Financial Mathematics.

Discrete Time Process:

A collection of Random Variables $\{X_t\}_{t\in T}$ on a common probability space $(\Omega, \mathbb{F}, P)$ where the set T of time is usually $\{0,1,2,\ldots\}$, could be finite $\leq T$ or infinite.
is discrete.

$\mathcal{F}_t$: A collection of events "observed" by time t.

Filtration:

Defined as an increasing sequence of sub-σ-fields $\mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F}_2 \ldots \subset \mathbb{F}$.
We say that a Stochastic Process $\{X_t\}_{t\geq 0}$ is adapted to filtration $\mathbb{F}$ if for any $t = 0,1,2,\ldots$ the random variable X_t is $\mathcal{F}_t$-measurable i.e.

$$\{X_t \in \mathbb{B}\} \in \mathcal{F}_t \quad \text{for any Borel Set } \mathbb{B} \in \mathcal{B}$$

$\{\Omega, F, \mathbb{F}, P\}$: Filtered Probability Space.

Martingale:

Given a filtered probability space $(\Omega, F, \mathbb{F}, P)$, one is given a stochastic process $\{X_t\}_{t\geq 0}$ that is adapted to the filtration $\mathbb{F}$. Then $\{X_t\}$ is a Martingale iff

$$\boxed{E(X_{t+1} \mid \mathcal{F}_t) = X_t.} \quad \text{provided } E|X_t| < \infty \quad \forall\, t = 0,1,2,\ldots$$

Sub-Martingale : $E(X_{t+1} \mid \mathcal{F}_t) \geq X_t \quad \forall\, t = 0,1,2,\ldots$
Super-Martingale : $E(X_{t+1} \mid \mathcal{F}_t) \leq X_t \quad \forall\, t = 0,1,2,\ldots$

Some Properties of Conditional Expectations:

1. $E(aX + bY \mid G) = aE(X\mid G) + bE(Y\mid G)$.
2. If Z is G-measurable then $E(ZX\mid G) = Z\,E(X\mid G)$.
   G contains info about Z. behaves like a constant.
3. If X is independent of G then $E(X\mid G) = E(X)$
   In particular then if $X =$ const or $G = \{\phi, \Omega\}$ then $E(X\mid G) = X$.
4. If $G_0 \subset G_1 \subset F$ are σ-fields then smaller subset "eats"
   $$E[E(X\mid G_1)\mid G_0] = E[X\mid G_0]$$
5. If $X_1 \leq X_2$ then $E(X_1\mid G) \leq E(X_2\mid G)$.

For a Martingales:

$F_t \subset F_{t+s-1}$

$$E[X_{t+s} | F_t] = E[\underbrace{E(X_{t+s} | F_{t+s-1})}_{X_{t+s-1}} | F_t]$$

$\forall s \geq 1$

$$= \dots = E(X_{t+1} | F_t) = X_t.$$

So given the past an MG always stays on a constant level on an average.

$\Rightarrow$ For a Continuous Time Martingale we have: $E(X_{t+s} | F_t) = X_t$ for any $s,t \geq$ $(s,t \geq 0)$

Examples of Martingales:

① Random Walks:

$X_0 = 0$

$X_n = Y_1 + Y_2 + \dots Y_n$ where Y_i's are i.i.ds and $E|Y_1| < \infty$

If no filtration is mentioned then we shall use the natural filtration $\{F_n\}_{n \geq 0}$ $\{X_n\}_{n \geq 0}$

We first verify integrability:

$$E|X_n| \leq E(|Y_1| + \dots |Y_n|) = nE|Y_1| < \infty.$$

Now. $E(X_{n+1} | F_n)$

$$\underbrace{Y_1 + Y_2 \dots}_{X_n} + Y_{n+1} = X_n + Y_{n+1}.$$

$$= E(X_n | F_n) + E(Y_{n+1} | F_n).$$

this is known!

$$= X_n + E(Y_1) \quad \text{(Ans)}$$

If $E(Y_1) = 0$ then $\{X_n\}_{n \geq 0}$ is a Martingale.

If $P(Y_j = 1) = 1 - P(Y_j = -1) = p$

then $\{X_n\}$ is a MG if only iff $\boxed{p = \frac{1}{2}}$ $2p-1$ should be 0. or $E(Y_1) = 0$

(2) Geometric Random Walk:

$$X_n = X_0 e^{Y_1 + Y_2 + \dots + Y_n}, \quad n \geq 1.$$

Y_i's are i.i.ds

When $\{X_n\}_{n\geq 0}$ a M.G. against the filtration $F_t = \sigma(Y_1, Y_2, \dots Y_n)$.

Verification of integrability.

$$E|X_n| = X_0 E(e^{Y_1 + \dots Y_n}) = X_0 E(e^{Y_1 n}) < \infty \text{ iff } \underline{\varphi_Y(1) = E(e^{Y})} < \infty$$

Now.

$$E(X_{n+1} | F_t) = E(\underbrace{X_n e^{Y_{n+1}}}_{\text{known wrt } F_t} | F_t) = X_n E(\underbrace{e^{Y_{n+1}} | F_t}_{\text{independent}}) = X_n \underline{E(e^{Y_{n+1}})} = X_n \varphi_Y(1)$$

$$E(e^{Y_1}) \to \varphi_Y(1).$$

which is equal to X_n iff $\boxed{\varphi_Y(1) = 1}$.

$\underline{E(e^{Y_1}) = 1}$

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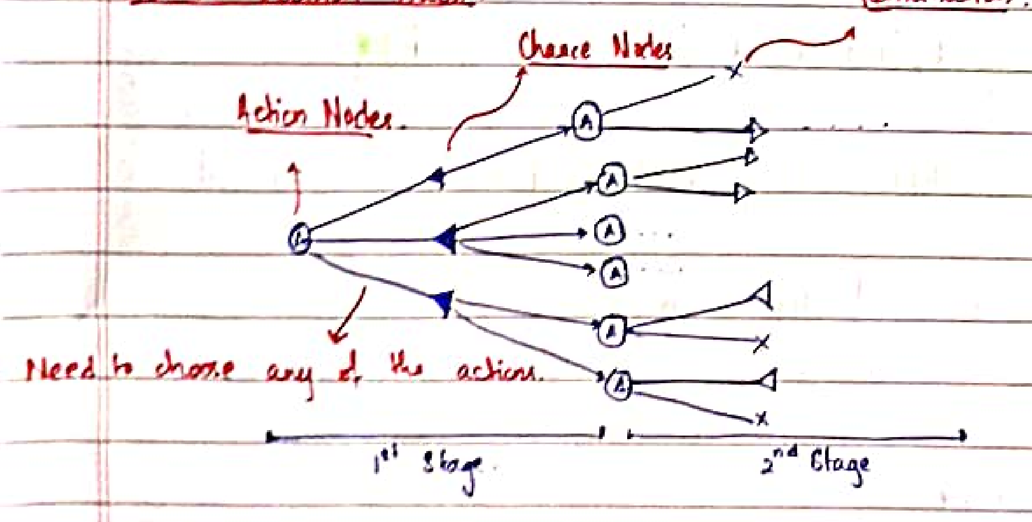
Markov Decision Process:



Basically, at each action node we have to base our decision on the info about the evolution of the system up to that node only.

Finite Stage Models:

(1) Process $X_t \to t \in \{0, 1, 2, \dots T\}$.

(2) Finite set of actions : $\{a \in A\}$

$p_{ij}(a)$: prob. that state $X_{t+1} = j$ such we chose action a & $\underline{X_t = i}$

(3) Reward function. If $X_t = i$ & action a is chosen then $\underline{R(i,a) \text{ is earned as reward}}$

(4) Policy: $\{a_t\}$: Rule for choosing actions at respective times.
If the policy is stationary ⇒ one's action at time t depends on X_t only.
⇒ $\{X_t\}$ is a time-homogenous MC with transition probabilities $P_{ij}(f(i))$ and the Process is a Markov Decision Process. (Q1)

(5) Objective: Maximise the expected value of the sum of rewards over a time length T

$$E\left[\sum_{t=1}^{T} R(X_t, a_t)\right] \longrightarrow \max_{a_1, a_2, \dots a_T}$$

So basically initial value = i — Optimum Value Function Solⁿ:

$$E\left[\sum_{t=1}^{T} R(X_t, a_t) \,\middle|\, X_1 = i\right] \longrightarrow \max_{a_1, a_2, \dots a_T} =: V_T(i).$$

So we need to find $V_n(i)$ for $n = T$ & the optimal policy for which this is obtained. This can be done using DP.

$$V_n(i) = \max_{a \in A} \left\{ R_i(i,a) + E_a\left(V_{n-1}(X_2) \mid X_1 = i \right) \right\}$$

$$= \max_{a \in A} \left\{ R(i,a) + \sum_j P_{ij}(a) V_{n-1}(j) \right\}$$ Understood.

$i = T - n + 1$ $n \rightarrow$ no. of steps to go.
We can ignore this since the process is homogenous.

Example: Selling a house:
A person has to sell their house urgently. Three buyers are going to offer her one after the other their prices which are iid.

$$P(Z_j = 100) = 0.3, \quad P(Z_j = 110) = 0.5, \quad P(Z_j = 120) = 0.2$$

$j = 1,2,3$ (Z_j → price)

If the seller rejects an offer, the offer is gone.
Find the max. expected selling price & the optimal policy to do that.

Solⁿ:

Markov Decision Process (Contd)

Q1) Selling a house A person moving overseas has to sell her house urgently. There are 3 buyers who are going to offer one after the other their prices Z_j, $j = 1,2,3$ (i.i.ds) with prob.

$P(Z_j = 100) = 0.3$

$P(Z_j = 110) = 0.5$

$P(Z_j = 120) = 0.2$.

Find the max. expected price of selling the house.

Sol^n: $a = 1$ means selling — Defining the actions
$a = 0$ means not sold.

Define, $X_t = \begin{cases} Z_t & \text{if the house is NOT sold yet.} \\ 0 & \text{else.} \end{cases}$

$R(\underset{\text{state}}{x}, \underset{\text{action}}{1}) = x$; $R(x, 0) = 0$.

