Random Numbers :

Pseudorandom Number (PRN) generation :

Seed x_0 $\quad x_{n+1} = a x_n \bmod m$

$a, m \geq 0$ $\quad m \overline{)\, a x_{n-1}\,(}\ 1$, $\quad x_n \leftarrow R$

$x_n = 0, 1, \ldots, m-1$

eg $X_i = (5 X_{i-1} + 1) \bmod 8$

$X_0 = 0$

$\rightarrow 1, 6, 7, 4, 5, 2, 3, 0$ $\quad 8 \overline{)\,1\,(}\ 0$, $\quad 1 = X_1$

random numbers $\frac{1}{8}, \frac{6}{8}, \frac{7}{8}, \ldots$

32 bit $\quad m = 2^{31} - 1, \quad a = 7^5$

36 bit $\quad m = 2^{35} - 31, \quad a = 5^5$

Using random numbers to evaluate integrals

$$\rightarrow \quad \theta = \int_0^1 g(x)\,dx = \boxed{E[g(U)]}$$

$U \sim U(0,1)$

$$f_U(u) = \begin{cases} 1, & 0 < u < 1 \\ 0 & \text{o.w.} \end{cases}$$

$U_1, \ldots, U_k$ indep $U(0,1)$ r.v.s

$g(U_1), \ldots, g(U_k)$ indep r.v.s

SLLN

$$\frac{1}{k} \sum^{k} g(U_i) \rightarrow E[g(U)] = \theta$$

$k \quad i=1$

$\rightarrow \quad \theta = \int_a^b g(x)\,dx = \int_0^1 \underbrace{(b-a)\,g(a+(b-a)y)}_{h(y)}\,dy$

$\left[\; y = \dfrac{x-a}{b-a} \quad ; \quad dy = \dfrac{dx}{b-a} \right.$

$= \int_0^1 h(y)\,dy = E[h(U)]$

$\rightarrow \quad \theta = \int_0^\infty g(x)\,dx$

$y = \dfrac{1}{x+1} \quad , \quad dy = -\dfrac{dx}{(1+x)^2} = -y^2\,dx$

$\theta = \int_0^1 \underbrace{\dfrac{1}{y^2}\, g\left(\dfrac{1}{y}-1\right)}_{h(y)}\,dy = E(h(U))$

$\rightarrow \quad \theta = \int_0^1 \int_0^1 \cdots \int_0^1 g(x_1, \ldots, x_n)\,dx_1 \ldots dx_n$

$= E[g(U_1, \ldots, U_n)]$, where

$U_1, \ldots, U_n$ indep $U(0,1)$

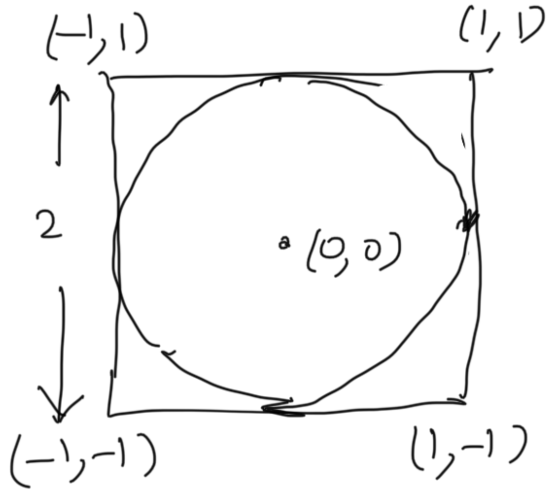
uni

indep $\left\{\begin{array}{ll} U_1^{(1)}, \ldots, U_n^{(1)} & g(U_1^{(1)}, \ldots, U_n^{(1)}) \\ U_1^{(2)}, \ldots, U_n^{(2)} & g(U_1^{(2)}, \ldots, U_n^{(2)}) \\ \vdots & \vdots \\ U_1^{(k)}, \ldots, U_n^{(k)} & g(U_1^{(k)}, \ldots, U_n^{(k)}) \end{array}\right.$

$$U_1, \ldots, U_n \quad g(U_1, \ldots, U_n)$$

$$\frac{1}{k}\sum_{i=1}^{k} g(U_1^{(i)}, \ldots, U_n^{(i)}) \text{ estimate } \theta.$$

<u>Example</u> estimation of π



$P((X,Y) \text{ is in circle})$

$$= P(X^2+Y^2 \le 1)$$

$$= \frac{\text{area of circle}}{\text{area of square}}$$

$$= \frac{\pi \times 1^2}{2 \times 2} = \frac{\pi}{4}$$

$$U \sim U(0,1) \Rightarrow 2U-1 \sim U(-1,1)$$

generate U_1 and U_2

$$X = 2U_1 - 1$$

$$Y = 2U_2 - 1$$

$$I = \begin{cases} 1 & \text{if } X^2+Y^2 \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(I) = P(X^2+Y^2 \le 1) = \frac{\pi}{4}$$

$$(2u_1-1)^2 + (2u_2-1)^2 \le 1 \leftarrow u_1, u_2 \qquad \frac{800}{1000} = \frac{\pi}{4}$$

1000

Probability Integral Transform

$$E\,g(X), \qquad X \sim F_X(x)$$

Theorem: Let X be a continuous r.v. with cdf $F_X(x)$. Define $Y = \boxed{F_X(X)}$. Then $Y \sim U[0,1]$.

