Tutorial

1. Show $\{C[0,1], d_\infty\}$ is a metric space.

where $d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$.

2. Show $\{\mathbb{R}^n, d_\infty\}$ and $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$ are metric spaces.

3. Let X be a nonempty set. Let $B(X)$ be the set of all bounded real valued function. Then $(B(X), d_\infty)$ is a metric sp. Verify!

4. $(C[0,1], d_2)$, $(C[0,1], d_1)$ are metric spaces where $d_2(f,g) = \left(\int_0^1 |f(t) - g(t)|^2 dt\right)^{\frac{1}{2}}$.

and $d_1(f,g) = \int_0^1 |f(t) - g(t)| dt$

5. ~~Show~~ Let (X, d) be a metric space.

Define $\delta_1(x,y) = \min\{1, d(x,y)\}$

$$\delta_2(x,y) = \frac{d(x,y)}{1 + d(x,y)},$$

Show (X, δ_1) and (X, δ_2) are metric spaces.

6. Let (X, d), (Y, d') be two metric spaces

show that

$$d\big((x_1, y_1), (x_2, y_2)\big) := \max\{d(x_1, x_2), d'(y_1, y_2)\}$$

defines a metric on $X \times Y$.

7. Let $l^2 = \left\{ \{x_n\} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$

Let $d_2\big(\{x_n\}, \{y_n\}\big) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^2 \right\}^{\frac{1}{2}}$.

Show (l^2, d_2) is a metric space.

8. Prove that an open interval in $\mathbb{R}$ is an open set and a closed interval is a closed set.

9. Show that $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ is open in $\mathbb{R}^2$.

10. Show that a finite subset of a metric space is closed set.

~~Tutorial on Metric Space~~

~~1. Please try to choose~~

Ex 11 Find all closed sets in a discrete metric space.

12. Show that arbitrary intersection of closed sets in a metric space is closed.

13. Show that a closed ball $B[0,1]$ is a closed set.

14. $A = \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$ is open and closed in $\mathbb{Q}$. Verify.

15. Limit of a sequence in a metric space is unique. Verify!

16. Let $f_n(x) = x^n$, $x \in [0,1]$, $n \in \mathbb{N}$. Show that $\{f_n\}$ converges in $(C[0,1], d_1)$ but not in $(C[0,1], d_\infty)$

17. Show that $(M_{n\times n}(\mathbb{R}), d_1)$ is a metric space. Is it complete?

18. Show that any Cauchy sequence in a metric space, may not converge.

19. Let X be any metric space. Assume $\{x_n\}$ and $\{y_n\}$ be Cauchy. Show that $d(x_n, y_n)$ is convergent in $\mathbb{R}$.

20. Show that $f : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$

$$f(A) = A^3,$$

is a continuous map. Hence show that $\{A \in M_{n\times n}(\mathbb{R}) : A^3 = 0\}$ is closed in $M_{n\times n}(\mathbb{R}^n)$.

21. A matrix $A \in M_{n\times n}(\mathbb{R})$ is called nilpotent if $\exists\, K \in \mathbb{N}$ s.t. $A^K = 0$.

[Note: If A is nilpotent, then $K \le n$ by Caley-Hamilton theorem in Linear alg]

Show that set of all nilpotent matrix is closed.

22. A function $f \in C(\mathbb{R})$: set of all continuous function on $\mathbb{R}$, is called compactly supported if $f = 0$ outside a compact set.

· Check, if the set of all compactly supported continuous functions are closed in $(B(\mathbb{R}), d_\infty)$ or not.. $B(\mathbb{R})$ set of all bounded functions

23. Check if continuously diff'able functions on $[0,1]$ is complete in d_∞.

24. Let $A, B \subseteq \mathbb{R}^n$ be such that A is compact B is closed. Show $A + B$ is closed.

25. Show that $SL(n, \mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : \det A = 1 \}$ is closed subset in $M_{n \times n}(\mathbb{R})$.

26. Let $f_n \longrightarrow f$ in $(B(\mathbb{R}), d_\infty)$ If f_n's are uniformly continuous check if f is also uniformly continuous or not.