Option pricing in discrete time:

Let $(S_t)_{t\geq 0}$ price process of a security (stock) that is traded in the market

Let risk free, fixed interest investment $(B_t)_{t\geq 0}$ (bond) with interest rate $r \geq 0$,

value of bond at time t $\quad B_t = (1+r)^t B_0 \quad$, $B_0 \to$ value of bond at time $t=0$

security/stock, derivative (forwards/futures/options such as calls and puts)

Forward and Futures:

Forwards are contracts that give the market participant the right to buy or sell an underlying or financial asset at a time T in the future or in the future time period $[T, T']$ at a fixed price K.

long position: entering into a contract to buy.

Short position: entering into a sale contract.

While futures are traded on financial markets, forwards are based on an individual agreement btw the participants without market intervention.

Derivative security (or contingent claim) with

... claim, with maturity (expiry) date T in our market is a fn

$$X = X(\omega) = g(S_T(\omega)) \geq 0.$$

of the underlying asset price S_T at time T.
The financial interpretation of X is that the contract will pay its owner the amount X at time T.

Binomial markets:

time $t = 0, 1, \dots, T$

- a bond (or bank account) yielding a riskless rate r of return in each time period

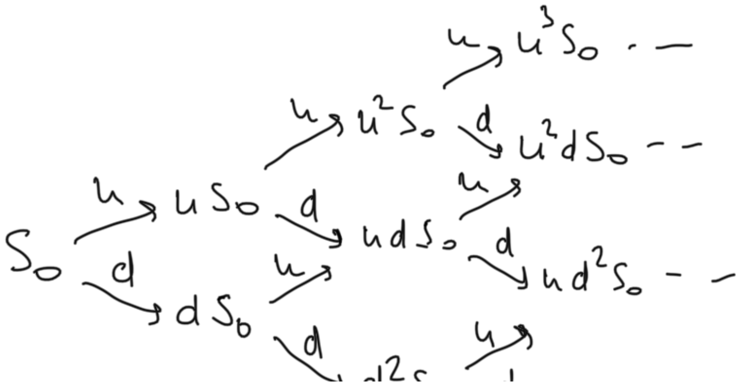
$$B_t = (1+r)^t \qquad , \quad t = 0, 1, \dots, T$$

- a risky asset (stock)

at time t, S_t

$$S_{t-1} \begin{cases} \nearrow S_t = u S_{t-1} \\ \searrow S_t = d S_{t-1} \end{cases} , \quad t = 1, 2, \dots, T$$

$$d < 1 + r < u$$



$$d^3 S_0 \quad - -$$

T time 0   1   2   3   - - -

$$\Omega = \{ \omega = (\nu_1, --, \nu_T) : \nu_t = u \text{ and } d, \ t = 1, 2, --, T \}$$

$$S_t = S_t(\omega) = \nu_1 \nu_2 - - \nu_t S_0$$ this price process is adapted to filtration $\{\mathcal{F}_t\}$

Single-period Binomial market:

trading strategy / portfolio (Δ, b) at time $t=0$

Δ: # of shares, b: # of bonds

at time t

$$V_t = V_t(\omega) = \begin{cases} \Delta S_0 + b B_0 = \Delta S_0 + b, & \text{at time } t=0 \\ \Delta S_1 + b B_1 = \Delta S_1 + b(1+r), & \text{at time } t=1 \end{cases}$$

The portfolio will be a hedge given that

$$V_1(\omega) \geq X(\omega), \quad \omega \in \Omega = \{u, d\}, \ T=1$$

$$X(\omega) = \{X_u, X_d\}$$

hedge condition

$$\Delta u S_0 + b(1+r) \geq X_u$$

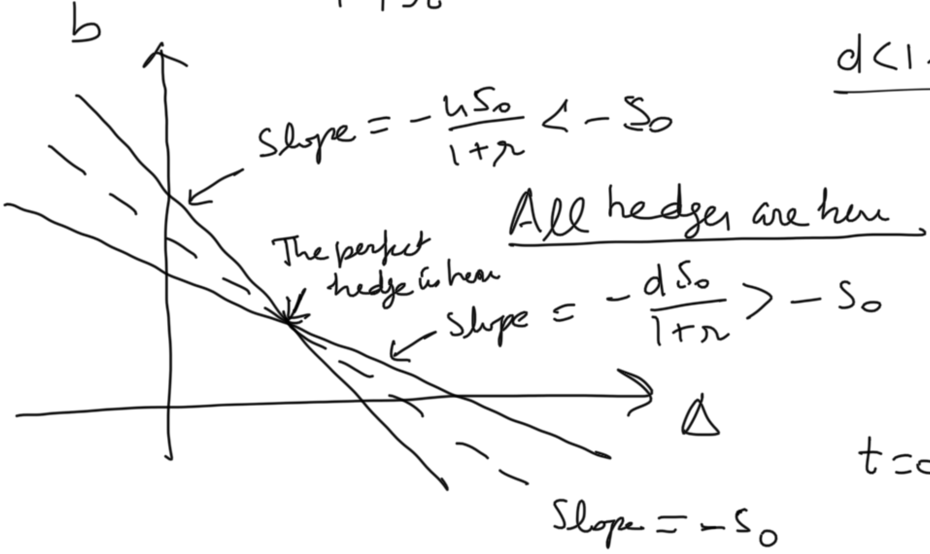
$$\Delta d S_0 + b(1+r) \geq X_d$$

$$\equiv \quad b \geq -\frac{u S_0}{1+r} \Delta + \frac{X_u}{1+r} \quad \text{--- (1)}$$

