Toric Varieties and their Euler characteristic

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Introduction

Definition

The affine variety $(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ is called the *algebraic n-torus*. It acts on itself by componentwise multiplication.

Definition

A *Toric Variety* is a normal variety X that contains an algebraic torus $\mathbb{T}$ as a Zariski open dense subset together with an action $\mathbb{T} \times X \to X$ that extends the action of $\mathbb{T}$ on itself.

Cones

Definition

Let $u_1, \ldots, u_k$ be points in $\mathbb{Z}^n$. Subsets of $\mathbb{R}^n$ of the following form are called *cones*:

$$\sigma = \mathrm{Cone}(u_1, \ldots, u_k) = \{r_1 u_1 + \ldots + r_k u_k : r_i \geq 0 \ \forall i\}.$$

The *dual cone* of σ is the set

$$\sigma^\vee = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \ \forall y \in \sigma\}.$$

We also assume σ contains no line through the origin (so the origin is an apex). A *face* of σ is a set of the form

$$\tau := \sigma \cap u^\perp = \{v \in \sigma \mid \langle v, u \rangle = 0\}$$

for some $u \in \sigma^\vee$.

Affine Toric Varieties

- For any commutative monoid M there is a $\mathbb{C}$-algebra, denoted $\mathbb{C}[M]$, with basis elements $\chi^\mu, \mu \in M$ and multiplication defined by $\chi^\mu \chi^{\mu'} = \chi^{\mu+\mu'}$. The elements of $\mathbb{C}[M]$ are finite sums $\sum a_i \chi^{\mu_i}$ where $a_i \in \mathbb{C}$. If $m_1, \ldots, m_r$ are generators of M then $\mathbb{C}[M] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_r}]$.
- To a cone σ we associate the monoid $M_\sigma = \sigma^\vee \cap \mathbb{Z}^n$, where the operation is addition of coordinates. Then we define the affine toric variety associated to σ to be

$$U_\sigma = \operatorname{Spec} \mathbb{C}[M_\sigma].$$

- I'll let you identify $\operatorname{Spec}(\mathbb{C}[X_1, \ldots, X_n]/I)$ with $V(I)$. For example,

$$\operatorname{Spec} \frac{\mathbb{C}[X_1, X_2]}{(X_1^2 + X_2^2 - 1)} \approx \{(X_1, X_2) \in \mathbb{C}^2 \mid X_1^2 + X_2^2 = 1\}.$$

Procedure

1. Start with a cone σ.
2. Compute its dual $\sigma^\vee = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \ \forall y \in \sigma\}$.
3. Intersect with the lattice to get a monoid: $M_\sigma = \sigma^\vee \cap \mathbb{Z}^n$.
4. Find generators for M_σ so we can write down $\mathbb{C}[M_\sigma]$.
5. The affine toric variety U_σ is $\operatorname{Spec} \mathbb{C}[M_\sigma]$.

Let τ be a face of σ and consider the following:

- $\tau \subseteq \sigma$.
- $\sigma^\vee \subseteq \tau^\vee$.
- $\sigma^\vee \cap \mathbb{Z}^n \subseteq \tau^\vee \cap \mathbb{Z}^n$.
- $\mathbb{C}[M_\sigma] \subseteq \mathbb{C}[M_\tau]$.
- $U_\tau \to U_\sigma$ is dominant (has dense image).

So there is a natural map from U_τ into U_σ. From the case $\tau = 0$ we see that

Theorem

An algebraic torus is a dense subset of every affine toric variety.

Abstract Toric Varieties

Definition

A *fan* Δ is a set of cones satisfying the following conditions:

- Each face of a cone in Δ is also a cone in Δ.
- The intersection of two cones in Δ is a face of each.

The *abstract toric variety* $X(\Delta)$ is the disjoint union of all the $U_\sigma, \sigma \in \Delta$, glued together by the following rule: If $\sigma, \tau \in \Delta$ then $\sigma \cap \tau$ is a face of both. There are natural maps

$$i : U_{\sigma\cap\tau} \to U_\sigma$$

$$i' : U_{\sigma\cap\tau} \to U_\tau.$$

We glue U_σ to U_τ by identifying images of the natural maps above. Doing this for every pair of cones in Δ yields $X(\Delta)$.

Some Theorems

The geometry of the fan captures the geometry of the variety.

- $X(\Delta)$ is compact if and only Δ covers the Euclidean space it lies in.
- If σ has codimension k then $\pi_1(U_\sigma) = \mathbb{Z}^k$. If σ has codimension 0, then it is contractible.
- Let $\sigma = \mathrm{Cone}(u_1, \ldots, u_k)$. The affine toric variety U_σ is smooth if and only if $\{u_i\}$ can be extended to a basis of $\mathbb{Z}^n$.
    - If so, $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = \dim \sigma$.
    - If not, then its singularities are all rational.

