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Mathematical Analysis :

Continous Function :

A function f is continous at a point, $c \in \mathbb{R}$ (domain) if for each $\epsilon > 0$, there exist a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ when $|x - c| < \delta$.

$d(f(x), f(c)) < \epsilon$ whenever $d(x, c) < \delta$ domain $x \in \mathbb{R}$.

Example: Show that $f(x) = x^2$ is continous at $x = 1$

$x^2 > 0$.

Sol: Choose $\epsilon > 0$, $c = 1$.

$\Rightarrow$ let $\epsilon > 0$ and $c \in D$ and we need to search for $\delta = f(\epsilon)$

$|f(x) - f(c)| < \epsilon$

$\Rightarrow |x^2 - 1| < \epsilon$

$\Rightarrow |(x+1)(x-1)| < \epsilon$ in general could be d

Now if $|x+1| \leq 10$ then $|(x+1)(x-1)| \leq 10|x-1| < \epsilon$

$\Rightarrow 10|x-1| < \epsilon$

$\Rightarrow |x-1| < \frac{\epsilon}{10} = \delta$

$|x-1| < \frac{\epsilon}{|x+1|} < \frac{\epsilon}{c}$

$|x^2 - 1| < \epsilon$

$|(x+1)(x-1)| < \epsilon$

Therefore for each $\epsilon > 0$, there exists a $\delta > 0$ s/t $|f(x) - f(1)| < \epsilon$ when $|x-1| < \delta$

$\therefore f(x) = x^2$ is continous at $x = 1$.

Ordered Set :

well set

here comp. elements can be done

Let S be a set. An order on S is a relation denoted by '$<$' with the following properties.

① If $x, y \in S$ then one & only & the statements is true: $x < y$, $y < x$ or $x = y$.

② Let $x, y, z \in S$ then if $x < y$ & $y < z$ then $x < z$.

So an ordered set is an nonempty set on which an order is defined.

Notn.

$x < y$: x is less than y.

$x \leq y$: x is less than or equal to y.

This is NOT E.

Upper bound :

Let S be a non.empty ordered set (elements are comparable) and let $E \subseteq S$. If there exist $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, then we say that E is bounded above and call β an upper bound of E.

Least Upper Bound:

Let S be an non empty ordered set, $E \subseteq S$ and E is bounded above.

S not E. (by def^n)

Suppose there exists an $\boxed{\alpha \in S}$ with the following property
① α is an upperbound of E. ($\alpha \in S$ and $x \leq \alpha \quad \forall x \in E$)
② If $\gamma < \alpha$, then γ is NOT an upperbound of E $(\gamma \in S)$

Then $\alpha \rightarrow$ least-upper-bound or supremum of E in S.
$\boxed{\alpha = \sup E}$

$\Rightarrow$ If supremum exists it has to be unique.

Ordered Field:

Let $(F, +, \cdot)$ be a field. We say F is an ordered field if : it this is an ordered set.
① If $y < z \Rightarrow x + y < x + z \quad \forall x, y, z \in F$.
② ~~$xy > 0$ if $x > 0$,~~ $y > 0$. If $x > 0$ then & $x < y$ then $xz < yz \quad \forall x, y, z \in F$

Theorem-1: There exists an ordered field $\mathbb{R}$, which has the least upper bound p. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a sub field. Members of $\mathbb{R}$ are called Real Numbers

Archimedian Property: If $x, y \in \mathbb{R}$, $x > 0$ then there exists a positive integer such that $nx > y$

Proof:

It is NOT true & hence $\forall n$, $nx \leq y$ & $x > 0$.
let $E = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$

Then clearly y is an upper bound of E.
Now since $E \subseteq \mathbb{R}$ which has the least upper-bound property, hence there exists some $\alpha \in \mathbb{R}$ such that $\boxed{\alpha = \sup E}$
Then $nx \leq \alpha \quad \forall n \in \mathbb{N}$
Then $\alpha - x$ is NOT an upper bound.
This means there exists some $n_0 \in \mathbb{N}$ such that $\boxed{n_0 x > \alpha - x}$
$\Rightarrow (n_0 + 1)x > \alpha$. which is a contradiction.

Theorem : If $a_1, a_2, \ldots a_n \in \mathbb{R}$ and $b_1, b_2 \ldots b_n \in \mathbb{R}$ then :

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$$\boxed{\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$$ Cauchy-Schwartz Inequality.

$\Sigma(a_i + t b_i)^2 \geq 0.$

$\Rightarrow D \leq 0$

Euclidean Space :

$\mathbb{R}^k$ · Set of ordered k-by tuples =

eg: $x = (x_1, x_2, \dots x_k) \in \mathbb{R}^k$

$x_i \in \mathbb{R} \Rightarrow$ co-ordinate of x.

The elements of $\mathbb{R}^k$ are called points.

① $y, x \in \mathbb{R}^k$ then $x+y = (x_1+y_1, x_2+y_2 \dots x_k+y_k) \in \mathbb{R}^k$.

② If $c \in \mathbb{R}$ then $cx = (cx_1, cx_2 \dots cx_k) \in \mathbb{R}^k$

We define inner product : $\langle x.y \rangle = \sum_{i=1}^{k} x_i y_i$

norm of x : $|x| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$

So. $\mathbb{R}^k$ with inner product & norm = Euclidean Space.

Theorem: let $x, y, z \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$.

① $|x| \geq 0$

② $|x| = 0 \iff x = 0$

③ $|\alpha x| = |\alpha| . |x|$

④ $|x.y| \leq |x| . |y|$

⑤ $|x+y| \leq |x| + |y|$

⑥ $|x-z| \leq |x-y| + |y-z|$) Add subtract y & apply Δ- inequality.

Sequence: A Sequence is a function $f: \mathbb{N} \to \mathbb{R}$.

If $f(n) = x_n \ \forall n \in \mathbb{N}$. Then we denote the sequence by $\{x_n\}$

converging point

Convergence: A Sequence is said to converge if there exists $x \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that,

$$\boxed{|x_n - x| < \epsilon \quad \text{for all } n \geq N.}$$

$d(x_n, x) < \epsilon \quad \forall n \geq N \Rightarrow x_n \in B(x, \epsilon) \ \forall n \geq N.$

Notation: $\boxed{x_n \to x \quad \text{as} \quad n \to \infty}$ or $\boxed{\lim_{n\to\infty} x_n = x.}$

Ex: $\{x_n\} = \left\{\frac{1}{n}\right\}$ Here limiting point is $x = 0$

Choose $\epsilon > 0$ Then $N = \left[\frac{1}{\epsilon}\right] + 1.$

$|x_n - x| = \left|\frac{1}{n} - 0\right| < \epsilon.$

Proof:

let $\epsilon > 0$ be given:

Then by Archimedian property $\exists$ $N, (N \in \mathbb{N})$ such that $\frac{1}{N} < \epsilon.$, we got the upper bound from Archimedian property.

Now of course when $n \geq N$ we have $\boxed{\frac{1}{n} \leq \frac{1}{N} < \epsilon.}$

Therefore for each +ve ϵ we can find $N \in \mathbb{N}$ s/t for $\forall$ $n \geq N$, $\left|\frac{1}{n} - 0\right| < \epsilon.$ [Hence proved]

Subsequence:

Consider a given sequence $\{x_n\}$

let $\{n_j\}$ be such that $n_1 < n_2 < n_3 \ldots$

then $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$.

Note: If $\lim_{n\to\infty} x_{n_j}$ exists, then we call it subsequential limit of $\{x_n\}$.

Cauchy Sequence:

A sequence $\{x_n\} \to$ Cauchy Sequence if for every $\epsilon > 0$, there exist $N \in$ such that $|x_n - x_m| < \epsilon$ $\forall$ $m, n \in \mathbb{N}$.

$n, m \geq N \in \mathbb{N}$

In $\mathbb{R}$, every Cauchy Sequence is convergent & vice-versa.

Sequence of function:

$f: \mathbb{N} \to C(\mathbb{R})$

We say $\{f_n\}$ converges pointwise if $\{f_n(x)\}$: Sequence of nos converges at each point x in the domain of $\{f_n\}$.

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Not^n: If $\{f_n\}$ converges & $f_n(x) \to f(x)$ for each x. Then $\lim_{n\to\infty} f_n(x) = f(x)$.

Example: $f_n(x) = \frac{x}{n}$, $x \in \mathbb{R}$ domain

Clearly. $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in \mathbb{R}$.

So $f(x) = 0$ is the limit of the sequence $\{f_n\}$. Pointwise Convergence def^n:

Def^n: for each $\epsilon > 0$, for each $x \in$ domain ~~for each x~~ there exists a $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$

Note that: $N = G(\epsilon, x)$. function of

Q) $f_n(x) = x^n \quad x \in [0,1]$

$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in [0,1]$

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$\therefore f_n(x)$ converges to $f(x)$ pointwise.

Uniform Convergence:

let $E \subset \mathbb{R}$ (E may or may not be closed interval)

let $f_n : E \to R$ for each $n \in \mathbb{N}$
(E ↳ domain)

Then we say $\{f_n(x)\}$ is uniformly convergent

If there exists $f : E \to \mathbb{R}$ such that
for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ (N depends on ϵ only) such that
$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$ for all $x \in E$.

Some use cases of uniform convergence:

① let $f_n(x) \to f(x)$ the ? true

for each $n \in \mathbb{N}$ $\lim_{t\to x_0} f_n(t) = f_n(x_0)$ and Prove this $\boxed{\lim_{t\to x_0} f(t) = \lim f(x_0)}$.

Proof:
$$\boxed{\lim_{t\to x_0} \lim_{n\to\infty} f_n(t) = \lim_{t\to x_0} f(t)}.$$

② $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$
$= \int_a^b f(x)\,dx$

Questions:

Ex 1: Find the limit of f_n where $f_n(x) = \frac{x^2}{(1+x^2)^n}$ $x \in \mathbb{R}$

Sol: $x = 0$: $f_n(x) = 0$ $\forall n$

So, $\lim_{n\to\infty} f_n(0) = 0 = f(x)$

If $x \neq 0$, $f_n(x) = \underbrace{x^2}_{\text{fix it}} \cdot \frac{1}{(1+x^2)^n} \to 0$ as $n \to \infty$.

So, limiting function: $\boxed{f(x) = 0}$

Ex 2: $f_n(x) = \frac{nx}{1+nx}$ $x \geq 0$

$x = 0$: $f_n(0) = 0$

$x \neq 0$ $f_n(x) \longrightarrow 1$ (L' Hospital Rule). or divide by n both sides.

So,

$\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \geq 0$

where $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$

Theorem: Cauchy's Criterion for Uniform Convergence:

The sequence of functions $\{f_n\}$ ~~depend~~ defined on E converges uniformly on E, if and if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ s/t $|f_n(x) - f_m(x)| < \epsilon$ $\forall x \in E$ $m, n \geq N$

Proof: To prove that the seq. is Cauchy.

$\boxed{\text{Necessary Part}}$ let's say $f_n \to f$ uniformly on E.

This means for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that.

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \forall x \in E.$$

Hence $|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

$\therefore |f_n(x) - f_m(x)| < \epsilon.$ $m, n \geq N_\epsilon.$

Sufficient Part: let for $\epsilon > 0$, $\exists\ N_\epsilon \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \epsilon \quad \forall\ x \in E$, $m, n \geq N_\epsilon$

We fix x.

Now $\{f_m\}$ is nothing but a Cauchy sequence.

We know that Cauchy Sequences are convergent. (in a complete space i.e. $\mathbb{R}$)

let $f_n \to f$ pointwise.

So. $|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq N_\epsilon$ for fixed x.

Now since our choice of x was arbitrary so,

$$|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq N_\epsilon \quad \forall\ x.$$

So $f_n \to f$ uniformly.

M_n-Test for convergence of sequences.

Theorem: Suppose $f_n \to f$ pointwise on E.
let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$

New def^n of uniform convergence.

Then $f_n \to f$ uniformly on E if and only if $\underline{M_n \to 0 \text{ as } n \to \infty}$.

Proof: let $M_n \to 0$ as $n \to \infty$, we need to prove that $f_n \to f$ uniformly.
for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that:

bi-directional proof.

$\iff |M_n - 0| < \epsilon \quad \forall\ n \geq N_\epsilon$. Def^n of $M_n \to 0$.

$\iff \sup_{x \in E} |f_n(x) - f(x)| < \epsilon$.

$\iff |f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq N_\epsilon \quad \forall\ x \in E$.

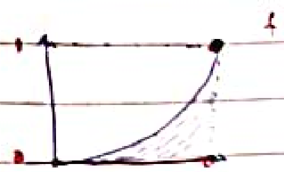
Uniformly Convergent.

Examples:

① $f_n(x) = x^n$, $x \in [0,1]$ ② $f_n = \frac{x}{n}$. Yes

find the limit of f_n & check if it is uniformly convergent.

Sol^n: $\lim_{n \to \infty} f_n(x) = f(x)$ where $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1. & x = 1 \end{cases}$



$M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)|$ ↳ $\frac{1}{n}$.

NOT uniformly convergent. We have $\sup|f_n(x) - f(x)| \geq \left(1 - \frac{1}{n}\right)^n$ Tends to non-zero limit as $n \to \infty$.

if a seq of functions $f_n(x)$ defined on E converges uniformly to a function $f(x)$ and if each $f_n(x)$ is continuous on E, then the limit function is also continuous.

<u>Uniform Convergence and Continuity</u>:

Theorem: $f_n \to f$ converges uniformly on E. let $x \in E$ be a limit point of E; suppose that $\lim_{t \to x} f_n(t) = A_n \quad \forall n$.

<u>Part (i)</u> Then $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n = A$.

Part i

Next part.

$\Rightarrow$ Prove $\{A_n\}$ is convergent

$\Rightarrow |f(t) - A| < \epsilon$

<u>Proof</u>: Acc to def^n for each $\epsilon > 0$, $\exists$ $N_\epsilon \in \mathbb{N}$ such that

$$|f_n(t) - f(t)| < \epsilon \quad \forall n \geq N_\epsilon \quad \forall t \in E.$$

Taking $t \to x$ — Cauchy Criterion

$$|f_n(t) - f_m(t)| < \epsilon \quad \forall m, n \geq N_\epsilon \quad \forall t \in E$$

$$\lim_{t \to x} |f_n(t) - f_m(t)| < \epsilon \quad \forall m, n \geq N_\epsilon \quad \forall t \in E.$$

$$|A_n - A_m| < \epsilon \quad \forall m, n \geq N_\epsilon \quad \forall t \in E.$$

So $\{A_n\}$ is a Cauchy Sequence and hence convergent.

Let $\lim_{n \to \infty} A_n = A$.

