A stochastically-risky alternative asset:

invest \$1 at time t → get \$$r$ at time ~~t~~ $t+1$

$\exists$ a risky asset

invest \$1 at time t → get \$$Z_t$ " " $t+1$

where $Z_t > 0,\ t \geq 0$ are i.i.d. r.v's.

$$
\begin{tikzcd}
& X_t \arrow[dl, "\text{consume}"'] \arrow[dr] & & \\
C_t & & X_t - C_t \arrow[dl, "\text{fraction } w_t"'] \arrow[dr, "\text{fraction } 1-w_t"] & \\
& \text{invest in risky asset} & & \text{invest in safe asset}
\end{tikzcd}
$$

$\therefore\ X_{t+1} = (X_t - C_t)\left((1-w_t)r + w_t Z_t\right),\quad t = 0, 1, \ldots, T-1$ —— $\star$, X_0, X_T are given

Optimization problem

$$E \sum_{t=0}^{T-1} \alpha^t u(C_t) \to \max_{\{C_t, w_t\}} \quad \text{subject to} \quad C_t = X_t - \frac{X_{t+1}}{(1-w_t)r + w_t Z_t}$$

assume initial wealth X_0 is given, and, for simplicity sake, that $X_T = 0$ (no bequest)

$$V_n(x) = \max_{C_{T-n}, w_{T-n}} \left[u(C_{T-n}) + \alpha E\left(V_{n-1}\left(X_{T-n+1} \mid X_{T-n} = x\right)\right)\right]$$

$$V_0(x) = 0 \quad \text{(as no bequest situation)}$$

$V_1(x) = u(C_{T-1}) = u(X_{T-1})$ since $C_{T-1} = X_{T-1}$

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| no bequest

$$V_2(x) = \max_{C_{T-2}, w_{T-2}} \left[u(C_{T-2}) + \alpha E\left(u(X_{T-1}) \mid X_{T-2} = x \right) \right]$$

$$= \max_{C_{T-2}, w_{T-2}} \left[\underbrace{u(C_{T-2})} + \underbrace{\alpha E u\left(\left((1-w_{T-2}) r + w_{T-2} Z_{T-2} \right)(x - C_{T-2}) \right)}_{\text{long } \bigstar_1} \right]$$

To max , solve

$$\frac{\partial}{\partial C_{T-2}} [\cdots] = \frac{\partial [\cdots]}{\partial w_{T-2}} = 0$$

On next step, we find $V_3(x)$ and so on.

Special case of Bernoulli utility $u(x) = \log x$

$$V_2(x) = \max_{C, w} \left[\log C + \alpha E \log\{ ((1-w) r + w Z)(x - C) \} \right]$$

$$= \max_{C} \left[\log C + \alpha \log(x - C) \right] + \alpha \underbrace{\max_{w} E \log((1-w) r + w Z)}_{\bigstar_2}$$

For 1st term $\quad 0 = \frac{\partial}{\partial C} \left[\log C + \alpha \log(x - C) \right] = \frac{1}{C} - \frac{\alpha}{x - C}$

$$\Rightarrow C = \frac{x}{1+\alpha}$$

For 2nd term

$$\underbrace{0 = \frac{\partial}{\partial w} E \log((1-w) r + w Z)} = E\left(\frac{Z - r}{(1-w) r + w Z} \right)$$

Optimal action is

$$C_{T-2} = \frac{X_{T-2}}{1+\alpha} , \quad w_{T-2} = w^*$$

Optimal value is (from $\star_2$)

$$\underline{V_2(x)} = \log \frac{x}{1+\alpha} + \alpha \log\left(x - \frac{x}{1+\alpha}\right) + \alpha \log r^*$$

$$= (1+\alpha)\log x + \alpha \log \alpha - (1+\alpha)\log(1+\alpha) + \alpha \log r^*$$

$$= (1+\alpha)\log x + k_1, \longleftarrow \star_3$$

where $\log r^* = E \log\left((1-w^*)r + w^* Z\right)$

$$V_3(x) = \max_{c_{T-3}, w_{T-3}} \left[u(c_{T-3}) + \alpha E\left(V_2(X_{T-2}) \mid X_{T-3} = x\right)\right.$$

$$X_{T-2} = (X_{T-3} - c)\left((1-w)r + wZ\right) \longrightarrow \star_4$$

$$V_3(x) = \max_{c,w} \left[\log c + \alpha E\left[(1+\alpha)\log\left(\left((1-w)r + wZ\right)(x-c)\right) + k_1\right]\right]$$

| using $\star_3$ & $\star_4$

$$= \max_{c} \left[\log c + \alpha(1+\alpha)\log(x-c)\right]$$

$$+ \alpha(1+\alpha) \underbrace{\max_{w} E \log\left((1-w)r + wZ\right)}_{\log r^*} + \alpha k_1$$

with the optimal value $w = v_{T-3} = w^*$

1st term

$$0 = \frac{\partial}{\partial c}\left[\log c + \alpha(1+\alpha)\log(x-c)\right]$$

$$= \frac{1}{c} - \frac{\alpha(1+\alpha)}{x-c}$$

$$\Rightarrow C = C_{T-1} = \frac{x}{1+\alpha+\alpha^2}$$

Optimal consumption decision is

$$C_t = \frac{X_t}{1+\alpha+\cdots+\alpha^{T-t+1}} = \frac{1-\alpha}{1-\alpha^{T-t}} X_t \ , \ t=0,1,\ldots,T-1$$

Optimal portfolio decision is always

$$w_t = w^* \ , \ t=0,1,\ldots,T-1$$