Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on $\mathscr{A}$, namely, that $\mathscr{A}$ be *self-adjoint*. This means that for every $f \in \mathscr{A}$ its complex conjugate $\bar{f}$ must also belong to $\mathscr{A}$; $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

7.33 Theorem *Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense $\mathscr{C}(K)$.*

Proof Let $\mathscr{A}_R$ be the set of all real functions on K which belong to $\mathscr{A}$.

If $f \in \mathscr{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since $\mathscr{A}$ is self-adjoint, we see that $u \in \mathscr{A}_R$. If $x_1 \neq x_2$, there exists $f \in \mathscr{A}$ such that $f(x_1) = 1, f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that $\mathscr{A}_R$ separates points on K. If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathscr{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$; hence $\mathscr{A}_R$ vanishes at no point of K.

Thus $\mathscr{A}_R$ satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on K lies in the uniform closure of $\mathscr{A}_R$, hence lies in $\mathscr{B}$. If f is a complex continuous function on K, $f = u + iv$, then $u \in \mathscr{B}$, $v \in \mathscr{B}$, hence $f \in \mathscr{B}$. This completes the proof.

EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.
3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

5. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \dfrac{1}{n+1}\right), \\ \sin^2 \dfrac{\pi}{x} & \left(\dfrac{1}{n+1} \le x \le \dfrac{1}{n}\right), \\ 0 & \left(\dfrac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series Σf_n to show that absolute convergence, even for all x, does not imply uniform convergence.

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

7. For $n = 1, 2, 3, \ldots$, x real, put

$$f_n(x) = \frac{x}{1+nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

8. If

$$I(x) = \begin{cases} 0 & (x \le 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\Sigma |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \le x \le b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

10. Letting (x) denote the fractional part of the real number x (see Exercise 16, Chap. 4, for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \text{ real}).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

11. Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

(*a*) Σf_n has uniformly bounded partial sums;

(*b*) $g_n \to 0$ uniformly on E;

(*c*) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\Sigma f_n g_n$ converges uniformly on E. *Hint*: Compare with Theorem 3.42.

12. Suppose g and $f_n (n = 1, 2, 3, \ldots)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^{\infty} g(x)\, dx < \infty.$$

Prove that

$$\lim_{n \to \infty} \int_0^{\infty} f_n(x)\, dx = \int_0^{\infty} f(x)\, dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathscr{R}$. (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179–186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182–187.)

13. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on R^1 with $0 \leq f_n(x) \leq 1$ for all x and all n.

(*a*) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

for every $x \in R^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem.*)

(*b*) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$, for any $x \in R^1$, to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points. This proves (*a*). To prove (*b*), modify your proof of (iii) appropriately.