$$(X_1, \dots, X_n) \overset{d}{=} (X_{1+h}, \dots, X_{n+h}), \quad \forall h \in \mathbb{Z}, \ \forall n$$

$\{X_t\}$ stationary

X_t weakly stationary $\begin{cases} E(X_t) = \text{const (indep of } t) \\ \mathrm{Cov}(X_t, X_{t+h}) = \gamma_X(h) \end{cases}$ ← autocovariance fn ACVF

ACF

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

AR

$$X_t = \alpha X_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2)$$

AR(p)

$$X_t = \sum_{i=1}^{p} \alpha_i X_{t-i} + \epsilon_t \iff \left(I - \sum_{i=1}^{p} \alpha_i B^i\right) X_t = \epsilon_t \iff \phi_\alpha(B) = \epsilon_t$$

$$V(X_t) = \sigma^2 + \alpha^2 \sigma^2 + \dots + \alpha^{2(k-1)} \sigma^2 + \sigma^{2k} \mathrm{Var}(X_{t-k})$$

sum converges if $|\alpha| < 1$ AR(1)

AR(p)

MA

$$X_t = \epsilon_t + \beta \epsilon_{t-1}$$

MA(q) $X_t = \epsilon_t + \sum_{j=1}^{q} \beta_j \epsilon_{t-j}$

$$\iff X_t = \left(I + \sum_{j=1}^{q} \beta_j B^j\right) \epsilon_t \iff X_t = \phi_\beta(B) \epsilon_t$$

$$V(X_t) = \sigma^2 + \sum_{j=1}^{q} \beta_j^2 \sigma^2$$

Backshift operator (B)

$$B X_t = X_{t-1}$$

$$B^2 X_t = B(BX_t) = X_{t-2}$$

MA(1) $\quad X_t = \epsilon_t + \beta \epsilon_{t-1}$

$$\mathrm{Var}(X_0) = r_0 = (1+\beta^2)\sigma^2$$

ACVF $\quad r_1 = \mathrm{Cov}(X_0, X_1) = \beta\sigma^2$

$$r_k = 0 \quad , k \geq 2$$

ACF $\quad \rho_0 = 1 , \ \rho_1 = \dfrac{\beta}{1+\beta^2} , \ \rho_k = 0, \ k \geq 2$

Differencing operator ∇

$$\nabla X_t = X_t - X_{t-1}$$

$$\nabla^2 X_t = \nabla(\nabla X_t) = \nabla(X_t - X_{t-1})$$

$$= X_t - 2X_{t-1} + X_{t-2}$$

MA(q)

ACVF $\quad r_k = \begin{cases} \sigma^2 \sum\limits_{i=0}^{q-|k|} \beta_i \beta_{i+|k|} & , |k| \leq q \\ 0 & , |k| > q \end{cases}$

ACF $\quad \rho_k = \dfrac{r_k}{r_0}$

Plot of ACF should show a sharp drop to near zero after the q^{th} coefficient.

AR(p):

AR(p):

$$X_t = \sum_{i=1}^{p} \alpha_i X_{t-i} + \epsilon_t$$

$E(X_t) = 0$

$$r_k = \sum_{i=1}^{p} \alpha_i r_{k-i}, \quad k > 0.$$

$$\rho_k = \sum_{i=1}^{p} \alpha_i \rho_{k-i}, \quad k > 0$$

Yule-Walker equations

AR(p) process has ρ_k decaying smoothly as k increases, which can be difficult to recognise in a plot of ACF.

PACF

$$\rho_k = \sum_{i=1}^{p} \alpha_i \rho_{|k-i|}, \quad k = 1, \ldots, p$$

$$r_k = \sum_{i=1}^{p} a_{i,p}\, r_{|k-i|}, \quad k = 1, \ldots, p$$

p^{th} PACF is a_{pp}

$k = 0$; $a_{00} = 0$, $a_{11} = \rho(1)$

$$a_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1} a_{j,k-1}\, \rho_{k-j}}{1 - \sum_{j=1}^{k-1} a_{j,k-1}\, \rho_j}$$

$$a_{j,k} = a_{j,k-1} - a_{k,k}\, a_{k-j,k-1}, \quad j = 1, \ldots, k-1$$

Levinson – Durbin recursion.

$a_{k,k}$ kth sample PACF

Gaussian process

$$a_{k,k} = \mathrm{corr}(X_t, X_{t-k} \mid X_{t-1}, \ldots, X_{t-k+1})$$

If a process X_t is genuinely AR(p) process

then $a_{k,k} = 0$ for $k > p$.

—— x ——

→ sample ACVF

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \quad -n < h < n$$

→ sample ACF

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \quad -n < h < n$$