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New formulations of the primitive equations of atmosphere and applications

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Abstract. The primitive equations are the fundamental equations of atmospheric dynamics. With the purpose of understanding the mechanism of long-term weather prediction and climate changes, we study in this paper as a first step towards this long-range project what is widely considered as the basic equations of atmospheric dynamics in meteorology, namely the primitive equations of atmosphere. The mathematical formulation and attractors of the primitive equations, with or without vertical viscosity, are studied. First of all, by integrating the diagnostic equations we present a mathematical setting, and obtain the existence and time analyticity of solutions to the equations. We then establish some physically relevant estimates for the Hausdorff and fractal dimensions of the attractors of the problems.

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Introduction

In order to understand the turbulent behaviour of long-term weather prediction and climate changes we can use historical records and numerical computations to detect the future weather and the possible global changes. Alternatively, a complementary method and source of information was advocated and developed by the pioneers of meteorology such as V Bjerknes, L F Richardson and others. This consists of studying the mathematical equations and models governing the motion of the atmosphere, and to solve them numerically. What was a dream at the time of Richardson becoming closer to reality due to the considerable increase in computer capacity in recent years and that to come, and also to the real-time information

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system allowing access to data. From the mathematical point of view, the problems we address here are how well posed the initial value problem is for the equations of meteorology, and the long-time behaviour of their solutions.

The general equations describing the motion and states of the atmosphere, that is, a specific compressible fluid, are the hydrodynamic and thermodynamic equations with Coriolis force. Because the resulting flow for the atmosphere is amazingly rich in its organization and complexity, the full governing equations are extremely complicated and seem to defy analysis, at least for the time being.

Fortunately, since the vertical scale of the global atmosphere is much smaller than the horizontal one, scale analysis, meteorological observations and historical data reveal that the large-scale atmosphere satisfies the quasi-static equilibrium equation (1.7), which is also called the hydrostatic equation. This equation provides a relationship between pressure and density. Based on this relationship, the original equations of atmosphere are reduced to the so-called primitive equations, which we will study in this paper. Because of its good accuracy, the quasistatic equilibrium equation is well accepted as a fundamental equation of the atmosphere, and is considered as a starting point for studying the extremely complicated atmospheric phenomena, and for predicting the weather and possible climate changes. These equations were proposed and used by Richardson (1922) in his historic forecast [1]. However, at that time they were considered too complicated, and were replaced by simplified models. Owing to the advances in meteorology and computing power since then, there is now a tendency to turn from the simplified models used by Charney, von Neumann and Philips in the early 1950s, and other simplified models, back to the equations of Richardson. The primitive equations are also called the predictive equations or the balance equations [2, ch 3, p 74; 3]. Since the primitive equations are the fundamental equations of the atmosphere, we intend to show in a subsequent article how they can be derived from the general conservation equations of physics and mechanics by an asymptotic procedure, using the fact that the height of the atmosphere is very small compared to the radius of the earth. We will also show in a subsequent paper that all other meteorologically interesting equations of the atmosphere, such as the linear balanced model, the 2D model, and the quasigeostropheric and the barotropic equations are derived from the primitive ones.

As we have already briefly said, the objective of this paper is to establish the mathematical foundations and to develop a rigorous mathematical framework for these equations, which does not seem to have been done before. We will also study the long-time behaviour of the solutions of these equations using some concepts recently developed in the mathematical theory of (infinite-dimensional) dynamical systems. Long-time aspects ought to be considered since the length of time for which weather prediction is sought is very long compared to the natural time attached to the equations and the physical parameters.

It is well known that if the forecast period is to be extended to 3 and more days, the diabatic heating and the friction of the atmosphere will have to be taken into account. Therefore, we have to consider the primitive equations with viscosity. On the other hand, the quasistatic equilibrium equations, as we have mentioned, is a fundamental property of the atmosphere. So we consider two types of viscosity for the primitive equations. First, we emphasize the importance of the quasistatic equilibrium equations and neglect the viscosity in the vertical direction. More specifically, keeping the quasistatic equilibrium relations unchanged, we only attach the viscosity to the horizontal velocity and the temperature. For simplicity, the resulting equations will still be called the primitive equations (PEs). The second type of viscosity is introduced in such a way that in addition to the viscosity terms given for the horizontal velocity and the temperature, we also keep the viscosity terms for the vertical velocity ω in the vertical direction. We call the resulting equations the primitive equations with vertical viscosity (PEV^{2}s).

Owing to the quasistatic equilibrium relation, the pressure p is a decreasing function of the vertical coordinate z and we can make a coordinate transformation from the usual spherical coordinate system (θ, φ, z) to the p-coordinate one (θ, φ, p). We know that the atmosphere is made up of a compressible fluid. The primitive equations in the p-coordinate system, however, possess a similar structure to the equations of an incompressible fluid. More precisely, the continuity equation takes the same form as that of an incompressible fluid. This is one of the advantages of the quasistatic equilibrium relation, and therefore, of the p-coordinate system. Of course, due also to this relation, the resulting equations are highly degenerate for the vertical velocity ω. So we have to overcome the difficulty caused by this degeneracy. In conclusion, the PEs using the p-coordinate system, are similar to but more complicated than the Navier–Stokes equations of incompressible fluids. They are, however, much simpler and more tractable than the equations of completely general compressible fluids from which they are derived.

For both the PEs and the PEV^{2}s, the prognostic feature for the vertical velocity ω is lost, and there are two diagnostic equations involving the geopotential Φ and ω. In both cases the vertical velocity ω changes with time, but only as specified by the diagnostic equations. Moreover, the vertical momentum equation is replaced by the quasistatic equilibrium relation, in which the vertical velocity ω does not appear explicitly. It is worth pointing out that using the quasistatic equilibrium equation does not imply that ω vanishes. It means, rather, that it must be found by diagnostic techniques and that the resulting values are precisely those needed to maintain the quasistatic equilibrium. From the mathematical point of view, both the PEs and PEV^{2}s do not appear as evolution equations due to the diagnostic ones. Owing to the specific form of the diagnostic relations they are integrated, and then yield two evolution systems with a non-local constraint (1.52) for the PEs and the PEV^{2}s, respectively. We cannot, however, use the usual techniques to handle these problems because the constraints are not local. The specific considerations to circumvent the difficulty caused by the non-local constraint are important aspects of this paper.

To be more specific, we consider the non-dimensional forms (1.41)–(1.44) of the PEs, and (4.1)–(4.4) of PEV^{2}s. Obviously, the vertical velocity ω and the geopotential Φ are given by the diagnostic equations (1.42) and (1.43). Since there are two boundary value conditions for ω, integrating the continuity equation (1.42) eliminates ω from the system, but yields a non-local constraint (1.52). On the other hand, if we consider the geopotential Φ at the isobaric surfaces as a given function, then integrating the quasistatic equilibrium equation provides an overdetermined system. Surprisingly, if we introduce a two-variable unknown function, Φ_s, as the function of the geopotential at the isobaric surface $p = P$, then we obtain two reformulation, (1.50)–(1.52), of the PEs, and (4.8)–(4.10) of the PEV^{2}s. It is remarkable from the mathematical point of view that Φ_s is exactly the Lagrange multiplier of the non-local constraint, and that the resulting systems are 3D evolution systems. Moreover, we can simply use the Galerkin method to solve the weak formulations (problems 2.1 and 4.1) of the systems. In [11], however, we did not find the reformulations (1.50)–(1.52) of PEs and (4.8)–(4.10) of the PEV^{2}s and, therefore, we had to construct some special kinds of function spaces for the vertical velocity of ω, in which the elements possess more regularity in the p-direction. At the same time, we had to use the semi-discretization method to prove the existence of weak solutions because we were not able to use the Galerkin procedure. It is also very useful from the meteorological point of view that our reformulations and the methods to solve them provide an effective means of determining the geopotential at the surface $p = P$. Moreover, we would like to point out that Φ_s also provides some information about the topography of the earth.

In the functional frameworks of our reformulated problems, we have to establish a regularity result concerning an elliptic system with the non-local constraint. The result is obtained by combining the difference quotient method of Nirenberg and our systematic study for the function spaces constructed in this paper. The result is not only crucial for our problem, but also possesses its own independent interest.

Another difficulty arises from the high nonlinearity of our problems. The nonlinear terms take the same form as that of the nonlinear terms in 3D Navier–Stokes equations. However, they possess a quite different nature. In our problems, due to the absence of the time derivative term $\partial\omega/\partial t$ of ω, all the information provided by this term is lost here. Thus, the situation is more complicated than that of 3D Navier–Stokes equations. The information for the PEs from those viscosity terms of the vertical velocity is also lost.

This paper is divided into three parts. In the first part we consider the PEs. For completeness, in section 1, we present a brief derivation of the primitive equations. Then, by integrating the diagnostic equations, we introduce another formulation of the equations in the last part of this section. In section 2 we construct some special function spaces for our problem, in which the non-local relation (1.52) appears as a constraint. Then, after establishing some functionals and their associated operators, we obtain the weak formulation of the PEs. We also prove a regularity result on solutions of an elliptic system with the non-local constraint, which, on one hand, possesses its own meaning and, on the other, is used in the remaining part of the paper. Finally, in section 3, we establish an existence theorem of global weak solutions for PEs.

Part 3 deals with the PEV^{2}s. As in section 2, we obtain in section 4 a mathematical formulation of the problem. Then, the existence of global weak solutions and local strong solutions, and time analyticity of the strong solutions, are established in section 5.

The last part of the article is devoted to attractors and their dimensions for both PEs and PEV^{2}s. Some physically relevant estimates for the Hausdorff and fractal dimensions of the attractors for PEV^{2}s are given in section 6. Moreover, the existence and dimensions of global attractors for both the PEs and PEV^{2}s are also reported in the last section of this paper. In forthcoming articles [5, 6] we will study the case where the pressure variable is allowed to vary in the interval $[0, P]$; here we restrict ourselves, more classically, to an interval $[p_0, P]$, $p_0 > 0$ (see section 1). Subsequently, we will investigate the PEs of the ocean where the independent variables are the original (natural) ones (θ, φ, z and t).

The principle notation used in this paper is listed in the appendix.

CONTENTS

Part I. The primitive equations of atmosphere (PEs)

In this part, we study the mathematical formuation and the existence of solutions for the PEs. As we mentioned in the introduction, regarding long-term weather prediction and climate changes, we have to consider the equations together with viscosity. The quasistatic equilibrium equation, however, provides a fundamental property of the atmosphere. So in this part we emphasize the importance of this relation, and consider the specific PEs in which we keep the quasistatic equilibrium relation unchanged and only attach the viscosity to the horizontal velocity and the temperature.

1. The PEs in meteorology

1.1. General equations of atmosphere

If we use a non-inertial coordinate system rotating with the earth, then atmospheric motion is described by the following general hydrodynamic and thermodynamic equations of compressible fluids with Coriolis force:

Momentum equations

$$\frac{\mathrm{d}\boldsymbol{V}_3}{\mathrm{d}t} = \text{pressure gradient} + \text{gravity} + \text{Coriolis force} + \text{dissipative force}$$

$$= -\frac{1}{\rho}\operatorname{grad}_3 p + \boldsymbol{G} - 2\boldsymbol{\Omega} \times \boldsymbol{V}_3 + \boldsymbol{D}. \tag{1.1}$$

Continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div}_3 \boldsymbol{V}_3 = 0. \tag{1.2}$$

The first law of thermodynamics

$$c_p \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{1.3}$$

The equation of state

$$p = R\rho T. \tag{1.4}$$

In the above equations, $\boldsymbol{D}$ denotes the viscosity terms, which will be specified later, and $\mathrm{d}Q/\mathrm{d}t$ is the heat flux per unit density in a unit time interval, which includes molecular or turbulent, radiative and evaporative heating. In other words $\mathrm{d}Q/\mathrm{d}t$ represents the solar heating, the albedo of the earth, and the molecular heating. With appropriate expressions of $\boldsymbol{D}$ and $\mathrm{d}Q/\mathrm{d}t$, (1.1)–(1.4) are the usual equations of compressible fluids with the additions of Coriolis and gravitational forces (e.g. see [7, vol. 1]).

Since we want to study the atmosphere around the earth, the spherical coordinate system obviously describes atmospheric motion most naturally. Let θ ($0 \leqslant \theta \leqslant \pi$) denote the colatitude of the earth, φ ($0 \leqslant \varphi \leqslant 2\pi$) the longitude of the earth and r the radial distance; $z = r - a$ is the height above sea level. Then we have

$$\boldsymbol{V}_3 = v_\theta e_\theta + v_\varphi e_\varphi + v_r e_z = r\dot{\theta} e_\theta + r \sin\theta \dot{\varphi} e_\varphi + \dot{r} e_z,$$

where e_θ, e_φ and e_z are unit vectors in θ-, φ- and z-directions. Using the geometrical notation, they are defined by

$$e_\theta = \frac{1}{a}\frac{\partial}{\partial\theta} \qquad e_\varphi = \frac{1}{a\sin\theta}\frac{\partial}{\partial\varphi} \qquad e_z = \frac{\partial}{\partial z}.$$

Then (1.1)–(1.4) can be rewritten as

$$\left.\begin{aligned}
&\frac{\mathrm{d}v_\theta}{\mathrm{d}t} + \frac{1}{r}\underline{(v_r v_\theta - v_\varphi^2\cot\theta)} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + 2\Omega\cos\theta v_\varphi + D_\theta \\
&\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{1}{r}\underline{(v_r v_\varphi + v_\theta v_\varphi\cot\theta)} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} - 2\Omega\cos\theta v_\theta - \underline{2\Omega\sin\theta v_r} + D_\varphi \\
&\frac{\mathrm{d}v_r}{\mathrm{d}t} - \underline{\frac{1}{r}(v_\theta^2 + v_\varphi^2)} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + \underline{2\Omega\sin\theta v_\varphi} + D_r \\
&\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}\right) = 0 \\
&c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \\
&p = R\rho T
\end{aligned}\right\} \tag{1.5}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} + v_r\frac{\partial}{\partial r} \tag{1.6}$$

and $\boldsymbol{D} = (D_\theta, D_\varphi, D_r)$. The underlined terms on the left-hand sides of (1.5) arise from the curvature of the earth.

1.2. The PEs

The scale analysis, meteorological observations and historical data show that the atmosphere satisfies the following quasistatic equilibrium equation:

$$\frac{\partial p}{\partial r} = -\rho g \tag{1.7}$$

which provides a relation between pressure and density. As we mentioned in the introduction, this relation is highly accurate for the large-scale atmosphere; thus, it has become a fundamental equation in atmospheric science and serves as the starting point of all theoretical investigations and practical weather predictions. As indicated in the introduction, a rigorous mathematical justification of (1.7) will be given elsewhere. This equation is derived from (1.5) by asymptotic expansions, using the fact that the height of the atmosphere is small compared to the radius of the earth.

Replacing the vertical velocity equation (the third equation) in (1.5) by the quasistatic equilibrium equation (1.7), we can obtain the following PEs of atmosphere in the coordinate system (t, θ, φ, z):

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{v_\varphi^2}{a}\cot\theta = -\frac{1}{\rho a}\frac{\partial p}{\partial\theta} + 2\Omega\cos\theta v_\varphi + D_\theta \tag{1.8}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{v_\theta v_\varphi}{a}\cot\theta = -\frac{1}{\rho a\sin\theta}\frac{\partial p}{\partial\varphi} - 2\Omega\cos\theta v_\theta + D_\varphi \tag{1.9}$$

$$\frac{\partial p}{\partial r} = -\rho g \tag{1.10}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_r}{\partial r}\right) = 0 \tag{1.11}$$

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{1.12}$$

$$p = R\rho T \tag{1.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{v_\theta}{a}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial}{\partial\varphi} + v_r\frac{\partial}{\partial r}. \tag{1.14}$$

Since the vertical scale is much smaller than the horizontal one, we have replaced r in (1.5) and (1.6) by the radius of the earth a and, moreover, we have omitted some terms underlined in (1.5), which are thought to be small.† However, we still keep the terms

$$-\frac{v_\varphi^2}{a}\cot\theta \qquad \frac{v_\theta v_\varphi}{a}\cot\theta$$

representing the force contributed by the curvature of the sphere S_a^2.

1.2.1. Pressure vertical coordinate. It is easy to see from (1.7) that p is decreasing with respect to the third independent variable z, and therefore the coordinate transformation

$$(t, \theta, \varphi, z) \to (t^*, \theta^*, \varphi^*, p = p(t, \theta, \varphi, z)) \tag{1.15a}$$

can be made with

$$t^* = t \qquad \theta^* = \theta \qquad \varphi^* = \varphi.$$

The inverse transformation is

$$(t, \theta, \varphi, p) \to (t, \theta, \varphi, z = z(t^*, \theta^*, \varphi^*, p)). \tag{1.15b}$$

We have

$$p = p(t, \theta, \varphi, z(t^*, \theta^*, \varphi^*, p)). \tag{1.16}$$

While z varies from 0 (the surface of the earth or of the ocean) to infinity, the pressure p varies from p_s (the (unknown) pressure on the sea or the earth level) to 0 (the pressure in the high atmosphere). However, as we will see later, we will restrict p to an interval $[p_0, P]$, where $p_0 > 0$ and P is a given approximate value of p_s; the case $p_0 = 0$ leading to some special mathematical difficulties is studied in future papers in this series [5, 6].

† If we keep all the terms, most of what follows remains true but the proofs become a little more complicated.

We now derive the PEs written using the new system of variables, which is called the pressure coordinate system.