<u>~~Part (ii)~~</u> $\boxed{\text{If } f_n \text{ is continuous } \forall n \text{ then } f(x) \text{ is also continuous}}$

Now we have:

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \quad \text{—} (0)$$

Now we choose large n, let $\epsilon > 0$, $\exists$ N_ϵ and $\delta > 0$ such that:

$$|f(t) - f_n(t)| < \frac{\epsilon}{3} \quad \forall n \geq N_\epsilon \quad \forall t \in E \quad \text{—} (1)$$

$$|A_{N_\epsilon} - A| < \frac{\epsilon}{3} \quad \forall n \geq N_\epsilon \quad \text{—} (2)$$

and also, $|f_n(t) - A_n| < \frac{\epsilon}{3} \quad \forall \; 0 < |x - t| < \delta$ [from the def^n of continuity] —(3)

Putting (1), (2), (3) in (0)

$$|f(t) - A| < \epsilon \quad \forall \quad 0 < |x - t| < \delta$$

$$\therefore \lim_{t \to x} f(t) = \lim_{n \to \infty} A_n = A.$$

Ex $f_n(x) = x^n$, $x \in [0, 1]$

[Converse is NOT true] if we have a seq of continuous functions converging to continuous function pointwise then it doesn't imply uniform convergence.

$$\overset{\text{def}^n}{\longrightarrow} \quad \overline{\int_{a}^{b}} f(x)\,dx = \underline{\int_{a}^{b}} f(x)\,dx.$$

Uniform Convergence and Integrability:

Theorem: Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. Suppose $f_n \to f$ uniformly on $I = [a,b]$. Then f is also Riemann integrable and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

we can interchange $\int$ & $\lim$

will be proved from sup def^n

Proof

Let $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ — Using def^n of sup

$$|f_n(x) - M_n| \le |f(x)| \le |f_n(x) + M_n|$$

$$\Rightarrow \int_a^b (f_n(x) - M_n)\,dx \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} (f_n(x) + M_n)\,dx$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f_n(x)\,dx - \underline{\int_a^b} f_n(x)\,dx + 2M_n(b-a). \quad (1)$$

do $n\to\infty$

Clearly $M_n \to 0$ as $n \to \infty$.

$\Rightarrow$ Hence

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx \quad \text{or} \quad f \text{ is Riemann Integrable}$$

the integral both sides

from (1) as $n\to\infty$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le M_n(b-a). \quad \xrightarrow{n\to\infty}$$

Taking $n\to\infty$

we have
$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.}$$

Rudin's Book Pg. 165 (1-11)

Example: Check the uniform convergence of $f_n(x) = n^2(x)(1-x^2)^n$, $0 \le x \le 1$, using the above integrability.

Uniform Convergence and Differentiability:

Theorem: Suppose $\{f_n\}$ be a sequence of differentiable functions on $[a,b]$, such that $\{f_n(x_0)\}$ sequence converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$.

Then (1) $f_n \to f$ for some f uniformly on $[a,b]$

(2) $f'(x) = \lim_{n\to\infty} f_n'(x) \quad \forall x \in [a,b]$.

Proof: let $\epsilon > 0$ be given. We choose N s/t $n, m \geq N$ such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$$

and

$$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)} \qquad a \leq t \leq b$$

Applying MVT :

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x-t|\,\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}$$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$$\Rightarrow |f_n(x) - f_m(x)| < \epsilon \qquad \text{Proved } (1)$$

[Pg-153] (Ignore this part → we'll prove this later).

Series of functions :

let $\{f_n\}$ be a sequence of functions.

We say the series $\sum_{n=1}^{\infty} f_n$ converges to a function F if the partial sum,

$S_n(x) = \sum_{j=1}^{n} f_j(x)$ converges to $F(x)$ for each x.

Def^n:

$$\left[\sum_{n=1}^{\infty} f_n \longrightarrow f \text{ uniformly on } E \text{ if } \{S_n\} \longrightarrow f \text{ uniformly on } E\right]$$

Cauchy's Criteria :

$\sum f_n \longrightarrow f$ uniformly if and only if for each $\epsilon > 0$, $\exists N_\epsilon$ such that

$$|S_m(x) - S_n(x)| = \left|\sum_{j=n+1}^{m} f_j(x)\right| < \epsilon \qquad \forall n, m \geq N_\epsilon \quad \forall x \in$$

Weierstrass-M-Test (Sufficient Cond^n of uniform convergence of a series of functions)

Suppose $|f_n(x)| \leq M_n \quad \forall n \geq 1$ & $\forall x \in E$, also $\sum_{n=1}^{\infty} M_n$ converges ~~uniformly~~

Then :

The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

T_n Cauchy

then show $|S_n - S_m| \leq$

Proof: Check ipad notes for proof → just use def^n of convergence of seq.

Problems for Practice :

Q.1) let $\sum_{n=1}^{\infty} u_n \to u$ uniformly on $[a,b]$ Let v be a bounded function on $[a,b]$ then show that $\sum_{n=1}^{\infty} u_n v \to vu$ uniformly on $[a,b]$

using def^n

$$|S_n(x) - S_m(x)| < \frac{\epsilon}{c} \quad \forall\ n,m \geq N_\epsilon \ \forall x \quad \text{and} \quad |v| \leq c. \text{ bounded.}$$

$$\Rightarrow \left| \sum_{j=n+1}^{m} f_j(x) \right| < \frac{\epsilon}{c} \quad \forall\ n,m \geq N_\epsilon \ \forall x \in [a,b]$$

Since $|v(x)| \leq c \quad \forall\ x \in [a,b]$

We can have:

$$\left| \sum_{j=1}^{m} f_j(x) v(x) \right| < \epsilon \quad \forall\ n,m \geq N_\epsilon \ \forall x \in [A,B]$$

$\Rightarrow u_n v \to uv$ uniformly on $[a,b]$.

Q.2) $\sum_{n=1}^{\infty} a_n$ converges absolute. Show that $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $\mathbb{R}$

Using M_n test:

$$|a_n \sin nx| \leq |a_n| \ \forall x \ \ \& \ \ \sum_{n=1}^{\infty} |a_n| < \infty$$

$\therefore$ By M_n test: $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $\mathbb{R}$.

Result from M_n Test:

If $\{f_n\}$ is a sequence of $\mathbb{R}$-integrable functions on $[a,b]$ and $\sum_{n=1}^{\infty} f_n \to f$ uniformly on $[a,b]$ then,

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx \qquad \Sigma \ \& \ \lim \text{ can also be chang}$$

Q.3) Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^p (1+x^n)}$ converges uniformly $\forall\ p > 1$, $\forall\ x \in \mathbb{R}$.

$$\left| \frac{(-1)^{n+1} x^n}{n^p (1+x^n)} \right| \leq c \cdot \frac{1}{n^p} \ \forall x \quad \text{and we have } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent iff } p > 1$$

Hence by Weierstrass M_n Test $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^p (1+x^n)}$ converges uniformly $\forall x$

Q.4) Find $\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$ We need to prove $\sum f_n \to f$ then we can exchange Σ & lim.

$$\left| \frac{\cos nx}{n(n+1)} \right| \leq \frac{1}{n(n+1)} \text{ convergent}$$

now exchange Σ & lim.

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Q5) Let $f_n, g_n : E \to \mathbb{R}$. Let $f_n \to f$, $g_n \to g$ uniformly on E.
Show that $f_n + g_n \to f+g$ uniformly on E.

Proof: $f_n \to f$ uniformly $\Rightarrow \{f_n\}$ Cauchy

So for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s/t

$$|f_n - f_m| < \frac{\epsilon}{2} \quad \forall x \in E \quad \forall m,n \geq N.$$

same for g

$$|g_n - g_m| < \frac{\epsilon}{2} \quad \forall x \in E \quad \forall m,n \geq N_1$$

$N' = \max(N, N_1)$.

Now,

$$|(f_n+g_n) - (f_m+g_m)| = |(f_n - f_m) + (g_n - g_m)| \leq |f_n - f_m| + |g_n - g_m| < \epsilon \quad \forall n,m \geq N'$$

$\therefore \{f_n + g_n\}$ converges to $f+g$ uniformly.

take $f_n = g_n = x + \frac{1}{n}$ then $f = g = x$

Note that if $f_n \to f$ & $g_n \to g$ the $f_n g_n \not\to fg$ uniformly !! ($x^2 + \frac{2}{n}x + \frac{1}{n^2}$)

doesn't converge uniformly on R

Q6) $f_n(x) = x\left(1 + \frac{1}{n}\right)$, $x \in \mathbb{R}$.

(see after ordering) $g_n(x) = \begin{cases} \frac{1}{n} & \text{If } x = 0 \text{ or } x = \text{irrational.} \\ b + \frac{1}{n} & \text{If } x \text{ is rational where } x = \frac{a}{b},\ b > 0. \end{cases}$

Prove that $\{f_n\}$, $\{g_n\}$ converges uniformly on any $[a,b]$

Solⁿ

① $\{f_n\} \to$ Use M_n Test

$$\sup_{a \leq x \leq b} |f_n(x) - f(x)| = \sup_{a < x < b} \left|\cancel{x} + \frac{x}{n} - \cancel{x}\right| = \sup_{a < x < b} \left|\frac{x}{n}\right| \leq \frac{b}{n}$$

$M_n = \frac{b}{n}$ So as $n \to \infty$ $M_n \to 0$ hence $\{f_n\}$ converges uniformly [for

② Find $g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = \text{irrational} \\ b & \text{if } x \in \mathbb{Q} \text{ where } x = \frac{a}{b} \text{ \& } b > 0. \end{cases}$

$$\sup_{a < x < b} |g_n(x) - g(x)| = \left|\frac{1}{n} - 0 \text{ or } \frac{1}{n}\right| = \frac{1}{n} = M_n \text{ as } n \to \infty.\ \frac{1}{n} \to 0$$

↳ Take two cases for two diff. x.

So $\{g_n\}$ converges uniform

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Now $\{f_n g_n\} \xrightarrow{x \in \mathbb{Q}} x\left(1+\frac{1}{n}\right)\left(b+\frac{1}{n}\right) = bx + \frac{x}{n} + \frac{bx}{n} + \frac{x}{n^2}.$

$\xrightarrow{x \notin \mathbb{Q}} \frac{x}{n} + \frac{x}{n^2}.$

Check if this converges to fg uniformly.

Theorem: (Repeat) Suppose $\{f_n\}$ be a sequence of differentiable f.ns defined on $[a,b]$, such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'(x)\}$ converges uniformly on $[a,b]$ ~~to a function f and~~

Pg-152.

Then $\{f_n\}$ also converges uniformly on $[a,b]$ to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x)$ $\forall x \in [a,b]$

Proof: Let the point at which f_n converges be x_0

First part: If $\{f_n(x_0)\}$ is convergent, then the sequence is also Cauchy!

Clearly $\exists\, N_1 \in \mathbb{N}$ s/t

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2 \quad \forall\, m,n \geq N_1 \qquad \text{(1)}$$

Also since $\{f_n'(x)\}$ converges uniformly we also have: (It'll also be Cauchy).

$\exists\, N_2 \in \mathbb{N}$ s/t

$$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)} \quad m,n \geq N_2 \quad \forall\, t \in [a,b].$$

$\in [a,b]$

$f(b)-f(a)$
$b-a$

We now apply Mean Value Theorem to $f_n - f_m \longrightarrow$ interval $[t,x]$

Let $N = \max(N_1, N_2)$

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x-t|\,\epsilon}{2(b-a)} \leq \frac{\epsilon}{2} \quad \forall\, x,t \in [a,b]$$

if $n,m \geq N$. (2)

Now focus:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall\, n,m \geq N.$$

$< \epsilon$

$\therefore$ Hence $f_n \to f$ uniformly on $[a,b]$.

fix

Second Part: To prove: $f'(x) = \lim_{n\to\infty} f_n'(x) \quad \forall x \in [a,b]$

game b/w t & x

Let $\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}$; $\phi(t) = \frac{f(t) - f(x)}{t-x}$.

Result for n^{th} eq:

Claim: $\phi_n \to \phi$ uniformly on $[a, b]$ $\quad \lim_{t \to x} \phi_n(t) = f'_n(x)$.

Using Cauchy Criterion:

from ②

$$|\phi_n(t) - \phi_m(t)| < \frac{\epsilon}{2(b-a)} \quad \forall n, m > N \ \forall t. \quad (\{\phi_n\} \Rightarrow U$$

on

This proves the claim.

Now we have: interchange of limits is NOW possible as $\{\phi_n\}$ n U.C.

$$\lim_{n\to\infty} \lim_{t \to x} \phi_n(t) = \lim_{t \to x} \phi(t).$$

Proved. Pg- 149, 152.

Weirstrass Theorem:

Let f be a continous function (could be real or complex valued)

So. there exists a sequence of polynomial P_n such that $\{P_n\}$ uniformly c

to f on $[a, b]$.

Proof:

WLOG we may assume $f(0) = f(1) = 0$, $a = 0$, $b = 1$

and define: $g(x) = f(x) - f(0) - x(f(1) - f(0))$

Now $g(1) = g(0) = 0$

We extend f to $\mathbb{R}$ by defining: $f(x) = 0 \ \forall x \in [0,1]^c$.

Now we define: $Q_n(x) = C_n(1-x^2)^n$, $n = 0, 1, 2, \ldots$ and $x \in [-1,$

where C_n's are defined as: $\int_0^1 Q_n(x)\,dx = 1$

let $P_n(x) = \int_{-1}^{1} f(x+t) Q_n(t)\,dt$. $\quad f * Q$

convolution.

This is an sequence of polynomials

Claim-1: $P_n(x)$ are polynomials in x.

Claim-2: $P_n \to f$ uniformly on $[0,1]$

Proof of Claim-1.

let $x + t = t'$

$$P_n(x) = \int_{x-1}^{x+1} f(t') Q_n(t'-x)\, dt'$$

If $0 \le x \le 1$ then

$$P_n(x) = \int_0^1 f(t) Q_n(t-x)\, dt.$$

Then $\{P_n\}$ is clearly a sequence of polynomials in x. [Proved]

Example: Let $f \in C([0,1])$ and $\int_0^1 f(x) x^n dx = 0 \quad \forall\, n = 0, 1, 2, \ldots$

Show that $f \equiv 0$ on $[0,1]$

Let $\epsilon > 0$, then there exists a polynomial p such that:

$$|P_n(x) - f(x)| < \frac{\epsilon}{M} \quad \forall\, n \ge N \quad \forall\, x \in [0,1]$$

$\int_0^1 f(x)\, dx$

P_n is of the form:

$$P_n(x) = a_0 + a_1 x + \ldots\; a_k x^k \qquad a_i \in \mathbb{R}.$$

Now, $\int_0^1 f(x) P_n(x)\, dx = \sum a_i \int_0^1 f(x) x^i dx = 0$ (given)

To show $f \equiv 0$ we need to now show that $f^2 \equiv 0$

$$\int_0^1 f^2(x)\, dx = \int f(x) \left(f(x) - P_n(x) + P_n(x) \right) dx$$

$$= \int f(x) \left| f(x) - P_n(x) \right| dx + \cancel{\int_0^1 f(x) P_n(x)\, dx}^{\,0}$$

$$< \epsilon \int_0^1 f(x)\, dx$$

$< \epsilon$.

So, $\int_0^1 f^2(x)\, dx = 0$

so $\boxed{f = 0}$

See the proof using Bernstein Polynomials.

