capacity. Strictly speaking, however, we can't do that, since the capacity function has to have the form $E \to [0, +\infty)$ according to our definition of a network.[3]

However, the only reason to give these edges infinite capacity is just to make sure they don't appear in any minimum cut. We can accomplish the same thing by simply giving these edges a sufficiently large finite capacity. Note that there is already a finite-capacity s-t cut, namely (S, T) where $S = \{s\}$ and $T = L \cup R \cup \{t\}$. The capacity of this cut is the number of vertices in L, $|L|$. Hence any edge with capacity larger than $|L|$ cannot appear in a minimum cut. Thus if we give the edges in E a capacity of $|L| + 1$, then they will behave identically to infinite capacity edges for our purposes. So for now on, we will simply regard these edges as having infinite capacity.

This completes our construction of the flow network G'. We now wish to formally argue that G' realizes our intuition that units of flow correspond to matches in matchings. We formalize this in the following claim:

Claim 2. *There is a one-to-one and onto correspondence between matchings in G and integral flows in G'. Furthermore, maximal matchings correspond to maximal flows and vice-versa.*

Proof. We give both directions of the correspondence, and then observe that the two directions are inverses of each other, which implies the correspondence is one-to-one and onto. We then inspect

[3] It is possible and quite reasonable to extend the notion of a network to allow for infinite capacity edges. We didn't do that, because it requires adding a special case to all of the duality theory to handle the possibility of infinite flow.

the correspondence to see that matchings of size k correspond with flows of value k, which implies that the maximal matchings correspond to maximal flows. Figure 10 serves as a useful visual aide for understanding the correspondence.

Matchings → Flows This direction of this correspondence is a straightforward interpretation of our intuition that units of flow are the same as matches. Let M be any matching in G. We create a flow f through the network G' whose value is the number of matches in M.

For each edge (u, v) in M, we want f to have a unit of flow across the edge (u, v) in G'. Of course, this flow has to come from s and end in t. The only way to do this is to have a unit of flow going from s to u and going from v to t. So let $f_{(u,v)}$ represent this single unit of flow from s to u to v to t. Note that it is itself a flow through G', since it satisfies the capacity and conservation constraints.

To construct the flow f, we simply superimpose all of these flows. Symbolically, $f = \sum_{e \in M} f_e$. The function f is indeed a flow:

- Since it is a sum of flows that satisfy the conservation constraints, it satisfies the conservation constraints as well.
- The only way to violate the capacity constraints is to send multiple units of flow along the same edge. But since M is a matching, each vertex appears in at most one edge in M. This means each edge of the form (s, u) or (v, t) has positive flow in at most one f_e, so the capacity constraints on these edges are satisfied.

Moreover, the flow f is integral, because it is a sum of integral flows.

Flows → Matchings For the other direction of the correspondence, we want to give an inverse to the above map from matchings to integral flows. We can do this by observing the following structure in our network:

- For every vertex u in L, there is only one incoming edge (from s), and this incoming edge has capacity one.
- For every vertex v in R, there is only one outgoing edge (to t), and this outgoing edge has capacity one.

These conditions imply that there is at most one unit of flow entering each vertex in L, and at most one unit of flow leaving each vertex in R. By conservation, this means that there is at most one unit of flow leaving each vertex in L, and at most one unit of flow entering each vertex in R. Thus, in any *integral* flow, each vertex in L sends positive flow to *at most one* vertex in R, and each vertex in R receives positive flow from at most one vertex in L.

This tells us how to define our matching: Let f denote any integral flow through G'. Then we define M to be the set of pairs (u, v) with u in L and v in R so that f sends positive flow from u to v. This is a matching by the properties we stated above. It is a matching in G because the units of flow in f can only be sent along edges present in G.

It's easy to check that these two correspondences are inverses of each other: if we start with a matching, apply the first transformation, then apply the second transformation, we get the

matching we started with; if we start with a flow, apply the second transformation, then apply the third transformation, we get the flow we started with.

Finally, a straightforward verification realizes that flows of value k correspond with matchings of size k. □

Thus, with Claim 2 in hand, we have a complete algorithm for bipartite matching. The algorithm is simply to construct the flow network G' above, and then run an algorithm for network flow. Using Ford–Fulkerson with a linear-time augmenting path finding subroutine, we get a running time bound of $O(M \cdot (n+m))$ when G has n vertices and m edges, and where M denotes the size of the maximal matching. Since $M \leq n$, we can eliminate the dependence on M to achieve a running time bound of $O(n(n+m))$.

Note that, for this specific problem, this application of Ford–Fulkerson out-performs the strongly polynomial-time bounds for general network flow.

2.1 Hall's marriage theorem

In the previous section, we discussed how maximum flow gives us maximum matchings. We now see how the dual to maximum flows—minimum cuts—tell us in the matching setting. We will focus on matchings which are *perfect*. In a bipartite graph $G = (L \sqcup R, E)$, a perfect matching is a matching $M \subseteq E$ so that every vertex in L is matched to some vertex in R, and every vertex in R is matched to some vertex in L. Naturally, this requires $|L| = |R|$. We set $n = |L| = |R|$.