Answer is basically $\sum_{t=1}^{3} R(X_t, a_t)$.

We start from the end

$V_0(x) = 0$ (All the three buyers are refused

$V_1(x) = \max_a R(x, a) = x$.

If the house isn't sold yet it should be sold to the last buyer ($= 120$).

We need to find $V_2(x)$.

$V_2(0) = 0$ (nothing to sell we have already sold)

$$V_2(x) = \max\Big[\underset{a=1}{R(x,a)} + \underset{a=0}{E_a\big(V_1(X_3) \mid X_2 = x \big)} \Big]$$

$120 \times 0.2 + 0.5 \times 110 + 100 \times 0.3$
$= 24 + 55 + 30$
$= 109$

$= \max\{x, 109\}$.

$\Rightarrow$ Hence the optimal action when the second offer has is to sell if $x > 109$ else wait.

$E(V_2(X_2)) = 109 \times 0.3 + 110 \times 0.5 + 120 \times 0.2 = 111.7$.

$V_3(x) = \max(x, 111.7)$.

Finally
$E(V_3(X_1)) = 111.7 \times 0.3 + 111.7 \times 0.5 + 120 \times 0.2 = 113.36$.

$\therefore$ Maximum expected selling price = 113.36 (Pg-133)

Theorem: Suppose you have an option to buy one share of stock at a given fixed price c & you have T days in which you can exercise the option.
Now $c = s_1 \le s_2 \dots \le s_T$ such that if there are n days to go, then one should exercise the option if the present price $\ge s_n$.

Proof: [Pg - 135 & 136]

Actions

$a = 1$: exercise the option

$a = 0$: don't exercise the option.

$$R(s,a) = \begin{cases} \underset{\text{sell}}{0} & \text{if } a = 0 \\ \underset{\text{buy}}{s - c} & \text{if } a = 1. \end{cases}$$

$\Rightarrow V_n(s) = \max\{s-c, 0\}$.

If there are n days to go & the current price is s then we do not exercise the option if $V_n(s) > s - c$.

$\Rightarrow \underbrace{V_n(s) - s}_{\text{non-increasing function}} > -c$.

Discounted Dynamic Programming: (Pg: 140, 141, 142)

Discounted return $= \sum_{t=0}^{T} \alpha^t R(X_t, a_t)$

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Markov Decision Process (Contd)

Discounting return : $\sum_{t=0}^{T} \alpha^t R(x_t, a_t)$. α : Discount factor

The Goal for MDP

$$\underset{\text{actions}}{E}\left[\sum_{t=0}^{T} \alpha^t R(x_t, a_t) \,\middle|\, X_0 = i\right] \longrightarrow \underset{\text{actions}}{\max}$$

$$V_n(i) = \max_a \left[R(i,a) + \alpha E_a \left(V_{n-1}(X_1) \mid X_0 = i \right) \right]$$

no. of periods to go

$$= \max_a \left[R(i,a) + \alpha \sum_j P_{ij}(a) \underline{V_{n-1}(j)} \right]$$ future steps

Example: Lifetime portfolio selection:

X_t : Wealth of an individual at the beginning of the t^{th} time period.

C_t : No of units consumed by the individual during that time period.

1. Now he invests $X_t - C_t$ in a non-risky asset at an interest r.

$$X_{t+1} = r(X_t - C_t)$$

wealth of the individual at the beginning of $(t+1)^{th}$ period

Objective: Maximise the total discounted utility.

$$C_t = X_t - r^{-1} X_{t+1}$$

$u(C_t)$.

$$\phi = \max_{\{X_t\}} \sum_{t=0}^{T-1} \alpha^t u\left(X_t - \frac{1}{r} X_{t+1} \right)$$

utility function

one step back

$\frac{\partial \phi}{\partial X_t} = 0$ look at terms containing X_t

previous term

$$= -\frac{\alpha^{t-1}}{r} u'\left(X_{t-1} - \frac{1}{r} X_t \right) + \alpha^t u'\left(X_t - \frac{X_{t+1}}{r} \right).$$

current term.

$$\Rightarrow \boxed{u'\left(X_{t-1} - \frac{X_t}{r} \right) = \alpha r \, u'\left(X_t - \frac{X_{t+1}}{r} \right)}.$$

let $u(x) = \ln x$; $u'(x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{X_{t-1} - \gamma^{-1} X_t} = \alpha\gamma \frac{1}{X_t - \gamma^{-1} X_{t+1}}.$$

$$\Rightarrow \boxed{X_{t+1} - \gamma(1+\alpha) X_t + \alpha\gamma^2 X_{t-1} = 0}$$

$\boxed{X_t = \lambda^t}$. $\lambda_1 = \gamma$, $\lambda_2 = \alpha\gamma$ } Remember this!

deposit — consume & deposit.

$$\boxed{X_t = b_1 \gamma^t + b_2 \alpha^t \gamma^t}$$

Remember the steps & derive!

$X_0 = b_1 + b_2$

$X_T = b_1 \gamma^T + b_2 \alpha^T \gamma^T$. } putting boundary cond^n

$$C_t = X_t - \frac{\alpha X^{t+1}}{\gamma_t} = b_2(1-\alpha)(\alpha\gamma)^t.$$

find b_1 & b_2. $\rightarrow$ $b_2 = \dfrac{X_0 - \frac{X_T}{\gamma^T}}{1-\alpha^T}$ and $b_1 = X_0 - b_2$

No bequest cond^n: $b_2 = X_0$, $b_1 = 0$. $\boxed{X_T = 0}$

2. Stochastically risky alternative asset :

Safe asset $\hookrightarrow$ \$1 $\Rightarrow$ \$$\gamma$.

Risky Asset $\hookrightarrow$ \$1 $\Rightarrow$ \$$Z_t$.

We invest w_t fraction of $X_t - C_t$ into risky asset and $1-w_t$ of $X_t - C_t$ into safe asset.

$V_4 = 0$

V_1

V_2

V_3

$$X_{t+1} = (X_t - C_t)\left((1-w_t)\gamma + w_t Z_t\right)$$

$$E \sum \alpha^t u(C_t) \quad \text{where} \quad C_t = X_t - \frac{X_{t+1}}{(1-w)\gamma + wZ_t}.$$

Optimal consumption decision :

we'll arrive here

$$C_t = \frac{X_t}{1+\alpha+\alpha^2 \dots \alpha^{T-t+1}} = \boxed{\frac{1-\alpha}{1-\alpha^{T-t}} X_t}$$

Optimal Portfolio decision : $w_t = w^*$ $\hookrightarrow$ same value $\forall t$

Financial Mathematics.

Consider a filtered prob. space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. A random T on that space is said to be the stopping time if ω has

$$\underbrace{\{T \le t\}}_{\text{event}} \in \underbrace{\mathcal{F}_t}_{\text{info already known}} \quad \text{for } t = 0, 1, 2, \ldots$$

Note:

$$\{T = t\} = \underbrace{\{T \le t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{T \le t-1\}^c}_{\in \mathcal{F}_{t-1} \subset \mathcal{F}_t} \in \mathcal{F}_t \quad \text{for } t = 0, 1, 2, \ldots$$

$$\Rightarrow \{T \le t\} = \bigcup_{s=0}^{t} \underbrace{(T = s)}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t}$$ because all times are discrete.

It is the time when we decide to stop doing something.
That when the event $\{T = t\}$ occurs, you act on the basis of what you already know by that time.

Q.1) First hitting times :

X_t : adaptive process.

u_t : boundary function. $t = 0, 1, 2 \ldots$ whenever it crosses the boundary.

Show that first hitting time

$$T := \inf\{t \ge 0 : X_t \ge u_t\}.$$

instead of min: when $X_t < u_t \ \forall t$, the set of t values on RHS is empty & $\inf \phi = \infty$ we get $T = \infty$

Proof:

for any $t = 0, 1, 2, \ldots$

$$\{T \le t\} = \bigcup_{s=0}^{t} \underbrace{\{X_s \ge u_s\}}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t} \in \mathcal{F}_t.$$

Q.2) let $\{X_t\}_{t \ge 0}$ be an MG & T be a ST on a common filtered prob. space. Then $Z_t = X_{t \wedge T}$ min $t = 0, 1, 2 \ldots$ is also an MG on that space.

Proof.

$$Z_{t+1} = \sum_{k=0}^{t} \underbrace{X_k 1_{\{T = k\}}}_{\le |X_k|} + \underbrace{X_{t+1} 1_{\{T > t\}}}_{\le |X_{t+1}|}$$

This yields.

$$E|Z_{t+1}| \le E\sum_{k=0}^{t+1} |X_k| = \sum_{k=0}^{t+1} E|X_k| < \infty \quad \text{(Integrability)}$$

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Date: __.__.__

$$E(Z_{t+1} \mid \mathcal{F}_t) = E\Big[\underbrace{\sum_{k=0}^{t} X_k 1_{\{\tau=k\}}}_{\text{known}} + X_{t+1} 1_{\{\tau > t\}} \,\Big|\, \mathcal{F}_t\Big]$$

$$= \sum_{k=0}^{t} X_k 1_{\{\tau=k\}} + E\Big[X_{t+1} \underbrace{1_{\{\tau>t\}}}_{\text{known by def}^n} \,\Big|\, \mathcal{F}_t\Big]$$

$$= \sum_{k=0}^{t} X_k 1_{\{\tau=k\}} + 1_{\{\tau>t\}} \underbrace{E[X_{t+1} \mid \mathcal{F}_t]}_{X_t \text{ MG.}}$$

$$= \sum_{k=0}^{t} X_k 1_{\{\tau=k\}} + 1_{\{\tau>t\}} X_t = Z_t.$$

Theorem (Optional Stopping Theorem).

Let $\{X_t\}_{t\geq 0}$ be an MG & a τ a bounded ST. $(\tau < C)$

Then

$$E(X_\tau) = E(X_0).$$

Thus in a fair game one cannot invent a new rule for quitting the game that would beat the system: the game will remain fair.

Put $t = C$ in previous theorem.