Proof of Claim-2:

It is enough to show that $\exists\, \epsilon > 0$, $\exists\, P_{n_0}$ s.t.

$$|P_{n_0} - f| < \epsilon \quad \forall\, x \in [0,1]$$

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$P_n(x)$. $\quad$ 1

Def^n $= \left| \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt - f(x) \int_{-1}^{1} Q_n(t)\, dt \right|$

$= \int_{-1}^{1} \left| (f(x+t) - f(x))\, Q_n(t) \right| dt$

$\leq \int_{-1}^{1} | f(x+t) - f(x) |\, Q_n(t)\, dt$

Since f is uniformly continous on $\mathbb{R}$, we have $\delta > 0$ such that:

$| f(x+t) - f(x) | < \epsilon \quad \forall \; |t| < \delta$. Continuity def^n.

$| P_n(t) - f(x) | \leq \int_{-1}^{-\delta} | f(x+t) - f(x) |\, Q_n(t)\, dt$ —— ① $\quad < 2M\epsilon$.

$+ \int_{-\delta}^{\delta} | f(x+t) - f(x) |\, Q_n(t)\, dt$ —— ② $\quad < \epsilon$.

$+ \int_{\delta}^{1} | f(x+t) - f(x) |\, Q_n(t)\, dt$ —— ③ $\quad < 2M\epsilon$

Claim

integ $\int_{-1}^{1} (1-x^2)^n\, dx > \frac{1}{\sqrt{n}}$.

$\geq \int_0^{1/\sqrt{n}} (1-x^2)^n\, dx$

$> \int_0^{1/\sqrt{n}} (1-nx^2)\, dx$

$> \frac{1}{\sqrt{n}}$

So $c_n < \sqrt{n}$.

Fact $c_n < \sqrt{n}$

$Q_n = c_n (1-x^2)^n$

$\leq \sqrt{n}(1-\delta^2)^n \quad$ if $\quad \delta \leq |x| \leq 1$

$= M_n$ (say)

$\int_{-0}^{1} Q_n\, dx$

$\int_0 c_n (1-x^2)^n$

$\leq c_n (1 - nx)$

See P_n -1

Clearly $M_n \to 0$ as $n \to \infty$.

This implies $Q_n \to 0$ uniformly on $[-1, 1]$

So, as $Q_n \to 0$ uniformly on $[-1, 1]$ we get n_0 large enough such that.

③ $< \epsilon \left| \int_{\delta}^{1} f(x+t) - f(x) \right| dt$

$< 2M\epsilon$ if $M = \sup_{-1 \leq x \leq 1} |f(x)|$

We get

$| P_{n_0} - f(x) | \leq 4M\epsilon + \epsilon$ (Proved) [ipad notes]

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Algebra: A family $\mathcal{A}$ of real functions defined on a set E is said to be an algebra if:

1. $f+g \in \mathcal{A}$ if $f, g \in \mathcal{A}$
2. $fg \in \mathcal{A}$ if $f, g \in \mathcal{A}$
3. $cf \in \mathcal{A}$, if $c \in \mathbb{R}$, $f \in \mathcal{A}$

$\Big\}\ f, g : E \to \mathbb{R}$.

* $\mathcal{A}' \subseteq \mathcal{A}$ is said to be a sub-algebra if $\mathcal{A}'$ is an algebra w.r.t same operation.

Uniformly closed algebra:

$\mathcal{A}$ is called Uniformly closed algebra if $f_n \to f$ uniformly & $f_n, f \in \mathcal{A}$.

Uniform Closure of $\mathcal{A}$:

Set $\mathcal{B}$ of all functions which are uniform convergent limits of seq of functions in $\mathcal{A}$.

Note:

1. An algebra $\mathcal{A}$ is said to "separate points" in E if for every pair of distinct points $x_1, x_2 \in E$, there exists a function $f \in \mathcal{A}$ s/t $f(x_1) \neq f(x_2)$.
2. $\mathcal{A}$ is said to vanish at no point in E, if for each $x \in E$, there exists a function $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Theorem: Let A' be a sub-algebra of set of all continuous functions defined on a closed & bounded interval $[a,b]$. $\mathcal{A}'$ separates points in $[a,b]$ and vanishes at no points in $[a,b]$. Then uniform closure of A' is the set of all real continuous function on $[a,b]$.

(the book pg-37, 38)

Q.1) Let A be the collection of real polynomials in x^2 i.e $f(x) = a_0 + a_1x^2 + a_2x^4 \dots a_n x^{2n}$

a) Show that every $f : [0,1] \to \mathbb{R}$, continuous is uniform limit of a polynomial in A

b) Find $f : [-1,1] \to \mathbb{R}$, continuous s/t it is not a uniform limit of polynomial in A.
x^2. $f(x)$ is even so cannot separate pts

separates points.

$A \to$ algebra: $f+g$, fg, cf.

interval is $[0,1]$ take $f(x) = x^2$

not vanishing at no point in $[0,1]$ $f(x) = 1 + x^2$.

Q.2) Prove $A' = \{ f : f(e^{it}) = \sum_{n=0}^{N} c_n e^{int} ; t \in \mathbb{R}[0, 2\pi) \} = A$ on $S' = \{ z \in \mathbb{C} : |z| =$ separates points & vanishes at no pt in S'.

Sol^n Take $f(.) = e^{it}$

then $\forall$ $z_1, z_2 \in S'$ we have $e^{it_1} \neq e^{it_2}$

Vanishing case:

Take $f(.) = e^{it}$ $\Rightarrow$ doesn't vanish on S'.

$\Rightarrow$ Prove that A {Set of all continuous functions defined over S' to $\mathbb{C}$} is ~~the~~ NOT the uniform closure of A'. (Refer pg- 40).

Theorem:

Let A' be a sub-algebra of set of all complex valued continuous functions defined compact set K in $\mathbb{C}$ let A' separate points & vanishes at no point in K. If is self-adjoint or $\overline{A'}$ = A' then uniform closure of A' is equal to set of all complex valued conjugate continuous functions on K.

earlier it was closed & bounded

Calculus on Manifolds

Linear Transformation:

$T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be linear if $T(cv+w) = cT(v) + T(w)$ where $v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Also $T(\tilde{x}) = T(1)\,\tilde{x}$; $\tilde{x} \in \mathbb{R}^n$

So, T is basically a matrix : $m \times n$ dimensions

$$\boxed{y_i = T(x) = \sum_{j=1}^{n} A_{ij}\, x_j} \quad i = 1, 2, \ldots m$$

Note: eg $T: \mathbb{R}^n \to \mathbb{R}^m$ then T is a $m \times n$ matrix, if $x \in \mathbb{R}^n$ is a vector $(\overset{n\times 1}{\ })$ and $y \in \mathbb{R}^m$ is a $m \times 1$ vector.

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

let $A \subseteq \mathbb{R}^n$ then we define

$$f: A \to \mathbb{R}^m$$
$$= (f_1(\tilde{x}), f_2(\tilde{x}) \ldots f_m(\tilde{x})) \qquad \underline{\tilde{x} \in \mathbb{R}^n}$$

where each $f_i : \mathbb{R}^n \to \mathbb{R}$
(A) in this case

Composition:

let $f: A \to \mathbb{R}^m \quad A \subseteq \mathbb{R}^n$

$g: B \to \mathbb{R}^k \quad B \subseteq \mathbb{R}^m$

then $g \circ f(\tilde{x}) = g(f(\tilde{x}))$ is k-dimensional.

Limit of a function:

$\lim_{\tilde{x} \to \tilde{a}} f(\tilde{x}) = \tilde{b}$ $\qquad f: A \to \mathbb{R}^m \quad A \subseteq \mathbb{R}^n$

$\tilde{a} \in \mathbb{R}^n$ & $b \in \mathbb{R}^m$

For every $\epsilon > 0$, there exists a $\delta > 0$ such that
such that $\| f(\tilde{x}) - \tilde{b} \| < \epsilon$ ~~then~~ $\forall$ $\| \tilde{x} - \tilde{a} \| < \delta$, $x \neq \tilde{a}$

where $\| a \| = \sqrt{a_1^2 + a_2^2 \ldots a_n^2}$ $\qquad$ Replace $\tilde{b}$ with $f(\tilde{a})$

Continuity: Here $\lim_{\tilde{x} \to \tilde{a}} f(\tilde{x}) = f(\tilde{a})$

A function is said to be continuous if it continuous at all points in the domain.

Differentiation :

Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $\tilde{a} \in \mathbb{R}^n$

f is said to be differentiable at $\tilde{a}$ if there exists an $(m \times n)$ matrix al called a linear transformation $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\|\tilde{h}\| \to \tilde{0}} \frac{\| f(\tilde{a}+\tilde{h}) - f(\tilde{a}) - \lambda(\tilde{h}) \|}{\|h\|} = 0 \quad \text{—— (1)}$$

If f is differentiable then λ is unique.

Proof of uniqueness :

Let there exist μ, λ both for a point at $\tilde{a}$ which satisfies (1)

Calculate

$x = h$ now

add subtract stuff now

$$\lim_{\|\tilde{x}\| \to 0} \frac{\| \mu(\tilde{x}) - \lambda(\tilde{x}) \|}{\|\tilde{x}\|} = \lim_{\|\tilde{h}\| \to 0} \frac{\| \left(f(\tilde{a}+\tilde{h}) - f(\tilde{a}) - \lambda(\tilde{h})\right) - \left(f(\tilde{a}+\tilde{h}) - f(\tilde{a}) \ldots \right.}{\|h\|}$$

$$\leq \frac{\| f(\tilde{a}+\tilde{h}) - f(\tilde{a}) - \lambda(\tilde{h}) \|}{\|h\|} + \left(\ldots \right.$$

$$\leq 0$$

$$\therefore \lim_{\|\tilde{x}\| \to 0} \frac{\| \mu(\tilde{x}) - \lambda(\tilde{x}) \|}{\|\tilde{x}\|} = 0$$

Now let $h \xrightarrow{\text{replacement}} ch_0$ s/t $c \to 0$ $c \in \mathbb{R}$

$$\lim_{\|c\| \to 0} \frac{\| \mu(c\tilde{h}_0) - \lambda(c\tilde{h}_0) \|}{\|c\| \|h_0\|} = 0 \Rightarrow \frac{\cancel{\|c\|} \| \mu(\tilde{h}_0) - \lambda(\tilde{h}_0) \|}{\cancel{\|c\|} \|h\|} = 0$$

as the choice of h_0 was arbitrary

$\mu(\tilde{x}) = \lambda(\tilde{x})$ (Proved)

Prove that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at all $a \in \mathbb{R}^n$. Then f continuous at a.

Proof: $\lim_{h\to 0} f(a+h) - f(a)$

$$\lim_{\|h\|\to 0} \frac{f(a+h) - f(a)}{\|h\|} \times \|h\|$$ ↓ reverse proving.

$$\leq \frac{\|f(a+h) - f(a) - \lambda(h) + \lambda(h)\|}{\|h\|} \times \|h\|$$

$$\leq \lim_{\|h\|\to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|}^{\to 0} \times \|h\| + \lim_{\|h\|\to 0} \lambda(h)^{\to 0}$$

$\leq M\|h\|$.

≤ 0.

$$\therefore \lim_{h\to 0} f(a+h) - f(a) = 0$$

$$\Rightarrow \boxed{\lim_{h\to 0} f(a+h) = f(a)}$$ (Proved)

Result: $\|\lambda(x)\| \leq M\|x\|$ for some M

where $\lambda : \mathbb{R}^n \to \mathbb{R}^m$.

Proof:

$$y_i = \sum_{j=1}^{n} a_{ij} h_j \quad \text{—①} \quad i = 1 \text{ to } m.$$

use that theorem you learnt

$$y_i^2 = |y_i^2| \leq (a_{i1}(h_1) + a_{i2}(h_2) \ldots a_{in}(h_n))^2 \leq n\alpha^2 \left(\sum_{j=1}^{m} h_j\right)^2 \leq n\alpha^2 |h|^2$$

$\alpha = \max(a_{i1}, a_{i2} \ldots a_{in})$ norm

Now $|\lambda(h)|^2 = \sum_{i=1}^{m} (y_i)^2 \leq \underbrace{mn\alpha^2}_{\approx M^2} |h|^2 = M^2|h|^2$.

$\Rightarrow |\lambda(h)| \leq M|h|$.

Theorems:

1) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a constant $k \in \mathbb{R}^m$. Then $Df(a) = 0 \ \forall a \in \mathbb{R}^n$.

$$\lim_{\|h\|\to 0} \frac{\|\cancel{f(a+h)}^{k} - \cancel{f(a)}^{k} - Df(a)(h)\|}{\|h\|} = 0$$

$$\Rightarrow Df(a) \frac{\cancel{\|h\|}}{\cancel{\|h\|}} = 0$$

$$\Rightarrow \boxed{Df(a) = 0}$$

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2) $f: \mathbb{R}^n \to \mathbb{R}^m$ Then $Df(a) = f \quad \forall\, a \in \mathbb{R}^n$

linear transformation.

Proof:

$$\lim_{\|h\| \to 0} \frac{\| f(a+h) - f(a) - Df(a)(h) \|}{\| h \|} = 0$$

$$= \frac{\| \cancel{f(a)} + f(h) - \cancel{f(a)} - D\cancel{f(a)} f(h) \|}{\| h \|} = 0$$

Hence $\boxed{Df(a) = f}$ if f is a linear transformation.

only operator !!

3.) If $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $a \in \mathbb{R}^n$.

then

(i) $D(f+g)(a) = Df(a) + Dg(a)$.

(ii) $D(fg)(a) = g(a)(D(f)(a)) + f(a)(Dg(a))$.

(iii) $D\left(\frac{f}{g}\right)(a) = \dfrac{g(a)(D(f(a)) - f(a)(Dg(a))}{(g(a))^2}$ $\neq 0$.

4.) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ and $g: \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $f(a) \in \mathbb{R}^m$ then $g \circ f(x) : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at a and:

$$\underbrace{Dg(f(a))}_{g \circ f\,(a)} = Dg(f(a)) \circ Df(a).$$

Proof:

let $b = f(a)$; $\lambda = Df(a)$; $\mu = Dg(f(a)) = Dg(y)$.

Now we define

$\phi(x) = f(x) - f(a) - \lambda(x-a) \to x$

$\psi(y) = g(y) - g(b) - \mu(y-b) \to y$

$\rho(x) = g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x-a)$.