Differentiating (1.16) with respect to p, θ^*, φ^*, respectively, we obtain

$$\left.\begin{aligned} &\frac{\partial p}{\partial z}\frac{\partial z}{\partial p}=1\\ &\frac{\partial p}{\partial \theta}+\frac{\partial p}{\partial z}\frac{\partial z}{\partial \theta^*}=0\\ &\frac{\partial p}{\partial \varphi}+\frac{\partial p}{\partial z}\frac{\partial z}{\partial \varphi^*}=0.\end{aligned}\right\} \tag{1.17}$$

Replacing $\partial p/\partial z$ by $-\rho g$ in (1.17) provides

$$\left.\begin{aligned} &\frac{\partial gz}{\partial p}=-\frac{1}{\rho}\\ &\frac{\partial gz}{\partial \theta^*}=\frac{1}{\rho}\frac{\partial p}{\partial \theta}\\ &\frac{\partial gz}{\partial \varphi^*}=\frac{1}{\rho}\frac{\partial p}{\partial \varphi}.\end{aligned}\right\} \tag{1.18}$$

Moreover, it is easy to see that

$$\left.\begin{aligned} &\frac{\partial}{\partial z}=\frac{\partial p}{\partial z}\frac{\partial}{\partial p}\\ &\frac{\partial}{\partial \theta}=\frac{\partial}{\partial \theta^*}+\frac{\partial p}{\partial \theta}\frac{\partial}{\partial p}\\ &\frac{\partial}{\partial \varphi}=\frac{\partial}{\partial \varphi^*}+\frac{\partial p}{\partial \varphi}\frac{\partial}{\partial p}.\end{aligned}\right\} \tag{1.19}$$

On the other hand, the material differentiation in the new coordinate system is given by

$$\begin{aligned}\dot F=\frac{\mathrm{d}F}{\mathrm{d}t^*}&=\left(\frac{\partial}{\partial t^*}+\dot\theta^*\frac{\partial}{\partial \theta^*}+\dot\varphi^*\frac{\partial z}{\partial \varphi^*}+\dot p\frac{\partial}{\partial p}\right)F\\ &=\left(\frac{\partial}{\partial t^*}+\dot\theta\frac{\partial}{\partial \theta}+\dot\varphi\frac{\partial z}{\partial \varphi}+\dot p\frac{\partial}{\partial p}\right)F.\end{aligned} \tag{1.20}$$

These expressions may be used to form the gradient of a scalar function A and, similarly, the 2D divergence of a vector $\boldsymbol{B}$ with the results

$$\mathrm{grad}_p\, A=\mathrm{grad}_z\, A+\frac{\partial A}{\partial p}\frac{\partial p}{\partial z}\mathrm{grad}_p\, z$$

$$\mathrm{grad}_p\cdot \boldsymbol{B}=\mathrm{grad}_z\cdot \boldsymbol{B}+\frac{\partial \boldsymbol{B}}{\partial p}\frac{\partial p}{\partial z}\cdot\mathrm{grad}_p\, z$$

where grad_p and grad_z denote the horizontal gradient operators in the p-coordinate and z-coordinate systems, respectively. With these formulae the horizontal pressure forces become

$$-\frac{1}{\rho}\operatorname{grad}_z p = -\operatorname{grad}_p \Phi$$

where $\Phi = gz$ is the geopotential. So the first two components of the momentum equations (1.8) and (1.9) become

$$\left.\begin{aligned}\frac{\mathrm{d}v_\theta}{\mathrm{d}t^*} &= -\frac{1}{a}\frac{\partial gz}{\partial \theta^*} + 2\Omega\cos\theta^* v_\varphi + D_\theta \\ \frac{\mathrm{d}v_\varphi}{\mathrm{d}t^*} &= -\frac{1}{a\sin\theta^*}\frac{\partial gz}{\partial\varphi^*} - 2\Omega\cos\theta^* v_\theta + D_\theta\end{aligned}\right\} \tag{1.21}$$

with $\mathrm{d}/\mathrm{d}t^*$ given by (1.20).

The first equation in (1.18) and the equation of state (1.13) yield

$$\frac{\partial\Phi}{\partial p} + \frac{RT}{p} = 0. \tag{1.22}$$

Now we consider the continuity equation (1.11). By (1.7), the continuity equation (1.11) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial p}{\partial z}\left(\frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_r}{\partial r}\right) = 0. \tag{1.23}$$

Now we check each term of (1.23):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial p}{\partial z}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right) + v_\theta\frac{\partial}{a\,\partial\theta}\left(\frac{\partial p}{\partial z}\right) + \frac{v_\varphi}{a\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial p}{\partial z}\right) + v_r\frac{\partial}{\partial r}\left(\frac{\partial p}{\partial z}\right) \\ &= \frac{\partial}{\partial z}\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right) - \frac{\partial v_\theta}{\partial z}\frac{\partial p}{a\,\partial\theta} - \frac{\partial v_\varphi}{\partial z}\frac{1}{a\sin\theta}\frac{\partial p}{\partial\varphi} - \frac{\partial v_r}{\partial z}\frac{\partial p}{\partial r}.\end{aligned}$$

Owing to (1.19) we have

$$\frac{\partial}{\partial z}\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right) = \frac{\partial p}{\partial z}\frac{\partial}{\partial p}\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)$$

$$\begin{aligned}-\frac{\partial v_\theta}{\partial z}\frac{\partial p}{a\,\partial\theta} + \frac{\partial p}{\partial z}\frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} \\ &= \frac{\partial p}{\partial z}\left(-\frac{\partial v_\theta}{\partial p}\frac{\partial p}{a\,\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}\right) \\ &= \frac{\partial p}{\partial z}\left[-\frac{\partial v_\theta}{\partial p}\frac{\partial p}{a\,\partial\theta} + \frac{1}{a\sin\theta}\left(\frac{\partial}{\partial\theta^*} + \frac{\partial p}{\partial\theta}\frac{\partial}{\partial p}\right)(v_\theta\sin\theta)\right] \\ &= \frac{\partial p}{\partial z}\frac{\partial v_\theta\sin\theta^*}{a\sin\theta^*\,\partial\theta^*}.\end{aligned}$$

Similarly,

$$-\frac{\partial v_\varphi}{\partial z}\frac{\partial p}{a\sin\theta\,\partial\varphi} + \frac{\partial p}{\partial z}\frac{1}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} = \frac{\partial p}{\partial z}\frac{\partial v_\varphi}{a\sin\theta^*\,\partial\varphi^*}.$$

So (1.23) implies

$$\frac{\partial}{\partial p}\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right) + \frac{1}{a\sin\theta^*}\left(\frac{\partial v_\theta\sin\theta^*}{\partial\theta^*} + \frac{\partial v_\varphi}{\partial\varphi^*}\right) = 0. \tag{1.24}$$

Since the total derivative does not depend on the coordinate system, we can define

$$\omega = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}t^*} = \dot{p}. \tag{1.25}$$

Then, omitting the asterisk for the independent variables of the new coordinate system, we can easily obtain the following PEs of atmosphere in the coordinate system (t, θ, φ, p), which is called the p-coordinate system:

$$\left.\begin{aligned}
&\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{v_\varphi^2}{a}\cot\theta = -\frac{1}{a}\frac{\partial\Phi}{\partial\theta} + 2\Omega\cos\theta v_\varphi + D_\theta \\
&\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{v_\theta v_\varphi}{a}\cot\theta = -\frac{1}{a\sin\theta}\frac{\partial\Phi}{\partial\varphi} - 2\Omega\cos\theta v_\theta + D_\varphi \\
&\frac{\partial\Phi}{\partial p} + \frac{R}{p}T = 0 \\
&\frac{\partial\omega}{\partial p} + \frac{1}{a\sin\theta}\left(\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{\partial v_\varphi}{\partial\varphi}\right) = 0 \\
&c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\omega = \frac{\mathrm{d}Q}{\mathrm{d}t}
\end{aligned}\right\} \tag{1.26}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{v_\theta}{a}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial}{\partial\varphi} + \omega\frac{\partial}{\partial p} \tag{1.27}$$

$$\begin{aligned}
v_r = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial t} + \frac{v_\theta}{a}\frac{\partial z}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial z}{\partial\varphi} + \omega\frac{\partial z}{\partial p} \\
&= \frac{\partial z}{\partial t} + \frac{v_\theta}{a}\frac{\partial z}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial z}{\partial\varphi} - \frac{\omega}{\rho g}.
\end{aligned} \tag{1.28}$$

Remark 1.1. It is well known that the atmosphere is made up of a compressible fluid. Here in the p-coordinate system the continuity equation in (1.26), however, takes the same form as that of an incompressible fluid. This is one of the advantages of the p-coordinate system.

Remark 1.2. There are other alternative ways of treating the vertical coordinate. By taking the orography into consideration, for instance, we can use $\sigma = p/p_s$ as the vertical coordinate, where p_s stands for the surface pressure on the earth. The resulting coordinate system is called σ-coordinate system. Even though it has some drawbacks (see p 79 of [2]), the σ-system is also widely accepted and practicely used. We will study the primitive equations on this coordinate system elsewhere.

We now want to make a further simplification of the term $RT\omega/p$ in the last equation of (1.26). The reason is two-fold. Firstly, the resulting equations are more accessible from a mathematical point of view. Secondly, the 'energy' provided by the simplified term will be balanced by the term RT/p in the third equation of (1.26).

Let $\bar{T} = \bar{T}(p) \in C^\infty([p_0, P])$ be a given vertical distribution of the standard temperature C^∞ on the interval $[p_0, P]$ such that

$$C^2 \doteq R\left(\frac{R\bar{T}}{c_p} - p\frac{\partial \bar{T}}{\partial p}\right) = \text{constant}. \tag{1.29}$$

$\bar{T}$ can be considered as the climate average value of the temperature on isobaric surfaces. Here $P > p_0 > 0$. Since we want to study the 3D large-scale atmospheric motion, as a first step, we can assume that the surface of the earth is nearly an isobaric surface given by $p = P$ where P denotes the approximate value of the pressure on the surface of the earth. We also assume in this paper that the isobaric surface $p = p_0$ provides the upper boundary of the atmosphere, where p_0 is a small positive number.

Moreover, we choose a standard distribution $\bar{\Phi} = \bar{\Phi}(p)$ of Φ defined by

$$\frac{\partial \bar{\Phi}}{\partial p} + \frac{R\bar{T}}{p} = 0. \tag{1.30}$$

Then, by some straightforward computations, it is easy to see that the unknown functions (v, ω), $\Phi' \doteq \Phi - \bar{\Phi}$ and $T' \doteq T - \bar{T}$ satisfy the following equations (for simplicity we omit the primes):

$$\begin{aligned}
&\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{v_\varphi^2}{a}\cot\theta = -\frac{1}{a}\frac{\partial \Phi}{\partial \theta} + 2\Omega\cos\theta v_\varphi + D_\theta \\
&\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{v_\theta v_\varphi}{a}\cot\theta = -\frac{1}{a\sin\theta}\frac{\partial \Phi}{\partial \varphi} - 2\Omega\cos\theta v_\theta + D_\varphi \\
&\frac{\partial \Phi}{\partial p} + \frac{R}{p}T = 0 \\
&\frac{\partial \omega}{\partial p} + \frac{1}{a\sin\theta}\left(\frac{\partial v_\theta \sin\theta}{\partial \theta} + \frac{\partial v_\varphi}{\partial \varphi}\right) = 0 \\
&c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{c_pC^2}{pR}\omega = \frac{\mathrm{d}Q}{\mathrm{d}t}.
\end{aligned} \tag{1.31}$$

Here we have used the following meteorologically reasonable approximation (see [8] for more details):

$$\frac{RT}{p}\omega \cong \frac{R\bar{T}}{p}\omega. \tag{1.32}$$

We would like to point out that while the number $|T - \bar{T}|$ may not be small compared to $|T| + |\bar{T}|$, it appears (see [8]) that $|T - \bar{T}|$ is small compared to $p/R\omega$:

$$|T - \bar{T}| \ll \frac{p}{R\omega}.$$

Thus it is legitimate to replace, as a first approximation, T by $\bar{T}$ in (1.32).

Since the main purpose of this paper, as we have mentioned, is to study the mechanism of long-term weather prediction and climate changes, it is necessary to add (or replace (D_θ, D_φ) and $\mathrm{d}Q/\mathrm{d}t$ by) some viscosity terms in the equations above. According to some physical considerations, we can introduce the viscosity in such a way that the PEs take the following form (the viscosity terms have been underlined) (see [9, 10]):

$$\left.\begin{aligned}
&\frac{\partial v}{\partial t} + \nabla_v v + \omega\frac{\partial v}{\partial p} + 2\Omega\cos(\theta k)v + \operatorname{grad}\Phi \underline{- \mu_1\,\Delta v - \nu_1\frac{\partial}{\partial p}\left[\left(\frac{gp}{R\bar{T}}\right)^2\frac{\partial v}{\partial p}\right]} = 0\\
&\frac{\partial\Phi}{\partial p} + \frac{R}{p}T = 0\\
&\operatorname{div} v + \frac{\partial\omega}{\partial p} = 0\\
&\frac{R^2}{C^2}\left(\frac{\partial T}{\partial t} + \nabla_v T + \omega\frac{\partial T}{\partial p}\right) - \frac{R}{p}\omega \underline{- \mu_2\,\Delta T - \nu_2\frac{\partial}{\partial p}\left[\left(\frac{gp}{RT}\right)^2\frac{\partial T}{\partial p}\right] = \frac{R^2}{C^2}\frac{\varepsilon}{C_p}}
\end{aligned}\right\} \quad (1.33)$$

with space domain $S_a^2 \times (p_0, P)$, where ε stands for the diabatic heating, including mainly the radiative heating and the evaporative heating, such as the solar heating and the albedo of the earth. One of the underlined terms concerning T in (1.33) corresponds to the derivative

$$\nu_2\frac{\partial^2 T}{\partial z^2} = \nu_2\frac{\partial}{\partial p}\left(\left(\frac{gp}{RT}\right)^2\frac{\partial T}{\partial z}\right).$$

Furthermore T is replaced by $\bar{T}$ and this is legitimate as a first-order approximation, since ν_2 is small.

Moreover, $\nabla_v v$ and $\nabla_v T$ are covariant derivatives of v and T with respect to v (see also (2.17)) given by

$$\left.\begin{aligned}
\nabla_v v &= \left(\frac{v_\theta}{a}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi^2}{a}\cot\theta\right)e_\theta\\
&\quad + \left(\frac{v_\theta}{a}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\theta v_\varphi}{a}\cot\theta\right)e_\varphi\\
\nabla_v T &= \frac{v_\theta}{a}\frac{\partial T}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial T}{\partial\varphi}
\end{aligned}\right\} \quad (1.34)$$

where e_θ and e_φ are the unit vectors in the θ- and φ-directions, respectively. In (1.33), there are two horizontal Laplace–Beltrami operators for the scalar function T and the vector field on S_a^2, respectively. They are defined as follows:

$$\Delta T = \frac{1}{a^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\varphi^2}\right] \quad (1.35)$$

$$\begin{aligned}
\Delta v = \Delta\left(\frac{v_\theta}{a}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial}{\partial\varphi}\right) &= \left(\Delta v_\theta - \frac{2\cos\theta}{a^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{a^2\sin^2\theta}\right)e_\theta\\
&\quad + \left(\Delta v_\varphi + \frac{2\cos\theta}{a^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{a^2\sin^2\theta}\right)e_\varphi
\end{aligned} \quad (1.36)$$

where, in (1.36), Δv_θ and Δv_φ are given by (1.35).

1.2.2. Boundary value conditions. There are no conditions at the boundary of the

(θ, φ) domain except the obvious conditions resulting from periodicity. It remains to specify the conditions at the boundary of the p-domain. As indicated before, p belongs to the interval $[p_0, P]$ where $p_0 > 0$ is a small number and P is an approximate value of the pressure at the surface of the earth.

The boundary conditions at p_0 and P are written as if p_0 corresponds to the upper atmosphere and P to the surface of the earth. However, it will appear that the solution of the boundary value problem will provide the value of the geopotential $\Phi = gz$ at P and p_0, i.e. we find afterwards the heights of the isobaric surfaces $p = P$ (close to the earth) and $p = p_0$ (close to the upper atmosphere)

$$\Phi(\theta, \varphi, p = p_0, t) = gH(\theta, \varphi, t)$$

and $\Phi(\theta, \varphi, p = P, t)$ is a small function of t. The justification of these boundary conditions and the (delicate) passage to the limit $p_0 \to 0$ will be studied subsequently.

The boundary value conditions for (1.33)–(1.36) are

$$\left.\begin{aligned} &p = P: \qquad (v, \omega) = 0 \qquad \frac{\partial T}{\partial p} = \alpha_s(T_s - T) \\ &p = p_0: \qquad (v, \omega) = 0 \qquad \frac{\partial T}{\partial p} = 0 \end{aligned}\right\} \tag{1.37}$$

where α_s is a constant related to the turbulent transition on the surface of the earth including the albedo of the earth, T_s is the temperature on the surface of the earth (more precisely, it is the temperature on the land and the surface of the ocean, which we assume given). Without loss of generality, we assume that $T_s \in C^\infty(S_a^2)$. The boundary condition

$$\frac{\partial T}{\partial p} = \alpha_s(T_s - T)$$

comes from the natural boundary condition

$$\frac{\partial T}{\partial z} = \frac{gp}{RT}\frac{\partial T}{\partial p} = \alpha'(T_s - T) \quad \text{at} \quad p = P.$$

We then replace T by $\tilde{T}$ (see after (1.32)) and we set that

$$\alpha_s = \frac{\alpha' R\tilde{T}(P)}{gP}.$$

We assume that α_s is a constant which leads to some slight simplifications, but the case of α_s non-constant can be treated in exactly the same way.

We would like to indicate that other boundary value conditions can also be considered and treated by the methods of this paper. The following boundary value conditions, for instance, are also widely used in meteorology (see [10, 11]):

$$\left.\begin{aligned} &p = P: \qquad (v, \omega) = 0 \qquad \frac{\partial T}{\partial p} = \alpha_s(T_s - T) \\ &p = p_0: \qquad \left(\frac{\partial v}{\partial p}, \omega\right) = 0 \qquad \frac{\partial T}{\partial p} = 0. \end{aligned}\right\} \tag{1.37'}$$

It is also worth pointing out that in the second part of this paper we will consider equations with vertical viscosity, which are called the primitive equations with vertical viscosity (PEV^{2}s). Since (1.33) is obtained from the general equations of atmosphere by making coordinate transformations, it is not obvious how to introduce such vertical viscosity. We will discuss this in the first part of section 4. Moreover, we should mention that in the case of ocean dynamics we can consider the PEs in the usual (x, y, z) coordinates since water is an incompressible fluid: incompressibility is written in the form $\mathrm{div}_3\, V_3 = 0$ in the original (x, y, z) system, and the quasistatic equation (1.7) (or (1.22)) is a simplified form of the conservation of momentum equation in the z-variable. Therefore, the viscosity in the z-direction can be introduced very naturally. We will study elsewhere the equations of ocean dynamics and the coupled system of ocean–atmosphere.

1.3. Non-dimensional form of the PEs

It is usual, from the physical point of view, to introduce the non-dimensional form of the PEs. To this end, first of all, we introduce some non-dimensional unknown functions and parameters. let

$$\left.\begin{aligned} &v = v'U \\ &\omega = \frac{P - p_0}{a} U\omega' \\ &T = \bar{T}_0 T' \\ &\Phi = U^2 \Phi' \end{aligned}\right\} \tag{1.38}$$

$$\left.\begin{aligned} &t = \frac{a}{U} t' \\ &p = (P - p_0)\zeta + p_0 \\ &f' = 2\cos\theta \end{aligned}\right\} \tag{1.39}$$

$$\left.\begin{aligned} &\frac{1}{Re_1} = \frac{\mu_1}{aU} \qquad && \frac{1}{Re_2} = \frac{\nu_1 a g^2}{UR^2\bar{T}_0^2}\left(\frac{P}{P - p_0}\right)^2 \\ &\frac{1}{Rt_1} = \frac{\mu_2 \bar{T}_0^2}{aU^3} && \frac{1}{Rt_2} = \frac{\nu_2 a g^2}{U^3R^2}\left(\frac{P}{P - p_0}\right)^2 \\ &Ro = \frac{U}{a\Omega} && \\ &a_1 = \frac{R^2\bar{T}_0^2}{C^2U^2} && b = \frac{R\bar{T}_0(P - p_0)}{U^2 P} \\ &\bar{\alpha}_s = (P - p_0)\alpha_s && \bar{T}_s = \frac{T_s}{\bar{T}_0}. \end{aligned}\right\} \tag{1.40}$$

The parameters Re_1, Re_2, Rt_1 and Rt_2 are Reynolds numbers. Ro stands for the Rossby number, which measures the significant influence of the earth's rotation on the dynamic behaviour of the atmosphere. The $\bar{\alpha}_s$ is a positive constant related to the turbulent transition on the surface of the earth. Other notation is explained in the appendix.

