An american call is an option entitling the holder to buy a block of share ("exercise the option") of a given company at a stated price at any time during a stated time interval.

$\exists$ various type of call options

European (or vanilla) call that can only be exercised at the terminal point of the time interval (at the options maturity time)

→ invester hopes that the price of the stock he/she wants to buy may drop in the near future, but is not sure

Price $\downarrow$ ⇒ buy and ignore the option

Price $\uparrow$ ⇒ exercise the option

→ Speculator expects a sharp price rise to occur soon, but is not sure.

Price $\downarrow$ ⇒ does not exercise the option

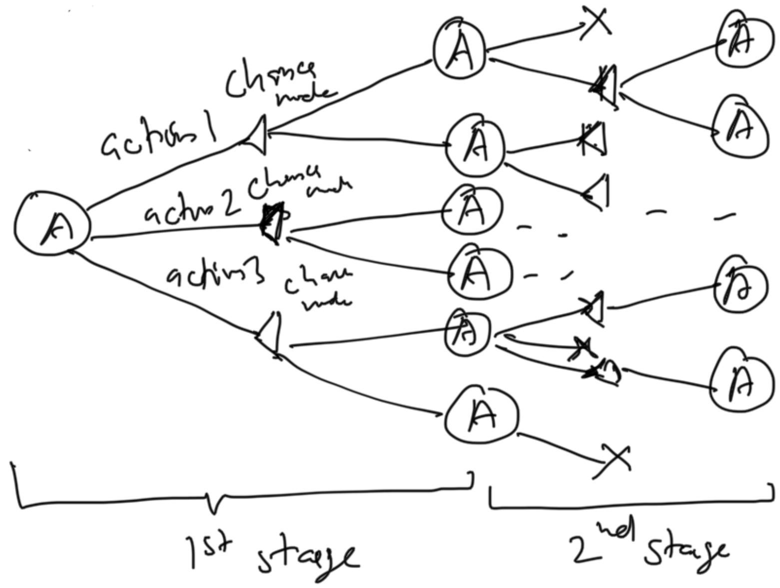
Price $\uparrow$ ⇒ exercise the option and resell the stock.

American derivative security (ADS)

Any adapted SP X_t, $t=0,1,\dots,T$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called an A.D.S.

Markov Decision process: We assume that we have a process describing a system evolving in a discrete time and at each step ...

time, and at each step, one is required to take an action



A decision tree

At each action node, we have to base our decision on the information about the evolution of the system upto that node only.
The task is to choose a sequence of actions optimising a given objective fn.

action $a \in A$

If $X_t = i$, an action $a \in A$ is chosen (based on the observed values of the process at time $\leq t$)

A reward fn $R(i,a)$, i state, a action

A policy $\{a_t\}$: which is a rule for choosing action at the respective times: at time t, the policy prescribes to take the action a_t

If the policy is stationary (one's action at time t depends on X_t only: $a_t = f(i)$ given $X_t = i$), then (X_t) is a time-homogeneous M.C. with transition probabilities $p_{ij}(f(i))$, and the process is called a Markov decision process

Objective

$$E\left(\sum_{t=1}^{T} R(X_t, a_t)\right) \to \max_{\{a_1, \dots, a_T\}}$$

$$E\left(\sum_{t=1}^{T} R(X_t, a_t) \,\Big|\, X_1 = i\right) \to \max_{(a_1, \dots, a_T)} =: V_T(i)$$

dynamic programming tech

and then compute $E(V_T(i))$

problem: find $V_n(i)$ for $n = T$ and the optimal policy for which this value is attained.

$$V_n(i) = \max_a E\left(\sum_{t=1}^{T} R(X_t, a_t) \,\Big|\, X_1 = i\right)$$

$$= \max_a \left(R(i,a) + E_a\left(\sum_{t=2}^{n} R(X_t, a_t) \,\Big|\, X_1 = i\right)\right)$$

$$= \max_a \left(R(i,a) + E_a\left(E\left[\sum_{t=2}^{n} R(X_t, a_t) \,\Big|\, X_2\right) \,\Big|\, X_1 = i\right)\right)$$

$$= \max_a \left(R(i,a) + E_a\left(V_{n-1}(X_2) \,|\, X_1 = i\right)\right)$$

max (R(...