Since f & g are differentiable we have:

$$\lim_{x \to a} \frac{\| \phi(x) \|}{\| x-a \|} = 0 \quad \text{and} \quad \lim_{y \to b} \frac{\| \psi(y) \|}{\| y-b \|} = 0.$$

To show that $\dfrac{\| \rho(x) \|}{\| x-a \|} = 0$

Now we have:

$$f(x) = g\circ f(x) - g\circ f(a) - \mu(\lambda(x-a))$$
$$= g\circ f(x) - g\circ f(a) - \mu\big(f(x) - f(a) - \phi(x)\big).$$
$$= \big[g\circ f(x) - g\circ f(a) - \mu(f(x) - f(a))\big] + \mu(\phi(x))$$
$$= \psi(f(x)) + \mu(\phi(x))$$

Now we need to show that:

$$\lim_{x\to a} \frac{\|\psi(f(x))\|}{\|x-a\|} = 0 \quad \text{and} \quad \lim_{x\to a} \frac{\|\mu(\phi(x))\|}{\|x-a\|} = 0$$

Now,

$$\lim_{x\to a} \frac{\|\mu(\phi(x))\|}{|x-a|} \leq M \frac{\|\phi(x)\|}{\|x-a\|} \to 0 \qquad \mu \to \text{linear transformation}$$

$$\therefore \lim_{x\to a} \frac{\mu(\phi(x))}{\|x-a\|} = 0$$

Now,

Claim: If $\epsilon > 0$ then there exist a $\delta > 0$ st:

$$\frac{\|\psi(y)\|}{\|y-b\|} < \epsilon \quad \forall \ \|y-b\| < \delta. \qquad \text{(This is true}$$

$y = f(x)$.

$$\Rightarrow \frac{\|\psi(f(x))\|}{\|y-b\|} < \epsilon \quad \forall \ \|f(x) - b\| < \delta \quad \forall \ |x-a\| < \delta_1$$

since f is continuous.

$f(a)$.

$$\therefore \|\psi(f(x))\| < \epsilon \|f(x) - b\|. \qquad \text{(Substituting the value of } f(x) - f(a))$$
$$= \epsilon \|\phi(x) + \lambda(x-a)\|$$
$$\leq \epsilon \|\phi(x)\| + M\|x-a\|. \to 0 \text{ as } y \to b.$$

$\therefore$

$$\lim_{x\to a} \frac{\|\psi(f(x))\|}{\|x-a|} = 0 \qquad \text{(Proved)}$$

Projection Mapping:

Let $\pi : \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x) = x \quad \forall \ x \in \mathbb{R}^n$

$\pi_j(x) = \pi_j\left((x_1, x_2 \ldots x_n)\right)$

$\Rightarrow \pi(x) = \left(\pi_1(x), \pi_2(x) \ldots \pi_n(x)\right)$. $\cancel{x \in} \mathbb{R}^n$.

$\pi_j : \mathbb{R}^n \to \mathbb{R}$.

Theorem: π_j is linear transformation.

$\pi_j(\tilde{x}) = x_j$

$\pi_j(\tilde{y}) = y_j$

$\pi_j(\tilde{x} + \tilde{y}) = x_j + y_j$ —①

$\pi_j(c\tilde{x}) = cx_j = c\,\pi_j(\tilde{x})$ —②

$\therefore \pi_j$ is linear.

$n \times 1$.

Matrix for π_j := $[0\ 0\ 0 \ldots 1 \ldots 0\ 0] = \pi_j$

j^{th} one.

Problems for practice:

Q1) let $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \sin x$

find $Df(a,b)$.

$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ 1×2 matrix

$= [\cos x \quad 0]$

Df is a matrix $(1 \times 2) = (d_1\ d_2) = Df$.

$Df(a,b) = (d_1\ d_2)\begin{pmatrix} a \\ b \end{pmatrix} = d_1 a + d_2 b$ —①.

$Df(a, x, y) = d_1 x + d_2 y$.

Take

$d_1 = \cos a$ and $d_2 = 0$

$Df(x,y)x = x\cos a$ —①

$$\lim_{\|h_1, h_2\| \to 0} \frac{\| \sin(a+h_1) - \sin(a) - h_1 \cos a \|}{\|(h_1, h_2)\|} \le \lim_{\|h_1\| \to 0} \frac{\|\sin(a+h_1) - \sin a - \ldots}{\|h_1\|}$$

$= 0$.

$\because (\sin)' a = \cos a$ and $\|(h_1, h_2)\| \ge \|h_1\|$ $\qquad \|(h_1, h_2)\| \ge \|h_1\|$

$\& \|h_2\|$ $\qquad \Rightarrow \frac{1}{\|(h_1, h_2)\|} \le \frac{1}{\|h_1\|}$

Here $Df(\cancel{a,b}) = (a\cos a, 0)$

Remember

$D\pi_j = \pi_j$ as π_j is a linear map

Q2) $f(x,y) = x+y$. find $Df(a,b)$ $\quad [1 \quad 1]$

f is a linear map.

Hence $Df = f$

So or $f(x,y) = \pi_1(x,y) + \pi_2(x,y)$.

$f = \pi_1 + \pi_2$

$\boxed{Df(a,b) = \pi_1 + \pi_2} \Rightarrow (1 \quad 1)$.

So ans $= \underline{a+b}$ $\quad Df(a,b)(x,y) = (1 \quad 1)_{1\times 2}\begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1} = \underline{x+y}$

Q3) $f(x,y) = xy$ find $Df(a,b)$ without (a,b).

Clearly $f(x,y) = \pi_1(x,y)\,\pi_2(x,y)$. only the map π_2 π_1

So,

$Df(a,b) = \pi_1(a,b)\ \underline{D\pi_2(a,b)} + \pi_2(a,b)\ \underline{D\pi_1(a,b)}$.

$= a\,\pi_2(\cancel{a,b}) + b\,\pi_1$.

$= \underline{a(0,1) + b(1,0)}$

$= \underline{(b,a)}$.

$Df(a,b)(x,y) = (b,a)\begin{pmatrix} x \\ y \end{pmatrix} = bx + ay$ $\quad$ (Ans)

Q4) $f: \mathbb{R}^2 \to \mathbb{R}$: $f(x,y) = \sin(x+y)$ find $Df(a,b)$ $\quad (\cos(x+y) \quad \cos(x+y))$

$\pi_1 + \pi_2$ $\quad \cos(\pi_1^{(a,b)} + \pi_2^{(a,b)})(\pi_1 + \pi_2)$

$f = g \circ h\,(x,y)$

where $g(x) = \sin(x)$

$h(x,y) = (x+y)$. $\quad Dh = h$. $\quad \underline{\cos(a+b)(\pi_1 + \pi_2)}$

Ans

So. $Df(x,y) = D(\sin(x+y))\,D(x+y)$.

$= \cos(a+b)\,(x+y)$.

Q5) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \cos y$ find $Df(a,b)$

$g(x) = \cos x$. $\Big\}$ $g \circ h$ $\quad$ Hence $D(g(x,y))\,.\,D(\pi_2(x,y))$

$h(x,y) = y = \pi_2(x,y)$ $\quad = -\sin b \quad \pi_2 \quad = -\sin b\ \pi_2$

Q.6) $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \sin(x\sin y)$ find $Df(a,b)$ DIY

$= \cos(a\sin y)\,(a\sin b + b\cos b)$.

Partial Derivatives:

let $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, then we define Partial derivative wrt i^{th} co-ordinate considering the function as a one variable function fixing all j^{th} co-ordinates (j

$$D_i f(a) = \lim_{h \to 0} \frac{f(a_1, a_2 \ldots a_i + h \ldots a_n) - f(a_1, a_2 \ldots a_i \ldots a_n)}{h}$$

eg: $F(x,y,z) = (x+y)\sin z$.

$D_1 f(x,y,z) = \sin z$

$D_2 f(x,y,z) = \sin z$

$D_3 f(x,y,z) = (x+y)\cos z$.

Conclusion:

If $f: A \to \mathbb{R}$. and if f is differentiable at 'a' then

$$D(f(a)) = (D_1 f(a) \quad D_2 f(a) \ldots \quad D_n f(a))_{1\times n}.$$

Cont.d (Q.6)

$f(x,y) = \sin(x\sin y)$.

$D_1 f(x,y) = \cos(x\sin y)\sin y$

$D_2 f(x,y) = \cos(x\sin y)\,x\cos y$.

$$\therefore \quad D(f(a,b)) = (\cos(a\sin b)\sin b \quad \cos(a\sin b)\,a\cos b)$$

New defn of differentiation:

If $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^2$. Then both partial derivatives ex- and

$$\lim_{(h,k)\to(0,0)} \frac{\| f(a+h, b+k) - f(a,b) - h D_1 f(a,b) - k D_2 (a,b) \|}{\sqrt{h^2+k^2}} = 0$$

let $S = \{(x,y), f(x,y),\ x,y \in A \subseteq \mathbb{R}^2\}$

then $z = f(a,b) + A(x-a) + B(y-b)$ is a tangent to S iff $A = D_1 f(a,b)$ and $B = D_2 f(a,b)$.

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Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^m$: then f is differentiable at $a \in \mathbb{R}^n$ if and only if each f_i is differentiable and

$$Df(a) = \begin{pmatrix} Df_1(a) \\ Df_2(a) \\ \vdots \\ Df_m(a) \end{pmatrix}_{m \times n}.$$

Proof:

To prove:

$$\lim_{h \to 0} \frac{\| f(a+h) - f(h) - \lambda(h) \|}{\|h\|} = 0$$

where $\lambda(h) = \begin{pmatrix} Df_1(a)h & Df_2(a)h & \dots & Df_n(a)h \end{pmatrix}$

Now $\dfrac{\| f(a+h) - f(h) - (\dots) \|}{\|h\|}$

$$= \frac{\| f_i(a+h) - f_i(h) - Df_i(a)h \|}{\|h\|} \quad \forall i$$

$= 0 + 0 + \dots 0$

$= 0$ (Proved).

Converse

$f_i = \pi_i \circ f$.

if f is differentiable at $a \Rightarrow f_i$ is also differentiable at a.

$Df_i(a) = D\pi_i(f(a))\, Df(a)$.

$= \pi_i\, Df(a)$.

General form of $Df(a)$:

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \dots & & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{m \times n} \quad \text{if } f : \mathbb{R}^n \to \mathbb{R}^m.$$

Jacobian Matrix.

Result:

If $f(x) = \int^{h(x)} g(t)\,dt$ then $\boxed{Df(x) = g(h(x))\, Dh(x).}$

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Mean Value Theorem:

Let $f: [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b) then there exists a c such that

$$\frac{f(b) - f(a)}{b-a} = f'(c).$$

Theorem:

Suppose f is continuous from $[a,b]$ to $\mathbb{R}^k$ and let f be differentiable in $(a,$
Then $\exists$ a $x \in (a,b)$ such that:

$$\| f(b) - f(a) \| \leq (b-a) \| f'(x) \|$$

Let

$$t = f(b) - f(a)$$

$$\phi(p) = t \cdot f(p) \qquad a \leq p \leq b$$

Major Step.

$t = f(b) - f(a)$
Apply MVT to
$\phi(p) = t f(p)$
p is the variable

Apply MVT to $\phi(p)$.

$$\phi'(x)(b-a) = \phi(b) - \phi(a).$$

$$\cancel{\phi(x)}\ (f(b) - f(a))\, f'(x)(b-a) = t\,(f(b) - f(a))$$

$$= t \cdot t$$

$$= \|t\|^2$$

$\Rightarrow \quad \|t\|^2 = |t\, f'(x)|(b-a).$

$\Rightarrow \quad \|t\|^{\cancel{2}} \leq \cancel{\|t\|}\, \|f'(x)\|(b-a)$

$\Rightarrow \quad \|f(b) - f(a)\| \leq \|f'(x)\|(b-a)$ (Proved)

Theorem:

Suppose $f: A \to \mathbb{R}^m$ where A is an open convex subset of $\mathbb{R}^n$, f is differentiable in A and there exists $M > 0$ such that:

$$\| Df(x) \| \leq M \quad \forall x \in A.$$

Then

$$\| f(b) - f(a) \| \leq M \| b - a \|. \qquad \forall a, b \in A.$$

Proof:

$$h(t) = (1-t)a + tb$$

Since A is convex, $h(t) \in A \quad \forall t \in [0,1]$

Define $\quad g(t) = f(h(t))$

$$g'(t) = f'(h(t))\, h'(t) = \underbrace{Df(h(t))}_{\leq M}(b-a)$$

$$\|g'(t)\| \leq M \|(b-a)\| \quad \text{--- (1)}$$

Applying MVT on g in $[0,1]$:

$$\|g(1)-g(0)\| \leq \|g'(t)\| \times 1$$
$$\leq M\|b-a\|.$$

$\therefore \quad \|f(b)-f(a)\| \leq M \|b-a\|$ (Proved).

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f_i(a)$ exists for $1 \leq i \leq m$, $1 \leq j \leq m$. Here $f = (f_1, \ldots, f_m)$. (Converse is NOT true)

$D_j f_i(a) = \dfrac{\partial f_i(a)}{\partial x_j}$.

partial derivatives must exist & should be continuous.
↳ then differentiable

Theorem:
Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$
Suppose $D_j D_i f$ and $D_i D_j f$ exist in a nbhd of a & are continuous at a. Then
$D_i D_j f(a) = D_j D_i f(a)$.

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f_i(x)$ exist in an open set containing a and if each of $D_j f_i$ is continuous at a.

Proof:
We know that $Df(a)$ exists if $Df_i(a)$ exists $\forall i \in [1, m]$

By hypothesis we have:
$D_j f(x)$ exists $\forall x \in N_\delta(a)$ and $D_j f$ is continuous at a.

To show that f is differentiable at a:

$$f(a+h)-f(a) = f(a_1+h_1, a_2 \ldots a_n) - f(a_1 \ldots a_n) + f(a_1+h_1, a_2+h_2 \ldots a_n) - f(a_1+h_1, a_2 \ldots a_n)$$

Now.
(by MVT) $f(a+h)-f(a) = \sum_{j=1}^{m} h_j D_j f(c)$ if $\xi_j \in (a_j, a_j+h_j)$

$= \sum_{j=1}^{n} h_j D_j f(a_1+h_1, a_2+h_2, \ldots, \xi_j, a_{j+1}, \ldots a_n)$

$- \sum_{j=1}^{n} h_j D_j f(a)$

$$= \sum_{j=1}^{n} h_j \left(D_j f(\xi_j) - D_j f(a) \right).$$

Now,

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - \sum_{j=1}^{n} h_j D_j f(a) \|}{\|h\|}$$

$$\leq \lim_{h \to 0} \sum_{j=1}^{n} \frac{\left(D_j f(\xi_j) - D_j f(a) \right) \cancel{|h_j|}}{\cancel{\|h\|}}$$

Clearly $h \to 0$ as $\xi_j \to a$

Since $D_j f$ is continuous at a, this implies $\to 0$ as $h \to 0$ (Hence proved).

See the problems
(Pg - 10 to 18).

Question to practice:

- ⇒ Rudin: Page - 165 → (Q.1 - Q.11)
- ⇒ Rudin Page - 239 → (Ex - 6, 7, 8, 9, The examples)
- ⇒ Spivak (Theorem - 2.8) → 2-4

Pg - 17 : ~~2-1~~, 2-5, ~~2-6~~, ~~2-7~~, ~~2-8~~

Pg - 23 : ~~2-10~~, ~~2-11~~, ~~2-12~~

Pg - 28 : ~~2-19~~, ~~2-18~~, ~~2-20~~, ~~2-21~~, ~~2-22~~, 2-24 MVT

Metric Spaces

Metric:

let $X \neq \phi$. A function $d : X \times X \to \mathbb{R}$ is said to be a metric/distance on X if it satisfies the following:

1. $d(x,y) \geq 0 \quad \forall x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$. (Non-negative & zero)
2. $d(x,y) = d(y,x)$ for all $x,y \in X$. (Symmetry)
3. $d(x,y) \leq d(y,z) + d(z,x)$ where $x,y,z \in X$ (Triangular inequality)

The pair (X, d) is called a Metric Space.