Brownian Motion Process.

$$X_i = \begin{cases} +1 & \text{if } i^{th} \text{ step of length } \Delta x \text{ is to right} \\ -1 & \text{if } i^{th} \text{ step of length } \Delta x \text{ is to left.} \end{cases}$$

$X(t)$ = Position of a particle at time t.

$= \Delta x \left(X_1 + X_2 + \dots + X_{\left[\frac{t}{\Delta t}\right]} \right)$ Greatest integer function.

Now we have: $\boxed{P(X_i = 1) = P(X_i = -1) = \frac{1}{2}}$ (Given).

First we analyze X_i:

$E(X_i) = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$.

$E(X_i^2) = 1 \times \frac{1}{2} + 1 \times \frac{1}{2} = 1$

$Var(X_i) = E(X_i^2) - (E(X_i))^2 = 1 - 0 = 1$

Next we analyze X_t.

$\Rightarrow E(X(t)) = E\left(\Delta x \sum_{i=1}^{\lfloor t/\Delta t \rfloor} X_i \right) = 0$.

$\Rightarrow Var(X(t)) = \Delta x^2 V\left(\sum_{i=1}^{\left[\frac{t}{\Delta t}\right]} X_i\right) = \Delta x^2 \frac{t}{\Delta t} = (\Delta x)^2 \left(\frac{t}{\Delta t}\right)$.

Some condition:

① If $\Delta x \to 0$, $\Delta t \to 0$ then $E(X(t)) = Var(X(t)) = 0$.

IMP → ② If $\Delta x = \sigma\sqrt{\Delta t}$ then $Var(X(t)) = \sigma^2 t$

Ⓐ $X(t) \sim N(0, \sigma^2 t)$

Ⓑ $X(t)$ has independent increments i.e. $X(t_1)$, $X(t_2) - X(t_1)$ are independent.

Ⓒ $X(t)$ has stationery increments (dependent on Δt) interval.

↳ the prob. dist^n depends on Δt & not on t itself.

Definition of Brownian Motion Process:

- A Stochastic Process is said to be a Brownian Motion Process if

① $X(0) = 0$

② $\{X(t)\}_{t \geq 0}$ has stationery & independent increments.

③ $\forall\ t > 0$, $X(t) \sim N(0, \sigma^2 t)$. and $X(t_2) - X(t_1) \sim N(0, \sigma^2(t_2 - t_1))$.

If $\sigma^2 = 1$ then $X(t) = W(t)$ ⟶ Wiener's Process or Standard BM Process.

Q.1) Is $W_t = W(t)$ a Martingale?

Ans: Yes!

~~$0 \le s \le t$~~

$F(F)$.

Main step
$\Rightarrow$ add & subtract W_t

$s \geq 0$

$\Rightarrow E(W_{t+s} \mid W_1, W_2, \dots W_t)$

$\Rightarrow E(W_{t+s} - W_t + W_t \mid W_1, W_2, \dots W_t)$ known

$\Rightarrow E(W_{t+s} - W_t \mid W_1, W_2 \dots W_t) + E(W_t \mid W_1, W_2 \dots W_t)$ independent

$\Rightarrow E(W_{t+s} - W_t) \sim N(0, \sigma^2 t) + W_t$ will behave as a constant.

$\Rightarrow 0 + W_t$

$\Rightarrow W_t$.

$\therefore E(W_{t+s} \mid F(s)) = W_t$. (Martingale).

Q.2) Prove that $Y_t = |W_t^2 - t|_{t \geq 0}$ is a M.G.

$E(Y_t) \le E(W_t^2) + t = t + t = 2t < \infty$ (finite expected value).

$E(Y_{t+s} \mid f_t)$ ~~indep~~ constant.

$= E(W_{t+s}^2 - (t+s) \mid f_t)$ add/subtract W_t.

$= E((W_{t+s} - W_t + W_t)^2 \mid f_t) - (t+s)$

$= E((W_{t+s} - W_t)^2 + W_t^2 + 2(W_{t+s} - W_t) W_t \mid f_t) - (t+s)$

known so const.

$= \underbrace{E((W_{t+s} - W_t)^2 \mid f_t)}_{\text{independent}} + \underbrace{E(W_t^2 \mid f_t)}_{\text{known}} + 2W_t E((W_{t+s} - W_t) \mid f_t)$ independent $- (t+s)$.

Variance $= E((W_{t+s} - W_t)^2) + W_t^2 + 2W_t E(W_{t+s} - W_t) - (t+s)$. $\sim N(0, s\sigma^2)$

$= \cancel{s} + W_t^2 + 2\cancel{W_t} \times 0 - t - \cancel{s}$

$= W_t^2 - t$.

$= Y_t$ (Hence proved).

find $(Z(t) \mid f(s))$ $(s<t)$

Q.3) Prove that $Z_t = \exp\left\{ \sigma W(t) - \frac{\mu^2 t}{2} \right\}_{t \geq 0}$ is a MG. $\sigma^2 > 0$.

MGF of normal dist $\exp\left(\frac{\sigma^2(1)}{2}\right)$

$W_t - W_s$ is independent of f_t

$E\left[\alpha e^{\sigma W_t - \sigma^2 t/2} \mid f_t\right] = \underbrace{E\left[e^{\sigma(W_t - W_s)}\right]}_{1} e^{\alpha W_s + \frac{\sigma^2 t}{2}} = Z_t$.

$\alpha = 1$

We know that $W_t \sim N(0, t)$, density function $f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$, $-\infty < x < \infty$.

Joint distribution function:

$$f_t(x_1, x_2, \ldots x_n) = \underbrace{f_{t_1}(x_1) \, f_{t_2 - t_1}(x_2 - x_1)}_{\text{independent}} \ldots \underbrace{f_{t_n - t_{n-1}}(x_n - x_{n-1})}_{\text{independent}}$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \ldots \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\}}{(2\pi)^{n/2}\left[t_1(t_2-t_1)\ldots\ldots(t_n - t_{n-1})\right]^{1/2}}$$

Q4) Find the conditional pdf of $[W(s) \mid W(t) = B]$; $s<t$.

$$f_{s|t}(x \mid B) = \frac{f_{s,t}(x, B)}{f_t(B)} = \frac{f_s(x)\, f_{t-s}(B-x)}{f_t(B)}$$

$$= k_1 \exp\left\{ \frac{-x^2}{2s} - \frac{(B-x)^2}{2(t-s)} \right\}$$

$$= k_1 \exp\left\{ -\frac{(t-s)x^2 + s(B-x)^2}{2s(t-s)} \right\}$$

look at this proof from pg-5.

$$= k_1 \exp\left\{ \frac{-tx^2 + \cancel{sx^2} - \cancel{sx^2} - sB^2 + 2sBx}{2s(t-s)} \right\}$$

$$= k_1 \exp\left\{ - \frac{x^2 + \frac{sB^2}{t} - \frac{2sBx}{t}}{\frac{2s(t-s)}{t}} \right\}$$

$$= k \exp\left\{ - \frac{\left(x - \frac{sB}{t}\right)^2}{\frac{2s(t-s)}{t}} \right\}$$

$\therefore \{W(s) \mid W(t) = B\}$ follows $N\left(\frac{sB}{t}, \frac{\sigma^2 s(t-s)}{t}\right)$.

μ Var(.)

if $Y(t) = \sigma W(t)$

input is a fract.

output → time.

Q5) We have a bicycle race b/w A and B. Let $Y(t)$ denote the amount of time (in sec) by which A is ahead when $100t$ % of the race is completed. Suppose $\{Y(t)\}_{t \geq 0}$ is modeled by Brownian Motion Process, with variance σ^2.

(i) If A is leading by 6 sec's at the mid-point of race, what is probability that he'll win the race?

Sol We need to find $P\left(Y(1) > 0 \mid Y\left(\frac{1}{2}\right) = 6\right) = ?$

$$= P\Big(\underbrace{Y(1) - Y\left(\tfrac{1}{2}\right)}_{\text{stationary}} \geq -6 \;\Big|\; Y\left(\tfrac{1}{2}\right) = 6\Big)$$

independent

$$\frac{Y(\frac{1}{2}) - 0}{\sqrt{\frac{\sigma^2}{2}}} : \frac{\sigma - 0}{\sqrt{\frac{\sigma^2}{2}}}$$

$(\sigma^2 t)$

$Y \sim N(0,1).$

$\frac{-\sigma - 0}{\sigma\sqrt{\frac{1}{2}}}$

LHS & RHS

$$\frac{X - \text{mean}}{\sqrt{Var(X)}}$$

$$= P\left(Y\left(\tfrac{1}{2}\right) > -\sigma \right)$$

starts, sign

$$= P\left(Z > \frac{-\sigma - 0}{\left(\frac{\sigma}{\sqrt{2}}\right)} \right)$$

$$Z = \frac{Y(\frac{1}{2}) - 0}{\sigma\sqrt{\frac{1}{2}}} \sim N(0,1)$$

whatever transf. you do to the LHS do it to RHS.

$$= 1 - P(Z \le -\sqrt{2})$$

$$= 1 - \Phi(-\sqrt{2}) = \Phi(\sqrt{2}) = \underline{0.9213} \quad \text{(Ans)}$$

(ii) If A wins the race by a margin of σ secs. What is the probability that A was ahead of B at the mid-point?

$$P\left(Y(\tfrac{1}{2}) > 0 \mid Y(1) = \sigma \right) = P(U > 0) \quad \text{—— (1)}$$

We already know that:

$$(W(s) \mid W(t) = \tfrac{c}{\sigma}) \sim N\left(\frac{s}{t}\frac{c}{\sigma}, \frac{s}{t}(t-s) \right).$$

Here $\boxed{Y(s) = \sigma W(s).}$

$$[Y(s) \mid Y(t) = c] \sim N\left(\frac{s}{t}c, \frac{\sigma^2 s(t-s)}{t} \right).$$

Here $s = \frac{1}{2}$, $t = 1$, $c = \sigma$

$$\text{So, } [Y(\tfrac{1}{2}) \mid Y(1) = \sigma] \sim N\left(\frac{\sigma}{2}, \frac{\sigma^2}{4} \right).$$

$$Z = \frac{U - \frac{\sigma}{2}}{\sqrt{\frac{\sigma^2}{4}}} \sim N(0,1).$$

$$P(U > 0) = P\left(Z > \frac{-\frac{\sigma}{2}}{\frac{\sigma}{2}} \cdot 1 \right) = 1 - P(Z \le -1)$$

$$= 1 - \Phi(-1)$$

$$= \Phi(1) = 0.8413$$

Drifted Brownian Motion:

Drifted Brownian Motion is such that:

(1) $X(0) = 0$, (2) $(X(t))_{t\ge 0}$ is stationary & independent increment.

