1. By definition (4.1),

$$V_n(i) = \max_{\{a_{T-n+1},\dots,a_T\}} \mathbf{E}\left[\sum_{t=T-n+1}^{T} R(X_t,a_t)\middle| X_{T-n+1}=i\right]$$

$$= \max_{\{\cdots\}}\left\{R(X_{T-n+1},a_{T-n+1}) + \mathbf{E}\left[\sum_{t=T-n+2}^{T} R(X_t,a_t)\middle| X_{T-n+1}=i\right]\right\},$$

where by the TPF the last expectation can be computed as

$$\sum_j \mathbf{E}\left[\cdots \mid X_{T-n+1}=i, X_{T-n+2}=j\right] \mathbf{P}\left(X_{T-n+2}=j \mid X_{T-n+1}=i\right)$$

$$= \sum_j \mathbf{E}\left[\cdots \mid X_{T-n+2}=j\right] p_{ij}(a_{T-n+1})$$

by Markov property. Complete the argument!

2. (i) $X_t = Z_t$ if has not bought yet, $X_t = 0$ otherwise, $t = 1,2,3,4$ (with $Z_4 = \infty$ to make the person purchase the land!). Actions: $a = 1$ is "buy"; $a = 0$ is "do nothing". Transition probabilities for $a = 0$ at different times t are specified by the table, while $a = 1$ always means transition to 0. Reward function: $R(x,0) = 0$, $R(x,1) = -x$.

(ii) As the process in non-homogeneous (in time), we have

$$V_n(i) = \max_a \Big[R(i,a) + \mathbf{E}_a(V_{n-1}(X_{5-n}) \mid X_{4-n}=i)\Big], \quad n = 1,2,3$$

(since $T = 3$, $T-n+1 = 4-n$ now) with $V_0(x) = -x$ (inflicting a huge penalty in case the person has not bought land during the three days).

(iii) Decision tree: DIY. Optimal policy: in week one, buy if the price is 2.2 else wait; in week two, buy if the price is 2.2 or 2.3, else wait; in week three buy if you haven't yet. The minimum expected price is 2.2574 (i.e., \$225,740).

3. The optimality equation becomes

$$V_n(x) = \max_{a\in[0,1]} \mathbf{E}\, V_{n-1}(x + axZ) = \max_{a\in[0,1]} \Big(pV_{n-1}(x(1+a)) + qV_{n-1}(x(1-a))\Big),$$

so that, since $V_0(x) = \log x$,

$$V_1(x) = \log x + \max_{a\in[0,1]} (p\log(1+a) + q\log(1-a)).$$

(i) If $p \le 1/2$, the function on the right-hand side is decreasing in $a \in [0, 1]$, so the optimal $a = 0$, $V_1(x) = \log x$. Repeating the argument, we derive that all $V_n(x) = \cdots = V_1(x) = V_0(x) = \log x$, and the optimal action is always $a = 0$.
(ii) If $p > 1/2$, the maximum is attained at $a^* = 2p - 1 = p - q$. As $V_1(x) = \log x + c$, $c = \mathbf{E}(1 + a^* Z) = \text{const}$, repeating the argument yields that a^* is optimal at each step.

4. (i) $X_t = Z_t$ if has not sold yet; $X_t = 0$ otherwise. Actions: $a = 1$ is "sell"; $a = 0$ is "do nothing". The evolution of $\{X_t\}$: given $X_t = 0$, $X_{t+1} = 0$ for any a; given $X_t = x > 0$, $X_{t+1} = 0$ if $a = 1$ and $X_{t+1} = Z_{t+1}$ if $a = 0$. Reward function: $R(x, 0) = 0$, $R(x, 1) = x$. The sum of one-step reward equals the only term ($\neq 0$) giving the selling price.
(ii) $V_n(x) = \max_{a=0,1}\big[R(x,a) + \mathbf{E}_a(V_{n-1}(X_1)|\, X_0 = x)\big] = \max\{\mathbf{E}_0(V_{n-1}(X_1)|\, X_0 = x),\, x\}$, where the subscript a indicates that the expectation is taken under action a. If $x = 0$, then $V_n(x) = 0$, so can only consider case $x > 0$, and then

$$V_n(x) = \max\{\mathbf{E}\, V_{n-1}(Z),\, x\}, \quad Z \sim U(0,1),$$

with the initial condition $V_0(x) = 0$ (as nothing can be gained after time $T = 4$). Solution:

$n = 1$: For $x > 0$, $V_1(x) = x$ and the optimal action (for which the maximum is attained) is always $a = 1$.

$n = 2$: $V_2(x) = \max\{\mathbf{E}\, V_1(Z),\, x\} = \max\{\mathbf{E}\, Z,\, x\} = \max\{1/2,\, x\}$. Optimal action: $a = 1$ iff $x > 1/2$.