Example: $d(x,y) = |x-y|$ on the set $\mathbb{R}$ (Verify all the characteristics)

→ $(\mathbb{C}, |.|)$ is also a metric space.

→ $(\mathbb{R}^n, d_1)$ where $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ where $x = (x_1, x_2 \dots x_n)$; $y = (y_1, y_2 \dots y_n)$

→ $(\mathbb{R}^n, d_\infty)$ where $d_\infty(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$ where $x = (x_1, x_2 \dots x_n)$; $y = (y_1, y_2 \dots y_n)$

→ $(\mathbb{R}^n, d_2)$ where $d_2(x,y)$. $\quad d_2(x,y) = \sum_{i=1}^{n} \left((x_1-y_1)^2 + (x_2-y_2)^2 + \dots (x_n-y_n)^2\right)^{1/2}$

$\longrightarrow \left(\sum (x_i - y_i)^2\right)^{1/2}$

Discrete Metric Space:

$$X \neq \phi, \quad d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

→ $X = C[0,1]$; set of all continuous functions on $[0,1]$ (domain)

$d_1(f,g) = \int_0^1 |f(t) - g(t)|\, dt$ is a metric

Proof:

1) $d(f,g) \geq 0 \quad \forall f,g \in X$

$d(f,g) = 0 \iff f(t) = g(t) \quad \forall t \in [0,1]$

2) $d(f,g) = d(g,f)$ ($\because$ There is a modulus)

3) $d(f,g) = \int_0^1 |(f-g)(t)|\, dt \leq \int_0^1 |f(t) - h(t)|\, dt + \int_0^1 |h(t) - g(t)| = d(f,h) + d(h,g)$

$\therefore$ d is a metric on X.

Result: $h \in [0,1]$ and $\int_0^1 |h(t)|\, dt = 0 \Rightarrow h \neq 0$ (Prove or Disprove)

→ Take x as a point
→ Consider δ nbhd s/t $f(x) < \epsilon$ for all $x \in (x-\delta, x+\delta)$
Split & do the Integration → apply inequality

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since $f \neq 0$, $\exists\, t_0$ s/t $f(t_0)$
let $\alpha = f(t_0)$ and $\epsilon = \alpha/2$

Thus,
$$\int_0^1 |f(t)|\,dt \geq \int_{t_0-\delta}^{t_0+\delta} f(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} \frac{\alpha}{2}\,dt.$$

let us assume $h \neq 0$

let $h(x) \neq 0$ for some $x \in [0,1]$ but $\int_0^1 |h(t)|\,dt = 0$

We find the δ-neighbourhood of x : $(x-\delta, x+\delta)$

such that $h(y) \geq \epsilon$ $\forall\, y \in (x-\delta, x+\delta)$ and $\epsilon = \frac{h(x)}{2}$

greater than or equal to

Now,
$$\int_0^1 |h(t)|\,dt = \int_0^{x-\delta} |h(t)|\,dt + \int_{x-\delta}^{x+\delta} |h(t)|\,dt + \int_{x+\delta}^1 |h(t)|\,dt$$

See this from the book.

Now $\int_{x-\delta}^{x+\delta} |h(t)|\,dt \geq \int_{x-\delta}^{x+\delta} \frac{h(x)}{2}\,dt = \delta h(x) > 0 \quad \delta > 0$

2δ.

$\therefore \int_0^1 |h(t)|\,dt > 0$

Result: let $f : [0,1] \to \mathbb{R}$ be continuous with $f(t) \geq 0$ $\forall\, t \in [0,1]$. Then $\int_0^1 |f(t)|\,dt$ if and only if $f(t) = 0$ $\forall\, t \in [0,1]$

6.1) $X = \mathbb{R}[0,1]$ $\quad d(f,g) = \int_0^1 |(f-g)t|\,dt$

Define :
$$g(t) = \begin{cases} f(t)+1 & t = x_0 \\ f(t) & t \in [0,1] \setminus \{x_0\} \end{cases}$$

$\therefore \int_0^1 |g(x) - f(x)|\,dx = 2\delta$ but since δ is very small
$= 0$

even if $f \neq g$ $\forall\, t$ we got 0.

$\therefore$ d is NOT a metric on $X = \mathbb{R}[0,1]$

→ $X = C[0,1]$, $d_\infty(f,g) = \sup_{x \in [0,1]} |(f-g)(t)|$ is a metric.

→ $X = C[0,1]$ $\quad d_2(f,g) = \left\{ \int_0^1 |(f-g)(t)|^2\,dt \right\}^{1/2}$ is a metric.

take $f = \frac{t}{1+t}$ if function is monotonic then $f(d(x,z)) \leq f(d \ldots$

Additional problems for practice:

1) Let (X,d) be a metric space. Define $\delta(x,y) = \frac{d(x,y)}{1+d(x,y)}$ $\forall\, x, y \in X$
Show that (X,δ) is also a metric space.

$d(x,y)$

$f(d(x,z)) \leq f(d \ldots$

Q. Let (X,d), (Y,d') be 2 metric spaces. Then prove that $\bar{d}((x_1,y_1),(x_2,y_2)) = \max(d_1, d_2)$ defines a metric on $X \times Y$ (Product Metric Space)

↳ Case study

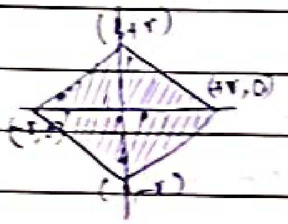
Open and Closed Sets:

Let (X,d) be a metric space, let $x \in X$ and $r > 0$.

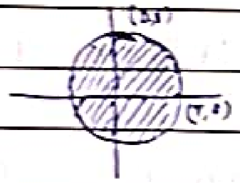
$B_d(x,r) = \{y \in X : d(x,y) < r\}$ is called the open ball centred at x & radius r

$B_d[x,r] = \{y \in X : d(x,y) \le r\}$ is called the closed ball centred at x & radius r

eg. Draw $B_{d_1}((0,0), r)$
$\quad |\cdot|^2$ ⟶



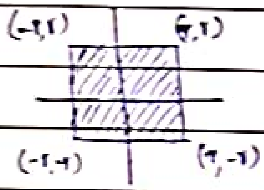
eg. Draw $B_{d_2}((0,0), r)$ ⟶



$x^2+y^2 < r^2$

eg. Draw $B_{d_\infty}((0,0), r)$

$\max(|x_1-x_2|, |y_1-y_2|)$ ⟶



eg. $(\mathbb{R}, |\,|)$

$B(x,r) = \{y \in \mathbb{R} : |x-y| < r\} = (x-r, x+r)$

eg. $X = C[0,1]$

$d_\infty(f,g) = \sup_{t \in [0,1]} |(f-g)(t)|$ — set of continuous & bounded functions

Then $B_{d_\infty}(0, \epsilon) = \{ f \in C[0,1] \mid \sup_{t\in[0,1]} |f(t)| < \epsilon \}$
radius

Similarly

$B_{d_\infty}(f, \epsilon) = \{ g \in X \mid \sup_{t \in [0,1]} |(f-g)(t)| < \epsilon \}$

Open Set:

A subset $U \subset X$ of a metric space is said to be d-open if for each $x \in U$, there $r > 0$ such that $B(x,r) \subset U$.

{for each x we need to find r s.t. $B(x,r) \subset U$}

Q.1) Prove that $(0,1)$ is an open set in $\mathbb{R}$

Ans. Let $x \in (0,1)$

To find $r > 0$, $B(x,r) \subseteq (0,1)$ ie $(x-r, x+r) \subset (0,1)$

r should satisfy $x+r < 1$ and $x-r > 0$

Given $0 < x < 1$

Then $1-x > 0$ and $x - 0 > 0$ ∴

$$\boxed{\text{We set : } r = \frac{1}{2}\min(1-x, x).}$$

$x > 0$

$x < 1$

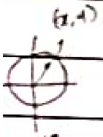
$r = \frac{1}{2}\min(x, 1-x)$

Then we have. $B(x,r) \subseteq (0,1) \Rightarrow (0,1)$ is open set in $\mathbb{R}$.

Q.2) Prove that $(0,1) \times (0,1)$ is open set in $(\mathbb{R}^2, d_2)$

Ans. let $(x,y) \in U$

$\Rightarrow$ $x \in (0,1)$ and $y \in (0,1)$ $\Rightarrow$ $x > 0$ and $x < 1$ & $y > 0$ and $y < 1$



let $r_1 = \min(x, 1-x)$

$r_2 = \min(y, 1-y)$

$r = \{1 - d(x,a),\ d(x,b)\}$ We take $r = \min(r_1, r_2)$

$x = ((x,y), r)$ is an open ball.

$0 < x < 1$ and $0 < y < 1$

$r_1 = \min(x, 1-x)$

$r_2 = \min(y, 1-y)$

$r = \min(r_1, r_2)$

Q.3) What are the open sets in discrete metric?

Ans. Every subset of X is both closed & open.

Proof. let $r > 1$

then $B(x,r) = X$ (Take everything).

$\cdot(x,y)$

$x \neq y$ $d(x,y) = 1$

Take everything

let $r < 1$

$B(x,r) = \{x\}$

Lemma: let (X,d) be a metric space and $x \in X$ & $r > 0$. Then $B(x,r)$ is an open set. of (X,d).

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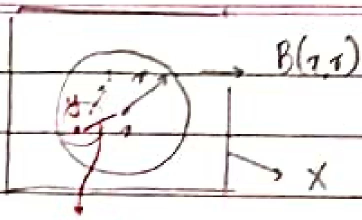
Proof:



let $y \in B(x,r)$

$d(x,y) < r$ for y $\exists$ some $s > 0$ such that

We need to show that $B(y,s) \subseteq B(x,r)$ for some $s > 0$

We need to find this.

let $z \in B(y,s) \Rightarrow z \in B(x,r)$

$\Rightarrow d(y,z) < s \Rightarrow d(x,z) < r$ Expand on this

↳ by def^n

Now $d(x,z) \leq d(x,y) + d(y,z) < r$

$d(x,z) \leq d(x,y) + s < r$

$d(y,z) < s \Rightarrow d(x,z) < r$

$d(x,z) < r$

$\underbrace{d(x,y)}_{s} + \underbrace{d(y,z)}_{s} < r$

Hence

$\boxed{r - d(x,y) > s}$ s exists

$\boxed{r - d(x,y) > s}$

$\therefore$ $B(x,r)$ is an open set.

Note:

ϕ is open set in (X,d)

X is an open set in (X,d)

Q4) Is $(-1,1)$ open in $(\mathbb{Q}, |.|)$

$x \in (-1,1) \cap \mathbb{Q}$

$r = \frac{1}{2}\min(1+x, 1-x))$

Sol^n let $x \in (-1,1) \cap \mathbb{Q}$

exists $\Rightarrow$ $(-1,1) \cap \mathbb{Q}$ is open

$B(x,r) = \{ y \in \mathbb{Q} \mid d(x,y) = |x-y| < r \} \subseteq \mathbb{Q}$

We choose $r = \min(1+x, 1-x)$

$\therefore (-1,1)$ is open in $\mathbb{Q}, |.|$

Q5) let (X,d) be a metric space and $a \in X$. Show that $U = \{ x \in X \mid d(x,a) > r \}$ is an open set in (X,d)

$d(x,a) > r$

Sol^n let $x \in U \Rightarrow d(x,a) > r$

let $r' = \frac{1}{2}(d(x,a) - r) > 0$

Take

$r = \frac{1}{2}(d(x,a) - r)$ > 0

Then $B(x,r') \subseteq U$

$\Rightarrow U$ is an open set in X

$\rightarrow$ Take x as centre

the set is in $\underline{\underline{U}}$

↳ $B(x,r)$

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Theorem:

Let (X, d) be a metric space and T be the collection of all open sets in X. Then it has following properties:

(i) $\phi, X \in T$ (Trivial Open Sets)

(ii) Arbitrary union of Open sets is open i.e $\bigcup_{i \in A} U_i \in T$ $\curvearrowright$ contra-positive

(iii) The intersection of arbitrary no. of closed sets is also closed. finite intersection of open sets is open

↳ (X, t) : Topological Space

Proof:

(i) Given any $x \in X$ and $r > 0$, by def^n $B(x, r) \subset X$. Hence X is open.
Now give me an element from ϕ, I will give you a r s/t $B(x,r) \subset X$ but you cannot, hence I win.

(ii) Let $\{U_i : i \in I\}$ be a family of open sets. (I → indexed set)

Easy! Let $x \in U_j$ for some $j \in I$. Since U_j is open so there exists $r > 0$ s/t $B(x, r) \subseteq U_j \subseteq \bigcup_{i \in I} U_i$. Since x is arbitrary, so $\bigcup_{i \in I} U_i$ is open in X

(iii) To show $\bigcap_{k=1}^{n} U_k$ is open.
Case-(i) The intersection is ϕ, then by default its open.
Case-(ii)
~~Let the intersection be U, then we choose $r = \min(r_1, r_2, \ldots r_n)$~~
Since U_k is open & $x \in U_k$ $\forall$ k. x belong to the intersection.
There exists $r_k > 0$ s/t $B(x, r_k) \subset U_k$
Let $r = \min\{r_k \mid 1 \le k \le n\}$
Clearly $B(x,r) \subseteq B(x, r_k) \subset U_k$ $\forall$ k

$\boxed{B(x, r) \subseteq \bigcap_{k=1}^{m} U_k}$

Interior Point:

Let $A \subseteq (X, d)$ and $x \in A$ is said to be an interior point of A if there exists $r > 0$ such that $B(x, r) \subseteq A$.

Every point is an interior point for open sets. $\boxed{\text{int } A \text{ or } A^\circ = A}$ if A is open.
↳ by def^n of open set

Also note that: A° or int A (A° = set of all interior pts in A) is the largest open set contained in A.

Closed Set:

A subset $E \subset (X,d)$ is called closed if E^c in X is an open set in X. ($E^c = X \setminus E$, complement of E)

$E^c = \{x \in X \mid x \notin E\}$

Problems for practice:

Q1) Prove that $[0,1]$ is closed in $(\mathbb{R}, |\cdot|)$ → Need to prove that its complement is open.

Sol^n $[0,1]^c = (-\infty, 0) \cup (1,\infty) = \{x \in \mathbb{R} \mid x < 0 \text{ or } x > 1\}$

let $y \in [0,1]^c$ — if $y \in (-\infty, 0) \cup (1,\infty)$

$\Rightarrow y < 0$ or $y > 1$ $\quad\Rightarrow\quad y < 0$ or $y > 1$

let $y < 0$ then $\boxed{r = \frac{0-y}{2} > 0}$ → $B(y,r) \subseteq (-\infty, 0)$

Then $(y-r, y+r) \subseteq (y-r, 0) \subset (-\infty, 0) \subseteq [0,1]^c$ → $(-\infty, 0)$ is open

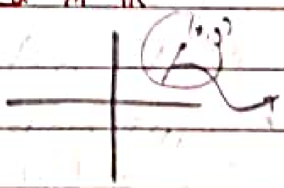
Similarly let $y > 1$ then $\boxed{r = \frac{y-1}{2}} > 0$

Then $(y-r, y+r) \subset [0,1]^c$

$\therefore$ $[0,1]^c$ is open.