(3) $X(t) \sim N(\mu t, \sigma^2 t)$.

Here in this case we have — Earlier there was NO involvement of prob.

$$X_i = \begin{cases} +1 & \text{if } i^{th} \text{ step is in the (+ve) dir}^n \text{ with probability } p. \\ -1 & \text{if } i^{th} \text{ step is in the (-ve) dir}^n \text{ with probability } 1-p. \end{cases}$$

$X(t) = \Delta x \left(X_1 + X_2 \ldots X_{\left[\frac{t}{\Delta t}\right]} \right)$ $\quad 1 \times p + 1 \times (1-p) = 1$

$E(X_i) = 2p-1$ and $Var(X_i) = 1 - (2p-1)^2$ $\quad \left[\because V(X) = E(X^2) - E(X)^2 \right]$

$\Rightarrow E(X(t)) = \Delta x \left[\frac{t}{\Delta t}\right] (2p-1)$ and $Var(X(t)) = \Delta x^2 \left[\frac{t}{\Delta t}\right] \left(1 - (2p-1)^2\right)$

Now $\Delta x = \sqrt{\Delta t}$, $p = \frac{1}{2}\left(1 + \mu\sqrt{\Delta t}\right)$ → then $X(t)$ converges to a BM with drift co-eff μ.
and as $\Delta t \to 0$

Then we have:

$E(X(t)) = \mu t$ and $Var(X(t)) = \sigma^2 t$. $\quad (\sigma = 1)$.

Probability that the process will hit A before $-B$, $A, B > 0$.

$P(x) = P\left(X(t) \text{ hits } A \text{ before } -B \mid \underbrace{X(0) = x}_{\text{current position}} \right)$ where $-B < x < A$.

Suppose $x = 0$

We have boundary cond^{n}s: $P(A) = 1$; $P(-B) = 0$

$-B \quad 0 \quad A$

Using Gambler's Ruin,

$$P(\text{up } A \text{ before down } B) = \frac{1 - \left(\frac{1-p}{p}\right)^{B/\Delta x}}{1 - \left(\frac{1-p}{p}\right)^{\frac{A+B}{\Delta x}}}$$

* let $p = \frac{1}{2}(1 + \mu \Delta x)$

Then $\left[\lim_{\Delta x \to 0} \left(\frac{1-p}{p}\right)^{1/\Delta x} = \lim_{\Delta x \to 0} \left(\frac{1 - \mu\Delta x}{1 + \mu \Delta x}\right)^{\frac{1}{\Delta x}} = \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu} \right]$ — ①

$$\boxed{\lim_{\Delta x \to 0} P(\text{up } A \text{ before down } B) = \frac{1 - e^{-2B\mu}}{1 - e^{-2(A+B)\mu}} = e^{2\mu A} \frac{e^{2B\mu} - 1}{e^{2(A+B)\mu} - 1}.}$$

remember this

Special Cases :

(1) If $\mu < 0$, $B \to \infty$

μ with sign.

P(process ever goes to A) = $e^{2\mu A}$.

(2) $\mu \to 0$

P(Process goes to A before down $-B$) = $\frac{B}{A+B}$.

In general.

Remember.

$$\boxed{P(x) = \frac{1 - e^{-2\mu(x+B)}}{1 - e^{-2\mu(A+B)}}}$$ given start pos = x.

prob. that process goes up A before down $-B$ given start pos = x.

(Q.6) Suppose we have the option of buying one unit of stock at fixed price A, independent of market price (taken to be 0). Suppose the price of the stock changes in accordance with a B.M.P with drift $-d$, $d>0$.

initial price

Question is when should we exercise the option. Expected gain in such a policy = $(x-A)$ P

$E(\cdot) = (x-A)e^{-2dx}$.

Here $\mu < 0$ and $B \to \infty$.

Optimal value of x is one maximising the gain

$$f(x) = e^{-2dx}(x-A).$$

prob. of moving stock to x = e^{-2dx}.

$f' = 0$

Then $\boxed{x = A + \frac{1}{2d}}$

Geometric Brownian Motion :

(i) $Y(t)$ has stationary & independent increments.

(ii) $Y(t) \sim N(\mu t, \sigma^2 t)$

We define $\boxed{X(t) = e^{Y(t)}}$ (Geometric Brownian Motion)

Refer ipad notes → Exponential Brownian Motion

Itô Integrals.

Definition of Itô's Integral:

let $\{W(t)\}_{t\geq 0}$ be a Standard Brownian Motion & f be a function having continuous derivatives. $\boxed{\sigma = 1}$

We define:

$$\int_a^b f \, dW(t) = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\left(W(t_i) - W(t_{i-1})\right)$$

$a = t_0 \; t_1 \; t_2 \; \dots \; t_{n-1} \; b = t_n$

$$= f(b)W(b) - f(a)W(a) - \sum_{i=1}^{n} W(t_i)\left[f(t_i) - f(t_{i-1})\right]$$

$\lim_{n\to\infty}$.

So finally we have:

$$\int_a^b f(t)\, dW(t) = f(b)W(b) - f(a)W(a) - \int_a^b W(t)\, df(t).$$

We also have $\boxed{\int_a^b f(t)\, dW(t) \sim N\Big(\underbrace{0}_{\text{mean}},\ \underbrace{\int_a^b f^2(t)\, dt}_{\text{variance}}\Big)}$ obtained from Itô's Isometry.

Q1) Find the distribution of $X := \int_0^t s \, dW(s)$

$$X \sim N\left(0, \int_0^t s^2 ds\right) = N\left(0, \frac{s^3}{3}\right).$$

Result (can be obtained from Itô's formula).

$$*\ \int_0^t W_s \, dW_s = \sum_j W_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right) = \frac{W_t^2}{2} - \frac{1}{2}\sum_j \left(W_{t_{j+1}} - W_{t_j}\right)^2 \longrightarrow \boxed{\frac{W_t^2}{2} - \frac{t}{2}}$$

Similarly we have:

$$\Rightarrow W_t^2 = \int_0^t ds + 2\int_0^t W_s \, dW_s.$$

$$\Rightarrow W_t^3 = 3\int_0^t W(s)\, ds + 3\int_0^t W_s^2 \, dW_s.$$

$$\Rightarrow \boxed{\frac{W_t^3}{3} - \int_0^t \frac{W(s)\, ds}{3} = \int_0^t W_s^2 \, dW_s.}$$

Itô's formula:

Let $\{X_t\}_{t\geq 0}$ be a Itô Process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ with a Brownian Motion Process $\{W_t\}_{t\in[0,T]}$ associated with it, then:

$$\underbrace{X_t = X_0 + \int_0^t a_s\, ds}_{\text{normal integral}} + \underbrace{\int_0^t b_s\, dW_s}_{\text{Itô integral}}, \quad t\in[0,T]$$

a_t, b_t are $\mathbb{F}$ adapted processes s/t $\int_0^t |a_s|\, ds < \infty$ & $\int_0^t E(b_s^2)\, ds < \infty$

In differential form we have:

$$\boxed{dX_t = a_t\, dt + b_t\, dW_t}$$

if $a_t = 0$ then $X_t = MG$

Shorthand Notations:

1. $dt \cdot dt = dt^2 = 0$
2. $dW(t)\, d(W(t)) = dt$ as $\sum(\Delta W_i^2) \to t$ from Quadratic variation
3. $dW(t)\, dt = 0$.

Taylor's formula in 1D for Itô Calculus:

$\{X_t\}_{t\geq 0}$ be an Itô Process and we have:

$$Y_t = f(X_t) \quad t\in[0,T]$$

As X_t is an Itô Process we have

$$\boxed{dX_t = a_t\, dt + b_t\, dW_t} \quad \text{—— (1)}$$

Now using Taylor's series:

$$a_t^2\, \cancel{dt^2}^{\,0} + b_t^2\, \cancel{dW_t^2}^{\,dt} + 2a_t b_t\, \cancel{dW_t\, dt}^{\,0}$$

$$dY_t = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= f'(X_t)[a_t\, dt + b_t\, dW_t] + \frac{1}{2} f''(X_t)\, b_t^2\, dt$$

$$= \left[a_t f'(X_t) + \frac{1}{2} b_t^2 f''(X_t)\right] dt + b f'(X_t)\, dW_t.$$

Questions:

Q.1) Find dY_t where $Y_t = \frac{1}{2} W_t^2$

classmate

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Use the $f(x)$ method.

$f(x) = x^2$.

$f'(x) = 2x$ and $f''(x) = 2$

$d W_t = dY_t = f'(W_t)\,dW_t + \frac{1}{2} f''(W_t)\,(dW_t^2)$ (Applying Taylors Series) only

$= 2W_t\,dW_t + \frac{1}{\cancel{2}} \times \cancel{2} \times dt$

$\Rightarrow \boxed{d\left(\frac{1}{2} W_t^2\right) = W_t\,dW_t + \frac{1}{2}\,dt.}$

Q.2) Calculate $d(e^{W_t})$

$f(x) = e^x$

$f'(x) = f''(x) = e^x$.

$$\boxed{dY_t = e^{W_t}\,dW_t + \frac{1}{2} e^{W_t}\,dt.}$$

Extending the logic of Taylor's Series to multiple variables:

let $F(t,x)$ have continuous partial derivative & $\partial_t f$, $\partial_x f$, $\partial_{xx} f$ are continuous & diff. in x.