Cohomology

- If $X(\Delta)$ is smooth and compact, then its odd cohomology is trivial and

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} d_{n-i}$$

where $b_i = \dim_{\mathbb{Q}} H^i(X(\Delta), \mathbb{Q})$ and d_j is the number of j dimensional faces in Δ.

For example, $\mathbb{P}^n(\mathbb{C}) = X(\Delta)$ where $\Delta = \{\text{Cone}(E) | E \subset \{e_1, \ldots, e_n, -(e_1 + \ldots + e_n)\}\}$ so we have $d_{n-i} = \binom{n+1}{i}$. We get

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \binom{n+1}{i+1} = \mathbf{1}_{0 \le k \le n}.$$

Toric decomposition

- $X(\Delta) = \bigsqcup_{\sigma \in \Delta} T_\sigma$ is the disjoint union of finitely many locally closed (open in its closure) subvarieties which satisfy $T_\sigma \cong (\mathbb{C}^*)^{\operatorname{codim} \sigma}$.

For example

$$\mathbb{P}^1(\mathbb{C}) = \{[1:0]\} \bigsqcup \{[0:1]\} \bigsqcup \{[1:x] \mid x \in \mathbb{C}^*\}$$

and

$$\{Y^2 = XZ\} = \{(0,0,0)\} \bigsqcup \{0\} \times \{0\} \times \mathbb{C}^* \bigsqcup \mathbb{C}^* \times \{0\} \times \{0\}$$

$$\bigsqcup \{(t_1, t_1 t_2, t_1 t_2^2) \mid t_1, t_2 \in \mathbb{C}^*\}.$$

Euler Characteristic

Definition

The Euler Chracteristic of a topological space X is

$$\chi(X) = \sum (-1)^i \mathrm{rank}\left(H^i(X,\mathbb{Z})\right).$$

Theorem

The Euler characteristic of a (compact) toric variety is $\chi(X(\Delta)) = d_n$. Here d_n is the number of codimension zero cones in Δ.

Cohomology with Compact Supports

Suppose we have a singular cochain complex $C^\bullet(X)$. We consider the cohomology of the subcomplex $C_c^\bullet(X)$ of cochains which have compact support (that is, they vanish outside some compact subset of X).

- $H_c^i(\mathbb{R}^n, \mathbb{Q}) = \mathbb{Q}$ if $i = n$ and zero otherwise.
- If $U \subseteq X$ is open and $C = X \setminus U$, then there is a long exact sequence:

  $$\cdots \to H_c^i(U) \to H_c^i(X) \to H_c^i(C) \to H_c^{i+1}(U) \to \cdots$$

  This gives
  $\sum(-1)^i \dim H_c^i(X) = \sum(-1)^i \left(\dim H_c^i(C) + \dim H_c^i(U)\right)$.
  So Euler characteristic with compact supports is an *additive function*:

  $$\chi_c(X) = \chi_c(U) + \chi_c(C).$$

χ_c is locally additive

- If $U \subseteq X$ is open and $C = X \setminus U$ then $\chi_c(X) = \chi_c(U) + \chi_c(C)$.
- If Y is locally closed (open in its closure) in X,

$$\chi_c(X) = \chi_c(X \setminus \overline{Y}) + \chi_c(\overline{Y}) = \chi_c(X \setminus \overline{Y}) + \chi_c(\overline{Y} \setminus Y) + \chi_c(Y).$$

Since $X \setminus \overline{Y}$ is open in $X \setminus Y$,

$$\chi_c(X \setminus Y) = \chi_c(X \setminus \overline{Y}) + \chi_c(\overline{Y} \setminus Y).$$

Lemma

Suppose Y is locally closed in X. Then

$$\chi_c(X) = \chi_c(X \setminus Y) + \chi_c(Y).$$

- $\chi_c(\mathbb{C}) = \chi_c(\{0\}) + \chi_c(\mathbb{C}^*)$ so $\chi_c(\mathbb{C}^*) = 0$.
- The Künneth formula

$$H_c^k(X \times Y) \cong \bigoplus_{i+j=k} \left(H_c^i(X) \otimes H_c^j(Y)\right)$$

  implies that $\chi_c((\mathbb{C}^*)^n) = 0$ for all $n \geq 1$.
- Since $X(\Delta) = \bigsqcup_{\sigma \in \Delta} T_\sigma$, $T_\sigma \cong (\mathbb{C}^*)^{\operatorname{codim} \sigma}$ locally closed, we have

$$\chi_c(X) = \sum_{\sigma \in \Delta} \chi_c\left((\mathbb{C}^*)^{\operatorname{codim} \sigma}\right).$$

- Therefore

$$\chi_c(X) = d_n.$$

$\square$