$$= \max_a \left(R(i,a) + \sum_j V_{n-1}(j)\, \underbrace{P(X_2=j \mid X_1=i)} \right)$$

$$= \max_a \Big(\underbrace{R(i,a)}_{\text{immediate reward}} + \underbrace{\sum_j p_{ij}(a)\, V_{n-1}(j)}_{\text{max exp. future reward}} \Big) \longrightarrow (2)$$

<u>Example</u>: (Selling a house) A person moving overseas has to sell her house urgently. Three different buyers are going to offer her, one after another, their prices, which are belived to be i.i.d. r.v.'s Z_j, $j=1,2,3$, with prob.

$P(Z_j = 100) = 0.3$, $P(Z_j = 110) = 0.5$, $P(Z_j = 120) = 0.2$

(Z_j' are given in thousand dollars). If the seller rejects an offer, then offer is lost.

The seller aims to max. the the expected price. The problem is to derive the optimal policy for selling the house and find the max. expected value of selling price.

$$X_t = \begin{cases} Z_t & \text{if house not sold yet} \\ 0 & \text{o.w.} \end{cases}, \quad t = 1,2,3 (=T)$$

$i,j \in S$
$a \in A = \{0,1\}$

$X_t \in \{0, 100, 110, 120\} = S$

$p_{ij}(a)$ $A \to \begin{cases} a=1 & \text{selling} \\ a=0 & \text{doing nothing / not selling} \end{cases}$

$p_{x0}(1) = 1$, $p_{00}(a) = 1$, $p_{xy}(0) = P(Z_j = y)$; $x \in \{100, 110, 120\}$

$R(x,1) = x, \quad R(x,0) = 0$

Total additive Reward

$$\sum_{t=1}^{3} R(X_t, a_t)$$

$V_0(x) := 0$ means if all the buyer's offers have already been refused, one can gain nothing

$x > 0$

$$V_1(x) = \max_a R(x,a) = x \quad ; \quad V_1(0) = 0$$

One has not sold the house to the first two buyers, the property should be sold to the last one, in that case $a_3 = 1$ whatever the price Z_3.

$$E(V_1(Z_3)) = E(Z_3) = 100 \times 0.3 + 110 \times 0.5 + 120 \times 0.2$$
$$= 109$$

$$E_a\left(V_1(X_3) \mid X_2 = x\right) = \begin{cases} E(V_1(Z_3)) = 109 \; ; a = 0 \text{ is already not sold in } 2^{nd} \text{ step} \\ E(V_1(0)) = 0 \; ; a = 1 \rightarrow \text{is already sold in } 2^{nd} \text{ step} \end{cases}$$

$$V_2(x) = \max_a \left[R(x,a) + E_a\left(V_1(X_3) \mid X_2 = x\right)\right]$$

$$= \max\{x, 109\}$$

Hence the optimal action when the 2^{nd} offer has been made to sell when $x > 109$ and wait otherwise

i.e. $a_2 = 1$ if $X_2 = 110$ or 120 and $a_2 = 0$ if $X_2 = 100$

$$E(V_2(Z_2)) = E(\max\{Z_2, 109\})$$

$$= 109 \times 0.3 + 110 \times 0.5 + 120 \times 0.2 = 111.7$$

$= 104 \times 0.3 + 110 \times 0.5 + 120 \times 0.2 = 111.7$

It remains to find the optimal action at time 1 and the fn $V_3(x)$

$X_1 = Z_1 > 0$

$$V_3(x) = \max_a \left[R(x,a) + E_a\left(V_2(X_2) \mid X_1 = x\right)\right]$$

$$= \max \{ x, 111.7 \}$$

So the optimal action is $a_1 = 1$ if $X_1 > 111.7$ (i.e. price is 120) and wait otherwise

$$E(V_3(X_1)) = E\left(\max(Z_1, 111.7)\right)$$

$$= 111.7 \times (0.3 + 0.5) + 120 \times 0.2 = 113.36$$

<u>Example</u> : <u>An american call option model</u>

X_t : price of a given stock on the n^{th} day.