If $(X_t)_{t \geq 0}$ is an Itô Process, then $Y_t = f(t, X_t)$ is also an Itô Process.

$$dY_t = \partial_t f(t,X_t)\,dt + \partial f_x(t,X_t)\,dX_t + \frac{1}{2}\partial_{xx} f(t,X_t)\,\underbrace{dX_t^2}_{b_t^2\,dt}.$$

Remember the Taylor Series:

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b)$$
$$+ \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Questions:

Q1) Geometric B.M. $Z_t = e^{\mu t + \sigma W_t} = f(t, W_t)$

$d(Z_t) = ?$

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$$d(Z_t) = \partial_t f(t, W_t)\, dt + \frac{\partial f}{\partial W_t} dW_t + \frac{1}{2} \partial_{xx} f(t, W_t) \underbrace{(dW_t)^2}_{dt}$$

$$= \mu Z_t\, dt + \sigma Z_t\, dW_t + \frac{1}{2} \sigma^2 Z_t\, dt$$

$$= \cancel{\mu Z_t} \cdot \left(\mu + \frac{\sigma^2}{2}\right) Z_t\, dt + \sigma Z_t\, dW_t \quad \underline{(Ans)}$$

Various Rules corr. to Stochastic Calculus:

Let X_t & Y_t be Itô Processes then we have:

① let $Z_t = X_t Y_t$

then $dZ_t = X_t\, dY_t + Y_t\, dX_t + \underbrace{dX_t\, dY_t}$

↳ Itô term (Correction)

② let $Z_t = X_t / Y_t$.

then $dZ_t = \left(\underbrace{\frac{Y_t\, dX_t}{Y_t^2} - \frac{X_t}{Y_t^2} dY_t}_{\text{usual terms}}\right) + \underbrace{\frac{X_t (dY_t)^2}{Y_t^3} - \frac{dX_t\, dY_t}{d\, Y_t^2}}_{\text{correction terms}}$.

$\times -1$

$\times -1$

Example:

→ $Z_t = \underset{X_t}{W_t}\, \underset{Y_t}{e^{W_t}}$. find dZ_t = ?

Given $d(e^{W_t}) = e^{W_t}\, dW_t + \frac{1}{2} e^{W_t}\, dt$.

$$dZ_t = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$$

$$= W_t\, d(e^{W_t}) + e^{W_t}\, dW_t + (de^{W_t})\, dW_t.$$

$$= W_t\left(e^{W_t}\, dW_t + \frac{1}{2} e^{W_t}\, dt\right) + e^{W_t}\, dW_t + e^{W_t}\, dt + 0$$

$$= e^{W_t}\, dW_t\, (1 + W_t) + \left(1 + \frac{W_t}{2}\right) e^{W_t}\, dt. \quad \underline{(Ans)}$$

Stochastic Differential Equations

SDEs are of the form:

$$dX_t = \underbrace{a(t, X_t)\, dt + b(t, X_t)}_{\text{non-random functions}} dW_t \qquad X(0) = x_0 \text{ and } t \in [0,T].$$

x_0 — non-random initial cond^n

Definition:

An Itô process $\{X_t\}_{t\geq 0}$ is said to be sol^n of the above SDE if

$$X_t = x_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dW_s \quad ; \quad t \in [0,T].$$

This eq^n has a unique sol^n if a & b are regular enough (measurable) & $\exists\, C$ s/t

$$\left.\begin{aligned} |a(t,x)| + |b(t,x)| &\leq C(1+x^2) \\ |a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| &\leq C|x-y| \end{aligned}\right\} \begin{aligned} &x, y \in \mathbb{R} \\ &t \in [0,T] \end{aligned}$$

Special SDE: Ornstein - Uhlenbeck Process: → solve then put it back.

$$\underset{\text{net velocity}}{dX_t} = \underset{\text{friction component}}{-\alpha X_t\, dt} + \underset{\text{impact because of molecular movem}}{\sigma\, dW_t}.$$

Since the sol^n of $dX_t = -\alpha X_t\, dt$ is $X_t = Ce^{-\alpha t}$

Here we put $X_t = C_t e^{-\alpha t}$ where C_t is an Itô process.

$$dX_t = e^{-\alpha t} dC_t - \alpha C_t e^{-\alpha t} dt = -\alpha X_t\, dt + e^{-\alpha t} dC_t$$

$$= -\alpha X_t\, dt + \sigma\, dW_t$$

$$\boxed{dC_t = \sigma e^{\alpha t} dW_t}$$

Putting $C_t = \int \sigma e^{\alpha t} dW_t + C_0$.

Finally we have

$$X_t = \left[\int_0^t \sigma e^{\alpha s} dW_s + C_0\right] e^{-\alpha t}$$

$$\Rightarrow \boxed{X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s.}$$

$$X_t \sim N\left(X_0 e^{-\alpha t}, \sigma^2 e^{-2\alpha t}\left(\int_0^t e^{2\alpha s} dt\right)\right)$$

$$\sigma^2 e^{-2\alpha t}\left(-1 + \frac{e^{2\alpha t}}{2\alpha}\right)$$

$$\frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)$$

$$\boxed{X_t \sim N\left(X_0 e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right)}$$

The Vasicek Interest Rate Model:

$$dr_t = a(b - r_t)\,dt + \sigma\, dW_t \quad , t \geq 0$$

Putting $X_t := -b + r_t$ we obtain the previous eq^n

So, we have $r_t \sim N\left(b + e^{-at}(r_0 - b), \frac{\sigma^2}{2a}(1 - e^{-2at})\right)$

But this is not accurate as $r_t < 0$ is also possible here.

Cox-Ingersoll-Ross interest model:

$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}\, dW_t$$

Correction

Geometric Brownian Motion: isolate this and solve.

N_t = price

α := volatility

r := interest

* Eq^n: $dN_t = rN_t\,dt + \alpha N_t\, dW_t$. $N(0) = N_0$.

Put $N_t = r \log N_t$ as the sol^n:

Put $f(t, x) = \log x$.

$\partial_t f = 0$

$\partial_x f = \frac{1}{x}$ $\quad \partial^2_{xx} f = \frac{-1}{x^2}$

from IS: $d(\log N_t) = \underbrace{\frac{1}{N_t} dN_t}_{r\,dt + \alpha\, dW_t} - \underbrace{\frac{1}{2N_t^2}(dN_t)^2}_{\alpha^2 N_t^2\, dt}$

$\therefore d(\log N_t) = \left(r - \frac{\alpha^2}{2}\right) dt + \alpha\, dW_t$.

$\therefore \log N_t - \log N_0 = \alpha W_t + \left(r - \frac{\alpha^2}{2}\right) t$.

$\Rightarrow N_t = N_0 \exp\left\{\left(r - \frac{\alpha^2}{2}\right) t + \alpha W_t\right\}$ is the unique sol^n of G.B.M.

Introduction to Finance.

Convert everything to $t=0$

Pricing Stock Options:

Present value $= v$

Value after time t $= ve^{-\alpha t}$; $\alpha \Rightarrow$ discount factor

$N=1$.

At $t=0$ Present Value $\$100$ → $\$200$ (P.V converted to $t=0$ by $\frac{}{e^{-\alpha t}}$)

→ $\$50$.

you have no stock now!!

premium $= c$.

Option ↷ to buy y shares of stock at $\$150$ per share. (Call option by default)

eg: If $S(1) = \$200$ then net profit $= \underline{\$50}$ for each y shares.

Convention:

$-ve$: selling

$+ve$: buying.

Target: find an appropriate value of c (unit cost of an option)

Sol^n:

At time $t=0$, we buy $+\underline{x \text{ units of stock}}$ and $+\underline{y \text{ units options}}$

Value with us at $t=0$ is $\underline{100x + cy}$.

Value with us at $t=1$ will

Value of what we have → $\$200$ → $200x + 50y$. (Benefit of exercising the option.) $\left[\underset{S(1)}{200} - \underset{150}{\text{strike price}}\right]$

→ $\$\cancel{100}\ 50$ → $50x$ (we don't exercise the option).

Now we choose y s/t the two values of holding is the same no matter what the actual price of the stock is. Then

$$200x + 50y = 50x$$

$$\boxed{y = -3x}$$

Gain = Value at $t=1$ − Value at $t=0$

Value of our holding $= \underline{50x}$ (always) at $t=1$

Gain $=$ ~~yc~~ $50x - (100x + yc) = -50x - yc = \boxed{-50x + 3xc}$.

(premium)

$\boxed{c \geq 50/3}$ [Arbitrage]

Arbitrage: A sure win betting scheme is called an Arbitrage.
[Gain = 0 $\rightarrow$ No Arbitrage cond^n]

Arbitrage Theorem:

Suppose we have an experiment with the following set of possible outcomes $S = \{1, 2, \dots m\}$ There are n wagers (betters). If the amount x is bet wager i, then $x r_i(j)$ is earned given j is the outcome of the experiment.

let $\vec{x} = (x_1, x_2 \dots x_n)$ (This is called a betting scheme).
s/t x_i bets on wager i
Then total return $= \sum_{i=1}^{n} x_i r_i(j)$

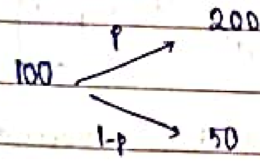
The theorem states exactly one of the following is possible.

(i) $\exists$ a probability vector $p = (p_1, p_2 \dots p_m)$ for which
$$\sum_{j=1}^{m} p_j r_i(j) = 0 \quad \forall\ i = 1, 2, \dots n. \quad \text{[No Arbitrage cond}^n\text{]}.$$

(ii) $\exists$ a betting scheme $x = (x_1, x_2, \dots x_n)$
$$\sum_{i=1}^{n} x_i r_i(j) > 0 \quad \forall\ j = 1, 2, \dots m.$$

(Q.1) Reconsider the previous case: Calculate unit option price for no arbitrage

Case (i): Consider return on only selling the stock to calculate p.



