Examples

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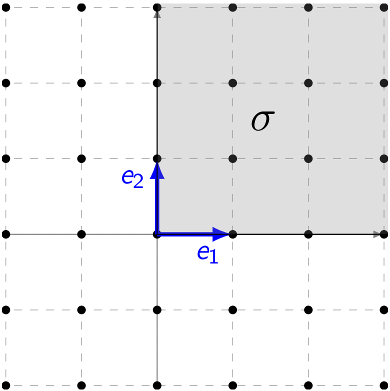
Example

Take $\sigma = \text{Cone}(e_1, e_2)$.



$$\begin{aligned}\sigma^\vee &= \{(x,y) \in \mathbb{R}^2 \mid \langle (1,0),(x,y)\rangle \geq 0 \text{ and } \langle (0,1),(x,y)\rangle \geq 0\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}.\end{aligned}$$

Affine Toric Varieties

Lemma

The affine toric variety corresponding to the zero cone is the n-torus.

Proof.

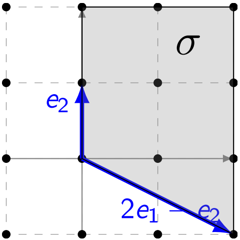
The dual is $\sigma^\vee = \mathbb{R}^n$. So $M_0 = \mathbb{R}^n \cap \mathbb{Z}^n = \mathbb{Z}^n$, which is generated by $e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n$. Therefore

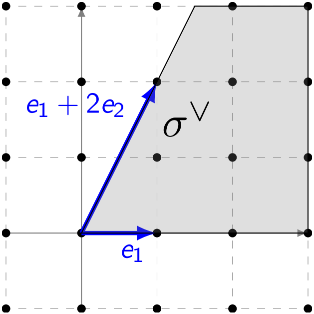
$$\begin{aligned}
\mathbb{C}[M_0] &= \mathbb{C}[\chi^{e_1}, \chi^{-e_1}, \chi^{e_2}, \chi^{-e_2}, \ldots, \chi^{e_n}, \chi^{-e_n}] \\
&\cong \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}] \\
&= \mathbb{C}[X_1, X_2, \ldots, X_n]_{X_1, X_2, \ldots, X_n}.
\end{aligned}$$

The spectrum of this is the set of points in $\mathbb{C}^n$ satisfying $x_i \neq 0 \ \forall i$. Therefore $U_0 = (\mathbb{C}^*)^n$. $\square$

Affine Toric Varieties

Example

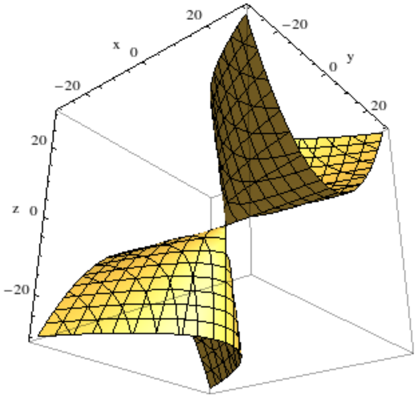




Since $(x, y) \in \sigma^\vee$ if and only if it is nonnegative on e_2 and $2e_1 - e_2$ i.e. $y \geq 0$ and $2x - y \geq 0$ we have the dual cone $\sigma^\vee = \text{Cone}(e_1, e_1 + 2e_2)$.

M_σ is generated by $e_1, e_1 + e_2$ and $e_1 + 2e_2$ so $\mathbb{C}[M_\sigma] = \mathbb{C}[X, XY, XY^2]$. The $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[U, V, W] \to \mathbb{C}[X, XY, XY^2]$ that sends $U \mapsto X, V \mapsto XY$ and $W \mapsto XY^2$ has kernel $I = (V^2 - UW)$. So

$$U_\sigma = \operatorname{Spec}\left(\mathbb{C}[U, V, W]/(V^2 - UW)\right).$$

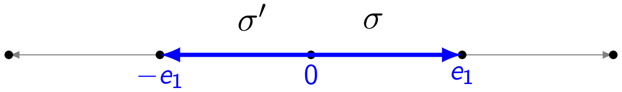


The elliptic cone $y^2 = xz$ in $\mathbb{R}^3$.

Abstract Toric Varieties

Example

Take $\Delta = \{0, \sigma, \sigma'\}$ as shown below:



We have $\sigma = \sigma^\vee = \text{Cone}(e_1)$ so M_σ is generated by e_1 so $\mathbb{C}[M_\sigma] = \mathbb{C}[X]$. Similarly, $\sigma' = (\sigma')^\vee = \text{Cone}(-e_1)$ so $M_{\sigma'}$ is generated by $-e_1$ and $\mathbb{C}[M_{\sigma'}] = \mathbb{C}[X^{-1}]$. We know $\mathbb{C}[M_0] = \mathbb{C}[X, X^{-1}]$.

Both U_σ and $U_{\sigma'}$ are copies of $\mathbb{C}$ and $X(\Delta)$ is obtained by gluing the images of the embeddings

$$\imath : U_0 = \operatorname{Spec} \mathbb{C}[X, X^{-1}] \to U_\sigma = \operatorname{Spec} \mathbb{C}[X]$$

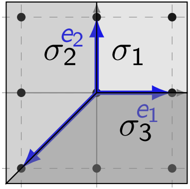
$$\imath' : U_0 = \operatorname{Spec} \mathbb{C}[X, X^{-1}] \to U_{\sigma'} = \operatorname{Spec} \mathbb{C}[X^{-1}].$$

Therefore we have

$$X(\Delta) = \frac{\mathbb{C} \bigsqcup \mathbb{C}}{(a \sim a^{-1})_{a \neq 0}}.$$

Recall $\mathbb{P}^1(\mathbb{C}) = \{[x : 1] \mid x \in \mathbb{C}\} \cup \{[1 : x] \mid x \in \mathbb{C}\}$ where the overlap is glued via the identification $[a : 1] = [1 : a^{-1}]$ for all $a \neq 0$. So $X(\Delta) = \mathbb{P}^1(\mathbb{C})$.

Generally, $\mathbb{P}^n(\mathbb{C}) = X(\Delta)$ where
$\Delta = \{\mathrm{Cone}(E) | E \subset \{e_1, \ldots, e_n, -(e_1 + \ldots + e_n)\}\}$.



More examples:

- $Y^{n+1} = XZ$ (Rational singularity of type A_n)
- $WZ = XY$ (Cone over a quadric)
- $Y^2 = X^3$ (Plane curve with a cusp)
- $V(WY - X^2, WZ - XY, XZ - Y^2) \subset \mathbb{P}^3(\mathbb{C})$ (Twisted cubic curve in $\mathbb{P}^3(\mathbb{C})$.)

$\mathbb{C}^n$, blow ups of projective space, Hirzebruch surfaces, rational normal scrolls ...