We can then obtain from (1.33) the following non-dimensional form of the PEs (for simplicity we omit the primes):

$$\frac{\partial v}{\partial t}+\nabla_v v+\omega\frac{\partial v}{\partial \zeta}+\frac{f}{Ro}k\times v+\operatorname{grad}\Phi-\frac{1}{Re_1}\Delta v-\frac{1}{Re_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2\frac{\partial v}{\partial \zeta}\right]=f_1 \tag{1.41}$$

$$\operatorname{div} v+\frac{\partial \omega}{\partial \zeta}=0 \tag{1.42}$$

$$\frac{\partial \Phi}{\partial \zeta}+\frac{bP}{p}T=0 \tag{1.43}$$

$$a_1\left(\frac{\partial T}{\partial t}+\nabla_v T+\omega\frac{\partial T}{\partial \zeta}\right)-\frac{bP}{p}\omega-\frac{1}{Rt_1}\Delta T-\frac{1}{Rt_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2\frac{\partial T}{\partial \zeta}\right]=f_2. \tag{1.44}$$

Here the covariant derivative operator ∇_v, the gradient operator grad, the divergence operator div and the Laplace–Beltrami operators Δ on S^2 are defined by (2.10)–(2.18). We have to point out that we use the same notation for these operators as that for S_a^2 listed in (1.34)–(1.36). The only difference between the operators given by (2.12)–(2.18) and those given by (1.34)–(1.36) is the factor a.

The space domain is now

$$M=S^2\times(0,1) \tag{1.45}$$

and the boundary value conditions are given by

$$\left.\begin{array}{lll}\zeta=1: & (v,\omega)=0 & \dfrac{\partial T}{\partial \zeta}=\bar{\alpha}_s(\bar{T}_s-T)\\[2ex] \zeta=0: & (v,\omega)=0 & \dfrac{\partial T}{\partial \zeta}=0.\end{array}\right\} \tag{1.46}$$

In the above equations, similar to (1.33), there are two horizontal Laplacian operators on S^2 for the scalar function T and the horizontal velocity field v, respectively. The explicit form of these two operators are given by (2.21) and (2.22) in section 2. Henceforth, we use Δ to denote these two Laplacians on S^2. Moreover, $\nabla_v v$ and $\nabla_v T$ denote the covariant derivatives of v and T with respect to v, given by (2.17).

For the real physical situation, f_1 can be taken as zero. From the mathematical point of view, however, we prefer to consider the general case. Moreover, for simplicity, we still use $(p/P)^2$ in the viscosity terms instead of using $[(P-p_0)\zeta+p_0/P]^2$.

1.4. Reformulation of the PEs

In (1.41)–(1.44) above, there are two diagnostic equations, (1.42) and (1.43). So, as we mentioned before, the whole system is not an evolution one. Therefore, in this section, we integrate the diagnostic equations, and then establish another formulation of the PES.

We would like to mention that it is significant to introduce the two-variable function Φ_s below. The reason is three-fold. Firstly, it is very nice from the mathematical point of view that Φ_s is exactly the Lagrange multiplier of the non-local constraint (1.52) (see section 2 below). Moreover, the whole system is a 3D evolution one, but Φ_s is only a function of the first two independent variables. Secondly, from a technical point of view we will see in section 3 that we can directly use the Faedo–Galerkin method to solve the weak formulation of the reformulated problem (1.50)–(1.54). In [11], however, we did not find this reformulation and, therefore, we had to construct some special kinds of function spaces for the vertical velocity of ω, in which the elements possess more regularity in the p-direction. In the meantime we had to use the semi-discretization method to prove the existence of weak solutions because we were not able to use the Galerkin procedure. Thirdly, from the meteorological point of view, the isobaric surface $p = P$, in general, is not just the surface of $z = 0$. In fact, the geopotential Φ at the isobaric surface $p = P$ should be considered as an unknown function rather than as a known function obtained from practical observations. This function also describes, at least partially, the topography of the earth.

More precisely, integrating (1.42), and taking the boundary value conditions for ω into account, we obtain

$$\left.\begin{aligned} &\omega(t;\theta,\varphi,\zeta) = W(v)(t;\theta,\varphi,\zeta) \doteq \int_\zeta^1 \operatorname{div} v(t;\theta,\varphi,\xi)\,\mathrm{d}\xi \\ &\int_0^1 \operatorname{div} v\,\mathrm{d}\zeta = 0. \end{aligned}\right\} \tag{1.47}$$

Suppose that the geopotential $\Phi = gz - \bar{\Phi}$ is equal to a certain function Φ_s at the isobaric surface $p = P$, then we can integrate (1.43) as

$$\Phi(t;\theta,\varphi,\zeta) = \Phi_s(t;\theta,\varphi) + \int_\zeta^1 \frac{bP}{p} T(t;\theta,\varphi,\zeta')\,\mathrm{d}\zeta' \tag{1.48}$$

which implies that

$$\operatorname{grad}\Phi = \operatorname{grad}\Phi_s + \int_\zeta^1 \frac{bP}{p}\operatorname{grad} T\,\mathrm{d}\zeta'. \tag{1.49}$$

Therefore, the PEs (1.41)–(1.44) can be rewritten as

$$\begin{aligned} &\frac{\partial v}{\partial t} + \nabla_v v + W(v)\frac{\partial v}{\partial \zeta} + \frac{f}{Ro} k \times v + \int_\zeta^1 \frac{bP}{p}\operatorname{grad} T\,\mathrm{d}\zeta' \\ &\qquad + \operatorname{grad}\Phi_s - \frac{1}{Re_1}\Delta v - \frac{1}{Re_2}\frac{\partial}{\partial\zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] = f_1 \end{aligned} \tag{1.50}$$

$$a_1\left(\frac{\partial T}{\partial t} + \nabla_v T + W(v)\frac{\partial T}{\partial \zeta}\right) - \frac{bP}{p}W(v) - \frac{1}{Rt_1}\Delta T - \frac{1}{Rt_2}\frac{\partial}{\partial\zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial T}{\partial \zeta}\right] = f_2 \tag{1.51}$$

$$\int_0^1 \operatorname{div} v\,\mathrm{d}\zeta = 0 \tag{1.52}$$

where $W(v)$ is defined by (1.47) and the physical parameters Ro, Re_1, Re_2, Rt_1 and Rt_2 are given by (1.40).

The boundary value conditions are

$$\left.\begin{array}{lll}\zeta=1: & v=0 & \dfrac{\partial T}{\partial \zeta}=\bar{\alpha}_s(\bar{T}_s-T) \\ \zeta=0: & v=0 & \dfrac{\partial T}{\partial \zeta}=0.\end{array}\right\} \tag{1.53}$$

Henceforth, we use Ξ to denote the pair (v, T), and then the initial value conditions can be given as

$$\Xi|_{t=0}=\Xi_0=(v_0, T_0).$$

In the remaining parts of this paper we will study the well-posedness and the long-term behaviour of the initial boundary value problem (1.50)–(1.54), and its modification (PEV^{2}s).

Intuitively, let (v, Φ_s, T) be a smooth solution of the problem (1.50)–(1.54) for $\bar{T}_s=0$. For any test function $\Xi_1=(v_1, T_1)\in C_0^\infty(TM \mid TS^2)\times C^\infty(\bar{M})$ satisfying (1.52), we can take the L^2 inner product between (1.50) and (1.51), and the test function Ξ_1, where the function spaces are defined in section 2. The resulting equations can be obtained by direct computation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Xi, \Xi_1)_H + a(\Xi, \Xi_1) + b(\Xi, \Xi, \Xi_1) + e(\Xi, \Xi) = ((f_1, f_2), \Xi_1)_{L^2} \tag{1.55}$$

where the functionals $a(\cdot,\cdot)$, $b(\cdot,\cdot,\cdot)$ and $e(\cdot,\cdot)$ are defined by (2.42), (2.48) and (2.53), respectively and the inner product $(\cdot,\cdot)_H$ is given by (2.31). As we mentioned before, the function Φ_s disappears in the resulting equations (1.55). Therefore, the basic idea of how to solve the problem is as follows. Firstly, we construct some function spaces for the unknown function $\Xi=(v, T)$ satisfying the constraint equation (1.52). Secondly, we prove that the function Φ_s is exactly the multiplier of the constraint. Then, we obtain that (1.55) is the variational formulation of the problem (1.50)–(1.54) for $\bar{T}_s=0$. Thus, we can solve the problem by the Faedo–Galerkin method. Moreover, to study the long-time behaviour, we can use the idea of [12] to estimate the Hausdorff and fractal dimensions of the attractor of the problem. Of course, in the case $\bar{T}_s\neq 0$, we have to find a function T^* satisfying the non-homogeneous boundary condition

$$\begin{array}{ll}\zeta=1: & \dfrac{\partial T^*}{\partial \zeta}=\bar{\alpha}_s(\bar{T}_s-T^*) \\ \zeta=0: & \dfrac{\partial T^*}{\partial \zeta}=0\end{array}$$

and inequality (2.58). This inequality is crucial to obtain some *a priori* estimates and, therefore, for solving the problem.

2. Mathematical setting of the PEs

2.1. *Some function spaces and their properties*

The space domain of our problem, $M=S^2\times(0,1)$, is a manifold with boundary $\Gamma=\Gamma_0\cup\Gamma_1$ given by

$$\Gamma_i=S^2\times\{i\} \qquad \forall i=0,1. \tag{2.1}$$

Noticing that M is the product of S^2 and the unit interval $(0,1)$, we see that the tangent space $T_{(q,\zeta)}M$ of M at $(q, \zeta) \in M$ can be decomposed into the product of T_qS^2 and $T_\zeta(0, 1) = \mathbb{R}$ as follows:

$$T_{(q,\zeta)}M = T_qS^2 \times T_\zeta(0, 1) = T_qS^2 \times \mathbb{R}. \tag{2.2}$$

Therefore, the Riemannian metric g_M on M is given by

$$g_M((q, \zeta); (v, \omega), (v_1, \omega_1)) = g_{S^2}(q; v, v_1) + \omega\omega_1 \quad \forall v, v_1 \in T_qS^2 \quad \omega, \omega_1 \in \mathbb{R} \tag{2.3}$$

where g_{S^2} is the Riemannian metric on S^2.

In particular, in the spherical coordinates $(x^1, x^2, x^3) = (\theta, \varphi, \zeta)$ on M, g_M is given by the matrix

$$(g_{ij}) = \begin{pmatrix} 1 & & \\ & \sin^2\theta & \\ & & 1 \end{pmatrix} \tag{2.4}$$

where

$$g_{ij} = g_M\left((\theta, \varphi, \zeta); \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

So the unit vectors in the θ-, φ- and ζ-directions are

$$e_\theta = \frac{\partial}{\partial\theta} \qquad e_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} \qquad e_\zeta = \frac{\partial}{\partial\zeta}. \tag{2.6}$$

We used in (2.6) a classical notation in geometry but throughout the article we use the notation e_θ, e_φ and e_ζ. For simplicity, we denote the inner product and norm in the tangent space $T_{(\theta,\varphi,\zeta)}M$ by

$$\left.\begin{aligned} &X \cdot Y = X^1Y^1 + X^2Y^2 + X^3Y^3 \\ &|X| = (X \cdot X)^{1/2} \end{aligned}\right\} \tag{2.7}$$

for any

$$\begin{aligned} X &= X^1e_\theta + X^2e_\varphi + X^3e_\zeta \\ Y &= Y^1e_\theta + Y^2e_\varphi + Y^3e_\zeta \qquad \in T_{(\theta,\varphi,\zeta)}M. \end{aligned}$$

After introducing this basic notation, we can consider functions on M. First, let $C^\infty(\bar{M})$ (resp. $C^\infty(T\bar{M})$) be the function space for all C^∞ functions from $\bar{M}$ into $\mathbb{R}$ (resp. C^∞ vector fields on $\bar{M}$). Then, by (2.2), $C^\infty(T\bar{M})$ possesses the following decomposition:

$$\begin{aligned} C^\infty(T\bar{M}) &= C^\infty(T\bar{M} \mid TS^2) \times C^\infty(\bar{M}) \\ &= \{v : \bar{M} \to TS^2 \in C^\infty \mid v(\theta, \varphi, \zeta) \in T_{(\theta,\varphi)}S^2\} \times \{\omega : \bar{M} \to \mathbb{R} \in C^\infty\}. \end{aligned} \tag{2.8}$$

Similarly, let $C_0^\infty(M)$ (resp. $C_0^\infty(TM)$) be the function space for all C^∞ functions from M into $\mathbb{R}$ (resp. C^∞ vector fields on M) with compact supports. Then, we have the following decomposition (similar to (2.8)) for $C_0^\infty(TM)$:

$$\begin{aligned} C_0^\infty(TM) &= C_0^\infty(TM \mid TS^2) \times C_0^\infty(M) \\ &= \{v : M \to TS^2 \in C^\infty \text{ with compact support } | v(\theta, \varphi, \zeta) \in T_{(\theta,\varphi)}S^2\} \\ &\quad \times \{\omega : M \to \mathbb{R} \in C^\infty \text{ with compact support}\}. \end{aligned} \tag{2.9}$$

Moreover, we consider $C^\infty(S^2)$ (resp. $C^\infty(TS^2)$) to be the space of all smooth functions (resp. smooth vector fields) on S^2.

We are now in a position to define some differential operators. First, the natural generalization of the directional derivative on the Euclidean space to the covariant derivative on S^2 is given as follows. Let $T \in C^\infty(S^2)$ and

$$v = v_\theta e_\theta + v_\varphi e_\varphi \qquad v_1 = (v_1)_\theta e_\theta + (v_1)_\varphi e_\varphi \qquad \in C^\infty(TS^2)$$

we then define the covariant derivative of v_1 and T with respect to v as follows:

$$\nabla_v v_1 = \left(v_\theta \frac{\partial (v_1)_\theta}{\partial \theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial (v_1)_\theta}{\partial \varphi} - v_\varphi (v_1)_\varphi \cot\theta\right) e_\theta$$
$$+ \left(v_\theta \frac{\partial (v_1)_\varphi}{\partial \theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial (v_1)_\varphi}{\partial \varphi} + v_\varphi (v_1)_\theta \cot\theta\right) e_\varphi \tag{2.10}$$

$$\nabla_v T = v_\theta \frac{\partial T}{\partial \theta} + \frac{v_\varphi}{\sin\theta}\frac{\partial T}{\partial \varphi}. \tag{2.11}$$

We now define some natural operators on functions and vector fields on M. Firstly, for any function $h \in C^\infty(M)$, we define the gradient $\mathrm{grad}_M\, h \in C^\infty(TM)$ of h by

$$\mathrm{grad}_M\, h = \frac{\partial h}{\partial \theta} e_\theta + \frac{1}{\sin\theta}\frac{\partial h}{\partial \varphi} e_\varphi + \frac{\partial h}{\partial \zeta} e_\zeta$$
$$= \mathrm{grad}\, h + \frac{\partial h}{\partial \zeta}\frac{\partial}{\partial \zeta} \tag{2.12}$$

where grad is the gradient on S^2 given by

$$\mathrm{grad}\, h = \frac{\partial h}{\partial \theta} e_\theta + \frac{1}{\sin\theta}\frac{\partial h}{\partial \varphi} e_\varphi. \tag{2.13}$$

Moreover, for any $X = X^1 e_\theta + X^2 e_\varphi + X^3 e_\zeta \in C^\infty(TM)$, we define the divergence of X, $\mathrm{div}_M\, X \in C^\infty(M)$, by

$$\mathrm{div}_M\, X = \mathrm{div}(X^1 e_\theta + X^2 e_\varphi) + \frac{\partial X^3}{\partial \zeta}$$
$$= \frac{1}{\sin\theta}\left(\frac{\partial X^1 \sin\theta}{\partial \theta} + \frac{\partial X^2}{\partial \varphi}\right) + \frac{\partial X^3}{\partial \zeta} \tag{2.14}$$

with div the divergence operator on S^2, which is given by

$$\mathrm{div}(X^1 e_\theta + X^2 e_\varphi) = \frac{1}{\sin\theta}\left(\frac{\partial X^1 \sin\theta}{\partial \theta} + \frac{\partial X^2}{\partial \varphi}\right). \tag{2.15}$$

The Laplace–Beltrami operator of a scalar function $h \in C^\infty(M)$ is

$$\Delta_M h = \mathrm{div}_M(\mathrm{grad}_M\, h)$$
$$= \mathrm{div}_M\left(\mathrm{grad}\, h + \frac{\partial h}{\partial \zeta} e_\zeta\right)$$
$$= \Delta h + \frac{\partial^2 h}{\partial \zeta^2}$$
$$= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial h}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 h}{\partial \varphi^2}\right] + \frac{\partial^2 h}{\partial \zeta^2} \tag{2.16}$$

where Δ is the Laplace–Beltrami operator on S^2 defined by

$$\Delta h = \operatorname{div}(\operatorname{grad} h)$$
$$= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 h}{\partial\varphi^2}\right] \tag{2.17}$$

Moreover, we need a horizontal Laplace–Beltrami operator for vector fields on S^2. For any $v = v_\theta e_\theta + v_\varphi e_\varphi \in C^\infty(TS^2)$ we define the horizontal Laplace–Beltrami operator, Δv of v on S^2 as

$$\Delta v = \left(\Delta v_\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{\sin^2\theta}\right)e_\theta + \left(\Delta v_\varphi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{\sin^2\theta}\right)e_\varphi \tag{2.18}$$

where Δv_θ and Δv_φ are given by (2.17).

By direct computation we can prove

$$\int_M (-\Delta v)\cdot v_1 \,\mathrm{d}M = \int_M (\nabla_{e_\theta} v \cdot \nabla_{e_\theta} v_1 + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} v_1 + v\cdot v_1)\,\mathrm{d}M. \tag{2.19}$$

We are now in a position to introduce the notation of Sobolev-type function spaces on M. First, let $L^2(M)$, $L^2(TM)$ and $L^2(TM \mid TS^2)$ be the Hilbert spaces of L^2 functions, L^2 vector fields and the first two components of L^2 vector fields on M, respectively. We denote the inner products and norms by the same notation $(\cdot,\cdot)$ and $|\cdot|$ given by

$$(h_1, h_2) = \int_M h_1 \cdot h_2 \,\mathrm{d}M \qquad |h| = (h, h)^{1/2} \tag{2.20}$$

for any $h, h_1, h_2 \in L^2(M)$, or $L^2(TM)$, or $L^2(TM \mid TS^2)$. Similarly, we can define the spaces $L^2(S^2)$ and $L^2(TS^2)$.

As usual, we can define the Sobolev spaces $H^s(S^2)$, $H^s(TS^2)$, $H^s(M)$, $H^s(TM)$, $H^s(TM \mid TS^2)$, $W^{m,p}(S^2)$, $W^{m,p}(TS^2)$, $W^{m,p}(M)$, $W^{m,p}(TM)$ and $W^{m,p}(TM \mid TS^2)$ for any $0 \le m = \text{integer} < \infty$. $1 \le p \le \infty$, $s \in \mathbb{R}$. More precisely, $H^s(S^2)$ and $H^s(TS^2)$ are defined by (see [13])

$$\left.\begin{aligned} H^s(S^2) &= D((-\Delta)^{s/2}) \\ H^s(TS^2) &= D((-\Delta)^{s/2}) \end{aligned}\right\} \tag{2.21}$$

where the first Laplace–Beltrami operator is the scalar one defined by (2.17); the second one is defined by (2.18) for vectors.