(1 Stock)

$$\text{Expected return} = \overset{200-100}{100p} + \overset{50-100}{-50}(1-p) = 0$$
$$= 100p - 50 + 50p = 0$$
$$\boxed{p = \frac{1}{3}}.$$

Case (ii) Consider return on only [gain from options]

[1 option]

$$C \begin{cases} \xrightarrow{p} & 200 - 150 = 50 \\ \xrightarrow{1-p} & 0 \end{cases}$$

(don't exercise) so no return

~~50-c~~

$$\text{Expected return} = p(50-c) + -c(1-p) = 0$$

$$\Rightarrow \frac{50-c}{3} - \frac{2c}{3} = 0$$

$$\Rightarrow \boxed{c = \frac{50}{3}} \quad \text{(Ans)}$$

(Refer the last example from the slides)

Financial Derivatives:
These are financial instruments which themselves have no intrinsic value, but they derive value from something else also called "the underlying".

Hedging: Investing with the aim of reducing the risk of adverse price movements in an asset.

Forwards: Contractual obligation to buy (long) / sell (short) an underlying at a pre-determined price on or before an expiration date.

Binomial Market Model: You have two kinds of assets in the market.

1) Bond (Bank Balance) $\longrightarrow$ provides riskless rate r of return.

$$B_t = B_0 (1+r)^t \qquad t = 0, 1, 2, \ldots$$

B_0 ↳ Usually taken to be 1.

2) Stock Prices (Stock - risky asset).

Let $u > d$

$$S_t \nearrow S_{t+1} = S_t u \qquad S_t \searrow S_{t+1} = S_t d$$

Consider the case: $\underline{d < 1+r < u}$. (NA condition)

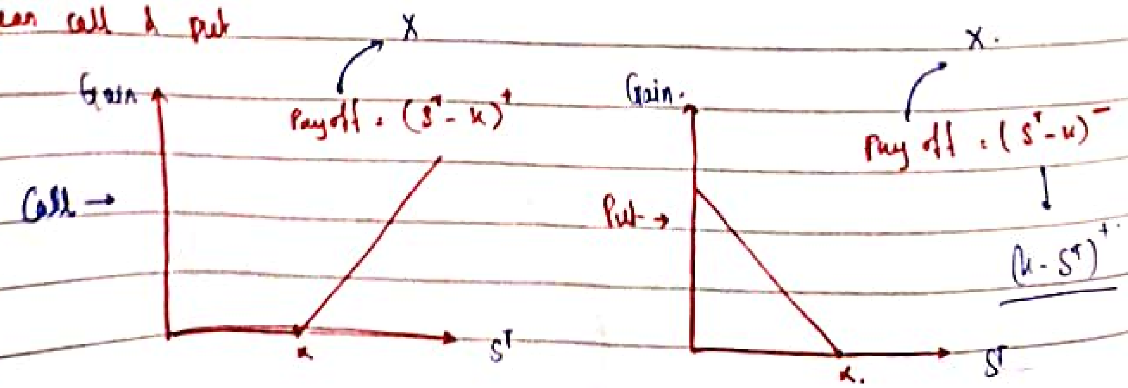
Short Selling: Selling of the stock that the seller doesn't own.

Derivative Security (X)

$$X : X(\omega) = g(S_T(\omega)) \geq 0$$

↳ the contract will pay the owner X at time T.

European call & put



Binomial - Single Period - Market Model :

(Δ, b)

no of shares   no of bond units.

$$\text{Value of the portfolio } V_t : \begin{cases} \Delta S_0 + b & \text{at } t=0 \\ \Delta S_1 + b(1+r) & \text{at } t=1. \end{cases}$$

ω: u or d.

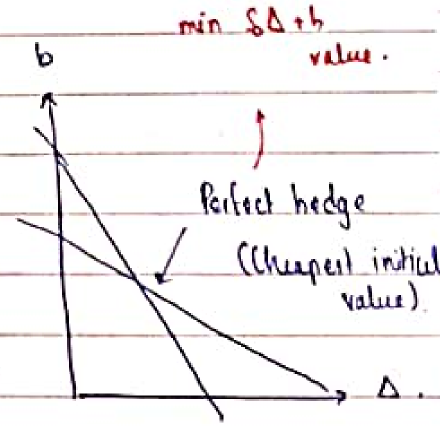
Portfolio will be given a hedge if $V_t \geq X_\omega$

Hedge Cond^n:

$$\Delta u S_0 + b(1+r) \geq X_u$$
$$\Delta d S_0 + b(1+r) \geq X_d$$

$$\Rightarrow \quad b \geq -\frac{uS_0}{1+r}\Delta + \frac{X_u}{1+r}$$

$$\Rightarrow \quad b \geq \frac{-dS_0\Delta}{1+r} + \frac{X_d}{1+r}.$$



Perfect Hedge condition:

$$\Delta = \frac{X_u - X_d}{(u-d)S_0}.$$

$$b = \frac{uX_d - dX_u}{(1+r)(u-d)} = \frac{X_u - u\Delta S_0}{1+r}.$$

Value of portfolio at perfect hedge $(t=0)$

$$V_0 = S_0\Delta + b$$

$$= \cancel{S_0}\left(\frac{X_u - X_d}{(u-d)\cancel{S_0}}\right) + \frac{uX_d - dX_u}{(1+r)(u-d)}$$

$$= \frac{1}{1+r}\left[\underbrace{\frac{1+r-d}{u-d}}_{p^*} X_u + \underbrace{\frac{u-(1+r)}{u-d}}_{1-p^*} X_d\right] \Rightarrow X^*.$$

$$= E^*\frac{X}{1+r} = X^* \quad \text{(Price of the claim)}$$

Calculate

$$p^* = \frac{1+r-d}{u-d}.$$

$$V_0 = \frac{1}{1+r}\left[\cancel{S_0} X_u p^* + X_d(1-p^*)\right]$$

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Q.1) Pricing European ~~Put~~ Call : find the price of the call option $\rightarrow$ V_0

$r = 0.25$, $u = 1.75$, $d = 0.5$, $S_0 = 1$, $K = 1$

$\rightarrow \Delta = \frac{X_u - X_d}{(u-d)S_0} = \frac{(uS_0 - K)^+ - (dS_0 - K)^+}{(u-d)S_0}$

X_d, X_u } Claims

$= \frac{(1.75 \times 1 - 1)^+ - (0.5 \times 1.75 - 1)^+}{1.25 \times 1}$

$= \frac{0.75 - 0}{1.25} = 0.6$

$\rightarrow b = \frac{X_u - \Delta u S_0}{1+r} = \frac{uX_d - dX_u}{(1+r)(u-d)} = \frac{1.75(0) - 0.75 \times 0.5}{1.25 \times 1.25}$

$= -0.24$

$\rightarrow p^* = \frac{1+r-d}{u-d} = 0.6$

$\therefore$ Price of the call (V_0) $= \frac{1}{1+r}\left[p^* X_u + (1-p^*) X_d\right]$ where $\boxed{p^* = \frac{1+r-d}{u-d}}$

$= 0.36$ $(0.6 \times 1 - 0.24 = \Delta S_0 + b)$.

$V_1(u)$ $\nearrow u\Delta S_0 + b(1+r) = -0.3 + 1.05 = 0.75$.

$\searrow \Delta d S_0 + b(1+r) = -0.3 + 0.3 = 0$.

Q.2) Pricing European Put :

C : European call price

P : European Put price.

K : Strike Price.

Portfolio $\rightarrow$ one share of stock

$\rightarrow$ one put one one share of

$\rightarrow$ selling of one call on share of stock

$V_0 = S_0 + \overset{\text{buy}}{P} - \overset{\text{sell}}{C}$.

$V_1 = S_1 + (S_1 - K)^- - (S_1 - K)^+$ $\boxed{x^+ - x^- = x}$

$= S_1 - (S_1 - K) = K$.

So at $t = 0$ $V_0 = \frac{V_1}{1+r}$.

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Check it.

$$\boxed{S_0 + P - C = \frac{K}{(1+r)^t}}$$ → Discrete Market

Put-Call Parity Eqⁿ.

So, if you know the price of one of the options, you can calc. the other one.

In general

$$\boxed{S_0 + P - C = Ke^{-r(T-t)}} \quad t \in [0, T].$$

→ Continuous Market

→ (American).

Finite Single-Period Markets

How to check for NA condition for a n-state model at time $t=1$?

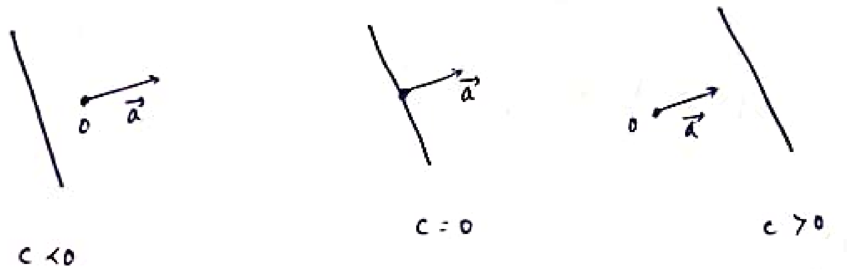
Recall:
A set $D \subset \mathbb{R}^m$ is said to be convex if for any $x_1, x_2 \in D$, all points on the straight line joining x_1 and x_2 must also lie in D.

or $\beta x_1 + (1-\beta) x_2 \in D \;;\; \beta \in (0,1)$.

We define:

$$X := \text{convex-set}(B) := \left\{ x \in \mathbb{R}^m : x = \sum_{j=1}^{N} a_j y_j \,,\; \alpha_j \geq 0 \,,\; \sum_{j=1}^{N} \alpha_j = 1 \right\}$$

L is a hyper-plane. then



$$H_L := \left\{ x \in \mathbb{R}^m : a \cdot (x - x_0) > 0 \right\}.$$

open-half space at which $\vec{a}$ is pointing

Theorem: (Separation Theorem).
D = convex set in $\mathbb{R}^m$
D_0 = relative interior of D.