It is easy to certify that a given G has a perfect matching—a perfect matching itself proves that G has one. The goal in this section is to see how we can give a similarly simple proof that G does *not* have a perfect matching. We will do this by going through duality.

We know by the previous section that G has a perfect matching if and only if the corresponding flow network has a maximum flow with value n. Hence, to prove that a graph does not have a perfect matching, we need to show that there does not exist a flow of value n. While superficially tricky, we can recast "does not exist a flow" as "exists a cut" by appealing to duality: a perfect matching does *not* exist in G if and only if the corresponding flow network for G has a cut of capacity less than n.

This is still unsatisfying though—giving a five-year-old a cut of capacity less than n won't convince him that G has no perfect matching. So let's translate what a cut of capacity less than n means into the language of matchings. Begin by considering Figure 11, where an s-t cut of capacity four (while $n = 5$) is given.

There are two useful observations we can draw from the figure. The first is that there are more green vertices in L than in R, and the second is that, for every vertex v in R, if v has a green neighbor in L, then v itself is green. Notice how they prove that the example graph does not have a perfect matching—we have more green vertices in L than green vertices in R, but green vertices in L can only be matched with green vertices in R!

It turns out that these two observations generalize. Suppose we have an s-t cut (S, T) with capacity $c(S, T) < n$. For a subset $A \subseteq L$, let $\Gamma(A)$ denote the vertices in R which are adjacent to some vertex in A.

Claim 3.

$$|L \cap S| > |R \cap S|$$

Proof. The capacity of the cut (S, T) is precisely the number of edges from the source to an element of $L \cap T$ plus the number of edges from an element of $R \cap S$ to the sink. *i.e.*, $c(S, T) = |L \cap T| + |R \cap S|$. Since $n > c(S, T)$, we can rearrange this to get $n - |L \cap T| > |R \cap S|$. Since S and T partition the vertices, we have $|L \cap S| + |L \cap T| = |L| = n$, and hence $n - |L \cap T| = |L \cap S|$. Putting these together yields $|L \cap S| > |R \cap S|$. □

Claim 4.

$$\Gamma(L \cap S) \subseteq (R \cap S)$$

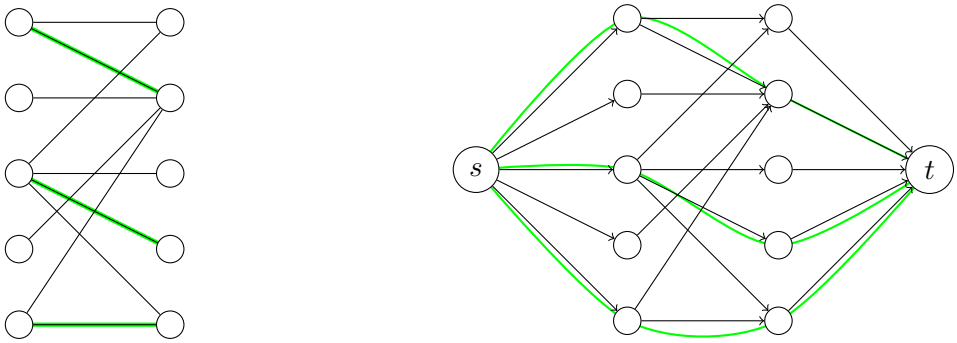
Proof. The edges from L to R have infinite capacity, and so cannot be in the cut (S, T). Hence for every edge (u, v) with $u \in L \cap S$, u is in S, so v must also be in S. □

Putting the two claims together, we get that $|L \cap S| > |\Gamma(L \cap S)|$. In other words, there is a set of vertices $A \subseteq L$ so that $|A| > |\Gamma(A)|$, and so G cannot have a perfect matching.

Thus cuts with capacity less than n in turn give us very simple proofs that G does not have a perfect matching. Since flows with value n give us perfect matchings, and since cuts of value less than n give us nonexistence of perfect matchings, we apply the strong duality of network flow to get Hall's marriage theorem:

Theorem 2. *(Hall's marriage theorem)*
G has a perfect matching if and only if every subset $A \subseteq L$ has $|\Gamma(A)| \geq |A|$.

Figure 10: Flow network for maximal matching

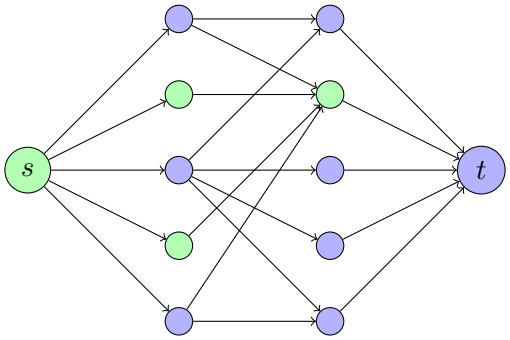


(a) Bipartite graph

(b) Flow network

The bipartite graph on the left is converted into a flow network on the right, with the property that matchings in the bipartite graph correspond to flows in the flow network. All the edges in the flow network which are incident to s or t have capacity 1; the remaining edges have infinite capacity. The matching in green on the left is represented by the green flow on the right.

