11.7 Problems

1. Let $X_1, X_2, \ldots$ be i.i.d. RVs with $\mathbf{E}\, X_1 = \mu$, $\mathrm{Var}\,(X_1) = \sigma^2 < \infty$. Put $S_0 := 0$, $S_n := S_{n-1} + X_n$, $n \geq 1$. Compute

   (i) $\mathbf{E}\,(S_{n+m} | \, S_n)$, $m, n = 0, 1, 2, \ldots$;

   (ii) $\mathbf{E}\,(X_1 | \, S_n)$, $n \geq 1$;

   (iii) $\mathbf{E}\,(S^2_{n+m} | \, S_n)$, $m, n = 0, 1, 2, \ldots$;

   (iv) $\mathbf{E}\,(S_m | \, S_n)$, $m = 0, 1, \ldots, n$.

   Hints. (ii) $\mathbf{E}\,(X_1 | \, S_n) = \mathbf{E}\,(X_2 | \, S_n)$ $(n \geq 2)$ *etc.* by symmetry. (iv) You may wish to use the result of one of the parts (i)–(iii) above.

2. Let $\{N_t\}_{t \geq 0}$ be a Poisson process with rate $\lambda > 0$, $\mathcal{F}_t = \sigma\{N_s, \, 0 \leq s \leq t\}$ the "history" of the process up to the time t. Using the properties of the Poisson process and conditional expectations, find

   (i) $\mathbf{E}\,(N_{t+s} | \, \mathcal{F}_t)$, $s, t \geq 0$;

   (ii) $\mathbf{E}\,(N^2_{t+s} | \, \mathcal{F}_t)$, $s, t \geq 0$;

   (iii) $\mathbf{E}\,(N_s | \, \mathcal{F}_t)$ and $\mathbf{E}\,(N^2_s | \, \mathcal{F}_t)$, $0 \leq s \leq t$;

   (iv) $\mathbf{E}\,(N_s | \, N_t)$ and $\mathbf{E}\,(N^2_s | N_t)$, $0 \leq s \leq t$.

   Hint. It is not much different from the previous problem, is it?

3. Let $\{X_t\}_{t=0,1,\ldots,T}$ be a positive SP adapted to a filtration $\mathbf{F} = \{\mathcal{F}_t\}$. In each of the following cases, say if the RV τ is an ST w.r.t. $\mathbf{F}$ (if the condition in the definition of the random time τ in (iii)–(iv) is never met for $t \leq T$, we just put $\tau := T$ to avoid any inconvenience). Explain (e.g., expressing events $\{\tau \leq t\}$ in terms of the RVs X_k, $k = 1, 2, \ldots$).

   (i) $\tau := m = \text{const}$;

   (ii) $\tau := \tau_1 \wedge \tau_2$, where τ_j are STs, $j = 1, 2$;

   (iii) $\tau := \min\{t \geq 0 : X_{t+1}/X_t > 1\}$;

   (iv) $\tau := \min\left\{t \geq 0 : \sum_{k=0}^{t} X_k > X_t^2\right\}$;

(v) $\tau := \max\{t \le T : X_t > 10\}$.

4. Let Y be an integrable RV (i.e., $\mathbf{E}|Y| < \infty$) on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\ge 0}, \mathbf{P})$. Show that $X_t := \mathbf{E}(Y|\mathcal{F}_t)$ is an MG.[16]

5. Let $\{X_t\}_{t\ge 0}$ be a square-integrable (i.e., $\mathbf{E}\,X_t^2 < \infty$) MG. Show that the process has *orthogonal increments* in the sense that, for any $0 \le t_1 \le t_2 \le t_3 \le t_4$, one has $\mathbf{E}(X_{t_2} - X_{t_1})(X_{t_4} - X_{t_3}) = 0$.

6. Let $S_0 := 0$, $S_n := Y_1 + \cdots + Y_n$, $n \ge 1$, Y_j being i.i.d. RVs with $\mathbf{E}\,Y_j = 0$, $\mathrm{Var}(Y_j) = \sigma^2 < \infty$. Show that $X_n := S_n^2 - n\sigma^2$, $n \ge 0$, is an MG (i) with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$; (ii) with respect to the natural filtration $\mathcal{F}'_n = \sigma(X_1, \ldots, X_n)$.