Proof $G_Y(y) = P(Y \le y)$

$$= P(F_X(X) \le y)$$

$$= P(X \le F_X^{-1}(y)) =$$

$$= F_X F_X^{-1}(y) = y$$

$$G_Y(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

So pdf $g_Y(y) = 1, \quad 0 \le y \le 1$

$\quad \left\{ 0, \quad \text{otherwise.} \right.$

So $\quad Y = F_X(X) \sim U[0,1]$.

Inverse Prob. Integral Transform

Let $U \sim U[0,1]$.

and F be the cdf of a continuous r.v.

Then $X = F^{-1}(U)$ has

a cdf $F(x)$.

Example: Let $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x}$$

$$y = F(x) = 1 - e^{-\lambda x}, \quad x > 0$$

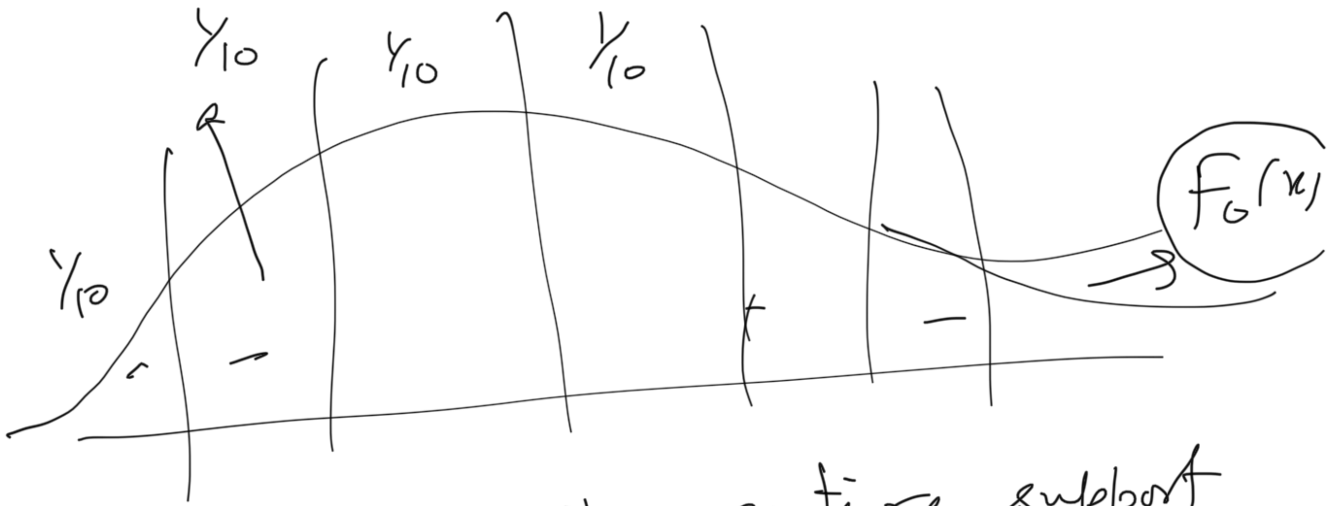
$$x = \boxed{-\frac{1}{\lambda} \log(1-y)}$$

Chi-square Test for Goodness of fit

We want to test if

$$H_0: F(x) = F_0(x) \quad \forall x$$

$$H_1: F(x) \neq F_0(x) \text{ at least for some } x$$



We divide the entire support of the distⁿ in k parts. For convenience we call these $I_1, \ldots, I_k$ and choose these such that $P(I_r) = \frac{1}{k}$, $r = 1 \ldots k$

Let us generate N number of observations from the distn.

Then define expected frequency of r^{th} interval I_r as

$$e_r = NP(X \in I_r), \quad r = 1 \ldots k.$$

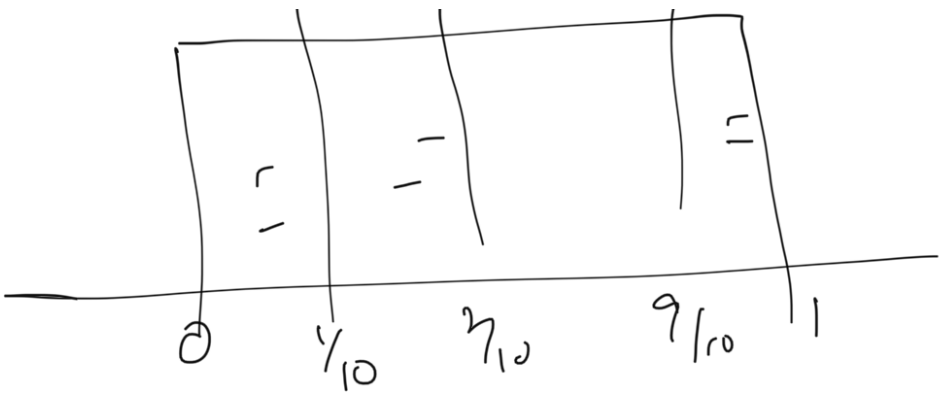
The observed frequency O_r of r^{th} interval is noted from the generated data.

$$W = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i} = \sum_{i=1}^{k} \frac{O_i^2}{e_i} - N$$

If each cell frequency is more than 5, then $W \sim \chi^2_{k-1}$.

So to test H_0 against H_1, the Chi square test is to

Reject H_0 if $W > \chi^2_{k-1, \alpha}$



$$F_0(x_1) = \frac{1}{10}, \qquad F(x_2) - F(x_1) = \frac{1}{10}$$

$$I_1 = (-\infty, x_1)$$

$$I_2 = (x_1, x_2)$$

$$\vdots$$

$$I_k = (x_{k-1}, \infty)$$