$$b \geq -\frac{d S_0}{\quad} \Delta + \frac{X_d}{\quad} \quad \text{--- (2)}$$

$1+r \qquad\qquad 1+r$

$d < 1+r < u$



$t=0 \quad \Delta S_0 + b = \text{const}$

Cheapest hedge, i.e., has the smallest value at $t=0$ $\quad \min\limits_{\text{all hedges }(\Delta, b)} (\Delta S_0 + b)$

$$\therefore \quad \Delta u S_0 + b(1+r) = X_u$$

$$\Delta d S_0 + b(1+r) = X_d$$

$$\Rightarrow \Delta = \frac{X_u - X_d}{(u-d)S_0} = \frac{g(uS_0) - g(dS_0)}{uS_0 - dS_0} \longrightarrow (3)$$

discrete version of "delta-hedging rule" for derivative security.

$$b = \frac{X_u - \Delta u S_0}{1+r} = \frac{uX_d - dX_u}{(1+r)(u-d)} \longrightarrow (4)$$

$$\Big[\ uX_d - dX_u = b(1+r)(u-d)$$

(3), (4) is a perfect hedge for X and its time $t=0$ value is

$$V_0 = \Delta S_0 + b = \frac{X_u - X_d}{(u-d)} + \frac{uX_d - dX_u}{(\ldots)}$$

$$(u-d) \qquad (1+r)(u-d)$$

$$= \frac{1}{1+r}\left[\underbrace{\frac{1+r-d}{u-d}}_{p^*} X_u + \underbrace{\frac{u-(1+r)}{u-d}}_{1-p^*} X_d\right]$$

$$= \frac{1}{1+r} E^*(X) = E^*\left(\frac{X}{1+r}\right) =: X^*$$

We maintain that thus calculated X^* is the fair price of the claim X at $t=0$.

$$E_{p^*}(S_1 | S_0) = p^* u S_0 + (1-p^*) d S_0$$

$$= \frac{1+r-d}{u-d} u S_0 + \frac{u-(1+r)}{u-d} d S_0$$

$$= (1+r) S_0$$

$$E_{p^*}\left(\frac{S_1}{1+r} \middle| S_0\right) = S_0$$

$$E^*\left(\frac{S_1}{1+r}\right) = S_0$$

<u>Example</u> (Pricing a <u>European Call</u>)

Let $S_0 = 1$, $k = 1$, $r = 0.25$, $u = 1.75$, $d = 0.5$

$$p^* = \frac{1+r-d}{u-d} = \frac{1.25 - 0.5}{1.75 - 0.5} = 0.6, \quad 1-p^* = 0.4$$

$$\therefore C = \frac{1}{1+r}[p^* X_u + (1-p^*) X_d]$$

$$= \frac{1}{1.25}\left[0.6\,(uS_0-k)^+ + 0.4\,(dS_0-k)^+\right]$$

$$= \frac{1}{1.25}\left[0.6 \times 0.75 + 0.4 \times 0\right]$$

$$= 0.36$$

replicating portfolio has the form

$$\Delta = \frac{X_u - X_d}{(u-d)S_0} = \frac{0.75-0}{(1.75-0.5)\times 1} = 0.6$$

$$b = \frac{uX_d - dX_u}{(u-d)(1+r)} = \frac{0 - 0.5\times 0.75}{(1.75-0.5)\times 1.25} = -0.24$$

Now check at $t=0$ value of replicating portfolio coincide with call options we computed above and its time $t=1$ value replicate the call payoff

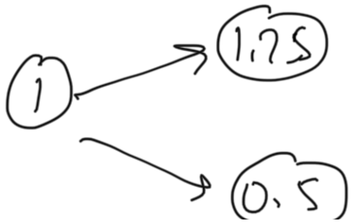
$$V_0 = \Delta S_0 + b = 0.36$$

$$V_1(u) = \Delta u S_0 + b(1+r) = 0.6\times 1.75\times 1 - 0.24\times 1.25 = 0.75$$

$$V_1(d) = \Delta d S_0 + b(1+r) = 0.6\times 0.5\times 1 - 0.24\times 1.25 = 0$$

In both cases, the value coincide with $(S_1-k)^+$ a perfect replication

Stock price



Call price