Figure 11: Small cuts in bipartite graphs

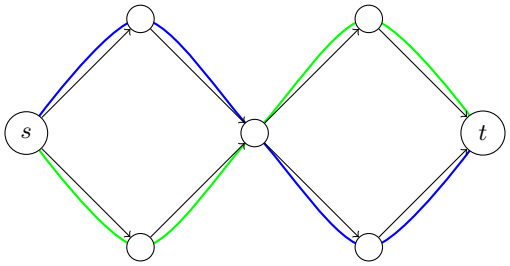


A bipartite graph with $|L| = |R| = 5$, with a cut of capacity 4 given. The s-side of the cut is the green vertices, and the t-side is the blue vertices.

3 Edge-disjoint paths

The edge-disjoint paths problem is essentially as its name implies: we are given a directed graph $G = (V, E)$ with distinguished vertices s and t. The goal is to select a maximal collection of paths from s to t so that no two chosen paths share any edge.

Figure 12: Edge-disjoint paths example



A simple input for the edge-disjoint paths problem. The green and blue paths give an example of a pair of edge-disjoint paths. Note that even though they share an intermediate vertex, the set of edges used by each path are disjoint.

This problem turns out to be essentially a special case of network flow. Forgetting capacity constraints for a moment, we can think of G as a flow network, with s the source and t the sink. Paths from s to t can be thought of as single units of flow pushed from s to t; a collection of paths is then the superimposition of these flows across all the paths in the collection. The value of this flow is simply the size of the collection of paths.

The edge-disjointness condition then translates to the property that every edge is transporting at most one unit of flow. Moreover, if we take *any* integral flow f with the property that every edge transports at most one unit of flow, then we can extract from f a collection of edge-disjoint paths whose size is the value of f. Intuitively, we follow a single unit of flow from s to t, take this as one path, subtract it from f, and repeat until f has value zero. The fact that f sends at most one unit of flow over each edge guarantees the edge-disjointness property.

Hence we have the following reduction: given the input $G = (V, E), s, t$, simply endow G with the capacity function $c : E \to \mathbb{R}_{\geq 0}$ as $c(e) = 1$ for all edges e. This is exemplified in Figure 13. We just argued that integral flows in this network induce collections of edge-disjoint paths and vice-versa, and the values of the flows equal the sizes of the corresponding collections of edge-disjoint paths. Hence a maximum integral flow gives us a maximum-size set of edge-disjoint paths.

Figure 13: Example flow network for edge disjoint paths

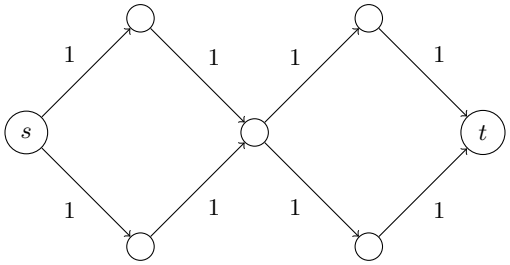


Figure 14: Circulation network examples

(Forthcoming)

4 Circulations

The maximum flow problem is an optimization problem. We can translate it into a "feasibility" version by introducing a parameter f in the input, and asking whether a given flow network has a flow with value f. While superficially not exciting, we can extend it a little further, by thinking of the source s as having a 'supply' of f units of flow, and the sink t as having a 'demand' of f units of flow, and then phrasing the question as whether we can transport the available flow from s to t. A generalization of this is to give every vertex a supply or demand, and ask whether there is a way to transport all of the supplied flow to where it is demanded. (For simplicity, we require that *all* the supply flow be used to satisfy demand.)

More formally, we can define *circulation networks*. These are a slight variation on flow networks, still involving a graph $G = (V, E)$, where instead of distinguishing two vertices as s and t, we instead give every vertex v a *demand* $d(v)$. We allow negative values of $d(v)$ to represent supplies at vertices. Circulation networks still have a non-negative capacity on each edge, $c : E \to \mathbb{R}_{\geq 0}$. The question is whether there is a flow, $g : E \to \mathbb{R}_{\geq 0}$, which satisfies the capacity constraints (*i.e.*, $g(e) \leq c(e)$ for all edges e), and which satisfies a modified conservation constraint at every vertex v:

$$g_{\text{in}}((u, v)) - g_{\text{out}}((v, u)) = d(v) \tag{2}$$

Any function $g : E \to \mathbb{R}_{\geq 0}$ is called a *circulation*; a circulation is called *feasible* if it satisfies the conservation and capacity constraints. Figure 14 gives a few example circulation networks.