The definition of $H^s(M)$, $H^s(TM)$ and $H^s(TM \mid TS^2)$ is as follows: when s is positive integer, these are the completed spaces of $C^\infty(\bar{M})$, $C^\infty(T\bar{M})$ and $C^\infty(T\bar{M} \mid TS^2)$ for the respective norms

$$\|h\|_{H^s} = \left(\sum_{j+k\le s} |(-\Delta)^{j/2}(\nabla_\zeta)^k h|_{L^2}^2\right)^{1/2}. \tag{2.22}$$

When s is zero, these spaces are the L^2 ones. When s is not an integer or s is negative, they are defined naturally by interpolation and duality (see [13]).

The norms in the $W^{m,p}$ spaces are given by

$$\|h\|_{W^{m,p}} = \left[\int_M \left(\sum_{1\le k\le m}\ \sum_{\substack{i_j=1,2,3\\ j=1,\dots,k}} |\nabla_{i_1}\dots\nabla_{i_k} h|^p + |h|^p\right)\mathrm{d}M\right]^{1/p} \tag{2.23}$$

for h in one of the spaces $W^{m,p}(M)$, $W^{m,p}(TM)$ and $W^{m,p}(TM \mid TS^2)$. For h in one of the spaces $W^{m,p}(S^2)$ and $W^{m,p}(TS^2)$, we define

$$\|h\|_{W^{m,p}} = \left[\int_{S^2} \left(\sum_{1 \leqslant k \leqslant m} \sum_{\substack{i_j = 1,2 \\ j=1,\ldots,k}} |\nabla_{i_1} \ldots \nabla_{i_k} h|^p + |h|^p \right) \mathrm{d}S^2 \right]^{1/p}. \tag{2.24}$$

Here we have used the notation

$$\nabla_1 = \nabla_{e_\theta} \qquad \nabla_2 = \nabla_{e_\theta} \qquad \nabla_3 = \frac{\partial}{\partial \zeta}. \tag{2.25}$$

Before proceeding, we emphasize that in the following we use the standard notation given below to denote the norms and inner products in the Sobolev-type function spaces defined above:

$$\|\cdot\|_{W^{m,p}} \qquad \|\cdot\|_{H^s} \qquad (\cdot, \cdot)_{H^s}. \tag{2.26}$$

Now we turn to the definition of the function spaces for the unknown function $\Xi = (v, T)$. Motivated by the studies for Navier–Stokes equations (e.g. [14, 15]), we construct the function spaces for v in such a way that (1.52), i.e.

$$\int_0^1 \operatorname{div} v \, \mathrm{d}\zeta = 0$$

appears as a constraint for v in the spaces, which plays the same role as the continuity equation $\operatorname{div} u = 0$ in the mathematical theory of the Navier–Stokes equations. More precisely, let

$$\mathscr{V}_1 = \left\{ v \in C_0^\infty(TM \mid TS^2) \middle| \int_0^1 \operatorname{div} v \, \mathrm{d}\zeta = 0 \right\}. \tag{2.27}$$

We then define

$$\left.\begin{aligned}
&V_1 = \text{the closure of } \mathscr{V}_1 \text{ for the } H^1 \text{ norm} \\
&V_2 = H^1(M) \\
&H_1 = \text{the closure of } \mathscr{V}_1 \text{ for the } L^2 \text{ norm} \\
&H_2 = L^2(M) \\
&V = V_1 \times V_2 \\
&H = H_1 \times H_2.
\end{aligned}\right\} \tag{2.28}$$

Before studying some properties of these spaces, we should define their norms and inner products. By definition, the inner product and norm in V_1 and V_2 are given by

$$\left.\begin{aligned}
&((v, v_1)) = \int_M \left(\nabla_{e_\theta} v \cdot \nabla_{e_\theta} v_1 + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} v_1 + \frac{\partial v}{\partial \zeta} \cdot \frac{\partial v_1}{\partial \zeta} + v \cdot v_1 \right) \mathrm{d}M \\
&\|v\| = ((v, v))^{1/2} \qquad \forall v, v_1 \in V_1 \\
&((T, T_1)) = \int_M (\operatorname{grad}_M T \cdot \operatorname{grad}_M T_1 + T T_1) \, \mathrm{d}M \\
&\|T\| = ((T, T))^{1/2} \qquad \forall T, T_1 \in V_2.
\end{aligned}\right\} \tag{2.29}$$

For simplicity, we use the same notation $\|\cdot\|$ and $((\cdot,\cdot))$ as for V_1 and V_2 to denote the norm and inner product of V given by

$$\left.\begin{aligned}&((\Xi,\Xi_1))=((v,v_1))+((T,T_1))\\&\|\Xi\|=((\Xi,\Xi))^{1/2}\qquad\forall\,\Xi=(v,T)\qquad \Xi_1=(v,T_1)\in V.\end{aligned}\right\}\tag{2.30}$$

Moreover, we use the same notation $(\cdot,\cdot)$ and $|\cdot|$ to denote the L^2 inner products and norms in the spaces $H_i\,(i=1,2)$ and H. However, since in (1.51) there is a coefficient a_1 for the time derivative term $\partial T/\partial t$, we introduce the following equivalent norm and inner product for the spaces H_2 and H:

$$\left.\begin{aligned}&(\Xi,\Xi_1)_H=(v,v_1)+(a_1T,T_1)\\&|\Xi\|_H=(\Xi,\Xi)_H^{1/2}\qquad\forall\,\Xi=(v,T)\qquad \Xi_1=(v,T_1)\in H\\&(T,T_1)_H=(a_1T,T_1)\\&|T|_H=(T,T)_H^{1/2}\qquad\forall\,T,T_1\in H_2.\end{aligned}\right\}\tag{2.31}$$

Henceforth, the norms and inner products in H_2 and H refer to those given in (2.31) unless otherwise stated.

Now, by the Riesz representation theorem, we can identify the dual space H' of H (resp. H_i' of H_i, $i=1,2$) with H (resp. H_i), i.e. $H'=H$ (resp. $H_i'=H_i$, $i=1,2$). We then have

$$V\subset H=H'\subset V'\tag{2.32}$$

where the two inclusions are compact continuous.

The following lemma characterizes the spaces H and V.

Lemma 2.1. Let $H_1^\perp$ be the orthogonal complement of H_1 in $L^2(TM\mid TS^2)$. Then

$$H_1^\perp=\{v\in L^2(TM\mid TS^2)\mid v=\operatorname{grad} l,\ l\in H^1(S^2)\}\tag{2.33}$$

$$H_1=\left\{v\in L^2(TM\mid TS^2)\Big|\int_0^1\operatorname{div}v\,\mathrm{d}\zeta=0\right\}\tag{2.34}$$

$$V_1=\left\{v\in H_0^1(TM\mid TS^2)\Big|\int_0^1\operatorname{div}v\,\mathrm{d}\zeta=0\right\}.\tag{2.35}$$

Proof.

(i) *Proof of* (2.22). It is easy to see that the right-hand side of (2.33) is included in $H_1^\perp$. Conversely, we claim that the inverse inclusion is also true. Indeed, for any $v\in H_1^\perp$, we have

$$\int_M vv_1\,\mathrm{d}M=0\qquad\forall v_1\in\mathcal{V}_1.\tag{2.36}$$

Obviously,

$$\mathcal{V}_1\supset\left\{\frac{\partial u}{\partial\zeta}\,\Big|\,u\in C_0^\infty(TM\mid TS^2)\right\}.\tag{2.37}$$

So it follows from (2.36) and (2.37) that

$$\int_M v\frac{\partial u}{\partial\zeta}\,\mathrm{d}M=0\qquad\forall u\in C_0^\infty(TM\mid TS^2).$$

That is, $\partial v/\partial\zeta = 0$ in the distribution sense. Therefore, v does not depend on ζ by theorem 3.1.4′ of chapter III of [16], which we will quote in part (*iii*) of lemma 2.2 below.

Now, choosing a function $h \in C_0^\infty(0, 1)$ such that

$$\int_0^1 h(\zeta)\,\mathrm{d}\zeta \neq 0$$

we can infer from (2.36) that

$$\int_M vu\,\mathrm{d}S^2 = 0 \qquad \forall u \in \{u \in C^\infty(TS^2) \mid \operatorname{div} u = 0\}.$$

Thus, lemma 2.2 below shows that there is a unique (up to a constant) function $l \in H^1(S^2)$ such that

$$v = \operatorname{grad} l \tag{2.38}$$

which proves (2.33).

(ii) Equations (2.34) and (2.35) can be proved similarly to the corresponding results for the Navier–Stokes equations (see [14]), and we omit the details. □

Lemma 2.2.

(*i*) Let $f \in \mathscr{D}'(TS^2)$, then

$$(f, v) = 0 \qquad \forall v \in \{v \in C^\infty(TS^2) \mid \operatorname{div} v = 0\} \tag{2.39}$$

if and only if there is an $l \in \mathscr{D}'(S^2)$ such that

$$f = \operatorname{grad} l. \tag{2.40}$$

(*ii*) If $f \in L^2(TS^2)$ satisfies (2.39), then the l obtained in (2.40) possesses the following regularity property:

$$l \in H^1(S^2). \tag{2.41}$$

(*iii*) If $v \in \mathscr{D}'(TM \mid TS^2)$ such that $\partial v/\partial\zeta = 0$ in the sense of distribution, then v is a distribution on S^2 independent of the third variable ζ. Here $\mathscr{D}'(TM \mid TS^2)$ represents the distribution space of the first two components of all the distribution vector fields on M.

For the proof of parts (*i*) and (*ii*) of this lemma, we refer to [14]. The last part of the lemma is the trivial generalization of theorem 3.1.4′ in chapter III of [16].

2.2. *Some functionals and their associated operators*

We are now in a position to define some functionals and their associated operators, which are related to different parts of the PEs.

First, concerning the principal parts of the PEs, we define three bilinear forms $a: V \times V \to \mathbb{R}$, $a_i: V_i \times V_i \to \mathbb{R}$ $(i = 1, 2)$, and their corresponding linear operators $A: V \to V'$, $A_i: V_i \to V_i'$ $(i = 1, 2)$ by

$$\begin{aligned} a(\Xi, \Xi_1) &= (A\Xi, \Xi_1)_H \\ &= (A_1 v, v_1) + (A_2 T, T_1)_H \\ &= a_1(v, v_1) + a_2(T, T_1) \end{aligned} \tag{2.42}$$

$$a_1(v, v_1) = (A_1 v, v_1)$$
$$= \int_M \left[\frac{1}{Re_1} (\nabla_{e_\theta} v \cdot \nabla_{e_\theta} v_1 + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} v_1 + v \cdot v_1) + \frac{1}{Re_2} \left(\frac{p\bar{T}_0}{P\bar{T}} \right)^2 \frac{\partial v}{\partial \zeta} \cdot \frac{\partial v_1}{\partial \zeta} \right] \mathrm{d}M \tag{2.43}$$

$$a_2(T, T_1) = (A_2 T, T_1)_H$$
$$= \int_M \left[\frac{1}{Rt_1} \operatorname{grad} T \cdot \operatorname{grad} T_1 + \frac{1}{Rt_2} \left(\frac{p\bar{T}_0}{P\bar{T}} \right)^2 \frac{\partial T}{\partial \zeta} \cdot \frac{\partial T_1}{\partial \zeta} \right] \mathrm{d}M + \int_{\Gamma_1} \frac{\bar{\alpha}_s}{Rt_2} \left(\frac{p\bar{T}_0}{P\bar{T}} \right)^2 T T_1 \, \mathrm{d}S^2 \tag{2.44}$$

for any $\Xi = (v, T)$, $\Xi_1 = (v_1, T_1) \in V$. Note that the appearance of the term $v \cdot v_1$ ensures full coercivity of a_1 on (2.43), this term arising from (2.19). For T the same is true but now the term involving TT_1 arises from the boundary value condition.

Lemma 2.3.

(*i*) a, a_i $(i = 1, 2)$ are coercive and continuous; therefore, $A: V \to V'$ and $A_i: V_i \to V_i'$ $(i = 1, 2)$ are isomorphisms. Moreover, we have

$$a(\Xi, \Xi_1) \leq c_0 \max\left\{ \frac{1}{Re_1}, \frac{1}{Re_2} \right\} \|v\| \cdot \|v_1\| + c_0 \max\left\{ \frac{1}{Rt_1}, \frac{1}{Rt_2}, \frac{\bar{\alpha}_s}{Rt_2} \right\} \|T\| \cdot \|T_1\| \leq \frac{1}{R_{\min}} \|\Xi\| \cdot \|\Xi_1\| \tag{2.45}$$

$$a(\Xi, \Xi) \geq c_0' \min\left\{ \frac{1}{Re_1}, \frac{1}{Re_2} \right\} \|v\|^2 + c_0' \min\left\{ \frac{1}{Rt_1}, \frac{1}{Rt_2}, \frac{\bar{\alpha}_s}{Rt_2} \right\} \|T\|^2 \geq \frac{1}{R_{\max}} \|\Xi\|^2 \tag{2.46}$$

where c_0, c_0' are absolute constants independent of the physically relevant constants $\bar{\alpha}_s$, Re_i, Rt_i, etc†. The parameters $R_{\min}$ and $R_{\max}$ are defined

$$\begin{aligned} R_{\max} &= \frac{1}{c_0'} \max\left\{ Rt_1, Rt_2, Re_1, Re_2, \frac{Rt_2}{\bar{\alpha}_s} \right\} \\ R_{\min} &= \frac{1}{c_0} \min\left\{ Rt_1, Rt_2, Re_1, Re_2, \frac{Rt_2}{\bar{\alpha}_s} \right\}. \end{aligned} \tag{2.47}$$

(*ii*) The isomorphism $A: V \to V'$ (resp. $A_i: V_i \to V_i'$ $(i = 1, 2)$) can be extended to a self-adjoint unbounded linear operator on H (resp. on H_i $(i = 1, 2)$), with compact inverse $A^{-1}: H \to H$ (resp. $A_i^{-1}: H_i \to H_i$ $(i = 1, 2)$), and with domain $D(A) = V \cap H^2(TM)$ (resp. $D(A_1) = V_1 \cap H^2(TM \mid TS^2)$, $D(A_2) = V_2 \cap H^2(M)$.)

† c_0, c_0' may depend on $\varepsilon = (P - p_0)RT_0/agP$ such that (4.19) holds true.

Proof. The proof of $D(A_1) = V_1 \cap H^2(TM \mid TS^2)$ can be obtained from lemma 2.7 below. The other parts of the proof are obvious, so we omit the details. $\square$

Now, related to the nonlinear terms appearing in the equations, we define three functionals $b : H \times H \times H \to \mathbb{R}$, $b_i : H_1 \times H_i \times H_i \to \mathbb{R}$ $(i = 1, 2)$ and their associated operators $B : H \times H \to H$, $B_i : H_1 \times H_i \to H_i$ $(i = 1, 2)$ by

$$\begin{aligned} b(\Xi, \Xi_1, \Xi_2) &= (B(\Xi, \Xi_1), \Xi_2)_H \\ &= (B_1(v, v_1), v_2) + (B_2(v, T_1), T_2)_H \\ &= b_1(v, v_1, v_2) + b_2(v, T_1, T_2) \end{aligned} \tag{2.48}$$

$$\begin{aligned} b_1(v, v_1, v_2) &= (B_1(v, v_1), v_2) \\ &= \int_M \left[\nabla_v v_1 + \left(\int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \right) \frac{\partial v_1}{\partial \zeta} \right] \cdot v_2 \, \mathrm{d}M \end{aligned} \tag{2.49}$$

$$\begin{aligned} b_2(v, T_1, T_2) &= (B_2(v, T_1), T_2)_H \\ &= a_1 \int_M \left[\nabla_v T_1 + \left(\int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \right) \frac{\partial T_1}{\partial \zeta} \right] T_2 \, \mathrm{d}M \end{aligned} \tag{2.50}$$

for any $\Xi = (v, T)$, $\Xi_i = (v_i, T_i) \in V$ or $D(A)$ $(i = 1, 2)$. We then have the following lemma.

Lemma 2.4.

(i) For any $\Xi \in V$, $\Xi_1 \in D(A)$,

$$b(\Xi, \Xi_1, \Xi_1) = b_1(v, v_1, v_1) = b_2(v, T_1, T_1) = 0. \tag{2.51}$$

(ii)
$$|b(\Xi, \Xi_1, \Xi_2)| \leqslant \begin{cases} C \, \|\Xi\|_{H^2} \cdot \|\Xi_1\|_{H^{3/2}} \cdot |\Xi_2| \\ C \, \|\Xi\|_{H^{3/2}} \cdot \|\Xi_1\| \cdot \|\Xi_2\|. \end{cases} \tag{2.52}$$

Proof.

(i) By definition, we have

$$\nabla_v |v_1|^2 = \nabla_v v_1 \cdot v_1 + v_1 \cdot \nabla_v v_1 = 2 \nabla_v v_1 \cdot v_1$$

which yields

$$\begin{aligned} b_1(v, v_1, v_1) &= \int_M \left[\nabla_v v_1 \cdot v_1 + \frac{1}{2} \left(\int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \right) \cdot \frac{\partial |v_1|^2}{\partial \zeta} \right] \mathrm{d}M \\ &= \int_M \left[\tfrac{1}{2} \nabla_v |v_1|^2 + \frac{1}{2} \left(\int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \right) \cdot \frac{\partial |v_1|^2}{\partial \zeta} \right] \mathrm{d}M \\ &= \frac{1}{2} \int_M \left[\operatorname{div}(|v_1|^2 v) - |v_1|^2 \operatorname{div} v + \left(\int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \right) \cdot \frac{\partial |v_1|^2}{\partial \zeta} \right] \mathrm{d}M \\ &= -\frac{1}{2} \int_M \left[|v_1|^2 \left(\operatorname{div} v + \frac{\partial W(v)}{\partial \zeta} \right) \right] \mathrm{d}M \\ &\quad \pm \int_{S^2} |v_1|^2 \left(\int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \right) \bigg|_{\zeta = 0, 1} \mathrm{d}S^2 \\ &= 0. \end{aligned}$$

Similarly, we can prove $b_2(v, T_1, T_1) = 0$ and, therefore, (2.51) follows.

(*ii*) Consider the typical terms in b_i as follows:

$$\left|\int_M \left[\left(\int_\zeta^1 \operatorname{div} v \,\mathrm{d}\zeta\right) \frac{\partial v_1}{\partial \zeta} \cdot v_2\right] \mathrm{d}M\right| \leqslant \begin{cases} C\,|v_2| \cdot \left|\dfrac{\partial v_1}{\partial \zeta}\right|_{L^3} \cdot |\operatorname{div} v|_{L^6} \\ C\,|v_2|_{L^6} \cdot \left|\dfrac{\partial v_1}{\partial \zeta}\right|_{L^2} \cdot |\operatorname{div} v|_{L^3} \end{cases}$$

$$\leqslant \begin{cases} C\,\|\Xi\|_{H^2} \cdot \|\Xi_1\|_{H^{3/2}} \cdot |\Xi_2| \\ C\,\|\Xi\|_{H^{3/2}} \cdot \|\Xi_1\| \cdot \|\Xi_2\|. \end{cases}$$

This gives (2.52). $\square$

As we mentioned in the introduction, the PEs possess some highly nonlinear terms like $(\int_\zeta^1 \operatorname{div} v \,\mathrm{d}\zeta)\, \partial v/\partial\zeta$, whose behaviour is the same as that of $|\nabla_i v|^2$ ($i = 1, 2, 3$), where ∇_i ($i = 1, 2, 3$) are the first-order differential operators given by (2.25). So, as far as existence and uniqueness of solutions are concerned, the PEs are even more complicated than the usual 3D Navier–Stokes equations, and we will consider some regularized forms of these equations later.