   Hint. (ii) Use the result of (i) and the fact that $\mathcal{F}'_n \subset \mathcal{F}_n$ (why does the last relation hold?).

7. Denote by $\{N_t\}_{t\ge 0}$ a Poisson process with rate $\lambda > 0$. Show that all three processes (i) $N_t - \lambda t$; (ii) $(N_t - \lambda t)^2 - \lambda t$; (iii) $\exp\{uN_t - \lambda t(e^u - 1)\}$ (u is a fixed real number) are MGs w.r.t. the filtration $\mathcal{F}_n = \sigma(N_s,\, s \le t)$.

8. Let $S_0 := 0$, $S_n := Y_1 + \cdots + Y_n$, $n \ge 1$, Y_j being i.i.d. RVs with $\mathbf{P}(Y_1 = 1) = 1 - \mathbf{P}(Y_1 = -1) = 1/2$. Denote by $\tau := \min\{n \ge 0 : S_n = a \text{ or } S_n = b\}$ the first time the RW S_n hits one of the (integer) barriers $a < 0 < b$. Use Theorem 11.2, without verifying its conditions in detail, to:

   (i) find the distribution of S_τ;[17]

   (ii) compute $\mathbf{E}\,\tau$.

   Hints. (i) Use the martingale $X_n := S_n$. (ii) Use the result of (i) and the MG from Problem 6.

9. Suppose you are playing a "fair game" betting \$1 at each play (in which you win/lose w.p. $\frac{1}{2}$, independently of the past). Then $\{X_n :=$ your fortune after n plays$\}_{n\ge 0}$ is an MG w.r.t. its "history" $\mathbf{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots\}$. Now suppose that, for each play $n = 1, 2, \ldots$, your stake can be an *arbitrary bounded amount* Y_n, but you have to decide how much to stake *before* that play, i.e., basing on the history up to play $n-1$ (inclusive). Mathematically, this means that, for any n, $|Y_n| \le C_n = \text{const} < \infty$, and $\{Y_n\}_{n\ge 1}$ is a predictable process (cf. p. 307).
   In that case, you win the amount $Y_n(X_n - X_{n-1})$ on play n (as it was $X_n - X_{n-1}$ when staking \$1 each time), and hence your total net gain after n plays is

$$Z_n := \sum_{k=1}^{n} Y_k(X_k - X_{k-1}), \quad n = 1, 2, \ldots; \quad Z_0 = 0.$$

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[16] This MG is referred to as a *Lévy martingale*. One can show that $X_t \to \mathbf{E}(Y|\mathcal{F}_\infty)$ a.s. as $t \to \infty$, where $\mathcal{F}_\infty := \sigma(\bigcup_{n\ge 0} \mathcal{F}_t)$ (generally speaking, $\mathcal{F}_\infty$ is NOT the same as $\bigcup_{n\ge 0} \mathcal{F}_t$!). In fact, *any uniformly integrable MG is the Lévy MG* for some Y.

[17] Note that we already solved that (gambler's ruin) problem using a different approach in Example 3.21 (see the second case in (3.44)).

The process $\{X_n\}_{n\geq 0}$ is called the *martingale transform*[18] of X_n by Y_n.

(i) Show that $\{Z_n\}_{n\geq 0}$ is an MG w.r.t. **F**.

(ii) The betting strategy of doubling when losing is called "martingale". You begin with a unit stake, win each play w.p. $\frac{1}{2}$ (regardless of the results of all the previous plays), and double your stake for the next play when losing. After each win, you reset the stake size to one. Represent your net gain process $\{Z_n\}$ when using this strategy as a martingale transform: specify the processes $\{X_n\}$ and $\{Y_n\}$.

(iii) Under the assumptions of part (ii), assume that you stop at the time τ of the first win. Find the distribution and expectation of the ST τ and verify if the statement of Theorem 11.2 holds for the process $\{Z_n\}$ and ST τ.