Then $x_0 \notin D_0$, there exists a vector $\vec{a} \in \mathbb{R}^m$ such $D_0 \subset$ hyper-plane with normal vector a

or $\boxed{a \cdot (x - x_0) > 0 \quad \forall\, x \in D_0.}$

Theorem: A finite single-period market is arbitrage-free if and only if there exists a prob. P^* on $(\Omega, 2^\Omega)$ such that $P^*(\omega) > 0$ for every $\omega \in \Omega$.
$\quad u, d, m \ldots$

and $\boxed{E^*\left(\frac{S_1}{1+r}\right) = S_0}$ $\rightsquigarrow$ M.G. $\left| \frac{1}{1+r}\left(P_1^* S_{\omega_1} + P_2^* S_{\omega_2} \ldots (1-P_1^*-P_2^*) S_{\omega_n}\right) = S_0 . \right.$

$\{ S_t (1+r)^{-t} \}_{0,1}$ is a M.G. under P^*.

Arbitrage Pricing Theorem:

In a single-period arbitrage-free market.

$$E^*\left(\frac{V_1}{1+r}\right) = V_0 \qquad V_0 = \text{Price of claim at } t=0$$

In particular if X is an attainable claim in the market then its $t=0$ value is

$$\boxed{X_0 = E^*\left(\frac{X}{1+r}\right).}$$

RHS doesn't depend on the choice of p^*.

Completeness Theorem:

A market with the property that any claim is attainable to it is called complete market.

single-period NA binomial market is always complete.

An arbitrage free market is complete if & only if $\exists$ exist a unique p^* values. (EMM)

Multiperiod Binomial Markets:

Calculate p^* values then you have.

$$V_{T-1}[s] = \frac{1}{1+r}\left[p^* g(us) - (1-p)^* g(ds)\right].$$

$$\Delta_{T-1}[s] = \frac{V_{T-1}[ds] - V_{T-1}[us]}{(u-d)S_0}.$$

$$b_{T-1}[s] = \frac{u V_{T-1}[ds] - d V_{T-1}[ds]}{(1+r)^{T\text{ period}}(u-d)}.$$

always

We move backwards.

Example to calculate multi-period binomial market:

Price and replicate a call with $K = 80$ on $(T=2)$ period market

$u = 1.5$

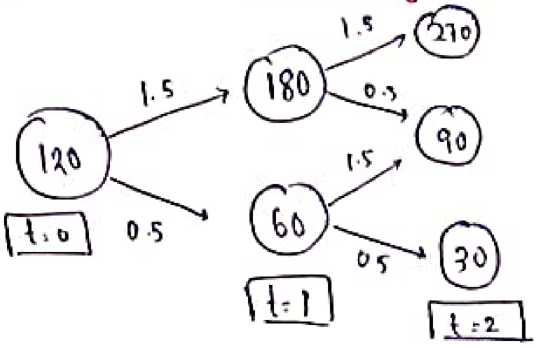
$d = 0.5$ $\qquad d < 1+r < u \Rightarrow$ NA.

$r = 0.$

$S_0 = 120.$

First of all we calculate $p^* = \frac{1+r-d}{u-d} = \frac{1-0.5}{1} = 0.5$

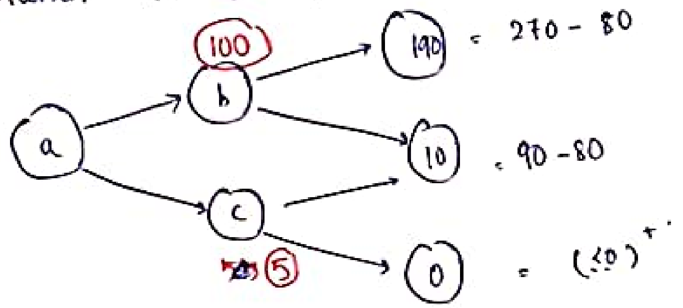
Next we draw the stock price diagram.



Draw it from left to right.

Next we compute the call prices from the last :

$\boxed{K = 80}$ last column do $(S^T - K)^+$.



[Call price Diagram]

For node labelled b:

$$\Rightarrow \Delta_b = \frac{190 - 10}{Us - ds} \qquad \frac{X_u - X_d}{(u-d) S_0}$$

$$= \frac{180}{180} = 1.$$

$$\Rightarrow b_b = \frac{1.5 \times 10 - 190 \times 0.5}{1^2 \cdot (1)} = -80$$

stock price at b

$$V_1[180] = 1 \times 180 - 80(1+r)$$

$$= 100.$$

t=1

t

don't forget this

For node labelled c:

$$\Delta_c = \frac{10 - 0}{60} = \frac{1}{6}.$$

$$b_c = \frac{1.5 \times 0 - 0.5 \times 10}{1^2 (1)} = -5.$$

$$V_1[60] = \frac{1}{\cancel{6}} \times \overset{10}{\cancel{60}} - 5(1+r)$$

$$= 5.$$

For node labelled a:

$$\Delta_a = \frac{100 - 5}{180 - 60} = \frac{\overset{19}{\cancel{95}}}{\underset{24}{\cancel{120}}} = \frac{19}{24}$$

$$b_a = \frac{1.5 \times 5 - 0.5 \times 100}{1^2 \times 1} = -42.5$$

(r+1)

$$V_0(a) = \frac{19}{\cancel{24}} \times \overset{5}{\cancel{120}} - 42.5$$

$$= \underline{\underline{52.5}} \quad \underline{\underline{(\text{Ans})}}.$$

Financial Mathematics Post Mids

Pricing European Call in Binomial Setting.

Price of the European call

$$C = \frac{1}{(1+r)^T} \sum_{j=0}^{T} {}^T C_j (p^*)^j (1-p^*)^{T-j} (u^j d^{T-j} S_0 - K)^+$$

Now we need to remove + so $u^j d^{T-j} S_0 \geq K$.

$$\Rightarrow j\ln u + (T-j)\ln d + \ln S_0 > \ln K$$

$$\Rightarrow \boxed{j \geq \frac{\ln K - \ln S_0 - T\ln d}{\ln u - \ln d}}$$

Taking $M = \frac{\ln K - \ln S_0 - T\ln d}{\ln u - \ln d}$.

$$\boxed{\hat{p} = \frac{up^*}{1+r}}$$

We have:

$$\Rightarrow C = S_0 \sum_{j=M}^{T} {}^T C_j \left(\frac{up^*}{1+r}\right)^j \Big(\underbrace{\frac{d(1-p^*)}{1+r}}_{1-\hat{p}}\Big)^{T-j} - \frac{K}{(1+r)^T} {}^T C_j (p^*)^j (1-p^*)^{T-j}$$

$$\Rightarrow C = S_0 P(\hat{U} \geq M) - \frac{K}{(1+r)^T} P(U^* \geq M)$$

Binomial market is crude & unrealistic.

$\hat{U} \sim B(n, \hat{p})$ $\quad$ $U^* \sim B(m, p^*)$.

Binomial RVs

Deriving Black Scholes Framework:

We split the given (continuous) time interval $[0, T]$ into a large no n of time periods of length $\delta_n = \frac{T}{n}$.

Then we proceed to use the binomial model.

$u = u_n$

$d = d_n$

$r = r_n$

We want to see what happens at $n \to \infty$.

Trading can occur now at $0, \delta_n, 2\delta_n, \ldots$

$$S_{(j-1)\delta_n} \to S_{j\delta_n} = u_n S_{(j-1)\delta_n}, \quad S_{j\delta_n} = d_n S_{(j-1)\delta_n}$$

where $u_n = e^{\sigma\sqrt{\delta_n}}$, $d_n = e^{-\sigma\sqrt{\delta_n}}$

σ = volatility.

As $n \to \infty$, $u_n = 1 + \sigma\sqrt{\delta_n} + O(\frac{1}{n})$

$d_n = 1 - \sigma\sqrt{\delta_n} + O(\frac{1}{n})$.

$$B_{j\delta_n} = e^{r\delta_n}\left(B_{(j-1)\delta_n}\right) = (1+r_n)\left(B_{(j-1)\delta_n}\right)$$

So, $\boxed{r_n = e^{r\delta_n} - 1}$

$\Rightarrow$ $r_n = r\delta_n + O\left(\frac{1}{n^2}\right)$ as $n \to \infty$.

Now,

$$S_n = S_0 e^{\sum_{i=1}^{n} Y_i} = S_0 e^{Z_n}$$

$Z_n = \sum_{i=1}^{n} Y_i$ is a random walk with iid jumps hence we define $Y_{n,k}$

$$Y_{n,k} = \begin{cases} \ln u_n = \sigma\sqrt{\frac{T}{n}} & \text{with prob } p^* = \frac{1+r_n-d_n}{u_n-d_n} \\ \ln d_n = -\sigma\sqrt{\frac{T}{n}} & \text{with prob } 1-p^* \end{cases}$$

As $n \to \infty$. $n \to \infty$ (co-effs come down).

$$p_n^* = \frac{1+r_n-d_n}{u_n-d_n} = \frac{e^{r\delta_n} - e^{-\sigma\sqrt{\delta_n}}}{e^{\sigma\sqrt{\delta_n}} - e^{-\sigma\sqrt{\delta_n}}} = \frac{r\delta_n + \sigma\sqrt{\delta_n}}{2\sigma\sqrt{\delta_n} + \cancel{\sigma\sqrt{\delta_n}}}$$

$$\boxed{p_n^* = \frac{r}{2\sigma}\sqrt{\frac{T}{n}} + \frac{1}{2}} \quad \text{using } \sqrt{\delta_n} = \sqrt{\frac{T}{n}}$$

If we put p_n^* and $\hat{p}_n = \frac{u p_n^*}{1+r}$ in the Call Price:

We obtain: Price of the Call.

$$\boxed{C = S_0\,\Phi(h) - Ke^{-rT}\,\Phi(h - \sigma\sqrt{T})}$$

where

$$\boxed{h = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}$$

Maturity $= T$
Strike $= K$

Black Scholes formula

Stock Price is modeled by $S_t = S e^{\mu t + \sigma W_t}$ and $B_t = e^{rt}$ $t \in [0,T]$ Continuous

$\left(\mu t - \frac{\sigma^2}{2}t\right)$

taking GBM makes the market complete & arbitrage free.

Martingales and Claim Pricing: different risky assets at diff t.