Unlike with the maximum flow problem, there is not always a feasible circulation in a circulation network. As we will prove in a bit, this can happen for two reasons. One is simple: if there is more flow supplied than demanded, or vice-versa, then it will be impossible to use all of the supply to satisfy all of the demand. The other is that the actual transportation of the flow may be impossible; some sort of cut exists to constrain the passage of flow from the supply to the demand.

We can find a feasible circulation (if it exists), or determine nonexistence of feasible circulations using a maximum flow algorithm. The reduction is fairly straightforward. First, we ensure the demands add up correctly. Note that if we add up all of the conservation constraints, we get 0 on the left, and $\sum_v d(v)$ on the right. Hence "add up correctly" means that the demands sum to 0, or more intuitively that the total supply equals the total demand. If the input circulation network does not satisfy this property, then clearly it does not have a feasible circulation. Otherwise, let D be the total amount of supply (or equivalently the total amount of demand):

$$D \doteq \sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$$

Next, we turn the circulation network G into a flow network G' to see if the supply can be transported to the demand. The basic idea is to construct G' so that flows with value D in G' correspond to circulations in G. We think of initializing G' to be identical to G. We then introduce two new vertices, s and t, which are the source and sink in G'. For each vertex v from the circulation network, we add an edge (s, v) if $d(v) < 0$), or an edge (v, t) if $d(v) > 0$. The capacity of these edges is $|d(v)|$.

Note that in G' the cuts with $S = \{s\}$ and $T = V \cup \{t\}$, and with $T = \{t\}$ and $S = V \cup \{s\}$ each have capacity exactly D. Hence every flow of value D has to saturate the edges crossing these cuts. In other words, in order to satisfy the conservation constraints, each supply vertex v has to export $-d(v)$ units of flow to the rest of the network, and each demand vertex v has to import $d(v)$ units of flow from the rest of the network. Since the capacity constraints for edges in the circulation network remain unchanged, it follows immediately that flows of value D correspond exactly with circulations in the original circulation network. More formally, we have the following claim:

Claim 5. *Let g be any feasible circulation in G. Then extending g to the flow f by $f(e) = g(e)$ for e in G, $f((s, v)) = -d(v)$ for vertices v with $d(v) < 0$, and $f((v, t)) = d(v)$ for vertices v with $d(v) > 0$), results in a flow of value D in G'.*

Second, let f be any flow of value D in G'. Then the circulation g defined by $g(e) = f(e)$ for every edge e in the circulation network is a feasible circulation.

Proof. As above. □

Claim 5 tells us everything we need to know about feasible circulations in G. A feasible circulation g exists iff the sum of the demands is zero and there is a flow of value D in G'. Following this, an efficient algorithm for finding a feasible circulation in G is just to check the condition that the demands sum to zero, and then construct G' and run a maximum flow algorithm on G'.

Adding lower bound constraints The advantage to the circulation framework over the network flow framework is that it has generalizations that are more difficult to express in the maximum flow setting. One generalization was just allowing multiple sources and sinks, and only allowing each source or sink to supply/demand a certain amount of flow, instead of being thought of as an infinite supplier. We can generalize further by introducing *lower bound constraints* on the edges in the network.

Hence, in addition to the network, $G = (V, E)$, capacity function $c : E \to \mathbb{R}_{\geq 0}$, and demands $d : V \to \mathbb{R}$, we also have a lower bound function $\ell : E \to \mathbb{R}_{\geq 0}$. The role of this lower bound function is that, in order for a circulation g to be considered feasible, it must have $g(e) \geq \ell(e)$ for every edge e, in addition to the conservation and capacity constraints. We will assume that $\ell(e) \leq c(e)$ for all edges e, since otherwise there is obviously no feasible circulation for G.

Introducing lower bounds into the circulation problem breaks our above reduction to network flow, since we don't know of a network flow algorithm that can handle lower bound constraints. Indeed, introducing lower bound constraints into the maximum flow problem may actually make it impossible for any flow to exist in the first place.

However, we can actually reduce the problem of finding circulations with lower bound constraints to finding circulations without lower bound constraints. The key idea is to handle the constraints "edge (u, v) needs to have at least $\ell((u, v))$ units of flow" by just pushing $\ell((u, v))$ units of flow from u to v, and then adjusting the demands at u and v to compensate. More specifically, for each edge (u, v), we decrease the capacity $c((u, v))$, increase the demand $d(u)$, and increase the supply $-d(v)$, all by $\ell((u, v))$. After doing this for each edge, we get a new circulation network G' through which circulations correspond to the "extra circulation" on top of that required by ℓ.