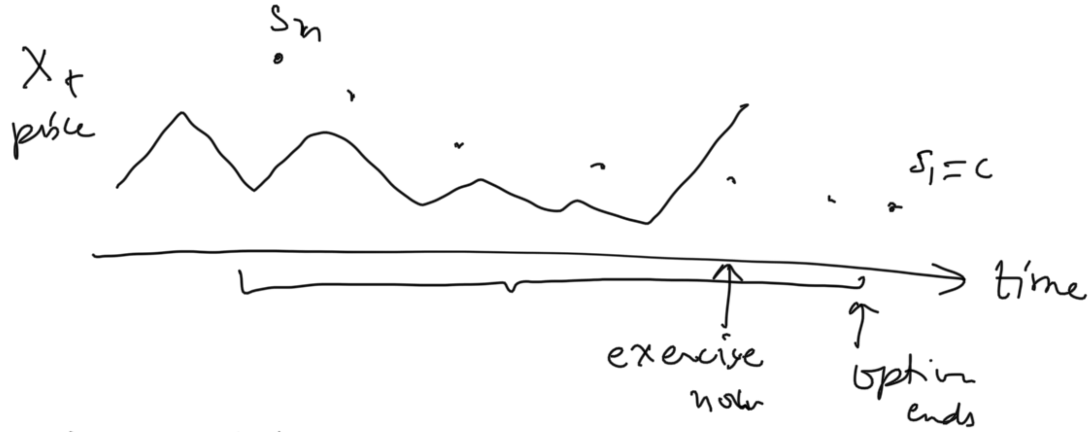
Assume that the dynamics of the price are given by the simple (absolute) random walk model

$$\rightarrow \quad X_{t+1} = X_t + Y_{t+1} = X_0 + \sum_{j=1}^{t+1} Y_j \longrightarrow (3)$$

where Y_j's i.i.d. r.v's with common d.f. F and having finite mean $\mu = E(Y_1)$

→ The optimal policy has the following form: there are increasing numbers $c =: s_1 \le s_2 \le \dots \le s_T$ s.t.

if there are n days to go, then one should exercise the option if the present price $\geq s_n$.



X_t state space $\mathbb{R}$

$X_t = -\infty$, when we exercise the option

$\mathbb{R} \cup \{-\infty\}$

On each day

$a = 1 \to$ exercise the option $\quad A = \{0, 1\}$

$a = 0 \to$ do not " ...

Reward fn

$$R(s, a) = \begin{cases} 0 & \text{if } a = 0 \\ s - c & \text{if } a = 1 \end{cases}$$

$$V_n(s) = \max_{a \in (0,1)} \left[R(s,a) + E_a\left(V_{n-1}(X_2) \mid X_1 = s\right) \right]$$

$$= \max \{ \underbrace{R(s,0)}_{=0} + E(V_{n-1}(s + Y_1)), R(s,1) + 0 \}$$

$$= \max \{ \underbrace{E(V_{n-1}(s + Y_1))}, \underbrace{s - c} \} \longrightarrow ✡$$

when a=0 when a=1

$$V_0(s) = 0$$

$$V_1(s) = \max\{ s-c, 0\} \quad ✓$$

If there are n days to go and the current price is s, then we do not exercise the option (see ✡)

$V_n(s) > s-c$ (the term corresponding to $a=1$ is not max).

$$\equiv V_n(s) - s > -c \longrightarrow \text{S}$$

LHS of S is ↓ (non-increasing) fn of s

$$V_1(s) - s \equiv \max\{-c, -s\}$$

assume true for n−1, try to show for n

$$V_n(s) - s = \max\{ E[V_{n-1}(s+Y_1) - s, -c\} \quad \text{| using ✡}$$

$$= \max\{ \underbrace{E[V_{n-1}(s+Y_1) - (s+Y_1)]}_{\downarrow \text{ in } s} + \mu, -c\}$$

$V_n(s) - s$ ↓ in s.