(iv) Compute the expectation $\mathbf{E}(Z_n;\, \tau > n)$ and hence verify if the general condition (11.11) sufficient for (11.9) is satisfied in this case.

10. Let $\{S_n\}_{n\geq 0}$ be an RW defined as follows: starting at $S_0 = 0$, the walking particle at each transition goes up 1 w.p. $p = \frac{6}{7}$ and down 2 w.p. $1 - p = \frac{1}{7}$.

(i) Show that $X_n := 2^{-S_n}$, $n = 0, 1, 2, \ldots$, is an MG.

(ii) Introduce the ST $\tau := \min\{n \geq 0 : X_n \leq 0.1\}$. Use Theorem 11.2 (without verifying its conditions) and the MG $\{Z_n := S_n - \mathbf{E}\, S_n\}$ to compute $\mathbf{E}\,\tau$.

Hint. (ii) First express the ST τ in terms of the random walk S_n. What are the possible values of S_τ? Note that if $S_n > S_{n-1}$, then $S_n = S_{n-1} + 1$.

11. Let $\{S_n\}_{n\geq 0}$ be a simple RW: starting at some initial point S_0, the walking particle at each transition goes up 1 w.p. $p \in (0,1)$ and down 1 w.p. $q = 1-p$. Assume that $p \neq \frac{1}{2}$.

(i) Show that $\{X_n := (q/p)^{S_n}\}$ is an MG.

(ii) Suppose the walk starts at $S_0 = 0$ and stops at the time $\tau := \min\{n \geq 0 : S_n = a \text{ or } S_n = b\}$, where $a < 0 < b$ are integers. Use Theorem 11.2 and the MG from part (i) to find the distribution of the RV S_τ, and then Theorem 11.2 and the MG $\{Z_n := S_n - n(p-q)\}$ to compute $\mathbf{E}\,\tau$.

12. Show that the set $\{U_1, U_2, \ldots\}$, where U_j are i.i.d. $U(0,1)$-RVs, is everywhere dense in $[0,1]$ with probability 1.

Hint. You may wish to use the Glivenko-Cantelli theorem (2.87).

13. Find the distribution of $X := 2W_{t_1} - W_{t_2}$, $0 < t_1 < t_2$.

14. Find the distribution of $X := W_0 + W_2 - W_3 + 2W_4$.

15. Derive the BM's FDD density $f_{t_1,\ldots,t_k}(x_1, \ldots, x_n)$ (see (11.22)) using (2.33) and the observation that $(W_{t_1}, W_{t_2}, \ldots, W_{t_n})$ is the result of a simple linear transformation of the vector $(W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$ with

[18] Martingale transforms are discrete analogues of stochastic integrals and play an important role in the mathematical theory of finance in discrete time.

independent components, so that the latter vector's density is just the product of the densities of the increments $W_{t_k} - W_{t_{k-1}}$, $k = 1, 2, \ldots, n$.

16. Note that the transformation of the standard normal vector $\boldsymbol{Z} \in \mathbf{R}^n$ into the vector of the values of the standard BM from the simulation algorithm on p. 320 can be written in the matrix form as $(W_{t_1}, \ldots, W_{t_n}) = \boldsymbol{Z}A$, $A \in \mathbf{R}^{n \times n}$. Specify the matrix A. How will the above matrix representation change if you were to directly simulate the vector $(X_{t_1}, \ldots, X_{t_n})$, where $\{X_t\}$ is the arithmetic BM (11.16)?

17. Compute the joint densities of (i) $(2W_3, W_5)$ and (ii) $(W_2, 2W_3, W_5)$.

18. Use Theorem 11.4 to show that $\{\widetilde{W}_t := tW_{1/t}\}_{t \geq 0}$ is a standard BM process proved that $\{W_t\}_{t \geq 0}$ is such.

19. Let $\tau := \min\{t > 0 : W_t = \pm\sqrt{a + bt}\}$ be the first time the BM crosses one of the two parabolic boundaries $\pm\sqrt{a + bt}$, $t \geq 0$, where $a > 0$ and $b \in (0, 1)$ are some constants. Use Theorems 11.6 and 11.2 to compute $\mathbf{E}\,\tau$.