Put another way, we construct a circulation network G' over the same vertex and edge sets, adjusting the capacities and demands. The desired effect is that circulations in G' are exactly shifts by ℓ of circulations in G. Specifically, we design G' so that g is a circulation in G if and only if $h = g - \ell$ is a circulation in G'. Pictorially, we have Figure 16 which demonstrates this.

Figure 15: Transforming G to G'

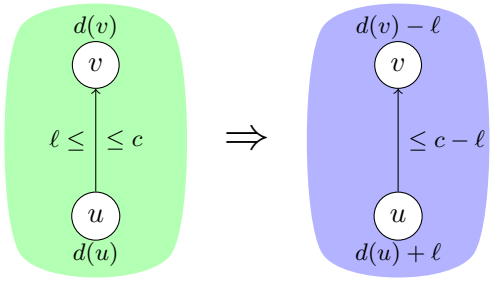
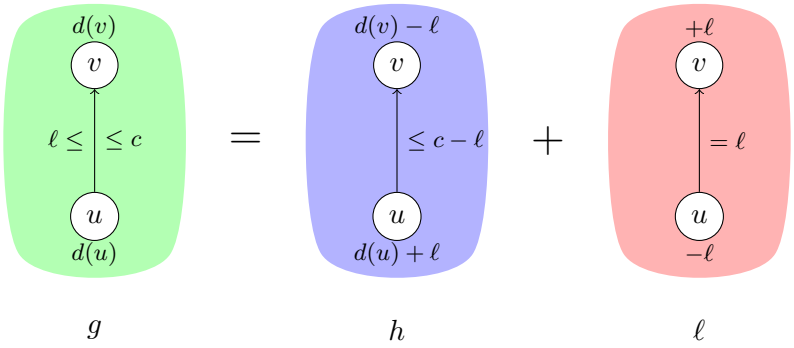


Figure 16: Circulations in G versus circulations in G'



Further notes on circulations Note that if an input circulation network has integral capacities and integral demands, then the induced flow network has all-integral capacities, and hence an integral maximum flow. This translates to the existence of an integral feasible circulation in the original circulation network whenever any feasible circulation exists. Moreover, in circulation networks with lower bounds, if the lower bounds are also integral, then there is an integral feasible circulation whenever there is any feasible circulation.

5 Survey design

In the survey design problem, we have a number of products and a number of customers that have purchased some products. We are interested in collecting reviews of each of the products from customers who have bought them, and to do so without overburdening any one customer. Formally, we have the following problem specification:

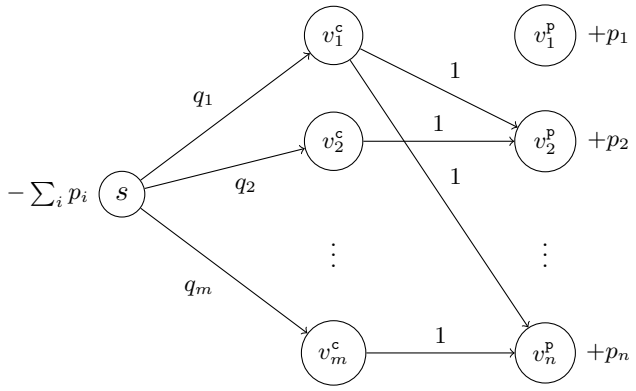
Input: There are n products and m customers. Each customer j has purchased some subset S_j of the products. Each product i requires p_i reviews. Customer j can only give q_j reviews, at most one for each distinct item he has purchased.

Output: A way to allocate the reviews for each item to customers so that no customer is overburdened and every item has the required number of reviews.

We can solve the survey design problem via reduction to the circulation problem. The idea is to think of each product i as having a demand of p_i reviews, and for there to be a supply of $\sum_i p_i$ reviews, which can only get to the products through customers. Each customer j has a capacity to handle a total of q_j reviews, and each customer can handle at most one review per item. Integral circulations will then correspond to ways of ways of assigning reviews to customers without violating any of the constraints.

Hence we form the circulation network G as follows: we make a vertex v_i^{p} for each product i, a vertex v_j^{c} for each customer j, and a review-supply vertex s. The demand of vertex v_i^{p} is the number of reviews required for product i, p_i. The demand of each customer v_j^{c} is zero. The supply at s is just $\sum_i p_i$ (so its demand is the negative of this). We add edges (s, v_j^{c}) for each customer j, with capacity q_j. We add edges $(v_j^{\mathsf{c}}), v_i^{\mathsf{p}})$ for each item i in S_j; each has capacity 1. Pictorially, we get a circulation network as in Figure 17.

Figure 17: Circulation network for survey design



From the figure, it's clear how integral circulations correspond to allocations of reviews to customers. or each unit of flow from s through customer j into product i, customer j is assigned one review for product i. The unit capacity constraints between v_j^{c} and v_i^{p} ensure that each customer reviews each product at most once, and the constraint between s and v_j^{c} ensures customer j reviews no more than q_j products. Reversing this process shows how feasible allocations of reviews to customers give feasible circulations.