$\therefore$ $[0,1]$ is closed.

Q2) Prove that $\underbrace{\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}}_{A}$ is closed in $\mathbb{R}^2$



$r = \frac{1}{2}\min(|x|, |y|)$

Sol^n Then $A^c = \{(x,y) : xy \neq 0\}$

let $r = \frac{1}{2}\min\{|x|, |y|\} > 0$

Then $B((x,y), r) \subseteq A^c$

$\Rightarrow A^c$ is open

$\Rightarrow A$ is closed.

Q3) $S = \{(x,y) \mid x^2 + y^2 = 1\}$ Show that S is closed in $(\mathbb{R}^2, d_2)$

$S^c = \{(x,y) \mid x^2 + y^2 \neq 1\}$

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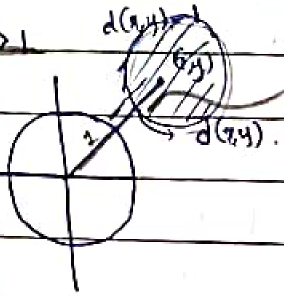
let $d(x,y)$ $x^2+y^2>1$

Similarly consider $x^2+y^2<1$



from (x,y)
$\frac{1}{2}\left(d(x,y)-1\right)$.

let $r = \frac{|d((x,y),(0,0))-1|}{2}$.

Then $B((x,y),r) \subseteq S_1^c$

$\therefore$ S_1^c is open.

$\therefore$ S_1 is closed.

Q4) Is $\mathbb{Q}$ closed in $(\mathbb{R}, |.|)$? No

Ans. No $\mathbb{Q}^c$ is not open in $(\mathbb{R}, |.|)$

let $y \in \mathbb{Q}^c$ and $r>0$ then $B(y,r) = (y-r, y+r) \not\subset \mathbb{Q}^c$.

$\Rightarrow$ $B(y,r) \not\subset \mathbb{Q}^c$

$\Rightarrow$ $\mathbb{Q}^c$ is NOT open set in $(\mathbb{R}, |.|)$

$\Rightarrow$ $\mathbb{Q}$ is NOT closed in $(\mathbb{R}, |.|)$

$y \in \mathbb{Q}^c$.
$(y-r, y+r) \cap \mathbb{Q} = \phi$ should be but not the case
intersection with $\mathbb{Q} \neq$ as it may contain rat

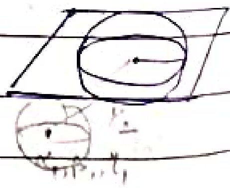
Similarly $\mathbb{Q}$ is also NOT open in $(\mathbb{R}, |.|)$

Q5) $P = \{(x,y,z) \mid ax+by+cz=d\}$ Show that P is closed in $(\mathbb{R}^3, d_2)$
square distance

let $((\alpha,\beta,\gamma),r) \in P$ then $a\alpha+b\beta+c\gamma = d$.

$B((\alpha,\beta,\gamma),r) \cap P \neq \phi$. (because of the plane)

$\Rightarrow$ Each point of P is a limit point of P (How?)



Let $(\alpha_1, \beta_1, \gamma_1) \notin P$ let $r = \frac{1}{2}\left(\frac{a\alpha_1+b\beta_1+c\gamma_1-d}{\sqrt{a^2+b^2+c^2}}\right) > 0$

Then $B((\alpha_1,\beta_1,\gamma_1),r) \cap P = \phi$.

$\therefore$ Any point outside P is NOT a limit point.

$\therefore$ P is closed.

P^c is open

$\perp$ distance from the po to the plane.

$E \to x$ $\quad r = \frac{1}{2}\min(1+x, 1-x)$

$\boxed{A \cap \mathbb{Q} \subset E} \Rightarrow E$ is open.

Q.6) Show that $E = \{x \in \mathbb{Q} \mid -1 < x < 1\}$ is open in $\mathbb{Q}$ but not closed in $\mathbb{Q}$.

Sol: let $y \in E \Rightarrow y \in \mathbb{Q}$ and $-1 < y < 1$

let us consider $r = \frac{1}{2}\min(y+1, 1-y)$.

Then $B(y,r) = (y-r, y+r) \cap \mathbb{Q}$ is an open ball contained in E.

$\Rightarrow$ E is open in $(-1,1)$.

$E^c = \{x \in \mathbb{Q} \mid x \geq 1 \text{ or } x \leq -1\}$ To prove that E^c is NOT open

Here 1 is NOT an interior point of E^c but $1 \in E^c$. Take $x = 1$

$\Rightarrow$ E^c is NOT open. no r exist

$\Rightarrow$ E is not closed in $(\mathbb{Q}, 1\cdot1)$.

Q.7) Show that $E = \{x \in \mathbb{Q} \mid -\sqrt{2} < x < \sqrt{2}\}$ is both open & closed in $\mathbb{Q}$.

Sol: let $y \in E \Rightarrow y \in \mathbb{Q}$ and $-\sqrt{2} < y < \sqrt{2}$.

let us consider $r = \frac{1}{2}\min(y+\sqrt{2}, \sqrt{2}-y)$

Then $B(y,r) = (y-r, y+r) \cap \mathbb{Q}$ is an open ball contained in E.

Hence E is open.

$E^c = \{x \in \mathbb{Q} \mid x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\}$. Take two cases.

Let $r = \frac{1}{2}(-\sqrt{2}-x)$ and let $x \in E^c$.

$B(y,r) \subset E^c$ (in both cases)

E^c is open.

$\therefore$ E is closed.

Theorem:

Let (X,d) be a metric space. E be a collection of closed sets in (X,d). Then:

(i) $\phi, X \in E$

(ii) $\bigcup_{i=1}^{n} U_i \in E$, if $U_i \in E$ for all $i = 1, 2, 3, \ldots n$ (finite union)

(iii) $\bigcap_{i \in \Lambda} U_i \in E$, if $U_i \in E$ for all $i = 1, 2, 3, \ldots n$. (arbitrary intersection)

Proof:

(i) $\phi^c = X$ (ϕ and X is open) $\Rightarrow$ ϕ is closed.
$X^c = \phi$ and ϕ is open $\Rightarrow$ X is closed
Therefore $\phi, X \in E$.

(ii) $U_i \in F$ for all $i = 1, 2, \ldots n$.

$\Rightarrow$ Each U_i^c is open.

$$\left(\bigcup_{i=1}^{n} U_i\right)^c = \bigcap_{i=1}^{n} U_i^c$$ and finite intersection of open sets is open.

$\Rightarrow \left(\bigcup_{i=1}^{n} U_i\right)^c$ is open in (X, d)

$\Rightarrow \bigcup_{i=1}^{n} U_i$ is closed in (X,d). $\Rightarrow$ $\boxed{\bigcup_{i=1}^{n} U_i \in F.}$

(iii) $\left(\bigcap_{i \in A} U_i\right)^c = \bigcup_{i \in A} U_i^c$ and arbitrary union of open sets is open.

$\Rightarrow \bigcap_{i \in A} U_i$ is closed in (X, d) $\Rightarrow$ $\boxed{\bigcap_{i \in A} U_i \in F.}$

Concept of Convergence:

A sequence in metric space is simply a function $x : \mathbb{N} \to X$ we use $x = \{x$…

Given a sequence $\{x_n\}$ in (X, d), a subsequence is a restriction of x on a…
$S \subset \mathbb{N}$ if $S = \{n_i\}$ where $n_1 < n_2 < n_3 \ldots$ (strictly) then we can write $\{x$…
is a subsequence. ($n_k \geq k \ \forall \ k \in \mathbb{N}$)

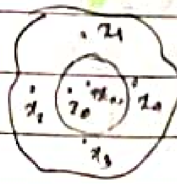
This can be proved by induction.

Def^n:

We say $\{x_n\}$ converges to $x \in X$ if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that when $x_n \in B(x, \epsilon) \ \forall \ n \geq N$ (Geometric)

We have:

(Analytic) $d(x_n, x) < \epsilon \quad \forall \ n \geq N$.



x is called the limit of $\{x_n\}$.

$x = \lim_{n \to \infty} x_n$ or $x_n \to x$ as $n \to \infty$. (Notation)

[Uniqueness Theorem] **Theorem:** If $\{x_n\}$ is a convergent sequence then

Let us assume that $x_n \to x$ and $x_n \to y$ and $x \neq y$.

If $x \neq y$, then $\delta = d(x,y) > 0$

We choose $r < \delta/2$ and consider $B(x,r)$ and $B(y,r)$. We claim that these balls are disjoint.

our choice so that we can get the balls as disjoint.

Let say $z \in B(x,r) \cap B(y,r)$

$\delta = d(x,y) < \underbrace{d(x,z)}_{=r} + \underbrace{d(z,x)}_{=r} < 2r < \delta$

Then

$\delta = d(x,y) \leq d(x,z) + d(z,y) < r + r \leq \delta$ (contradiction) $\delta < \delta$

So the balls are disjoint

Now since

$x_n \to x$. So $\exists\ n_1 \in \mathbb{N}$ s/t $x_n \in B(x,r)$ for $n \geq n_1$

As $x_n \to y$ So $\exists\ n_2 \in \mathbb{N}$ s/t $x_n \in B(y,r)$ for $n \geq n_2$

In particular, $\forall\ n \geq \max(n_1, n_2)$ we see that $x_n \in B(x,r) \cap B(y,r)$.

This contradicts the assumption that $B(x,r)$ and $B(y,r)$ are disjoint.

$\therefore$ $x = y$. or limit is unique.

Ex: If $\{x_n\}$ is convergent then every subsequence of $\{x_n\}$ is also convergent and the limit will be the same:

$\{x_n\} \longrightarrow x$ as $n \to \infty$

$\Rightarrow$ $d(x_n, x) < \epsilon \quad \forall\ n \geq N_\epsilon$. $\quad x_n \geq n$ $\quad$ We know that

$\Rightarrow$ $d(x_{n_k}, x) < \epsilon \quad \forall\ n_k \geq N_\epsilon$. for some $n_k \geq n$ $\quad \boxed{n_k \geq k}$ always

$\Rightarrow$ $d(x_{n_k}, x) < \epsilon \quad \forall\ k \geq N_\epsilon$.

Practice:

Q.1) Let $\{x_n\} \subseteq (\mathbb{R}^m, d_2)$ & $d_2(x,y) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$ and $\{x_n\} \to x$ ~~and $\{y_n\} \to y$~~ as $n \to \infty$

Show that $d_2(x_n, 0) \to d_2(x, 0)$ in $(\mathbb{R}, |.|)$. Is convergence true?

Q.2) Let X, Y be two metric spaces and $X \times Y$ is the product space. Show that $(x_n, y_n) \to (x,y)$ in $X \times Y$ if & only if $x_n \to x$ in X and $y_n \to y$ in Y.

Important result

Same as Hw Test.

Q1) let $X = C[0,1]$, sup $= d$ that is $f, g \in X$ $d(f,g) = \sup_{x \in [0,1]} |(f-g)(x)|$

let $f_n \to f$ in (X, d). Show that $f_n \to f$ uniformly. Converse is also true.

$\hookrightarrow$ $d(f_n, f) < \epsilon$ $\forall n \geq N_\epsilon \Rightarrow \sup |(f - f_n)x| < \epsilon$

Proof: $f_n \to f$ as $n \to \infty$ in (X, d) $\Rightarrow (f - f_n)x < \epsilon$ $\forall n \geq N$

$\Leftrightarrow$ For some $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d(f_n, f) < \epsilon$ for all n

$\Leftrightarrow$ $\sup_{x \in [0,1]} |(f_n - f)(x)| < \epsilon$ $\forall n \geq N_\epsilon$.

removing sup.

Converse $\Leftrightarrow$ $|(f_n - f)(x)| < \epsilon$ $\forall n \geq N_\epsilon$ $\forall x \in [0,1]$

$\Rightarrow$ $|f_n(x) - f(x)| < \epsilon$ $\forall n \geq N_\epsilon$ $\forall x \in [0,1]$

$\Rightarrow$ $\{f_n\}$ converges uniformly to $f(x)$ on $[0,1]$

$C[0,1]$

Conversely, it is true that if $\{f_n\} \to f$ uniformly $\Rightarrow f_n \to f$ in (X, d_∞).

sup. metric

Q2) let $f_n(x) = x^n$, $x \in [0,1]$. Show that f_n is convergent in $(C[0,1], d_1)$ wh

it is NOT convergent in $(C[0,1], \sup)$.

Proof: $d_1(f,g) = \int_0^1 |(f-g)(t)|\, dt$

$\boxed{0 \in C[0,1]}$ could be g

$d_1(f, 0) = \int_0^1 t^n\, dt = \left.\frac{t^{n+1}}{n+1}\right)_0^1 = \frac{1}{n+1} < \epsilon$. where $n \geq \frac{1}{\epsilon} - 1$

$\Rightarrow$ $d_1(f, 0) < \epsilon$ $\forall n \geq k$ where $k = \frac{1}{\epsilon} - 1$ $\Rightarrow n \geq \left[\frac{1}{\epsilon} - 1\right]$

$\rightarrow$ $\{f_n\}$ is convergent in $(C[0,1], d_1)$ k

$\because$ $\{f_n\}$ is NOT uniformly convergent on $[0,1]$. (from previous theorem)

$\Rightarrow$ f_n is NOT convergent in $(C[0,1], \sup)$.

Discrete Metric (X, d), $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

let $x_n \to x$ in discrete metric then,

$x_n = x$ $\forall n \geq N$. (Eventually constant sequence)

$\hookrightarrow$ $(x_1, x_2, x_3, \ldots x, x, x, x, x, \ldots)$

Limit Point:

Limit Points:

Let (X, d) be a metric space, and $A \subset X$. A point $x \in X$ is a limit point of A if for every $r > 0$, $B(x, r) \cap A \neq \phi$.

sticky point.

$\Rightarrow$ If $x \in A$ then x is a limit point of A. (Obvious)

Q1) Find limit point of $A = (0,1)$ in $\mathbb{R}$

$A^d = [0,1]$

Q2) Find all limit points of $\mathbb{Q}$ in $\mathbb{R}$

$\mathbb{Q}^d = \mathbb{R}$

Q3) Let $E \subseteq (X,d)$. A point x is a limit point of E if and only if there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$. Read from the book

If x is a limit point of E then we can have two cases: $x \in E$ and $x \notin E$.

Case-(i) $[x \in E]$ then $x_n = x \ \forall n \in E$ and $\{x_n\}$ is convergent.

Case-(ii) $[x \notin E]$

x is a limit point (Choose $r = \frac{1}{n}$)

$B(x, \frac{1}{n}) \cap E \neq \phi$

$\Rightarrow x_n \in B(x, \frac{1}{n}) \cap E$.