Concerning the linear terms in the equations, we define a bilinear functional $e: H \times H \to \mathbb{R}$ and the associated linear operator $E: H \to H$ by

$$e(\Xi, \Xi_1) = (E(\Xi), \Xi_1)_H$$
$$= \int_M \left[\frac{f}{Ro}(k \times v) \cdot v_1 + \left(\int_\zeta^1 \frac{bP}{p} \operatorname{grad} T \,\mathrm{d}\zeta\right) \cdot v_1 - \frac{bP}{p} W(v) \cdot T_1\right] \mathrm{d}M \tag{2.53}$$

for any $\Xi, \Xi_1 \in V$, where $W(v)$ is defined in (1.47). We then have the following lemma.

Lemma 2.5.

(*i*) **There is a constant c independent of the physically relevant parameters Re_i, Rt_i ($i = 1, 2$) and Ro, such that**

$$|(E(\Xi), \Xi_1)_H| \leqslant \begin{cases} \dfrac{c}{Ro} |v| \cdot |v_1| + c\,|\Xi| \cdot \|\Xi_1\| \\ \dfrac{c}{Ro} |v| \cdot |v_1| + c\,\|\Xi\| \cdot |\Xi_1|. \end{cases} \tag{2.54}$$

(*ii*) $$e(\Xi, \Xi) = 0 \qquad \forall \Xi \in V. \tag{2.55}$$

Proof. We only have to prove (2.55). The other parts of the proof are obvious. To this end, by definition we have

$$e(\Xi, \Xi) = \int_M \frac{f}{Ro}\left(-v_\varphi \frac{\partial}{\partial\theta} + \frac{v_\theta}{\sin\theta} \frac{\partial}{\partial\varphi}\right) \cdot v \,\mathrm{d}M$$
$$+ \int_M \left[\left(-\int_\zeta^1 \frac{bP}{p} T \,\mathrm{d}\zeta\right) \operatorname{div} v - \frac{bP}{p} W(v) \cdot T\right] \mathrm{d}M$$
$$= \int_M \left[\left(-\int_\zeta^1 \frac{bP}{p} T \,\mathrm{d}\zeta\right) \frac{\partial}{\partial\zeta}\left(\int_\zeta^1 \operatorname{div} v \,\mathrm{d}\zeta\right) - \frac{bP}{p} W(v) \cdot T\right] \mathrm{d}M$$
$$= \int_M \left(\frac{bP}{p} W(v) \cdot T - \frac{bP}{p} W(v) \cdot T\right) \mathrm{d}M$$
$$= 0,$$

which proves (2.55). $\square$

2.3. Functional setting of the PEs

First, we should homogenize the boundary value condition for T. To this end, solving

$$\frac{\partial \bar{T}^*}{\partial \zeta} = \bar{\alpha}_s(\bar{T}_s - \bar{T}^*)$$

yields

$$\bar{T}^* = \bar{T}_s[1 - \exp(-\bar{\alpha}_s\zeta)].$$

Then, we define a function T_ε^* by

$$T_\varepsilon^* = \bar{T}^*\theta_\varepsilon(\zeta) \qquad \forall \varepsilon \in (0, \tfrac{1}{2}) \tag{2.56}$$

where $\theta_\varepsilon \in C^\infty([0, 1])$ is given by

$$\theta_\varepsilon(\zeta) = \begin{cases} 1 & 1-\varepsilon \leqslant \zeta \leqslant 1 \\ \text{increasing} & 1-2\varepsilon \leqslant \zeta \leqslant 1-\varepsilon \\ 0 & 0 \leqslant \zeta \leqslant 1-2\epsilon. \end{cases}$$

Obviously, then T_ε^* satisfies the boundary value conditions for T given in (1.53). Moreover, we have the following lemma.

Lemma 2.6. For any $\delta > 0$, there is an $\varepsilon_0 \in (0, 1)$ such that

$$|b_2(v, T_\varepsilon^*, T)| \leqslant \delta \, \|v\| \cdot \|T\| \qquad \forall \Xi \in V. \tag{2.57}$$

Proof. By definition, we have

$$\begin{aligned}
|b_2(v, T_\varepsilon^*, T)| &= |b_2(v, T, T_\varepsilon^*)| \\
&= \left| a_1 \int_M \left[\nabla_v T + W(v)\frac{\partial T}{\partial \zeta} \right] T_\varepsilon^* \, \mathrm{d}M \right| \\
&= \left| a_1 \int_{S^2\times[1-2\varepsilon,1]} \left[\nabla_v T + W(v)\frac{\partial T}{\partial \zeta} \right] \bar{T}_s(1 - \mathrm{e}^{-\bar{\alpha}_s\zeta})\theta_\varepsilon(\zeta) \, \mathrm{d}S^2 \, \mathrm{d}\zeta \right| \\
&\leqslant a_1 \int_{S^2\times[1-2\varepsilon,1]} |v| \cdot |\mathrm{grad}\, T| \cdot |\bar{T}_s| \, \mathrm{d}S^2 \, \mathrm{d}\zeta \\
&\quad + a_1 \int_{S^2\times[1-2\varepsilon,1]} \left[\left(\int_{1-2\varepsilon}^1 |\mathrm{div}\, v| \, \mathrm{d}\zeta \right) \left| \frac{\partial T}{\partial \zeta} \right| \cdot |\bar{T}_s| \right] \mathrm{d}S^2 \, \mathrm{d}\zeta \\
&\leqslant a_1 \, |\bar{T}_s|_{L^\infty(S^2)} \left(\int_{S^2\times[1-2\varepsilon,1]} |v|^2 \, \mathrm{d}S^2 \, \mathrm{d}\zeta \right)^{1/2} \\
&\quad \times \left(\int_{S^2\times[1-2\varepsilon,1]} |\mathrm{grad}\, T|^2 \, \mathrm{d}S^2 \, \mathrm{d}\zeta \right)^{1/2} \\
&\quad + a_1 \, |\bar{T}_s|_{L^\infty(S^2)} \int_{S^2\times[1-2\varepsilon,1]} \left(\int_{1-2\varepsilon}^1 |\mathrm{div}\, v| \, \mathrm{d}\zeta' \right) \left| \frac{\partial T}{\partial \zeta} \right| \mathrm{d}S^2 \, \mathrm{d}\zeta
\end{aligned}$$

$$\leqslant a_1 |\bar{T}_s|_{L^\infty(S^2)} \|T\| \left\{\left(\int_{S^2\times[1-2\varepsilon,1]} \mathrm{d}S^2\,\mathrm{d}\zeta\right)^{1/2} \times \left(\int_{S^2\times[1-2\varepsilon,1]} |v|^4\,\mathrm{d}S^2\,\mathrm{d}\zeta\right)^{1/2}\right\}^{1/2}$$

$$+ a_1 |\bar{T}_s|_{L^\infty(S^2)} \int_{S^2} \left\{\left(\int_{1-2\varepsilon}^1 |\mathrm{div}\, v|\,\mathrm{d}\zeta\right)\left(\int_{1-2\varepsilon}^1 \left|\frac{\partial T}{\partial \zeta}\right|\mathrm{d}\zeta\right)\right\}\mathrm{d}S^2$$

$$\leqslant a_1 |\bar{T}_s|_{L^\infty(S^2)} \|T\| \cdot |S^2|^{1/4} (2\varepsilon)^{1/4} \cdot |v|_{L^4}$$

$$+ a_1 |\bar{T}_a|_{L^\infty(S^2)} \int_{S^2} \left\{(2\varepsilon)\left(\int_{1-2\varepsilon}^1 |\mathrm{div}\, v|^2\,\mathrm{d}\zeta\right)^{1/2} \times \left(\int_{1-2\varepsilon}^1 \left|\frac{\partial T}{\partial \zeta}\right|^2 \mathrm{d}\zeta\right)^{1/2}\right\}\mathrm{d}S^2$$

$$\leqslant a_1(|S^2|^{1/4}(2\varepsilon)^{1/4} + 2\varepsilon)\, |\bar{T}_s|_{L^\infty(S^2)} \|T\| \cdot \|v\|$$

which proves the inequality we need.

Now we choose $\varepsilon_0 \in (0, 1/2)$ such that $\Xi^* \doteq (0, T^*) \doteq (0, T^*_{\varepsilon_0})$ satisfies

$$|b(\Xi, \Xi^*, \Xi)| = |b_2(v, T^*, T)| \leqslant \frac{1}{2R_{\max}} \|v\| \cdot \|T\| \leqslant \frac{1}{2R_{\max}} \|\Xi\|^2. \tag{2.58}$$

And we define another unknown function $\mathscr{T}$ in terms of T as

$$\mathscr{T} = T - T^*. \tag{2.59}$$

Then we have the following equations describing the distribution of $\Xi = (v, \mathscr{T})$:

$$\frac{\partial v}{\partial t} + \nabla_v v + W(v)\frac{\partial v}{\partial \zeta} + \frac{f}{Ro} k \times v + \mathrm{grad}\, \Phi_s + \int_\zeta^1 \frac{bP}{p} \mathrm{grad}\, \mathscr{T}\,\mathrm{d}\zeta' - \frac{1}{Re_1}\Delta v - \frac{1}{Re_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] = \hat{f}_1 \doteq f_1 - \int_\zeta^1 \frac{bP}{p}\mathrm{grad}\, T^*\,\mathrm{d}\zeta \tag{2.60}$$

$$a_1\left\{\frac{\partial \mathscr{T}}{\partial t} + \nabla_v \mathscr{T} + W(v)\frac{\partial \mathscr{T}}{\partial \zeta}\right\} + a_1\left\{\nabla_v T^* + W(v)\frac{\partial T^*}{\partial \zeta}\right\} - \frac{bP}{p} W(v) - \frac{1}{Rt_1}\Delta \mathscr{T} - \frac{1}{Rt_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial \mathscr{T}}{\partial \zeta}\right] = \hat{f}_2 = f_2 + \frac{1}{Rt_1}\Delta T^* + \frac{1}{Rt_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial T^*}{\partial \zeta}\right] \tag{2.62}$$

$$\int_0^1 \mathrm{div}\, v\,\mathrm{d}\zeta = 0 \tag{2.62}$$

with the initial and boundary value conditions as follows:

$$\left.\begin{array}{lll}\zeta=1 & v=0 & \dfrac{\partial\mathcal{T}}{\partial\zeta}+\bar{\alpha}_s\mathcal{T}=0\\ \zeta=0 & v=0 & \dfrac{\partial\mathcal{T}}{\partial\zeta}=0\\ t=0 & \Xi=(v,\mathcal{T})=(v_0,\mathcal{T}_0). & \end{array}\right\}\tag{2.63}$$

We are in a position to state the functional formulation of the problem (2.60)–(2.63) or (1.50)–(1.54) as follows.

Problem 2.1. For $F \doteq (f_1, (1/a_1)f_2)$, $\Xi_0=(v_0, \mathcal{T}_0)\in H$ given, find $\Xi=(v, \mathcal{T})$ such that

$$\Xi \in L^2(0, \tau; V)\cap L^\infty(0, \tau; H) \qquad \forall \tau>0 \tag{2.64}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Xi, \Xi_1)_H+(A\Xi, \Xi_1)_H+(B(\Xi, \Xi), \Xi_1)_H+(B(\Xi, \Xi^*), \Xi_1)_H$$

$$+(E(\Xi), \Xi_1)_H=(F, \Xi_1)_H \qquad \forall \Xi_1\in D(A) \tag{2.65}$$

$$\Xi|_{t=0}=\Xi_0. \tag{2.66}$$

Obviously, (2.65a) is equivalent to the following operator equation in H:

$$\Xi_t+A\Xi+B(\Xi, \Xi)+B(\Xi, \Xi^*)+E(\Xi)=F. \tag{2.65b}$$

Moreover it is easy to see that for any solution $(v, \mathcal{T}, \Phi_s)$ of (2.60)–(2.63) or (1.50)–(1.54), which is sufficiently smooth, $\Xi=(v, \mathcal{T})$ is a solution of problem 2.1. Conversely, following the procedure of the proof of (2.33) (in lemma 2.1), we can obtain that, for any solution $\Xi=(v, \mathcal{T})$ of problem 2.1, there is a unique distribution (up to a constant) $\Phi_s\in\mathcal{D}'(S^2)$ such that $(v, \mathcal{T}, \Phi_s)$ is a solution of (2.60)–(2.63) or (1.50)–(1.54), at least in the sense of distributions. Thus, problem 2.1 is indeed the weak formulation of the PEs, (2.60)–(2.63) or (1.50)–(1.54).

2.4. Regularity of the linear stationary problem

In this subsection we want to prove some regularity result concerning the solutions of the linear operator equation in V_1:

$$A_1 v=f\in L^2(TM\mid TS^2). \tag{2.67}$$

By definition, (2.67) is equivalent to the following weak formulation:

$$\left.\begin{array}{ll}\text{find } v\in V_1 \text{ such that} & \\ a_1(v, v_1)=(f, v_1) & \forall v_1\in V_1.\end{array}\right\}\tag{2.68}$$

It follows from the Lax–Milgram theorem that there is a unique solution $v\in V_1$ for (2.68) or (2.67). Then, integrating by parts, we obtain

$$\left\langle -\frac{1}{Re_1}\Delta v-\frac{1}{Re_2}\frac{\partial}{\partial\zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2\frac{\partial v}{\partial\zeta}\right]-f, v_1\right\rangle=0 \qquad \forall v_1\in\mathcal{V}_1. \tag{2.69}$$

Similar to the proof of (2.33) (in the proof of lemma 2.1), we can obtain that, as a distribution,

$$-\frac{1}{Re_1}\Delta v-\frac{1}{Re_2}\frac{\partial}{\partial\zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2\frac{\partial v}{\partial\zeta}\right]-f$$

does not depend on the third variable ζ, and there is an L^2 function $l\in L^2(S^2)$ such that

$$\left.\begin{aligned}&-\frac{1}{Re_1}\Delta v-\frac{1}{Re_2}\frac{\partial}{\partial\zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2\frac{\partial v}{\partial\zeta}\right]+\operatorname{grad} l=f\\&\int_0^1\operatorname{div} v\,\mathrm{d}\zeta=0\\&v\in H_0^1(TM\mid TS^2)\qquad l\in L^2(S^2)\\&\int_{S^2} l\,\mathrm{d}S^2=0\end{aligned}\right\}\tag{2.70}$$

where l is uniquely determined because of the restriction $\int_{S^2} l\,\mathrm{d}S^2=0$. Under this constraint we have

$$|l|\leqslant c\,|f|.$$

The following lemma provides the regularity of the solution (v, l) obtained in (2.70).

Lemma 2.7. Let $f\in H^k(TM\mid TS^2)\ (k\geqslant 0)$ and $(v,l)\in V_1\times L^2(S^2)$ be a solution (2.70). Then

$$\left.\begin{aligned}&v\in H^{k+2}(TM\mid TS^2)\cap V_1\qquad l\in H^{k+1}(S^2)\\&\|v\|_{H^{k+2}}+\|l\|_{H^{k+1}}\leqslant c\,\|f\|_{H^k}\end{aligned}\right\}\tag{2.71}$$

where c is a constant independent of v, l and f.

Proof.

(*i*) The case $k=0$. We proceed by the difference quotient method of Nirenberg [17]. For some local coordinates $(x,y)\in U\subset S^2$, we consider a ball

$$B_R=\{(x,y)\in U\subset S^2\mid |x|^2+|y|^2<R^2\}.$$

Let η be a C^∞ function such that

$$\eta=\begin{cases}1 & \text{in } B_{R/4}\\ 0 & \text{in } B_R\backslash B_{R/4}.\end{cases}$$

Then $(\eta v, \eta l)$ satisfies

$$\left.\begin{aligned}&-\frac{1}{Re_1}\Delta(\eta v)-\frac{1}{Re_2}\frac{\partial}{\partial\zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2\frac{\partial(\eta v)}{\partial\zeta}\right]+\operatorname{grad}(\eta l)\\&\qquad=\eta f-\frac{1}{Re_1}(\Delta(\eta v)-\eta\,\Delta v)+l\operatorname{grad}\eta\\&\int_0^1\operatorname{div}(\eta v)\,\mathrm{d}\zeta=\int_0^1\operatorname{grad}\eta\cdot v\,\mathrm{d}\zeta.\end{aligned}\right\}\tag{2.72}$$

We now consider the difference quotient $D_{1,h}f$ and the translation $f^{1,h}$ of a function f defined by

$$D_{1,h}f(x, y, \zeta) = \frac{f(x+h, y, \zeta) - f(x, y, \zeta)}{h}$$

$$f^{1,h}(x, y, \zeta) = f(x+h, y, \zeta).$$

Then, for any $|h| < R/8$, let $w = D_{1,h}(\eta v)$. We want to cmpute $a_1(w, w)$. To this end, we observe by (2.72) that

$$\begin{aligned}
a_1(D_{1,h}(\eta v), D_{1,h}(\eta v)) = \int_M \Bigg\{\Bigg[& D_{1,h}(\eta f) \\
& + D_{1,h}\left(-\frac{1}{Re_1}(\Delta(\eta v) - \eta\,\Delta v) + l \operatorname{grad} \eta\right)\Bigg] \cdot D_{1,h}(\eta v)\Bigg\}\, \mathrm{d}M \\
& - \int_M \operatorname{grad}(D_{1,h}(\eta l)) \cdot D_{1,h}(\eta v)\, \mathrm{d}M \\
\leqslant & \; C\,|D_{1,-h}(D_{1,h}(\eta v))| \cdot (|f| + |v| + \|v\| + |l|) \\
& + \left| \int_M D_{1,h}(\eta l) \operatorname{div}(D_{1,h}(\eta v))\, \mathrm{d}M \right| \\
\leqslant & \; C\,|f| \cdot \|D_{1,h}(\eta v)\| + \left| \int_M D_{1,h}(\eta l) D_{1,h}(\operatorname{grad} \eta \cdot v + \eta \operatorname{div} v)\, \mathrm{d}M \right| \\
\leqslant & \; C\,|f| \cdot \|D_{1,h}(\eta v)\| + \left| \int_M D_{1,h}(\eta l) D_{1,h}(\operatorname{grad} \eta \cdot v)\, \mathrm{d}M \right| \\
\leqslant & \; C\,|f| \cdot \|D_{1,h}(\eta v)\| + \left| \int_M \eta l D_{1,-h} D_{1,h}(\operatorname{grad} \eta \cdot v)\, \mathrm{d}M \right| \\
\leqslant & \; C\,|f| \cdot \|D_{1,h}(\eta v)\| + \Bigg| \int_M l\{D_{1,-h} D_{1,h}[\operatorname{grad} \eta \cdot (\eta v)] \\
& - (D_{1,-h} D_{1,h} \eta) \cdot (\operatorname{grad} \eta \cdot v) - (D_{1,h}\eta)^{1,h} \cdot D_{1,-h}(\operatorname{grad} \eta \cdot v) \\
& - (D_{1,-h}\eta^{1,h}) \cdot D_{1,h}(\operatorname{grad} \eta \cdot v)\}\, \mathrm{d}M \Bigg| \\
\leqslant & \; C\,|f| \cdot \|D_{1,h}(\eta v)\| + C\,|l| \cdot (|D_{1,-h} D_{1,h}(\eta v)| + \|v\| + |D_{1,h} v| + |D_{1,-h} v|) \\
\leqslant & \; C\,|f| \cdot \|D_{1,h}(\eta v)\| + C\,|f|^2.
\end{aligned} \tag{2.73}$$