Complexity analysis The size of the circulation network constructed is $O(n+m)$ and has $O(m+\sum_i |S_i|)$ edges. Hence a sufficiently fast network flow implemention leads to a running time of $O((n+m)(m+\sum_i |S_i|))$.

6 Image segmentation

Our next application of network flow is the problem of image segmentation, where the goal is to partition a given image into foreground and background regions. We will see how this can be cast as a minimum cut problem which can then be solved using network flow.

Formally, the image segmentation problem is specified as follows:

Input: An image consisting of a collection of pixels, I. Each pixel i has associated to it a 'likelihood' that it is in the foreground, f_i, and a likelihood that it is in the background, b_i. These are nonnegative values, but do not necessarily have to sum to 1 (as probabilities would). The *separation cost*, c of putting two adjacent pixels into different regions.

Output: A partition $I = F \sqcup B$ of the pixels into foreground (F) and background (B) so that the total likelihood of this partition minus its separation cost is maximized:

$$\sum_{i \in F} f_i + \sum_{j \in B} b_j - c \cdot \#\{(i,j) \in F \times B : i \sim j\} \tag{3}$$

where $i \sim j$ denotes that i and j are adjacent in the image.

As a sidenote: when the separation cost is zero (or very close), there is a simple greedy algorithm for image segmentation: for each pixel i, put i into F or B according to whether f_i or b_i is larger. This can lead to segmentations that have many little islands of pixels that, to a human, are clearly not foreground, since they are small and far away from the actual foreground. On the other hand, if the separation cost is very large, then an optimal solution will have no separations, *i.e.*, there will be no foreground or no background pixels. Thus a successfull application of this model of image segmentation requires a good choice of c. We don't go into this here, and instead show how to solve the problem for an arbitrary setting of c.

As said above, we can solve the image segmentation problem by reducing it to the minimum cut problem. In some sense, this is a very natural reduction to consider, since we are thinking of partitioning—or cutting—the pixels into two regions. Hence we might expect to construct a flow graph where S corresponds to F and T corresponds to B in an s-t cut.

On the other hand, this seems strange, since image segmentation itself is a maximization problem, while min-cut is a minimization problem. However, we can address this with the following trick: instead of maximizing the likelihood of a partitioning, we think of minimizing the likelihood of incorrectly partitioning the pixels. More specifically, we will think of minimizing the following objective function:

$$\sum_{i \in F} b_i + \sum_{j \in B} f_j + c \cdot \#\{(i,j) \in F \times B : i \sim j\} \tag{4}$$

Note how this more closely resembles the objective function for minimum cut. But before delving into that, let's first see why minimizing (4) is the same as maximizing (3). This essentially boils down to some algebra. Note that for any function f, the equation $\max f = -\min(-f)$ holds, and the extrema for each are obtained at the same points. Moreover, $\min(\alpha + f) = \alpha + \min f$ (and

likewise for max), and that the extrema are obtained at the same points in each expression. Hence, we can go from (3) to (4) as follows:

$$\begin{aligned}
&\max\left(\sum_{i\in F} f_i + \sum_{j\in B} b_j - c\cdot\#\{(i,j)\in F\times B : i\sim j\}\right)\\
&= \max\left(\sum_{i\in I} f_i - \sum_{i\in I\setminus F} f_i + \sum_{j\in I} b_j - \sum_{j\in I\setminus B} b_j - c\cdot\#\{(i,j)\in F\times B : i\sim j\}\right)\\
&= \sum_{i\in I} f_i + \sum_{j\in I} b_j + \max\left(-\sum_{i\in I\setminus F} f_i - \sum_{j\in I\setminus B} b_j - c\cdot\#\{(i,j)\in F\times B : i\sim j\}\right)\\
&= \sum_{i\in I} f_i + \sum_{j\in I} b_j - \min\left(\sum_{i\in F} b_i + \sum_{j\in B} f_j + c\cdot\#\{(i,j)\in F\times B : i\sim j\}\right)
\end{aligned}$$

Thus by minimizing (4), we maximize (3).

We now see how to turn minimizing (4) into a minimum cut instance. Recall the idea from earlier, using S to represent F and T to represent B, where S and T are the two halves of an s-t cut in the network we will construct. This suggests we make the pixels into vertices of the flow network. In order to allow any pixel to be in S or in T, we will need to make two new vertices to play the role of s and t in the flow network; call them s and t. These are all the vertices we will need.

It remains to specify the edges and their capacities. We want to choose these so that each s-t cut (S,T), which we think of as inducing $F = S\setminus\{s\}$ and $B = T\setminus\{t\}$, has capacity given by the formula in (4). If we can do this, then the reduction will be complete, since then minimum cuts will correspond to maximum likelihood partitionings of the input image. We will do this by counting the contribution of the terms appearing in (4) separately.