$\Rightarrow d(x_n, x) < \frac{1}{n} \ \forall n$. (Proved).

x is a limit point & we want to prove that there exists a seq $\{x_n\} \to x$

Take $r = \frac{1}{n}$.

$d(a_n, x) < \frac{1}{n}$.

Arch. Prop. $N\epsilon > 1 \Rightarrow \epsilon > \frac{1}{N}$

$\Rightarrow$ Conversely, there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$.

For arbitrary $\epsilon > 0$ $\exists$, $k \in \mathbb{N}$ such that

$d(x, x_n) < \epsilon \ \forall n \geq k$

$\Rightarrow x_n \in B(x, \epsilon) \ \forall n \geq k$

$\Rightarrow B(x, \epsilon) \cap E \neq \phi$

$\Rightarrow x$ is a limit point of E.

$a_n \in \cancel{B(x, \frac{1}{n})} \subseteq B(x, \frac{1}{N})$

$\subseteq B(x, \epsilon)$

$\Rightarrow d(x, a_n) < \epsilon$.

For converse use the def^n

Q.4) $\{x_n\} \subseteq (\mathbb{R}^m, d_2)$ where $d_2(x,y) = \sqrt{\sum_{i=1}^{m}(x_i - y_i)^2}$

$\{x_n\} \to x$ as $n \to \infty$ we def^n.

Very simple proof

Show that $d_2(x_n, 0) \to d_2(x, 0)$. $\Rightarrow \sqrt{\sum_{i=1}^{m}(x_{n_i} - 0)^2} \to \sqrt{\sum x^2}$

Using Δ-ineq

$\because$ $\lim_{n\to\infty} x_n = x \Rightarrow$ for some arbitrary $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$d_2(x_n, x) < \frac{\epsilon}{2} \quad \forall\, n \geq k$

$\Rightarrow d_2(x_n, 0) \leq d_2(x_n, x) + d_2(x, 0)$

$\Rightarrow |d_2(x_n, 0) - d_2(x, 0)| \leq d_2(x_n, x) < \frac{\epsilon}{2} < \epsilon \quad \forall\, n \geq k$.

$\Rightarrow d_2(x_n, 0) \to d_2(x, 0)$ as $n \to \infty$ (Proved).

$d((x_1, y_1), (x_2, y_2)) = \max(d_1(x_1, x_2), d_2($

Q.5) $\{(x_n, y_n)\} \to (x, y)$ in $X \times Y$ if and only if $x_n \to x$ in X, $y_n \to y$

let

Read from iPad some props

$(x_n, y_n) \to (x, y)$ in $(X \times Y, d)$ Write the def^n.

$\Leftrightarrow d((x_n, y_n), (x, y)) < \epsilon \quad \forall\, n \geq k$. let us consider d_∞

$\Leftrightarrow \max\{d_1(x_n, x), d_2(y_n, y)\} < \epsilon \quad \forall\, n \geq k$.

$\Leftrightarrow d_1(x_n, x) < \epsilon \quad \forall\, n \geq k$ and $d_2(y_n, y) < \epsilon \quad \forall\, n \geq k$.

$\Leftrightarrow x_n \to x$ in X and $y_n \to y$ in Y.

Theorems related to limit point :

Theorem: A subset $E \subseteq (X, d)$ is closed if and only if E contains all of its limit

Proof. let $x \in E^c \Rightarrow$ there exists $r > 0$ such that $B(x, r) \subseteq E^c \Rightarrow x$ is no

$\Rightarrow \cancel{B(x,r) \cap E = \phi}$

$\Rightarrow \cancel{x \notin \bar{E}}$

$\therefore \cancel{x \in E^c \Rightarrow x \notin \bar{E}}$ } converse

if $\cancel{x \in \bar{E} \Rightarrow x \notin E^c \text{ i.e. } x \in E}$

$\Rightarrow \boxed{\cancel{\bar{E} \subseteq E}}$

$\Rightarrow \cancel{E \text{ contains all limit points}}$

Proof.

Assume E is closed To prove that any $x \in E^c$ is n

limit point.

lets say $x \in E^c$ is a limit p

$\Rightarrow$ for every $r > 0$

$B(x, r) \cap E \neq \phi$

as $B(\ldots$

Conversely let E contains all of its limit points.
To show that E is closed.
let $x \in E^c \Rightarrow x$ is NOT a limit point of E.
$\exists\, r > 0$ s/t $B(x,r) \cap E = \phi$. $\Rightarrow B(x,r) \subseteq E^c$.
$\Rightarrow B(x,r) \subseteq E^c$.
$\Rightarrow x$ is interior pt of E^c. (since the choice of x was arbitrary)
$\Rightarrow E^c$ is open.
$\Rightarrow E$ is closed.

Corollary:

See this again

- A subset $E \subseteq (X,d)$ is closed if and only if the following holds:
if $x_n \in E$ and $x_n \to x \Rightarrow x \in E$.

let E be closed $\Rightarrow E$ contains all its limit points.
$\Rightarrow$ let $\{x_n\} \subseteq E$ s/t $x_n \to x$.
$\Rightarrow x$ is a limit point
$\Rightarrow x \in E$.

for all sequences.
Conversely assume $x_n \in E$ and $x_n \to x \Rightarrow x \in E$
To show that E is closed.

If it is NOT closed then E^c is NOT open
then $\exists\, x \in E^c$ s/t $B(x,r) \not\subseteq E^c \quad \forall\, r > 0$
$\Rightarrow B(x,r) \cap E \neq \phi \quad \forall\, r > 0$
Take $r = \frac{1}{n} \; \forall\, n$
Then $B(x, \frac{1}{n}) \cap E \neq \phi$

Then $x_n \in B(x, \frac{1}{n}) \cap E$.
$\Rightarrow x_n \in E$ & $x_n \to x$.
$\Rightarrow x \in E \longrightarrow$ contradiction.
$\therefore E$ is closed.

Closure of a set:

Let (X,d) be a metric space and $E \subseteq X$

The closure of E is the set of all limit points of E.

We use the notation $\bar{E}$ to denote the closure of E.

Note:

(i) $E \subseteq \bar{E}$

(ii) $\bar{E}$ is the smallest closed set containing E.

(iii) Int A or A° is the largest open set contained in E.

Q.) Find $\bar{\mathbb{Q}}$, $\bar{\mathbb{Z}}$ in $(\mathbb{R}, |.|)$

$\bar{\mathbb{Q}} = \mathbb{R}$

$\bar{\mathbb{Z}} = \mathbb{Z} \cup \phi = \mathbb{Z}$.

Cauchy Sequence:

A sequence $\{x_n\}$ in (X,d) is called Cauchy Sequence if for every $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ $\forall$ $n, m \geq N$.

If $\{x_n\}$ is convergent then it must be Cauchy ~~& vice versa~~.

Q.1) $(X,d) = ((0,1], |.|)$ $x_n = \frac{1}{n} \in X$

Ans. $d(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right| < \underbrace{\frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2}}_{\Rightarrow \text{ from Archimedean Property}} = \epsilon$ $\forall$ $m, n \geq N = \left[\frac{2}{\epsilon}\right] + 1$

$\Rightarrow$ $\{x_n\}$ is Cauchy in X

$\lim_{n \to \infty} x_n = 0 \notin X \Rightarrow \{x_n\}$ is NOT convergent in X

$\therefore$ $\{x_n\}$ is Cauchy but NOT convergent in X

Q2) Show that Cauchy Sequence is bounded.

Show that Cauchy Sequence is bounded.

Sol. let $\{x_n\}$ be Cauchy in (X,d)

For $\epsilon = 1$ $\exists$ $k \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ $\forall$ $n, m \geq k$

for $m = k$, $d(x_n, x_k) < 1$ $\forall$ $n \geq k$. (Fix x_0)

let $M = \max\{ d(x_1, x_k), d(x_2, x_k) \ldots d(x_{k-1}, x_k), 1 \}$

$d(x_n, x_m) < M$ $\forall$ $n, m \in \mathbb{N}$

$\therefore$ $\{x_n\}$ is bounded.

Easy proof

Lemma: If a Cauchy Sequence has a convergent subsequence, then the sequence is convergent to the same limit.

Proof: let $\{x_n\}$ be a Cauchy Sequence in (X,d) and $\{x_{r_n}\}$ is a convergent subsequence and let $\lim_{n\to\infty} x_{r_n} = x$.

$\because$ $\{x_n\}$ is Cauchy in (X,d) $\Rightarrow$ x_{r_n} is Cauchy

Now we need to prove that $\{x_n\}$ converges to x

for arbitrary $\epsilon > 0$, $\exists$ $k_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$ $\forall$ $n, m \geq k_1$

Cauchy sequence

$\because$ $\lim_{n\to\infty} x_{r_n} = x$ $\Rightarrow$ for above $\epsilon > 0$. $\exists$ $k_2 \in \mathbb{N}$ such that

$d(x_{r_n}, x) < \frac{\epsilon}{2}$ $\forall$ $n \geq k_2$. Convergent subsequence

let $k = \max(k_1, k_2)$

Then $\forall$ $n, m \geq k$, $d(x_n, x_m) < \frac{\epsilon}{2}$ & $d(x_{r_n}, x) < \frac{\epsilon}{2}$

Putting $m = r_k$

$d(x_n, x_{r_k}) < \frac{\epsilon}{2}$ $(\because r_k \geq k)$

$d(x_n, x) \leq d(x_n, x_{r_n}) + d(x_{r_n}, x) < \epsilon$ $\forall$ $n \geq k$

$\Rightarrow$ $\{x_n\}$ is convergent to the same limit x.

Q. let $E \subseteq \mathbb{R}$, bounded and closed. Show that $\sup E \in E$ and $\inf E \in E$.

$\cdot$ Given E is bounded and closed, this implies that sup & inf exist.

let $c = \sup E$

For $\varepsilon > 0$, there exists $e \in E$ such that $c - \varepsilon < e < c$. — from def^n of sup

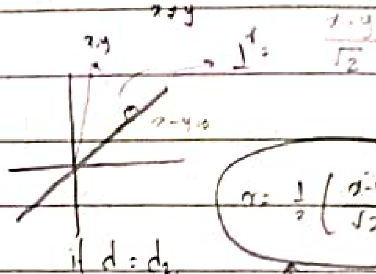
$\Rightarrow e \in (c-\varepsilon, c)$

$\Rightarrow \boxed{e \in B(c, \varepsilon)}$ and $e \in E$

$\Rightarrow B(c,\varepsilon) \cap E \neq \phi$.

$\Rightarrow c$ is a limit point of E.

$\because E$ is closed $\Rightarrow c \in E$. $\Rightarrow \boxed{\sup E \in E}$.



Similarly, $\boxed{\inf E \in E}$. $x > y$

$x = \frac{1}{2}\left(\frac{x-y}{\sqrt{2}}\right)$

H.W) Let (X, d) be a metric space. Let $E = \{(x,x) : x \in X\}$ be a subset of $(X$ … Show E is closed.

$\longrightarrow d \times d((x,y),(x',y')) = \max(d_0(x,x'), d(y,y'))$

Q2) a) Let $B[0,1] = \{ f : [0,1] \to \mathbb{R},\ f \text{ is bounded} \}$ Show that $(B[0,1], \|.\|$ … is a metric space.

See again

b) $C[0,1]$ is a closed subset of $B[0,1]$. under sup norm.

Sol^n We show whenever $f_n \in B[0,1]$ and $f_n \to f$, we have $f \in C[0,1]$

every convergent sequence has a limit in C

Now, $f_n \to f$ in $B[0,1]$

$\Leftrightarrow d(f_n, f) < \epsilon \quad \forall\ n \geq N_\epsilon$.

$\Leftrightarrow \sup\limits_{x \in [0,1]} |(f_n - f)(x)| < \epsilon \quad \forall\ n \geq N_\epsilon$.

$\Leftrightarrow f_n \to f$ uniformly on $[0,1]$ — from the "sup" theorem.

$\Leftrightarrow f \in C[0,1]$

$\Rightarrow C[0,1]$ is a closed subset of $B[0,1]$.

Q.3) Let $l^2 = \{ \{x_n\} : \sum\limits_{n=1}^{\infty} |x_n|^2 < \infty \}$, $l^\infty = \{ \{x_n\} : \sup\limits_n |x_n| < \infty \}$.

$l^1 = \{ \{x_n\} : \sum\limits_{n=1}^{\infty} |x_n| < \infty \}$

Verify (l^1, d_1), (l^2, d_2), (l^∞, d_∞) are ~~sp~~ metric spaces.

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Complete Metric Space:

A metric space (X, d) is complete if every Cauchy sequence in (X, d) converges to an element of X.

Ex: $X = (0,1)$ is NOT complete. eg: $\{\frac{1}{n}\}$ converges to $0 \notin (0,1)$

Ex: $\mathbb{R}^n$ is complete.

Theorem: Assume (X, d) is a complete metric and $A \subseteq X$. Then (A, d) is complete iff only if A is closed in X. (Easy proof)

Proof. Let (A, d) be complete.

To show that A is closed in X.

Let $\{x_n\}$ be a Cauchy sequence in A.

$\Rightarrow$ $\{x_n\}$ is convergent and converges to $x \in A$.

$\Rightarrow$ A is closed in X. (since the choice of x_n was arbitrary)

Conversely, A is closed in X. To show (A, d) is complete.

Let $\{x_n\}$ be a Cauchy sequence in (A, d)

$\Rightarrow$ $\{x_n\}$ is also a Cauchy sequence in (X, d)

$\because$ X is complete $\Rightarrow$ $\{x_n\} \to x$ in X

$x_n \in A$ and $x_n \to x$ $\Rightarrow$ x is a limit point of A.

As A is closed $\Rightarrow$ $x \in A$.

$\{x_n\}$ is convergent $\Rightarrow$ (A, d) is complete.

Continous functions:

Defn: Let (X, d) and (Y, d') be metric spaces. A function $f: X \to Y$ is said to be continous at $x \in X$ if and only if every sequence $\{x_n\}$ in X converging to x, we have $f(x_n) \to f(x)$ in Y.

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0 \ \forall x$.

Let $a \in \mathbb{R}$ Then

$$d(f(a), f(0)) = |f(a) - f(0)| = 0 < \epsilon \quad \text{when } |a - 0| < \delta$$

$\Rightarrow$ f is continous at a.

Since a is arbitrary f is continous on $\mathbb{R}$

Ex: $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x$

let $\{x_n\} \to x$

then $f(x_n) = x_n \to x = f(x)$.

Theorem: let (X, d) be a metric space. let $x \in X$

(i) If $f, g : X \to \mathbb{R}$ are continous at x, then $f+g$, αf. $(\alpha \in \mathbb{R})$ are also contin

↳ Hence $C(X, x) = \{ f : X \to \mathbb{R},\ f$ is continous at $x \}$ forms a vector space over $\mathbb{R}$

(ii) If $f: X \to \mathbb{R}$ is continous at $x \in X$ and if $f(x) \neq 0$ then $\exists\ r > 0$ s/t $f(y) \neq 0$ $\forall y \in B(x, r)$ and the function $g(y) = \frac{1}{f(y)}$ from $B(x, r)$ to $\mathbb{R}$ is continous at x

Practice:

1) $P(x_1, x_2, \ldots x_n)$ is continous function from $\mathbb{R}^n$ to $\mathbb{R}$.