20. Denote by $\tau := \min\{t > 0 : W_t \leq 2t - 4\}$ the first time the BM process crosses the boundary $v_t := 2t - 4$, $t \geq 0$. Using the three martingales of the Brownian motion (Theorem 11.6) and Theorem 11.2 (do not verify the conditions of the theorem), compute for the stopping time τ its:

    (i) mean value $\mathbf{E}\,\tau$;

    (ii) variance $\mathrm{Var}\,(\tau)$;

    (iii) Laplace transform $l_\tau(s) = \mathbf{E}\,e^{-s\tau}$, $s \geq 0$.

    (iv) Compute also $\mathbf{E}\,W_\tau$ and $\mathbf{E}\,W_\tau^2$.

21. Denote by $\tau := \min\{t > 0 : W_t = a \text{ or } W_t = b\}$ the first time the standard BM process takes one of the values a or b $(a < 0 < b)$. Using the three martingales of the BM and Theorem 11.2 (do not verify the conditions of the theorem),

    (i) find the distribution of W_τ;

    (ii) compute the mean value $\mathbf{E}\,\tau$;

    (iii) compute the Laplace transform $l_\tau(s) = \mathbf{E}\,e^{-s\tau}$, $s \geq 0$, in the case when $a = -1$, $b = 1$;

    (iv) use the result of part (iii) to compute the mean $\mathbf{E}\,\tau$ when $a = -1$, $b = 1$. Compare the result with that for question (ii).

    (v) Use the result of part (iii) to compute the variance $\mathrm{Var}\,(\tau)$ when $a = -1$, $b = 1$.

22. Let f_t and g_t be simple processes on $[0, T]$ given on a common filtered probability space, with a BM $\{W_t\}_{t \geq 0}$ given on it. Show by a direct calculation that $\mathbf{E}\,I_t(f)I_t(g) = \int_0^t \mathbf{E}\,f_s g_s\,ds$, $t \in [0, T]$.

23. Let g_t be a non-random function on $[0, T]$ satisfying $\int_0^T g_t^2 dt < \infty$. Use Itô's formula to show that the following process is an MG:

$$Y_t = \exp\left\{ \int_0^t g_s dW_s - \frac{1}{2} \int_0^t g_s^2 ds \right\}, \quad t \in [0, T].$$

24. Compute the stochastic differential $d \cos(W_t)$.

25. Put $X_t := t + W_t$, $t \geq 0$.

(i) Apply Itô's formula to compute the stochastic differential de^{-2X_t}.

(ii) Is the process $Y_t := e^{-2X_t}$, $t \geq 0$, a martingale? Explain.

26. The price S_t of a risky asset evolves according to the SDE

$$dS_t = 0.2 S_t dt + S_t dW_t, \quad t \geq 0, \quad S_0 = 5$$

(this is a special case of the so-called *Black–Scholes framework* to be discussed in Chapter 13).

(i) It is suspected that the SDE has a solution of the form $S_t = ce^{at+bW_t}$, where a, b and c are some constants. Use Itô's formula to verify this suspicion and find the values of the constants a, b and c in the solution.

(ii) Show that the process $X_t = 1/S_t$ satisfies the SDE

$$dX_t = 0.8 X_t dt - X_t dW_t, \quad t \geq 0, \quad X_0 = 0.2.$$

27. The "stochastic volatility" Heston model assumes that the "variance process" $\{V_t\}_{t \geq 0}$ follows the SDE

$$dV_t = (1 - V_t) dt + 2\sqrt{V_t}\, dW_t, \quad t \geq 0, \quad V_0 = 1.$$

(i) Derive an SDE for the "volatility process" $Z_t = \sqrt{V_t}$, $t \geq 0$, and find the initial condition for the SDE (i.e., the value Z_0).

(ii) Show that the SP $Z_t := e^{-t/2}\left(1 + \int_0^t e^{s/2} dW_s\right)$ satisfies the SDE and the initial condition you derived in part (i).