To handle the term $\sum_{i\in F} b_i$, we aim to construct edges (u,v) so that u is in S and v is in T if and only if i is in F, and assign it the capacity b_i. Doing this gives us exactly what we want. We can do this by setting $u\leftarrow i$ and $v\leftarrow t$, for each pixel i in I. This works, since i is in F if and only if i is on the s-side of the cut, and t is always on the t-side of the cut.

Handling the term $\sum_{j\in B} f_j$ is similar. We make an edge (s,j) for every pixel j in I, and assign it capacity f_j.

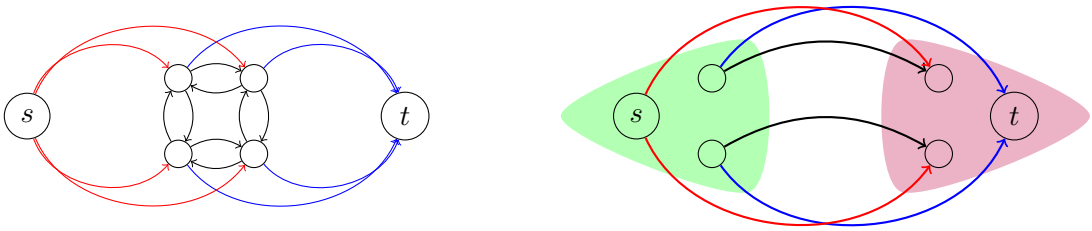
Finally, to handle the term $c\cdot\#\{(i,j)\in F\times B : i\sim j\}$ by first writing it as a sum of indicators as follows:

$$c\cdot\#\{(i,j)\in F\times B : i\sim j\} = \sum_{i,j\in I : i\sim j} c\cdot\chi\{i\in F, j\in B\}$$

where $\chi\{\cdots\}$ indicates the function which is 1 when $\cdots$ is true, and 0 otherwise. We can handle the contribution of the term corresponding to (i,j) by constructing an edge (u,v) so that u is in S and v is in T if and only if i is in F and j is in B, and then assign it the capacity c. Setting $u\leftarrow i$ and $v\leftarrow j$ works.

This completes the reduction. The end result is given pictorially in Figure 18. The resulting flow graph has $n+2$ nodes, when there are n input pixels, and has $2n+2m$ edges, where m is the total number of adjacencies of pixels. The reduction can be computed in time linear in $n+m$, and

Figure 18: Flow network for image segmentation



(a) Flow network arising from image segmentation

(b) Possible cut with highlighted edges

The input is a 2×2 image. The induced flow network is on the left, with the blue arrows to t, red arrows from s, and black arrows between pixels being the three sorts of edges we add during the reduction. The figure on the right gives an example cut, showing only the edges crossing the cut in the forward direction. The blue edges contribute b_i for each foreground pixel i, the red edges contribute f_j for each background pixel j, and the black edges contribute c for each pair of adjacent pixels with different classifications.

a minimum cut can be recovered from a maximum flow in linear time, so the overall running time for solving image segmentation is dominated by the maximum flow algorithm utilized to solve it. The most efficient known algorithms lead to $O(n^2 + nm)$ for the overall running time.

7 Project selection

The next problem we will reduce to network flow is the problem of Project Selection. The basic setup here is that some engineers have some projects that they would like to complete. Each project requires a set of tools in order to be completed, and these tools have costs associated with them that the engineers would like to keep small. However, tools can be re-used between projects, and have no additional cost for multiple uses.

Formally, there is a set of projects P and tools L, as well as a dependency relation, $R \subseteq P \times L$, where (p, ℓ) is in R if and only if the project p requires the tool ℓ to be completed. Each project p has a value v_p, and each tool ℓ has a cost c_ℓ. For subsets of projects $P' \subseteq P$, we let $\Gamma(P')$ denote the set of tools which are required by some project in P'; symbolically,

$$\Gamma(P') = \{\ell : \left(\exists p \in P'\right) (p, \ell) \in R\}$$

The goal of the engineers is to find a set of projects P^* so that the total value of the projects in P^* minus the total cost of the tools required by projects in P^* is maximized. Symbolically, we want to find a set P^* for which $P' = P^*$ achieves the maximum in the expression

$$\max_{P' \subseteq P} \sum_{p \in P'} v_p - \sum_{\ell \in \Gamma(P')} c_\ell \tag{5}$$

Reduction to network flow We will solve this problem via reduction to minimum cut. The intuition is that we want to partition the jobs and tools into those that we do use and those that we don't use, and cuts in graphs are a way of expressing this. The rest of the reduction is just figuring out how to represent our problem as a flow network so that s-t cuts correspond to ways of

selecting projects and tools such that finding a minimum s-t cut is equivalent to finding an optimal set of projects and tools.

A first idea to try is just to make a flow network G whose vertices are a special source and sink vertices, s and t respectively, as well as a vertex for each project and a vertex for each tool. We can then think of an s-t cut (S, T) as telling us to choose the projects and tools whose vertices appear in S. This will be the right idea for us, but we still have to specify the edges and their capacities in this flow network.