It follows that

$$\frac{1}{R_{\max}} \|D_{1,h}(\eta v)\|^2 \leqslant C\,|f| \cdot \|D_{1,h}(\eta v)\| + C\,|f|^2.$$

Hence

$$\|D_{1,h}(\eta v)\| \leqslant C\,|f|.$$

Passing to the limit $h \to 0$ in the above inequality, we find

$$\left.\begin{aligned}
& \frac{\partial(\eta v)}{\partial x} \in H^1(TM \mid TS^2) \\
& \left\| \frac{\partial(\eta v)}{\partial x} \right\|_{H^1} \leqslant C\,|f|.
\end{aligned}\right\} \tag{2.74}$$

Similarly, we can prove that (2.74) is still true if we replace $\partial(\eta v)/\partial x$ by $\partial(\eta v)/\partial y$. So we obtain

$$\left.\begin{aligned}&\nabla(\eta v)\in H^1(TM\mid TS^2)\\&\|\nabla(\eta v)\|_{H^1}\leqslant C\,|f|.\end{aligned}\right\}\tag{2.75}$$

Since S^2 is a compact manifold without boundary, the above argument yields

$$\left.\begin{aligned}&\nabla v\in H^1(TM\mid TS^2)\\&\|\nabla v\|_{H^1}\leqslant C\,|f|.\end{aligned}\right\}\tag{2.76}$$

Then, by applying the horizontal divergence operator to the first equation of (2.70), we have

$$\Delta l\in H^{-1}(S^2)$$

so

$$\left.\begin{aligned}&l\in H^1(S^2)\\&\|l\|_{H^1}\leqslant C\,|f|.\end{aligned}\right\}\tag{2.77}$$

Then, returning to (2.70), we have $f-\operatorname{grad} l\in L^2(TM\mid TS^2)$. Therefore, by the usual regularity of elliptic equations, we obtain

$$\left.\begin{aligned}&v\in H^2(TM\mid TS^2)\\&\|v\|_{H^2}\leqslant C\,|f|.\end{aligned}\right\}\tag{2.78}$$

The case $k=0$ has been proved.

(*ii*) The case $k>0$. This case can be shown more easily by using the results of the case $k=0$. As an example, we only prove the case $k=1$. To this end, integrating the first equation in (2.70) in $\zeta\in(0,1)$, we have

$$\begin{aligned}-\frac{1}{Re_1}\Delta\left(\int_0^1 v\,\mathrm{d}\zeta\right)&+\operatorname{grad} l\\&=(\text{by }(2.78))\\&=\int_0^1 f\,\mathrm{d}\zeta+\frac{1}{Re_2}\left[\left(\frac{\bar T_0}{\bar T(P)}\right)^2\frac{\partial v}{\partial\zeta}\bigg|_{\zeta=1}-\left(\frac{p_0\bar T_0}{P\bar T(p_0)}\right)^2\frac{\partial v}{\partial\zeta}\bigg|_{\zeta=0}\right]\\&\in H^{1/2}(TS^2).\end{aligned}\tag{2.79}$$

So the usual regularity result for the 2D Navier–Stokes equations on S^2 implies

$$\int_0^1 v\,\mathrm{d}\zeta\in H^{2+1/2}(TS^2)\qquad l\in H^{1+1/2}(S^2).\tag{2.80}$$

It follows that $f-\operatorname{grad} l\in H^{1/2}(TM\mid TS^2)$. Thus, by the first equation of (2.70), we have

$$v\in H^{2+1/2}(TM\mid TS^2).\tag{2.81}$$

Then

$$\int_0^1 f\,\mathrm{d}\zeta+\frac{1}{Re_2}\left[\left(\frac{\bar T_0}{\bar T(P)}\right)^2\frac{\partial v}{\partial\zeta}\bigg|_{\zeta=1}-\left(\frac{p_0\bar T_0}{P\bar T(p_0)}\right)^2\frac{\partial v}{\partial\zeta}\bigg|_{\zeta=0}\right]\in H^1(TS^2).\tag{2.82}$$

Thus, we infer from (2.79) that $l\in H^2(S^2)$ and then, by the first equation of (2.70) again, $v\in H^3(TM\mid TS^2)$. □

3. Existence of global weak solutions

The main result in this section is the following existence theorem of global weak solutions to problem 2.1.

Theorem 3.1. For any time interval $[0, \tau]$, there is at least one solution $\Xi = (v, \mathscr{T})$ to problem 2.1 such that

$$\left.\begin{aligned} &\Xi_t \in L^2(0, \tau; H^{-3}(TM)) \\ &\Xi \in C([0, \tau]; H_w) \end{aligned}\right\} \tag{3.1}$$

$$|\Xi(t)|_H^2 + \frac{1}{R_{\max}} \int_0^t \|\Xi(t)\|^2 \, dt \leqslant |\Xi_0|_H^2 + \int_0^t (F, \Xi(t))_H \, dt \qquad \text{a.e. } t \in [0, \tau] \tag{3.2}$$

where H_w is the space H endowed with the weak topology.

Proof. We proceed by the Faedo–Galerkin procedure. Since this method is now standard, we only need to present an outline of the proof. Let $\{\psi_i \mid i = 1, 2, \ldots\}$ be the eigenfunctions of A, then we look for an approximate solution $\Xi_m(t) \doteq \sum_{j=1}^{m} g_{jm}(t)\psi_j$ satisfying

$$\left.\begin{aligned} &\frac{d}{dt}(\Xi_m, \psi_j)_H + (A\Xi_m + B(\Xi_m, \Xi_m) + B(\Xi_m, \Xi^*) + E(\Xi_m), \psi_j)_H \\ &\qquad = (F, \psi_j)_H \qquad j = 1, \ldots, m \\ &\Xi_m|_{t=0} = P_m \Xi_0 \end{aligned}\right\} \tag{3.3}$$

where P_m is the projector from H onto the space spanned by $\{\psi_1, \ldots, \psi_m\}$. We then have from (3.3) that

$$\frac{1}{2}\frac{d}{dt}|\Xi_m(t)|_H^2 + \frac{1}{R_{\max}} \|\Xi_m(t)\|^2 \leqslant \frac{1}{2R_{\max}} \|\Xi_m(t)\|^2 + (F, \Xi_m(t))_H \tag{3.4}$$

which implies (3.2) (by passing to the limit) and

$$\Xi_m \in \text{a bounded set of } L^\infty(0, \tau; H) \cap L^2(0, \tau; V). \tag{3.5}$$

Since

$$\|B(\Xi, \Xi_1)\|_{H^{-3}} \leqslant C \, \|\Xi\| \cdot |\Xi_1|_H$$

we have

$$(\Xi_m)_t \in \text{a bounded set of } L^2(0, \tau; H^{-3}(TM)). \tag{3.6}$$

Equations (3.5) and (3.6) show that we can pass to the limit $m \to \infty$ to obtain a solution $\Xi = (v, \mathscr{T})$ satisfying (3.1); (3.2) follows from (3.4) by passing to the limit. □

Remark 3.1.

(*i*) A similar result was obtained in theorem 2.1 in chapter 2 of [11]. Since in [11] we did not find the reformulation (1.50)–(1.52) of the PEs we had to solve the original PEs (1.41)–(1.44). Therefore, we had to construct some special function spaces for the vertical velocity ω, and use different functional formulation. Moreover, we were not able to prove the result by the Galerkin method due to the degeneracy of the equations with respect to the vertical velocity ω. So we proved the result there by the semi-discretization method. Here we reformulate the PEs by integrating the vertical momentum and the continuity equations, so that the variable ω disappears. Therefore, we can simply use the Faedo–Galerkin method here to prove the existence of global weak solutions.

(*ii*) By directly considering (2.60)–(2.63) or (1.50)–(1.54), using the Leray–Schauder principle we can prove the existence and uniqueness of local strong solutions (in $L^2(0, \tau; H^4 \cap V) \cap C([0, \tau]; H^3 \cap V)$ for τ small) to the problem. Since the method and the result are very similar to those in [11], we omit them.

Part II. Primitive equations with vertical viscosity (PEV²s)

4. Mathematical setting of the PEV²s

4.1. The equations and some physical considerations

In this part, we consider the PEV²s as

$$\frac{\partial v}{\partial t} + \nabla_v v + \omega \frac{\partial v}{\partial \zeta} + \frac{f}{Ro} k \times v + \operatorname{grad} \Phi - \frac{1}{Re_1} \Delta v - \frac{1}{Re_2} \frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] = f_1 \tag{4.1}$$

$$\frac{\partial \Phi}{\partial \zeta} + \frac{bP}{p} T - \underline{\frac{\varepsilon^2}{Re_1} \Delta \omega - \frac{\varepsilon^2}{Re_2} \frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial \omega}{\partial \zeta}\right]} = 0 \tag{4.2}$$

$$\operatorname{div} v + \frac{\partial \omega}{\partial \zeta} = 0 \tag{4.3}$$

$$a_1\left(\frac{\partial T}{\partial t} + \nabla_v T + \omega \frac{\partial T}{\partial \zeta}\right) - \frac{bP}{p} \omega - \frac{1}{Rt_1} \Delta T - \frac{1}{Rt_2} \frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial T}{\partial \zeta}\right] = f_2 \tag{4.4}$$

with the same initial and boundary value conditions as those in part I. In (4.2) $\varepsilon = (P - p_0)R\bar{T}_0/agP$, indicating the ratio between the height of the atmosphere and a.

Comparing these equations with the PES in section 1, we can see that the only difference is the underlined terms in (4.2), representing the viscosity in the p-direction. It is reasonable from the meteorological point of view to have these viscosity terms for ω. Recalling the procedure of deriving the PES in section 1, we use the quasistatic equilibrium assumption $\partial p/\partial z = -\rho g$ to approximate the vertical momentum equation. So, in (1.10), we have omitted the term D_r, denoting the viscosity in the vertical direction. If we keep this term D_r, then (1.10) is replaced by

$$\frac{\partial p}{\partial z} = -\rho g + \rho D_r.$$

Since D_r is smaller than the term $-\rho g$, we can still make the coordinate transformation from $(\theta, \varphi, z; t)$ to $(\theta, \varphi, p; t)$. Then the first equation of (1.18) becomes

$$\begin{aligned}\frac{\partial gz}{\partial p} &= \frac{1}{-\rho + \rho D_r/g} \\ &= -\frac{1}{\rho} - \frac{D_r}{\rho g} + o\left(\frac{D_r}{g}\right)\frac{1}{\rho} \\ &\approx -\frac{1}{\rho} - \frac{D_r}{\rho g}.\end{aligned}$$

Keeping in mind that D_r is small, we can omit this term in the second and third equations of (1.18). Therefore, (1.28) is replaced by

$$\frac{\partial \Phi}{\partial p} + \frac{R}{p} T + \frac{1}{\rho g} D_r = 0$$

and (1.26) and (1.27), (1.29) and (1.32) remain unchanged. Finally, we can add, in the non-dimensional forms of the PE, those viscosity terms underlined in (4.2).

It is worth pointing out that for large-scale oceanic dynamics, owing to the incompressibility of sea water, the PEs are written in the usual $(\theta, \varphi, z; t)$ coordinate system rather than transforming the equations into the p-coordinate system. Therefore, we can consider more easily the PEV^{2}s large-scale oceanic circulation, and we will report this in a forthcoming paper.

4.2. *Reformulation of the PEV^{2}s*

Similar to section 1.4, integrating (4.2) and (4.3), and taking the boundary value conditions for ω into account, we have

$$\left.\begin{aligned} &\int_0^1 \operatorname{div} v \, \mathrm{d}\zeta = 0 \\ &\omega = W(v) = \int_\zeta^1 \operatorname{div} v \, \mathrm{d}\zeta \end{aligned}\right\} \tag{4.5}$$

$$\Phi = \Phi_s + \int_\zeta^1 \frac{bP}{p} T \, \mathrm{d}\zeta + \int_\zeta^1 LW(v) \, \mathrm{d}\zeta \tag{4.6}$$

where Φ_s is a function on S^2, and L is a linear differential operator given by

$$LW(v) = -\frac{\varepsilon^2}{Re_1} \Delta W(v) - \frac{\varepsilon^2}{Re_2} \frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial W(v)}{\partial \zeta}\right]. \tag{4.7}$$

Recalling the non-homogeneous boundary value conditions for T, we define

$$\mathcal{T} = T - T^*$$

where $T^* = T^*_\varepsilon$ given by (2.56) with ε such that (2.58) holds. We then obtain the reformulation of the PEV^{2}s as

$$\begin{aligned} &\frac{\partial v}{\partial t} + \nabla_v v + W(v) \frac{\partial v}{\partial \zeta} + \frac{f}{Ro} k \times v + \operatorname{grad} \Phi_s + \operatorname{grad} \int_\zeta^1 LW(v) \, \mathrm{d}\zeta \\ &\qquad + \int_\zeta^1 \frac{bP}{p} \operatorname{grad} \mathcal{T} \, \mathrm{d}\zeta - \frac{1}{Re_1} \Delta v - \frac{1}{Re_2} \frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] = \hat{f}_1 \end{aligned} \tag{4.8}$$

$$\begin{aligned} &a_1\left(\frac{\partial \mathcal{T}}{\partial t} + \nabla_v \mathcal{T} + W(v) \frac{\partial \mathcal{T}}{\partial \zeta}\right) + a_1\left(\nabla_v T^* + W(v) \frac{\partial T^*}{\partial \zeta}\right) \\ &\qquad - \frac{bP}{p} W(v) - \frac{1}{Rt_1} \Delta \mathcal{T} - \frac{1}{Rt_2} \frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial \mathcal{T}}{\partial \zeta}\right] = \hat{f}_2 \end{aligned} \tag{4.9}$$

$$\int_0^1 \operatorname{div} v \, \mathrm{d}\zeta = 0 \tag{4.10}$$

with the same initial and boundary value conditions (2.63) or (1.53) and (1.54) as those in the PEs in part I, since we have not changed the maximum order of the ζ-derivatives.

4.3. Functional setting of the PEV^2s

Now we begin with the mathematical formulation of the PEV^{2}s. To this end, recall that in (4.8) there is one term corresponding to the viscosity of ω as follows:

$$\operatorname{grad} \int_\zeta^1 LW(v)\,d\zeta.$$

If we take the inner product in L^2 between this term and v, then

$$\begin{aligned}
\int_M \left[\left(\operatorname{grad} \int_\zeta^1 LW(v)\,d\zeta\right) \cdot v\right] dM
&= -\int_M \left[\left(\int_\zeta^1 LW(v)\,d\zeta\right) \operatorname{div} v\right] dM \\
&= \int_M LW(v)\left(\int_\zeta^1 \operatorname{div} v\,d\zeta\right) dM \\
&= \int_M LW(v) \cdot W(v)\,dM \\
&= \int_M \left[\frac{\varepsilon^2}{Re_1} \operatorname{grad} W(v) \cdot \operatorname{grad} W(v)\right. \\
&\qquad \left. + \frac{\varepsilon^2}{Re_2}\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial W(v)}{\partial \zeta}\frac{\partial W(v)}{\partial \zeta}\right] dM.
\end{aligned} \tag{4.11}$$

Thus, it is natural that the space for v in this case should have the following regularity:

$$W(v) \in H^1(M).$$

Therefore, we can define the function spaces for v as follows.

First, for the 3D Navier–Stokes equations, we can define a function space

$$\tilde{V}_1^w = \left\{(v, \omega) \in H_0^1(TM) \mid \operatorname{div} v + \frac{\partial \omega}{\partial \zeta} = 0\right\} \tag{4.12}$$

which is the completion in H^1 norm of

$$\tilde{\mathscr{V}}_1^w = \left\{(v, \omega) \in C_0^\infty(TM) \mid \operatorname{div} v + \frac{\partial \omega}{\partial \zeta} = 0\right\}. \tag{4.13}$$

Then, it is easy to see that

$$\left.\begin{aligned}
\mathscr{V}_1^w &= \left\{v \in C_0^\infty(TM \mid TS^2) \mid W(v) = \int_\zeta^1 \operatorname{div} v\,d\zeta \in C_0^\infty(M)\right\} \\
V_1^w &= \left\{v \in H_0^1(TM \mid TS^2) \mid W(v) = \int_\zeta^1 \operatorname{div} v\,d\zeta \in H_0^1(M)\right\} \\
&= \left\{v \in H_0^1(TM \mid TS^2) \mid W(v) \in H^1(M), \int_0^1 \operatorname{div} v\,d\zeta = 0\right\}.
\end{aligned}\right\} \tag{4.14}$$

Here V_1^w is equipped with the following inner product and norm:

$$\left.\begin{aligned}&(v, v_1)_w = ((v, v_1)) + (W(v), W(v_1))_{H_0^1}\\ &\|v\|_w = (v, v)_w^{1/2} \qquad \forall v, v_1 \in V_1^w.\end{aligned}\right\} \tag{4.15}$$

We let

$$V^w = V_1^w \times H^1(M) = V_1^w \times V_2 \tag{4.16}$$

and we use the same notation $(\cdot, \cdot)_w$ and $\|\cdot\|_w$ as for V_1^w to denote the norm and inner product in V^w.

As in section 2 we define two bilinear functionals $a^w: V^w \times V^w \to \mathbb{R}$, $a_1^w: V_1^w \times V_1^w \to \mathbb{R}$, and their corresponding linear operators $A^w: V^w \to (V^w)'$, $A_1^w: V_1^w \to (V_1^w)'$ by

$$\begin{aligned}a^w(\Xi, \Xi_1) &= (A^w\Xi, \Xi_1)_H\\ &= (A_1^w v, v_1) + (A_2 T, T_1)_H\\ &= a_1^w(v, v_1) + a_2(T, T_1)\end{aligned} \tag{4.17}$$

$$\begin{aligned}a_1^w(v, v_1) &= (A_1^w v, v_1)\\ &= a_1(v, v_1) + \int_M \left[\frac{\varepsilon^2}{Re_1} \operatorname{grad} W(v) \cdot \operatorname{grad} W(v_1)\right.\\ &\quad \left. + \frac{\varepsilon^2}{Re_2}\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial W(v)}{\partial \zeta} \cdot \frac{\partial W(v_1)}{\partial \zeta}\right] \mathrm{d}M\end{aligned} \tag{4.18}$$

for any $\Xi = (v, T)$, $\Xi_1 = (v_1, T_1) \in V^w$. Then, as in lemma 2.3, we have the following lemma.