Show that π_i is continous.

↳ [Hint: $\pi_i : \mathbb{R}^n \to \mathbb{R}$ $(x_1, x_2 \ldots x_n) \mapsto x_i$ on $(\mathbb{R}^n, d_\infty)$ to $(\mathbb{R}, |.|)$ Use it to prove]

2) $M(n, \mathbb{R}) = \{ A_{n \times n} \mid a_{ij} \in \mathbb{R} \}$ set of all real matrices.

a) Show that $(M(n, \mathbb{R}), d_\infty)$ is a metric space.

b) $f: M(n, \mathbb{R}) \to \mathbb{R}$, $f(A) = \det A$ is continous f is poly.

Extra Notes on Continuity

Ex: Show that $\mathbb{R}^2 \to \mathbb{R}$ given by $\alpha: (x,y) \mapsto x+y$ and $\vartheta: (x,y) \to xy$ are continuous

Proof: let $(x_n, y_n) \to (x,y)$ in $\mathbb{R}^2$

$\Rightarrow$ This means $x_n \to x$ and $y_n \to y$.

Using algebra of convergent sequences

$$x_n + y_n \to x + y \Rightarrow \alpha \text{ is continuous}$$

A similar argument can be made for ϑ.

Space of Continuous functions:

let (X, d) be a metric space. and $x \in X$

(i) If $f, g : X \to \mathbb{R}$ are continuous at x then so are $f+g$, fg, αf $(\alpha \in \mathbb{R})$. Consequently the set of functions continuous at x forms a real vector space. (Proof by algebra of convergent sequences)

(ii) If $f: X \to \mathbb{R}$ is continuous at $x \in X$ and if $f(x) \neq 0$, then there exists $r > 0$ such that $f(x') \neq 0$ for all $x' \in B(x,r)$ and the function $g(x') = \frac{1}{f(x')}$ from $B(x,r)$ to $\mathbb{R}$ is continuous at x.

Ex: Polynomial functions $p(x_1, x_2, \ldots x_n)$ where $p: \mathbb{R}^n \to \mathbb{R}$ will be continuous.

Theorem: let X, Y, Z be metric spaces. let $f: X \to Y$ be continuous at $x \in X$ and $g: Y \to Z$ be continuous at $y = f(x)$. Then the composition map $g \circ f: X \to Z$ is continuous at $x \in X$

Proof. $x_n \to x$. Then $y_n = f(x_n) \to y = f(x)$

Since g is continuous at $y \Rightarrow g(y_n) \to g(y)$ or

$g(f(x_n)) \to g(f(x))$. (Hence proved).

Equivalent Characterisations of Continuity:

let X, Y be metric spaces. let $f: X \to Y$ be a function. Then the following are equivalent

(a) f is continuous at $x \in X$

(b) Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x') < \delta$ then $d(f(x), f(x')) < \varepsilon$

(c) Given an open set V containing $f(x)$ in Y, we can find an open set U containing x such that $f(U) \subset V$.

Proof. (a) $\Rightarrow$ (b).

We shall prove this by contradiction:

for given $\epsilon > 0$, there exists no δ that satisfies the required property.

let take $\delta = \frac{1}{n}$ and $x' = x_n$

Then there exists x_n a x' such that $d(x, x_n) < \frac{1}{n}$ but $d(f(x), f(x_n)) \geq \epsilon$

As $d(x_n, x) < \frac{1}{n}$ this implies $x_n \to x$

Since f is continuous $f(x_n) \to f(x)$ $\Rightarrow$ $d(f(x_n), f(x)) \to 0$ as $n \to \infty$

But this is a contradiction as $d(f(x), f(x_n)) \geq \epsilon$ (unbounded)

(b) $\Rightarrow$ (c).

take V as an open set containing $f(x)$.

let V be given such that V is open and $f(x) \in V$

This implies that there exist $\epsilon > 0$ such that $B(f(x), \epsilon) \in V$. (by open sets def^n)

Since we assume (b), for each $\epsilon > 0$ there exists a $\delta > 0$ s/t $d(x, x') < \delta$.

That is let $U := B(x, \delta)$, then U is an open set that contains x and is such that $\boxed{f(U) \subset V}$.

(c) $\Rightarrow$ (a):

let $x_n \to x$. We need to show that $f(x_n) \to f(x)$.

let $\epsilon > 0$ be given then $V = B(f(x), \epsilon)$ is an open set containing $f(x)$, then by (c) there exists an open set $U \subset X$ such that U contains x & $f(U) \subset V$

Since U is open, $\exists\, r > 0$ s/t $B(x, r) \subset U$ (by def^n of open set).

As $x_n \to x$ $\exists$ N s/t for all $n \geq N$ $x_n \in B(x, r)$ (Apply $f(\cdot)$ both side)

This implies $f(x_n) \in$ ~~B(f(~~ $f(B(x,r)) \subset f(U) \subset V = B(f(x), \epsilon)$

$\Rightarrow$ $f(x_n) \to f(x)$.

Def^n: let X, Y be topological spaces and $f: X \to Y$ be a map. We say that f is continuous at x if given an open set V containing $f(x)$ we can find another open set containing x s/t $f(U) \subset V$.

Practice Problems:

Q 1.) Let X, Y be metric spaces. Show that $f: X \to Y$ is continuous iff for every open set $V \subset Y$, $f^{-1}(V)$ is open in X.

Proof: Suppose f is continuous.

let G be an open set in Y. To prove that $f^{-1}(G)$ is an open set in X.

→ If $f^{-1}(G) = \phi$, $f^{-1}(G)$ is open.

→ If $f^{-1}(G) \neq \phi$

$x \in f^{-1}(G)$

⇒ $f(x) \in G$. (G is open). every point is an interior point.

$\exists\ r > 0$ s/t $B(f(x), r) \subset G$ —— ①

By def^n of continuity there exists an open ball $B(x, \delta)$ s/t

$f(B(x,\delta)) \subset B(f(x), \epsilon)$. (from ①).

⇒ $f(B(x,\delta)) \subset G$. (because G contains $f(x)$.

⇒ $B(x,\delta) \subset f^{-1}(G)$.

As $x \in f^{-1}(G)$ was arbitrary. and we prove that $\exists\ \delta$ s/t $B(x,\delta) \subset f^{-1}(G)$

Every point of $f^{-1}(G)$ is an interior point hence $f^{-1}(G)$ is open.

Conversely, assume that $f^{-1}(G)$ is open in X whenever G is open in Y.

To prove that f is continous.

let $B(f(x), \epsilon)$ be an open set in Y.

$f^{-1}(B(f(x), \epsilon))$ is open in X.

$x \in f^{-1}(B(f(x), \epsilon))$

x is an interior point of X / open — open set.

There exists an open ball $B(x,\delta) \subset f^{-1}(B(f(x), \epsilon))$

↳ def^n of continuity.

So, f is continous

Remark: Same thing for closed sets too.

Q.2) Show that the set of all invertible matrices ($Gl_n(\mathbb{R})$) in $M(2,\mathbb{R})$ is open.

A: $n \times n$ matrix.

By Cramer's Rule $\det(A)$ is a polynomial and hence it is continous.

$\det\ \mathbb{R}^{n^2} \to \mathbb{R} - \{0\}$ for invertible matrices.

$\mathbb{R} - \{0\}$ is an open set in $\mathbb{R}$, hence by continuity $f^{-1}(\mathbb{R} - \{0\})$ is open in $\mathbb{R}^2$.

So $Gl_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.

$\phi(A) = A^3$ and continuous $\phi^{-1}(\{0\})$ is closed

Q2) Show that the set of all nilpotent matrices in $M(n, \mathbb{R})$ is closed.

[Pg-8/23]

Uniform Continuity:

Let $f : (X, d) \to (Y, d)$ be a function. We say that f is uniformly continuous on X if for a given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that whenever $x_1, x_2 \in X$ s/t $d(x_1, x_2) < \delta$ we have $d(f(x_1), f(x_2)) < \epsilon$.

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Compactness.

Open Cover:

Let X be a metric space and $A \subset X$. A family of subsets $\{U_i \mid i \in I\}$ is called an open cover of A if each U_i is open and (index sets)

$$\boxed{A \subset \bigcup_{i \in I} U_i}$$

If $J \subset I$ and $A \subset \bigcup_{i \in J} U_i$ then and J is finite. Then we say that the given cover admits a finite sub-cover.

Ex: Consider the family $S = \{(1/n, 1) : n \in \mathbb{N}, n \geq 2\}$ We show that ~~is~~ S is an open cover of $(0,1)$

→ To show that $(0,1) \subset \bigcup_{i=2}^{\infty} (1/i, 1)$

let say $x \in (0,1)$ then we want to show $\frac{1}{n} < x < 1$.

By Archimedes principle there exists $N \in \mathbb{N}$ such that $Nx > 1$ or $N > \frac{1}{x}$ or $x > \frac{1}{N}$ and $x < 1$

$\therefore$ $(0,1) \subset \bigcup_{i=2}^{\infty} (\frac{1}{i}, 1)$. (Proved)

→ We claim that this open cover admits no finite subcover.

Suppose it does

$$(0,1) \subset \bigcup_{j=1}^{k} \left(\frac{1}{n_j}, 1\right).$$

$N = \max(n_j : 1 \leq j \leq k)$

Then we know $\boxed{\frac{1}{N} \in (0,1)}$ because $N \in \mathbb{N}$.

Also

$$\bigcup_{j=1}^{k} \left(\frac{1}{n_j}, 1\right) = \left(\frac{1}{N}, 1\right)$$

and $(0,1) \subset (\frac{1}{N}, 1)$

→ $\boxed{\frac{1}{N} \notin (0,1)}$ $\Rightarrow \Leftarrow$

$GL(n, R) \rightarrow$ not compact = No! (open)

$SL(n, R) \rightarrow$ not compact $\rightarrow$ (not bounded) $\begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$

Orthogonal matrices $\rightarrow$ compact

Def^n: A subset $K \subset X$ of a m.s. is said to be compact if every open cover of K admits a finite sub-cover.

$A = \{x_1, x_2, \dots x_n\}$

$A \subseteq \bigcup_{i \in I} U_i$

$x_i \in U_i$ for some $i \in I$

then $A \subseteq \bigcup_{i=1}^{n} U_i$

Theorem: Any finite subset of a metric space is compact.

Proof: ~~S = ϕ~~ For a finite subset of a metric space, the open cover itself will be finite. Hence it is compact.

Heine-Borel Theorem for $\mathbb{R}$

Let $J := [a, b]$ be a closed and bounded set in $\mathbb{R}$. Then any open cover of $[a, b]$ admits a finite sub-cover a in other words $[a, b]$ is compact.

Proof. S be a bounded set.

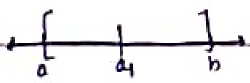
$S \subset [a, b]$

Let $C = \{ A_i \mid i \in \mathbb{N} \}$ be an open cover of $[a, b]$

$\Rightarrow [a, b] \subset \bigcup_{i \in \mathbb{N}} A_i$

Let us assume that C has NO finite sub-cover.

Then bisecting $I_0 = [a, b]$ at $\frac{a+b}{2} = a_1$



Let say there doesn't exist a finite subcover in atleast one of the intervals $[a, a_1]$ and $[a_1, b]$

~~let us~~ WLOG let us assume this interval by $I_1 = [a_1, b_1] = [a, a_1]$.

So, $[a_1, b_1]$ has no finite sub-cover.

$[a_2, b_2]$

Then we keep on repeating this process:

We get a sequence of nested interval

$I_0 \supset I_1 \supset I_2 \dots \supset I_n$ in which no finite sub-cover of any interval exist in C.

Clearly I_n^{th} interval has length $\frac{b-a}{2^n}$.

Now $\lim_{n\to\infty} l(I_n) = 0$.

By nested interval theorem we have: $\bigcap_{n=1}^{\infty} I_n \neq \phi$

let $x_0 \in \bigcap I_n$

i.e. for $\epsilon > 0$ and $\exists k \in N$.

~~$(x_0 - \epsilon, x_0 + \epsilon)$~~ $\boxed{I_k \subset (x_0 - \epsilon, x_0 + \epsilon)}$

Consequently $\exists$ a finite sub-cover for I_k in C.

$\Rightarrow$ Our assumption is false.

$\Rightarrow$ C has a finite subcover.

$[a, b]$ is compact (Hence proved).

Ex: Find all compact subsets of a discrete metric space. [All finite subsets of X]

(Pg-6) Theorem: Any compact subset of a metric space is closed and bounded.

Theorem: Any continuous function f from a compact metric space X to another metric space Y is bounded then $f(X)$ is a bounded subset of Y.

Proof: Fix $y \in Y$. for each $n \in \mathbb{N}$, $B(y, n)$ is an open set in Y

$U_n = f^{-1}(B(y, n)) = \{x \in X : d(f(x), y) < n\}$ is open

(by continuity def^n)

$U_n \subset U_{n+1}$

This is important $\Rightarrow$ So the collection $\{U_n : n \in \mathbb{N}\}$ is an open cover for compact space X.

If $\{U_{n_j} : 1 \leq j \leq m\}$ is a finite subcover & $N = \max(n_1, n_2, \ldots n_m)$

$\Rightarrow X = U_N$? take function both sides.

$\Rightarrow \boxed{f(X) \subset B(y, N)}$ Proved

$\hookrightarrow$ bounded.

Did proof Theorem: Any continuous function from a compact metric space to any other metric space is uniformly continuous.

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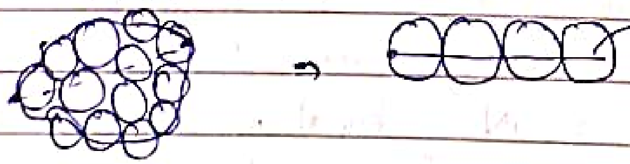
Def^n: Let A be a subset of the metric space. We say that A is totally bounded if for every $\epsilon > 0$ we can find a finite set of points x_i, $1 \le i \le n$ s/t $A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

⇒ Compact Space is totally bounded because of the finite sub cover
Pg-16. take open balls.

Theorem:
For a metric space, the following are equivalent.
(i) X is compact.
(ii) X is complete & totally bounded.
(iii) Every sequence has a convergent subsequence.

Theorem:
Every totally bounded set is bounded but the converse is NOT true.

Proof: $B \subseteq X$ be a totally bounded set.
For $\epsilon > 0$, there are n-open balls that cover B. no. of balls.



$\le 2n\epsilon$.

Take any two points then
$$d(x,y) < 2n\epsilon = r.$$

∞-set in discrete metric space.

Converse:
$M = \{ x \in l^\infty \mid \|x\| \le 1 \}$ is a closed unit ball in l^∞.
⇒ M is bounded subset of l^∞.
M is NOT totally bounded.

Theorem: Continuous image of a compact metric space is compact.