Our first step is to model the constraints of this problem. The constraints for this problem only come from the dependency relation R: for a project p and tool ℓ for which ℓ is required to complete p, we cannot allow s-t cuts (S, T) which have p in S but ℓ not in S. One way to do this is to introduce an infinite capacity edge from p to ℓ in G for every pair (p, ℓ) in R. This has the effect that, whenever an s-t cut (S, T) has p in S and ℓ in T, the capacity of this cut is infinite. As long as we design the network to have some finite-capacity cut, this means that no minimum cut will violate the constraints from R, which is good enough for our purposes.

Now it only remains to make sure that minimum cuts maximize the value in (5). Making a minimization objective achieve a maximization goal seems difficult at first, but this turns out to be a simple matter of algebra for our case. Note that for any real-valued function f, the expression "min f" has the same value as "$-\max(-f)$", and the expression "max $(f + \alpha)$" has the same value as "$\alpha + \max f$" for any constant α. Furthermore, the minima and maxima are attained at the same points of the domain of f. These facts give us a fair amount of flexibility in adjusting (5) to suit our situation.

Specifically, we'll use the first property to turn (5) into a maximization objective:

$$(5) = -\min_{P' \subseteq P} \sum_{\ell \in \Gamma(P')} c_\ell - \sum_{p \in P'} v_p \tag{6}$$

This can work well for us for the purposes of computing a minimum cut. We can account for the contribution from the first sum by simply adding edges from the vertices corresponding to tools to the sink vertex t. The capacity of the edge added from the tool ℓ to t is just c_ℓ. It's easy to verify that for any s-t cut (S, T), the contribution of these new edges to the capacity of this cut is exactly $\sum_{\ell \in S} c_\ell$. Since we are thinking of the S-part of s-t cuts as telling us which tools to select, *i.e.*, the tools in $\Gamma(P')$ for the projects P' selected by S, these new edges exactly account for the costs of tools.

The projects, however, seem less hopeful. The capacity of a cut is simply the sum over some positive quantities, where as $-\sum_{p \in P'} v_p$ is the negative of this. This is where our second property about min's and max's comes into play. The value $\sum_{p \in P} v_p$ is just a constant for the purposes of (5) and (6). This means we can rewrite (6) as

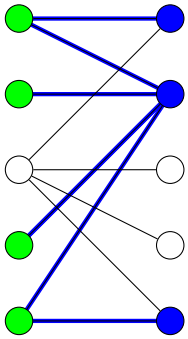
$$(6) = -\sum_{p \in P} - \min_{P' \subseteq P} \sum_{\ell \in \Gamma(P')} c_\ell + \sum_{p \in P \setminus P'} v_p \tag{7}$$

So now we can apply the same general idea as before: since we're thinking of the T-part of s-t cuts as telling us which projects *not* to select, we want to count the contribution of v_p to the value of a cut only if it puts p on the T-side of the cut. Thus we want to add edges from s to p for every project p, and give these edges capacity v_p.

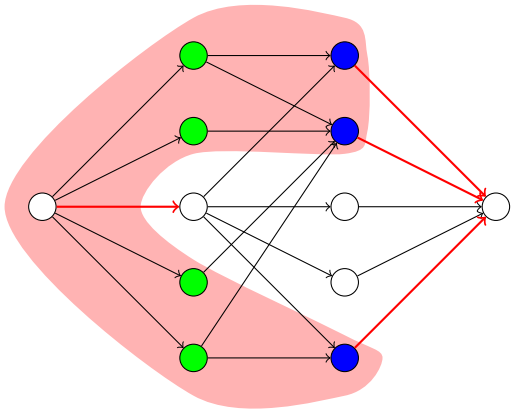
These are all the edges that we will need. They are represented pictorially in Figure 19. Our reduction from project selection to network flow is simply to create the flow network described

above and run a maximum flow algorithm on it. The value of the maximum flow, which is equal to the value of the minimum cut, isn't equal to the optimal project selection cost, but we can plug the value into (7) and determine the exact cost of the optimal project selection. To recover an optimal selection of projects and dependencies, we just find a minimum cut in the flow network, and using the S-side of the cut to tell us what projects to select.

Figure 19: Flow network for project selection



(a) A project/tool dependency graph



(b) Corresponding flow network

The bipartite graph on the left represents the project/tool dependency relation R. The vertices on the left are the projects, and the vertices on the right are tools. An edge (p, t) exists exactly when (p, t) is in R. An example selection of projects is given in green, and the corresponding tools are given in blue. The figure on the right gives the flow network corresponding to this instance. The edges of the form (s, p) have capacity v_p; the edges of the form (ℓ, t) have capacity c_ℓ; and the edges of the form (p, ℓ) have capacity $+\infty$. The cut corresponding to the selection of projects and tools is indicated in red.