$n = 3$: Here

$$V_3(x) = \max\{\mathbf{E}\, V_2(Z),\, x\} = \max\{\mathbf{E}\, \max\{1/2,\, Z\},\, x\} = \max\{5/8,\, x\}$$

since (using the hint)

$$\mathbf{E}\, \max\{c,\, Z\} = c\,\mathbf{P}(Z \le c) + \int_c^1 x\, dx = \frac{1}{2}(1 + c^2).$$

The optimal action: $a = 1$ iff $x > 5/8$.

$n = 4$: Now

$$V_4(x) = \max\{\mathbf{E}\, V_3(Z),\, x\} = \max\{\mathbf{E}\, \max\{5/8,\, Z\},\, x\} = \max\{89/128,\, x\}.$$

The optimal action: $a = 1$ iff $x > 89/128$.

(iii) Day 1: sell if $Z_1 > 89/128 \approx 0.695$. Day 2: sell if $Z_2 > 5/8 = 0.625$. Day 3: sell if $Z_1 > 1/2 = 0.5$. Day 4: sell. Maximum expected price: $\mathbf{E}\, V_4(X_1) = \mathbf{E}\, \max\{89/128,\, Z\} \approx 0.742$.

5. As we proved that $\{s_n\}$ is non-decreasing, it suffices to show that $s_2 = \infty$. Since $V_1 \ge s - c$, we get $\mathbf{E}\, V_1(s + Y_1) \ge \mathbf{E}\,(s + Y_1 - c) = s - c + \mu > s - c$, so from (4.4) with $n = 2$ we see that (4.5) holds for all s, i.e., $s_2 = \infty$.

6. (i) $\mathbf{E}\,Y = \sum_{j\le n}\lambda_j \mathbf{E}\,X_j = \mu$; $\mathrm{Var}\,(Y) = \sigma^2\sum_{j\le n}\lambda_j^2 = \sigma^2(\lambda_1^2+\cdots+\lambda_{n-1}^2+(1-\lambda_1-\cdots-\lambda_{n-1})^2)$. Solve $\partial(\cdots)/\partial\lambda_j = 0$, $j=1,\ldots,n-1$, to get $\lambda_j = 1/n$ for all j (and note that this is a minimum indeed!).

(ii) First show that $f(\boldsymbol{\lambda}) := \mathbf{E}\,u\Big(\sum_{j=1}^n \lambda_j X_j\Big)$ is a strictly concave function of $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)$: for any $\boldsymbol{\lambda}'$, $\boldsymbol{\lambda}''$ and $\alpha\in(0,1)$,

$$f(\boldsymbol{\lambda}) > \alpha f(\boldsymbol{\lambda}') + (1-\alpha) f(\boldsymbol{\lambda}''). \tag{13.46}$$

Next assume that the maximum of $f(\boldsymbol{\lambda})$ is attained at a point $\boldsymbol{\lambda}'$ such that $\lambda_i' \ne \lambda_j'$ for some $i\ne j$. Define $\boldsymbol{\lambda}''$ by setting $\lambda_j'' := \lambda_i'$, $\lambda_i'' := \lambda_j'$, and $\lambda_k'' := \lambda_k'$ for all $k\ne i,j$. As $\sum \lambda_k'' X_k$ has the same distribution as $\sum \lambda_k' X_k$ (since $X_1,\ldots,X_n$ are exchangeable), $f(\boldsymbol{\lambda}'') = f(\boldsymbol{\lambda}')$ is also a minimum, and by taking $\alpha = 1/2$ we see from (13.46) that for the midpoint $\boldsymbol{\lambda} = (\boldsymbol{\lambda}'+\boldsymbol{\lambda}'')/2$, $f(\boldsymbol{\lambda}) > (f(\boldsymbol{\lambda}')+f(\boldsymbol{\lambda}''))/2 = f(\boldsymbol{\lambda}')$, a contradiction! So must have $\lambda_i' = \lambda_j'$ for all i,j at the maximum point.

7. Since $u_n(x) = \max\{x,\mu_{n-1}\}$, assuming $\mu_{n-1}\le\mu_n$, we get

$$\begin{aligned}\mu_{n+1}-\mu_n &= \alpha\mathbf{E}\,(u_{n+1}(Z)-u_n(Z))\\ &= \alpha(\mu_n-\mu_{n-1})\,\mathbf{P}\,(Z\le\mu_{n-1}) + \alpha\mathbf{E}\,((Z-\mu_{n-1});\ \mu_{n-1}<Z\le\mu_n).\end{aligned}$$

Clearly, $0\le$ (right-hand side) $\le \alpha|\mu_n-\mu_{n-1}|$. Similar argument if $\mu_{n-1}>\mu_n$.