Lemma 4.1.

(*i*) a^w and a_1^w are coercive and continuous; therefore, $A^w: V^w \to (V^w)'$ and $A_1^w: V_1^w \to (V_1^w)'$ are isomorphisms. Moreover, we have

$$\left.\begin{aligned}&a^w(\Xi, \Xi_1) \leqslant \frac{1}{R_{\min}} \|\Xi\|_w \cdot \|\Xi_1\|_w\\ &a^w(\Xi, \Xi) \geqslant \frac{1}{R_{\max}} \|\Xi\|_w^2\\ &a_1^w(v, v_1) \leqslant \frac{1}{R_{\min}} \|v\|_w \cdot \|v_1\|_w\\ &a_1^w(v, v) \geqslant \frac{1}{R_{\max}} \|v\|_w^2.\end{aligned}\right\} \tag{4.19}$$

(*ii*) The isomorphism $A^w: V^w \to (V^w)'$ (or $A_1^w: V_1^w \to (V_1^w)'$) can be extended as a self-adjoint unbounded linear operator on H (or H_1), with compact inverse $(A^w)^{-1}: H \to H$ (or $(A_1^w)^{-1}: H_1 \to H_1$), and with domain $D(A^w) = V \cap (D(A_1^w) \times H^2(M))$ (or $D(A_1^w) = V_1^w \cap \{v \in H^2(TM \mid TS^2) \mid W(v) \in H^2(M)\}$.)

Proof. We only have to prove that

$$D(A_1^w) = V_1^w \cap \{v \in H^2(TM \mid TS^2) \mid W(v) \in H^2(M)\}.$$

To this end, we consider

$$A_1^w v = f \in L^2(TM \mid TS^2)$$

then following the same procedure as in section 2.4 we obtain the existence of a function $\Phi_s \in H^{-1}(S^2)$ such that

$$-\frac{1}{Re_1}\Delta v - \frac{1}{Re_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] + \operatorname{grad}\Phi_s + \operatorname{grad}\int_\zeta^1 LW(v)\,d\zeta = f$$

$$\int_0^1 \operatorname{div} v\, d\zeta = 0.$$

We define

$$\Phi = \Phi_s + L\int_\zeta^1 W(v)\,d\zeta.$$

We can then obtain that

$$-\frac{1}{Re_1}\Delta v - \frac{1}{Re_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] + \operatorname{grad}\Phi = f$$

$$\frac{\partial \Phi}{\partial \zeta} - \frac{1}{Re_1}\Delta W(v) - \frac{1}{Re_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial W(v)}{\partial \zeta}\right] = 0$$

$$\operatorname{div} v + \frac{\partial W(v)}{\partial \zeta} = 0$$

which is essentially the same problem as the 3D Stokes problem. By the method of the Cattabriga regularity theorem (see [14, 18]) the result follows. □

We now state the functional formulation of the PEV^{2}s as follows.

Problem 4.1. For $F \doteq (\hat{f}_1, \hat{f}_2/a_1)$, $\Xi_0 = (v_0, \mathcal{T}_0) \in H$ given, find $\Xi = (v, \mathcal{T})$ such that

$$\Xi \in L^2(0, \tau; V^w) \cap L^\infty(0, \tau; H) \qquad \forall \tau > 0 \tag{4.20}$$

$$\frac{d}{dt}(\Xi, \Xi_1)_H + (A^w\Xi, \Xi_1)_H + (B(\Xi, \Xi), \Xi_1)_H + (B(\Xi, \Xi^*), \Xi_1)_H$$

$$+ (E(\Xi), \Xi_1)_H = (F, \Xi_1)_H \qquad \forall \Xi_1 \in D(A^w) \tag{4.21}$$

$$\Xi|_{t=0} = \Xi_0. \tag{4.22}$$

Remark 4.1.

(*i*) Problem 4.1 is indeed a weak formulation to the initial boundary problem of the PEV^{2}s. That is, for any solution $(v, \mathcal{T}, \Phi_s)$ of the PEV^{2}s which is sufficiently smooth, $\Xi = (v, \mathcal{T})$ is a solution of Problem 4.1. Conversely, following the procedure of the proof of (2.33) (in lemma 2.1), we can prove that for any solution $\Xi = (v, \mathcal{T})$ of problem 4.1, there is a unique distribution (up to a constant) $\Phi_s \in \mathcal{D}'(S^2)$ such that $(v, \mathcal{T}, \Phi_s)$ satisfies the PEV^{2}s, at least in the sense of distribution.

(*ii*) Equation (4.21) is just the same as (2.65) in problem 2.1 with the operator A replaced by A^w. Formally, both possess the same nonlinear terms $B(\Xi, \Xi)$. But, in (4.21), the principal term $A^w\Xi$ provides some *a priori* estimates for $W(v) =$

$\int_{\zeta}^{1} \operatorname{div} v \, d\zeta$, the absence of which causes some serious difficulty for solving problem 2.1. In comparison to the 3D Navier–Stokes equations, we do not have the time derivative term of $W(v)$ here and, therefore, we cannot obtain some *a priori* estimates from $\partial W(v)/\partial t$ as in the case of 3D Navier–Stokes equations. Due to the absence of this term, for instance, we cannot use the Sobolev–Lieb–Thirring inequality when estimating the dimensions of the attractors of the equations (see section 6).

Before ending this subsection, we would like to write explicitly some estimates of the nonlinear term $B(\Xi, \Xi_1)$ in terms of $\|\cdot\|_w$ and A^w.

Lemma 4.2.

$$|b(\Xi, \Xi_1, \Xi_2)| \leqslant C \begin{cases} \|\Xi\|_w \cdot \|\Xi_1\| \cdot |\Xi_2|_H^{1/2} \cdot \|\Xi_2\|^{1/2} \\ \|\Xi\|_w \cdot \|\Xi_1\|^{1/2} |A\Xi_1|^{1/2} \cdot |\Xi_2|_H \\ \|\Xi\|_w \cdot |\Xi_1|_H \cdot \|\Xi_2\|_{H^{3/2}}. \end{cases} \tag{4.23}$$

Proof. The proof of (4.23) is the same as that for the corresponding result for 3D Navier–Stokes equations. □

Remark 4.2. We would also like to mention that in the case of 3D Navier–Stokes equations we can obtain some better estimates for the nonlinear terms in terms of the norms $\|\cdot\|$ and $|\cdot|$, which cannot be proved in our case here. The reason is that the norm $|\cdot|_H$ in the space H does not provide any information about the L^2 norm of $W(v)$, owing to the absence of the term $\partial W(v)/\partial t$ in the equations.

5. Existence of solutions and their properties

5.1. Existence of solutions

As for theorem 3.1, we can prove the following existence result of solutions for problem 4.1.

Theorem 5.1. For any time interval $[0, \tau]$ there is at least one solution $\Xi = (v, \mathcal{T})$ to problem 4.1 such that

$$\left.\begin{aligned} &\Xi_t \in L^2(0, \tau; H^{-3/2}(TM)) \\ &\Xi \in C([0, \tau]; H_w) \end{aligned}\right\} \tag{5.1}$$

$$|\Xi(t)|_H^2 + \frac{1}{R_{\max}} \int_0^t \|\Xi(t)\|_w^2 \, dt \leqslant |\Xi_0|_H^2 + \int_0^t (F, \Xi(t))_H \, dt \qquad \text{a.e. } t \in [0, \tau] \tag{5.2}$$

where H_w is the space H endowed with weak topology.

Moreover, we have the following theorem.

Theorem 5.2. If the initial value $\Xi_0 \in V^w$ then there is a time $\tau_1 = \tau_1(\|\Xi_0\|_w) > 0$, such that there is a unique solution $\Xi = (v, \mathcal{T})$ to problem 4.1 satisfying

$$\left.\begin{aligned} &\Xi_t \in L^2(0, \tau_1; H) \\ &\Xi \in L^2(0, \tau_1; D(A^w)) \cap C([0, \tau_1]; V^w). \end{aligned}\right\} \tag{5.3}$$

Here $\tau_1 = \tau_1(\|\Xi_0\|_w)$ can be given by

$$\tau_1(\|\Xi_0\|_w) = \frac{C}{(1+\|\Xi_0\|_w^2)^2}. \tag{5.4}$$

Proof of theorem 5.1. We proceed by the Faedo–Galerkin procedure. Let $\{\psi_i \mid i = 1, 2, \ldots\}$ be eigenfunctions of A^w. Let P_m be the orthogonal projector of H onto the space spanned by $\{\psi_1, \ldots, \psi_m\}$. Then, it is easy to find an approximate solution $\Xi_m(t) \doteq \sum_{j=1}^m g_{jm}(t)\psi_j$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Xi_m, \psi_j)_H + (A^w\Xi_m + B(\Xi_m, \Xi_m) + B(\Xi_m, \Xi^*) + E(\Xi_m), \psi_j)_H$$

$$= (F, \psi_j)_H \qquad j = 1, \ldots, m \tag{5.5}$$

$$\Xi_m\big|_{t=0} = P_m\Xi_0. \tag{5.6}$$

Then, by (5.5) and (5.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Xi_m(t)|_H^2 + \frac{1}{R_{\max}}\|\Xi_m(t)\|_w^2 \leq (F, \Xi_m(t))_H \tag{5.7}$$

which implies that

$$\Xi_m \in \text{a bounded set of } L^\infty(0, \tau; H) \cap L^2(0, \tau; V^w). \tag{5.8}$$

On the other hand, by (4.23), we have

$$\|B(\Xi_m, \Xi_m)\|_{H^{-3/2}} \leq C\,\|\Xi_m\|_w\,|\Xi_m|_H$$

which yields

$$(\Xi_m)_t \in \text{a bounded set of } L^2(0, \tau; H^{-3/2}(TM)). \tag{5.9}$$

The combination of (5.7)–(5.9) proves theorem 5.1. □

Proof of theorem 5.2. From (5.5) we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(A^w\Xi_m, \Xi_m)_H + |A^w\Xi_m(t)|_H^2 &= (F - B(\Xi_m, \Xi_m) - B(\Xi_m, \Xi^*) - E(\Xi_m), A^w\Xi_m)_H \\ &\leq |A^w\Xi_m|_H\cdot[|F|_H + |B(\Xi_m, \Xi_m)|_H + |B(\Xi_m, \Xi^*)|_H + |E(\Xi_m)|_H] \\ &\leq (\text{by } (4.23)) \\ &\leq \tfrac{1}{2}|A^w\Xi_m|_H^2 + C(|F|^2 + \|\Xi_m\|_w^2 + \|\Xi_m\|_w^6).\end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(A^w\Xi_m, \Xi_m)_H + |A^w\Xi_m(t)|_H^2 \leq C(1 + (A^w\Xi_m, \Xi_m)_H)^3. \tag{5.10}$$

Let

$$y(t) = 1 + (A^w\Xi_m, \Xi_m)_H$$

then

$$y(t) \leq \frac{y(0)}{\sqrt{1 - 2y^2(0)Ct}} \qquad \text{for } 0 \leq t < \frac{1}{2y^2(0)C}.$$

So we can choose

$$\tau_1 = \tau_1(\|\Xi_0\|_w) = \frac{C}{(1 + \|\Xi_0\|_w^2)^2} \tag{5.11}$$

such that whenever $0 \leqslant t \leqslant \tau_1$,

$$(A^w\Xi_m, \Xi_m)_H \leqslant 2(1 + (A^w\Xi_0, \Xi_0)_H) \tag{5.12}$$

which proves that the solution $\Xi = (v, \mathcal{T})$ satisfies

$$\Xi \in L^\infty(0, \tau; V^w). \tag{5.13}$$

By (5.10) and (5.13) we have

$$\Xi \in L^2(0, \tau_1; D(A^w)). \tag{5.14}$$

and then, by (4.21), we obtain

$$\Xi_t \in L^2(0, \tau_1; H). \tag{5.15}$$

Remark 5.1. Here we have only shown the existence of local strong solutions. In a forthcoming paper we intend to prove the existence of global strong solutions to the problem using the smallness of the vertical scale in comparison to the horizontal scale of the global atmosphere.

5.2. Time-analyticity of the solutions

The main objective in this subsection is to prove that the strong solutions of problem 4.1 given by theorem 5.2 are analytic in time $D(A^w)$-valued functons. To this end, let $H^{\mathbb{C}}$, $D(\bar{A}^w)^{\mathbb{C}}$ and $(V^w)^{\mathbb{C}}$ be the complex spaces of H, $D(A^w)$ and V^w. We use the same notation as for A^w, B, E, etc., to denote the naturally extended complex operators on the corresponding complex spaces. Moreover, the complex equation of (4.21) become (here z is the complex time variable)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}(\Xi(z), \Xi_1)_{H^{\mathbb{C}}} &+ (A^w\Xi(z) + B(\Xi(z), \Xi(z)) + B(\Xi(z), \Xi^*), \Xi_1)_{H^{\mathbb{C}}} \\ &+ (E(\Xi(z)), \Xi_1)_{H^{\mathbb{C}}} = (F, \Xi_1)_{H^{\mathbb{C}}}, \qquad \forall \Xi_1 \in D(A^w)^{\mathbb{C}}.\end{aligned} \tag{5.16}$$

Then, our main result in this subsection is as follows.

Theorem 5.3. Let $\Xi_0 \in V^w$. Then there exists a $\tau_0 = \tau_0(\|\Xi_0\|_w))$ given by

$$\tau_0 = \tau_0(\|\Xi_0\|_w) = \frac{C}{(1 + \|\Xi_0\|_w^2)^2} \leqslant \tau_1(\|\Xi_0\|_w) \tag{5.17}$$

such that the complex problem (5.16) and (4.22) possesses a unique solution $\Xi = (v, \mathcal{T})$, which is analytic from $\Delta(\|\Xi_0\|_w)$ into $D(A^w)^{\mathbb{C}}$, and which coincides with the unique solution of problem 4.1 when restricted in the positive real interval $[0, \tau_1(\|\Xi_0\|_w)]$. Here $\Delta(\|\Xi_0\|_w)$ is given by

$$\Delta(\|\Xi_0\|_w) \doteq \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{ll} |\mathrm{Im}\, z| \leqslant \mathrm{Re}\, z & \text{if } 0 < \mathrm{Re}\, z \leqslant \tau_0 \\ |\mathrm{Im}\, z| \leqslant \tau_0 & \text{if } \tau_0 \leqslant \mathrm{Re}\, z \leqslant \tau_1 \end{array} \right\}. \tag{5.18}$$

Proof. We proceed by applying the Galerkin method, following the ideas of Foias and Temam (see [19, 20]). Since the procedure is now standard, we only have to present some formal *a priori* estimates, which will become rigorous if the procedure works out in detail.

First, we take Ξ_1 in (5.16) to be $A^w\Xi$, and we multiply the resulting equation by $e^{i\theta}$ (when $z = re^{i\theta}$), and then we consider the real part. This yields ($|\theta| < \pi/2$)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}&(A^w\Xi(re^{i\theta}), \Xi(re^{i\theta}))_H + \cos\theta\, |A^w\Xi(re^{i\theta})|_H^2\\ &= -Re[e^{i\theta}(B(\Xi(z), \Xi(z)) + B(\Xi(z), \Xi^*) + E(\Xi(z)) - F, A^w\Xi(z))_H]\\ &\leqslant \tfrac{1}{2}\cos\theta\, |A^w\Xi(re^{i\theta})|_H^2 + \frac{C}{\cos\theta}(1 + \|\Xi(re^{i\theta})\|_w^2)^3.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}&(A^w\Xi(re^{i\theta}), \Xi(re^{i\theta}))_H + \cos\theta\, |A^w\Xi(re^{i\theta})|_H^2\\ &\leqslant \frac{C}{\cos\theta}[1 + (A^w\Xi(re^{i\theta}), \Xi(re^{i\theta}))_H]^3. \end{aligned}\tag{5.19}$$

On integrating (5.19) we have

$$(A^w\Xi(re^{i\theta}), \Xi(re^{i\theta}))_H \leqslant C(1 + \|\Xi_0\|_w^2)\tag{5.20}$$

as long as

$$\begin{aligned}&0 \leqslant r \leqslant \frac{C}{(1 + \|\Xi_0\|_w^2)^2}\\ &|\theta| \leqslant \frac{\pi}{4}.\end{aligned}\tag{5.21}$$

Following the method developed by Foias and Temam [19] we conclude that the solution Ξ of (5.16) and (4.22) is analytic from a region in $\mathbb{C}$, which comprises $\Delta(\|\Xi_0\|_w)$ given by (5.18), into $D(A^w)^{\mathbb{C}}$. The other parts of the proof follow the standard procedure we mentioned before. □

Part III. Attractors and their dimensions

6. Existence and dimensions of attractors for the PEV^{2}s

The main purpose of this section is to estimate the Hausdorff and fractal dimensions of the attractors and/or functional invariant sets for the problem in terms of some physically relevant parameters, using the CFT theory developed by Constantin, Foias and Temam [12]. Here we follow Temam [4].

6.1. Absorbing sets and attractors

As we have mentioned before, it is not known in general that there exist global strong solutions of problem 4.1, even though we have proved the local existence of strong solutions in theorem 5.2. Therefore, in this article we restrict ourselves to the study of attractors and functional invariant sets associated with global strong solutions for the PEV^{2}s, when such solutions exist.

According to theorem 5.2, we can define the solution operator $S(t)$ ($t>0$ given), whenever it makes sense, that maps the initial value Ξ_0 to the solution $\Xi(t)$ of problem 4.1. For any $t>0$, $S(t)$ is an operator from V^w into V^w. Since, generally speaking, $S(t)$ ($t>0$) is only defined and continuous on some part of V^w into V^w, we define

$$\left.\begin{aligned} &D(S(t)) = \bigcup_{\rho>0} D_\rho(S(t)) \\ &D_\rho(S(t)) = \{s_0 \in V^w \mid \|S(t)s_0\|_w \leqslant \rho,\ 0 \leqslant \tau \leqslant t\}. \end{aligned}\right\} \tag{6.1}$$

Obviously, we have the following semigroup properties of $S(t)$:

$$\left.\begin{aligned} &S(0) = I \qquad D(S(0)) = V^w \\ &S(t+s) = S(t)S(s) \text{ on } D(S(t+s)) \qquad \forall t, s \geqslant 0. \end{aligned}\right\} \tag{6.2}$$

Moreover, for the sake of convenience, we quote the definitions of functional invariant sets and attractors from [4] as follows.

Definition 6.1.

(i) A set $X \subset V^w$ is a functional invariant set for the semigroup $\{S(t)\}_t$ if

$$\left.\begin{aligned} &S(t)\Xi_0 \qquad \text{exists } \forall \Xi_0 \in X \qquad \forall t \geqslant 0 \\ &S(t)X = X \qquad \forall t \geqslant 0. \end{aligned}\right\} \tag{6.3}$$

(ii) An attractor set is a set $\mathscr{A} \subset V^w$ such that:

(a) $\mathscr{A}$ is an invariant set;

(b) $\mathscr{A}$ is the ω-limit set of one of its open neighbourhoods U.†

As we mentioned at the beginning of this section, the main objective in this section is to estimate Hausdorff and fractal dimensions of the functional invariant sets and attractors for the PEV^{2}s. To this end, we assume the existence of a particular solution $\Xi = (v, \mathscr{T})$ of problem 4.1 satisfying the following boundedness property:

$$\sup_{t>0} \|\Xi\|_w < \infty. \tag{6.4}$$

We have then the following theorem.