Assume a price process $\{S_t := (S_t^1, \dots S_t^n)\}_{t \geq 0}$

Risk-free security will be given by:

$$B_t = \begin{cases} (1+r)^t & t = 0,1,2. \quad \text{(Discrete)}. \\ e^{rt} & t \geq 0 \quad \text{(continous)}. \end{cases}$$

Key-Results of Arbitrage Pricing theory:

(i) If $\exists$ an EMM P^* such that (i) $P^*(A) > 0$, $A \in \mathcal{F}$.

$$S_t^* = \left.\begin{cases} (1+r)^{-t} S_t. & \text{(discrete case)} \\ e^{-rt} S_t & \text{(continous case)} \end{cases}\right\}$$ is an Martingale then market is arbitrage free.

(ii) For an attainable claim X with maturity T, the arbitrage free price is as follo

$$P_t(X) = \begin{cases} E^*\left((1+r)^{t-T} X \mid F_t \right) & \text{(discrete case)} \\ E^*\left(e^{-r(t-T)} X \mid F_t \right) & \text{(continous case)}. \end{cases}$$

$(t-T)$ in the case...

(iii) If the EMM P^* is unique then the market is arbitrage-free and complete i.e. any claim is always attainable.

The Black Scholes Framework:

Black Scholes Market
- Bond (Riskless): B_t $\boxed{B_t = e^{rt}}$ $(B_0 = 1)$. $dB_t = r B_t \, dt$
  "b": no. of bonds
- Risky asset: S satisfying:
  $\boxed{dS_t = \mu S_t \, dt + \sigma S_t \, dW_t}$

Equation of Stocks:

[Just remember this eq^n]

$$\frac{dS_t}{S_t} = \underbrace{\mu dt}_{\text{Systematic return}} + \underbrace{\sigma dW_t}_{\text{specifies the effect of noise on the return.}}$$

normal white noise dW_t.

We had observed in case of Itô Formula that:

$$d(\ln S_t) = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}(dS_t)^2$$ (Application of Ito's formula to $f(x) = \ln x$)

Putting the values

~~taking~~ $d(\ln S_t) = \mu dt + \sigma dW_t - \frac{1}{2\cancel{S_t^2}} \times \sigma^2 \cancel{S_t^2} dt$

$$\Rightarrow d(\ln S_t) = \mu dt + \sigma dW_t - \frac{\sigma^2}{2} dt$$

$$\Rightarrow \ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$\Rightarrow \boxed{S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}}$$ (Hence proved). —— (1)

↳ basically GBM.

Major Question:

Q1) Is the Black Scholes Market arbitrage free?

Q2) Is the Black Scholes Market complete?

To answer this

What kind of prob. dist^n will be equivalent to (1).

Ans: $Z_t = S_0 \exp\left(\int_0^t a_s ds + \sigma W_t\right)$. $t \in [0,T]$

To check if $e^{-rt} Z_t$ is a MG. It is a MG if co-efficient of $dt = 0$

$$d(e^{-rt} Z_t) = d\left(\underbrace{\exp\left\{\int_0^t a_s ds\right\}}_{= X_t} \times \underbrace{\exp\left\{\sigma W_t - rt\right\}}_{= Y_t}\right).$$

$X_t = e^{\int_0^t a_s ds}$

$dX_t = X_t a_t dt$

$Y_t = e^{\sigma W_t - rt}$

$dY_t = (\sigma dW_t - r dt)$

$$= X_t dY_t + Y_t dX_t.$$

$$= \left[\left(\frac{\sigma^2}{2} - r\right)Y_t dt + \sigma Y_t dW_t\right] + \left[Y_t (X_t a_t dt)\right]$$

$$= \sigma X_t Y_t dW_t + X_t Y_t \underbrace{\left(a_t - r + \frac{\sigma^2}{2}\right)}_{= 0} dt$$ → T.S expansion.

Theorem: ∃ a unique EMM in Black Scholes Market, under prob. P^*

$dS_t = r S_t dt + \sigma S_t dW_t$.

makes the market Arbitrage free and complete.

Pricing European Calls in the Black Scholes Market :

Note that : (After integration stuff)

$$\boxed{P_0(X) = S_0\,\Phi(h) - Ke^{-rT}\,\Phi(h - \sigma\sqrt{T})}$$

where

$$\boxed{h = \frac{\ln\left(\frac{S_0}{K}\right) + \left(\frac{\sigma^2}{2} + r\right)T}{\sigma\sqrt{T}}}$$

Using the fact $\frac{W_t}{\sqrt{T}} \sim N(0,1)$

Concept of Self-Financing :

In discrete time a model (Δ_t, b_t) is said to be self financing if

$$V_t = \Delta_t S_t + b_t(1+r)^t = \Delta_{t+1} S_t + b_{t+1}(1+r)^t$$

Putting this in place of V_t

$$\Rightarrow \boxed{V_{t+1} - V_t = \Delta_{t+1}(S_{t+1} - S_t) + b_{t+1}\Big[\underbrace{(1+r)^{t+1} - (1+r)^t}_{B_{t+1} - B_t}\Big]}$$

A bi-variate process is said to be a trading strategy if it is predictable & $\int_0^u \Delta_t \, dS$ are well-defined.

A trading strategy (Δ_t, b_t) is said to be self-financing if :

$$\boxed{dV_t = \Delta_t\, dS_t + b_t\, dB_t}$$

Absence of arbitrage = No self financing
- 0 - initial value.
- $V_T \geq 0$ at time T.

So, replicating a claim X means generating a self-financing trading strategy that would generate the same cash flow as X at expiry.

Time claim value $\boxed{P_t(X) = V_t = f(t, S_t)}$ if not equal then arbitrage.

Value of portfolio.

$dV_t = dP_t(x) = df(t, S_t)$ (Applying Ito's Formula) to $f(t,S_t)$

$$= \frac{\partial_t}{\partial t} f(t,S_t)\partial t + \frac{\partial_s}{\partial S} f(t,S_t)\partial S + \frac{1}{2}\frac{\partial^2_{ss}}{\partial S^2} f(t,S_t).\partial S^2$$

Now we know $\boxed{dV_t = \Delta_t\, dS_t + b\, dB_t}$ from this eqⁿ.

So $\boxed{\Delta_t = \frac{\partial f(t,S_t)}{\partial S}}$ and $\boxed{b_t = \frac{f(t,S_t) - \Delta_t S_t}{B_t}}$

Delta.

Sensitivities of Claim Price : (Also called greeks)

Gamma : $\frac{\partial^2 f(t,S_t)}{\partial S^2}$

Vega : $\frac{\partial P_t(X)}{\partial \sigma}$

θ : $\frac{\partial P_t(X)}{\partial T}$ ($\theta \to t$ sounding)

ρ = $\frac{\partial P_t(X)}{\partial r}$ (r sounding)

Example :

Given $C = S_0 \Phi(h) - Ke^{-rT}\phi(h - \sigma\sqrt{T})$

find Δ, vega, Γ, θ, ρ

$\boxed{\Phi(h) = \frac{e^{-h^2/2}}{\sqrt{2\pi\sigma^2}}}$

$$\Delta = \frac{\partial C}{\partial S} = \Phi(h) + S_0\Phi'(h)\frac{\partial h}{\partial S} - Ke^{-rT}\Phi'(h - \sigma\sqrt{T})\frac{\partial h}{\partial S}$$

$$= \Phi(h) + \frac{\partial h}{\partial S}\left(S_0\Phi'(h) - Ke^{-rT}\Phi'(h - \sigma\sqrt{T})\right)$$

$$= \Phi(h)$$

$\boxed{\Delta = \Phi(h)}$ for put:

Using Put-Call Parity Eqⁿ.

$$\boxed{\frac{\partial P}{\partial S_0} = \frac{\partial C}{\partial S} - \frac{\partial S}{\partial S} = \Phi(h) - 1}$$

$$\Gamma = \frac{\partial^2 C}{\partial S_0^2} = \frac{\partial}{\partial S}\bar{\Phi}(h) \cdot \phi(h)(\sigma\sqrt{T}S_0)^{-1}$$

$$\nu = \phi(h) S_0 \sqrt{T}$$ (no σ is present here).

Put Call Parity Eqⁿ used:

$$\boxed{S_t + P_t - C_t = Ke^{-r(T-t)}}$$ T-C+S ⟹ Post Credits Scene with alternate t, -

Δ for call ≥ 0.
Δ for put < 0.
} Issuer of the call should deliver the stock in case the owner decides to exercise the option.

Value at Risk: (Delta-Normal method).

Return for each of the institutions assets are normally distributed (Assumption)

$$Var(\tilde{R}) = \sum_i \sum_j w_i w_j \sigma_{ij} = \sum_i \sum_j w_i w_j \rho_{ij} \sigma_i \sigma_j$$

w_i : fraction of total portfolio value consisting of asset i, s.t. $\sum_i w_i = 1$.
σ_{ij} = cov. of asset i's return with asset j's return.
σ_i : s.d. of asset i's return.
ρ_{ij} = corelation of asset i's return with asset j's return.

Example:
Consider a portfolio consisting of three assets. a.) Currency swap because of the change in the exchange rate since the swap was first entered into, the swap now has a value of \$2 Million (8.7 % of portfolio's value).
b) A bond : Market value of bond is \$17 Million (73.9 % of portfolio's value)
c) A stock : 10,000 shares are worth \$4 Million (17.4 % of portfolio value

Var-Cov Matrix is as follows (σ_{ij} value)

	swap	bond	stock
swap	0.009	-0.0008	0.00007
bond	-0.0008	0.0004	-0.0001
stock	0.00007	-0.0001	0.003

of daily return

two times occuring

$Var(\tilde{R}) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2w_1 w_2 \sigma_{12} + 2w_2 w_3 \sigma_{23} + 2w_1 w_3 \sigma_{13}$

$= 0.0002822.$

S.d of daily return $= \sqrt{0.0002822} = 0.0168 = \underline{1.68\ \%}$.

⇒ 1. s.d of \$ loss from portfolio value of \$ 23 Million = 1.68 x 23 M = \$ 386375

VaR = Z_α x 1sd of loss from portfolio value.

5 % prob that the one day loss = (1sd value) x $Z(5)$ ~ $\Phi(5)$

386375 x 1.645