Theorem 6.1.‡ Let $\Xi = (v, \mathscr{T})$ be a solution of problem 4.1 with initial value $\Xi_0 = (v_0, \mathscr{T}_0)$, satisfying (6.4). Then there exists a functional invariant set $X = X(\Xi_0)$ for the semigroup $S(t)$, which is bounded in $D(A^w)$, and $\Xi(t)$ converges to X as $t \to \infty$.

† That is, $\mathscr{A} = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)U}$, the closure being taken in V^w.

‡ If it is true that

$$S(t)\Xi_0 \text{ exists } \forall \Xi_0 \in V^w \qquad \forall t \geqslant 0 \tag{6.4a}$$

then we have the following theorem.

Theorem 6.1′. Assuming (6.4a), then the semigroup $\{S(t)\}_{t\geqslant 0}$ possesses an absorbing set in V^w and a maximal attractor $\mathscr{A} \subset D(A^w)$, which is bounded in $D(A^w)$.

This is the direct application of theorem 1.2 of [4, p 382].

In order to estimate the dimensions of the attractor $\mathscr{A}$ given by theorem 6.1, we should study the linearized problem of problem 4.1. Thus, consider a solution $\Xi = (v, \mathscr{T})$ of problem 4.1 satisfying

$$\Xi \in L^\infty(0, \infty; V^w). \tag{6.5}$$

The linearized equation of (4.21) is

$$\begin{aligned} &U' + A^w U + B(\Xi, U) + B(U, \Xi) + B(U, \Xi^*) + E(U) = 0 \\ &U(0) = \xi \in H. \end{aligned} \tag{6.6}$$

We quote a direct application of property 2.1 of [4, p 387] without proof as follows.

Theorem 6.2.

(i) There exists a unique solution U of (6.6) satisfying

$$U \in L^2(0, \tau; V^w) \cap L^\infty(0, \tau; H) \qquad \forall \tau > 0. \tag{6.7}$$

(ii) For any $t_1 > 0$, $\rho > 0$, $D_\rho(S(t))$ is open in V^w, and $S(t_1)$ is differentiable in $D(S(t_1))$ equipped with the norm of H. The differential of $S(t_1)$ at a point Ξ_0 of $D_\rho(S(t_1))$ is the mapping

$$\xi \in H \rightarrow L(t_1; \Xi_0)\xi = U(t_1) \tag{6.8}$$

where U is the solution of (6.6).

6.2. *Dimensions of the attractors*

We are now in a position to estimate the Hausdorff and fractal dimensions of the functional invariant set X given by theorem 6.1, or the attractor $\mathscr{A}$ given by theorem 6.1′, of the semigroup $S(t)$, or problem 4.1. Before we proceed, we would like to mention that, in the case where (6.4a) holds, we can obtain the exponential decay of the volume elements (see the footnote to theorem 6.3). For the general case, however, we are not able to establish the exponential decay of the volume elements.

We follow the notation and procedures of [4]. Let $U_1, U_2, \ldots, U_m$ be m solutions of (6.6) with initial values $\xi_1, \xi_2, \ldots, \xi_m$, respectively. Then, we have

$$|U_1(t) \wedge \ldots \wedge U_m(t)|_{\wedge^m H} = |\xi_1 \wedge \ldots \wedge \xi_m|_{\wedge^m H} \exp \int_0^t \operatorname{Tr} F'(S(\tau)\Xi_0) \circ Q_m(\tau)\, \mathrm{d}\tau \tag{6.9}$$

where $Q_m(\tau)$ is the orthogonal projection in H onto

$$Q_m(\tau)H = \operatorname{Span}\{U_1(\tau), \ldots, U_m(\tau)\}.$$

Choosing $\{\psi_j(\tau)\}_{j=1}^m$ to be an orthonormal basis of $Q_m(\tau)H$, we have

$$\operatorname{Tr}[F'(\Xi(\tau)) \circ Q_m(\tau)] = \sum_{j=1}^m (F'(\Xi(\tau))\psi_j(\tau), \psi_j(\tau))_H \tag{6.10}$$

where $F'(\Xi(\tau))$ is defined by

$$F'(\Xi(\tau))U = -A^w U - B(\Xi(\tau), U) - B_1(U, \Xi(\tau)) - B(U, \Xi^*) - E(U). \tag{6.11}$$

Moreover, we introduce

$$q_m(t) = \sup_{\Xi_0 \in X} \sup_{\substack{\xi_i \in H \\ |\xi_i|_H \leqslant 1 \\ i=1,\ldots,m}} \left(\frac{1}{t}\int_0^t \operatorname{Tr}[F'(\Xi(\tau)) \circ Q_m(\tau)]\,d\tau\right) \tag{6.12}$$

$$q_m = \limsup_{t \to \infty} q_m(t). \tag{6.13}$$

Then, we quote the main result of [12] from [4], which will be used in the remaining part of this section.

Theorem 6.3.† If

$$q_j \leqslant -\alpha j^{\theta} + \beta \qquad \forall j \geqslant 1 \tag{6.14}$$

then the Hausdorff dimension of X and the fractal dimension of X satisfy

$$d_H(X) \leqslant m \qquad d_F(X) \leqslant 2m \tag{6.15}$$

$$m - 1 < \left(\frac{2\beta}{\alpha}\right)^{1/\theta} \leqslant m. \tag{6.16}$$

In order to apply this theory to estimate dimensions of X, we should estimate the right-hand side of (6.10). To this end, for simplicity, we omit temporarily τ, then

$$\begin{aligned}
\operatorname{Tr}(F'(\Xi) \circ Q_m) &= -\sum_{j=1}^{m} (A^{w}\psi_j, \psi_j)_H \\
&\quad - \sum_{j=1}^{m} (B(\Xi, \psi_j) + B(\psi_j, \Xi) + B(\psi_j, \Xi^*) + E(\psi_j), \psi_j)_H \\
&= -\sum_{j=1}^{m} (A^{w}\psi_j, \psi_j)_H - \sum_{j=1}^{m} (B(\psi_j, \Xi) + B(\psi_j, \Xi^*), \psi_j)_H.
\end{aligned} \tag{6.17}$$

Before we check each term on the right-hand side of (6.17), we define a number, which is related to the energy dissipation, as follows:

$$\gamma = R_{\max} \sup_{\Xi \in X} \|\Xi\|^2. \tag{6.18}$$

Now, for simplicity, let

$$\psi_j = (v_j, \mathscr{T}_j) \qquad \Xi = (v, \mathscr{T})$$

† In the case where (6.4*a*) holds, we can obtain further that, if for some m and $t_0 > 0$.

$$q_m(t) \leqslant -\delta < 0 \qquad \forall t \geqslant t_0$$

then the volume element $|U_1(t) \wedge \ldots \wedge U_m(t)|_{\wedge^m H}$ decays exponentially as $t \to \infty$, uniformly for $\Xi_0 \in \mathscr{A}$, $\xi_1, \ldots, \xi_m \in H$,

$$|U_1(t) \wedge \ldots \wedge U_m(t)|_{\wedge^m H} \leqslant C \exp(-\delta t).$$

then

$$|b(\psi_j, \Xi, \psi_j)| = \left|\int_M \left[\left(\nabla_{v_j} v + W(v_j)\frac{\partial v}{\partial \zeta}\right)v_j + \left(\nabla_{v_j} T + W(v_j)\frac{\partial T}{\partial \zeta}\right)T_j\right] \mathrm{d}M\right|$$
$$\leqslant C\,\|\Xi\|\,(|v_j|_{L^6} + |W(v_j)|_{L^6})\,\|\psi_j\|_{H^1}^{1/2}\,|\psi_j|_H^{1/2}$$
$$\leqslant C\,\|\psi_j\|_w^{3/2}\,\|\Xi\| \tag{6.19}$$

$$|b(\psi_j, \Xi^*, \psi_j)| = \left|\int_M \left\{\left(\nabla_{v_j} T^* + W(v_j)\frac{\partial T^*}{\partial \zeta}\right)T_j\right] \mathrm{d}M\right|$$
$$\leqslant C(|v_j|^2 + |T_j|^2)^{1/2}\,|W(v_j)|_{L^2}\,|\mathrm{grad}_M\, T^*|_{L^\infty}$$
$$\leqslant C\,\|\psi_j\|_w\,(1 + \bar{\alpha}_s). \tag{6.20}$$

Therefore, we have

$$\mathrm{Tr}[F'(\Xi(\tau))\circ Q_m(\tau)]$$
$$\leqslant -\frac{1}{R_{\max}}\sum_{j=1}^m \|\psi_j\|_w^2 + C\sum_{j=1}^m \|\Xi\|\,\|\psi_j\|_w^{3/2} + C\sum_{j=1}^m \|\psi_j\|_w\,(1+\bar{\alpha}_s)$$
$$\leqslant -\frac{1}{2R_{\max}}\sum_{j=1}^m \|\psi_j\|_w^2 + CmR_{\max}^3\,\|\Xi\|^4 + CmR_{\max}(1+\bar{\alpha}_s)^2$$
$$\leqslant -\frac{1}{2R_{\max}}\sum_{j=1}^m \|\psi_j\|_w^2 + CmR_{\max}^2\gamma\,\|\Xi\|^2 + CmR_{\max}(1+\bar{\alpha}_s)^2 \tag{6.21}$$
$$\leqslant \text{(by lemma 6.1 below)}$$
$$\leqslant -\frac{C}{R_{\max}} m^{5/3} + CmR_{\max}[\gamma R_{\max}\,\|\Xi\|^2 + (1+\bar{\alpha}_s)^2].$$

It follows that

$$q_m \leqslant -\frac{C}{R_{\max}} m^{5/3} + CmR_{\max}[\gamma\kappa + (1+\bar{\alpha}_s)^2] \tag{6.22}$$

where

$$\kappa = R_{\max}\limsup_{t\to\infty}\ \sup_{\Xi_0\in X}\frac{1}{t}\int_1^t \|\Xi\|^2\,\mathrm{d}t. \tag{6.23}$$

Integrating (5.2), we can easily obtain that

$$\kappa \leqslant CR_{\max}^3\,|F|_H^2 \tag{6.24}$$
$$\leqslant \text{(by definition of } F)$$
$$\leqslant CR_{\max}^3\left\{1 + |f_1|^2 + |f_2|^2 + \frac{1}{R_{\min}^2}(1+\bar{\alpha}_s)^2\right\}.$$

Therefore,

$$q_m \leqslant -\frac{C}{R_{\max}} m^{5/3} + CmR_{\max}[\gamma R_{\max}^3\,|F|_H^2 + (1+\bar{\alpha}_s)^2]$$
$$\leqslant -\frac{C}{R_{\max}} m^{5/3} + CR_{\max}^4[\gamma R_{\max}^3\,|F|_H^2 + (1+\bar{\alpha}_s)^2]^{5/2}. \tag{6.25}$$

Finally, by theorem 6.3, we have proved the following theorem.

Theorem 6.4.† We consider the dynamical system, problem 4.1 in H, and define m by

$$m-1<CR_{\max}[\gamma R_{\max}^3 |F|_H^2+(1+\bar{\alpha}_s)^2]^{3/2}\leqslant m \tag{6.26a}$$

or

$$m-1<CR_{\max}\left[\gamma R_{\max}^3\left(1+|f_1|^2+|f_2|^2+\frac{1}{R_{\min}^2}(1+\bar{\alpha}_s)^2\right)\right]+(1+\bar{\alpha}_s)^2\Big]^{3/2}\leqslant m \tag{6.26b}$$

where C is an absolute constant independent of the physically relevant parameters. Then

$$d_H(X)\leqslant m \qquad d_F(X)\leqslant 2m. \tag{6.27}$$

Remark 6.1.

(*i*) In (6.26*a*) or (6.26*b*), $R_{\max}$, $R_{\min}$ are given by (2.35), which represent the Reynolds numbers.

(*ii*) γ is given by (6.18). We do not know how to estimate γ explicitly in this paper. Basically, estimates of γ will depend on the external forcing (f_1, f_2), the Reynolds numbers $R_{\max}$, $R_{\min}$ and the Rossby number Ro (see (1.40) for the definition of Ro). The Rossby number represents a measure of the significant influence of the rotation of the earth on the dynamics (long-term behaviour and climate changes) of the atmosphere.

(*iii*) Since, in the equations, there is no time derivative term $\partial w(v)/\partial t$, we are not able to obtain any estimate for $|w(v)|_{L^2}$ from the norm of H. Therefore, in (6.19), we cannot use the Sobolev–Lieb–Thirring-type inequality to improve the estimates of the dimensions of the attractors.

Lemma 6.1. We have

$$\sum_{j=1}^m \|\psi_j\|^2\geqslant Cm^{5/3}. \tag{6.28}$$

Proof. Applying the generalized Sobolev–Lieb–Thirring inequality (theorem 4.1 of [4, p 466]), we obtain

$$\int_M\left(\sum_{j=1}^m (|v_j|^2+a_1|T_j|^2)\right)^{5/3}\mathrm{d}M\leqslant C\sum_{j=1}^m\|\psi_j\|^2. \tag{6.29}$$

On the other hand,

$$m=\sum_{j=1}^m|\psi_j|_H^2\leqslant C\left[\int_M\left(\sum_{j=1}^m(|v_j|^2+a_1|T_j|^2)\right)^{5/3}\mathrm{d}M\right]^{3/5}.$$

Equation (6.28) then follows. □

† In the case where (6.4*a*) holds, we also have that the volume element $|U_1(t)\wedge\ldots\wedge U_m(t)|_{\wedge^m H}$ decays exponentially as $t\to\infty$ in the phase space.

7. Global attractors for the PEV^{2}s

In this section we consider the PEs and the PEV^{2}s by adding to the equations some enhanced viscosity terms. First, corresponding to problem 2.1 (the weak formulation of PEs), we study briefly the following Cauchy problem obtained from (2.65*b*) with the enhanced viscosity term $\varepsilon A^r \Xi$:

$$\begin{aligned}&\Xi_t + \varepsilon A^m \Xi + A\Xi + B(\Xi, \Xi) + B(\Xi, \Xi^*) + E(\Xi) = F\\ &\Xi\big|_{t=0} = \Xi_0 \in H \qquad m > 1,\ \varepsilon > 0.\end{aligned} \tag{7.1}$$

As $\varepsilon \to 0$, (7.1) reduces to problem 2.1. The main result in this section is the following theorem.

Theorem 7.1. Assume $m \geqslant \frac{7}{2}$, then:

(*i*) There is a unique global solution Ξ of (7.1) satisfying

$$\Xi \in L^2(0, \tau; V \cap H^m(TM)) \cap C([0, \tau]; H) \qquad \forall \tau > 0. \tag{7.2}$$

Moreover, for any $t \geqslant 0$, the mapping $S^m(t): H \to H$, defined by $S^m(t)\Xi_0 = \Xi(t)$, the solution of (7.1) with initial value Ξ_0, is continuous.

(*ii*) If $\Xi_0 \in D(A^{m/2}) \subset V \cap H^m(TM)$, then the unique solution of (7.1) obtained in part (*i*) satisfies

$$\Xi \in L^2(0, \tau; V \cap H^{2m}(TM)) \cap C([0, \tau]; V \cap H^m(TM)) \qquad \forall \tau > 0. \tag{7.3}$$

(*iii*) There is a ball B_{ρ_0} with radius ρ_0 in H such that for any ball $B_R \subset H$ with radius R there is a time $t_0 = t_0(R) > 0$ satisfying

$$S^m(t)B_R \subset B_{\rho_0} \qquad \forall t \geqslant t_0. \tag{7.4}$$

B_{ρ_0} is called an absorbing ball in H. Similarly, there is an absorbing ball B_{ρ_1} in $D(A^{m/2})$. Moreover, the ω-limit set of B_{ρ_0},

$$\mathscr{A} = \bigcap_{\tau \geqslant 0} \bigcup_{t \geqslant \tau} S^m(t)B_{\rho_0} \subset D(A^{m/2}) \tag{7.5}$$

is a compact, connected global attractor of the semigroup $S^m(\cdot)(t \geqslant 0)$ in H.

(*iv*) $\mathscr{A}$ possesses finite Hausdorff and fractal dimensions.

Sketch of the proof. First, we claim that if m satisfies (7.2) then

$$B(\Xi, \Xi) \in L^2(0, \tau; H) \qquad \forall \tau > 0. \tag{7.6}$$

Indeed, by (2.40), we have

$$\begin{aligned}|B(\Xi, \Xi)|_H &\leqslant C\, \|\Xi\|_{H^2} \|\Xi\|_{H^{3/2}}\\ &\leqslant C\, \|\Xi\|_{H^m}^{7/2m} |\Xi|_H^{(4m-7)/2m}\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^\tau |B(\Xi, \Xi)|_H^2 &\leqslant C\, \|\Xi\|_{H^m}^{7/2m} |\Xi|_H^{(4m-7)/2m}\, d\tau\\ &\leqslant C \int_0^\tau (\|\Xi\|_{H^m}^2 + |\Xi|_H^{2(4m-7)/(2m-7)})\, d\tau < \infty.\end{aligned}$$

Thus, (7.6) follows.

The theorem can then be proved by the same procedure as that used for the similar results of the 2D Navier–Stokes equations (see [4]), and we omit the details. □

Remark 7.1. Consider the following Cauchy problem obtained from problem 4.1 (for the PEV^{2}s) by adding the same enhanced viscosity term $\varepsilon A^m \Xi$ to (4.21):

$$\begin{aligned} &\Xi_t + \varepsilon A^m\Xi + A^w\Xi + B(\Xi, \Xi) + B(\Xi, \Xi^*) + E(\Xi) = F \\ &\Xi|_{t=0} = \Xi_0 \in H \qquad m > 1 \qquad \varepsilon > 0. \end{aligned} \tag{7.7}$$

Then theorem 7.1 is true for this problem. Moreover, problem 7.7 will reduce to problem 4.1 as $\varepsilon \to 0$.

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Appendix. Principal notation

Functions

$f = 2\cos\theta$	Coriolis parameter
V_3	the 3D velocity
v	horizontal velocity
$\omega = \mathrm{d}p/\mathrm{d}t$	vertical velocity in p-coordinate system
T	temperature
$\Phi = gz$	geopotential
p	pressure
ρ	density
$\bar{T} = \bar{T}(p)$	average vertical distribution of the temperature
T_s	temperature on the surface of the earth
ϵ	diabatic heating

Constants and parameters

a	radius of the earth
R	gas constants for dry air
c_p	specific heat of dry air at constant pressure
$c_v = c_p - R$	
g	acceleration due to gravity
$U > 0$	standard wind velocity
$P > 0$	standard atmosphere pressure
$\bar{T}_0$	reference value of the temperature of T
$\Omega > 0$	angular velocity of the earth
$\mu_i, \nu_i (i = 1, 2)$	viscosity coefficients

Others

k	vertical unit vector
S^2	2D unit sphere
S_a^2	2D sphere of radius a
Δ	Laplacian on S^2 or S_a^2
grad	gradient on S^2 or S_a^2
grad_3	gradient in the 3D atmosphere

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