1. (i) $\mathbf{E}(S_{n+m}|\,S_n) = \mathbf{E}(S_n + X_{n+1} + \cdots + X_{n+m}|\,S_n) \overset{\text{CE1}}{=} \mathbf{E}(S_n|\,S_n) + \mathbf{E}(X_{n+1} + \cdots + X_{n+m}|\,S_n) \overset{\text{CE2,CE3}}{=} S_n + \mathbf{E}(X_{n+1} + \cdots + X_{n+m}) = S_n + m\mu$.

(ii) Using CE2 and the hint, $S_n = \mathbf{E}(X_1 + \cdots + X_n|\,S_n) = \sum_{k\le n} \mathbf{E}(X_k|\,S_n) = n\mathbf{E}(X_1|\,S_n)$. Hence $\mathbf{E}(X_1|\,S_n) = S_n/n$.

(iii) $\mathbf{E}(S_{n+m}^2|\,S_n) = \mathbf{E}((S_n + X_{n+1} + \cdots + X_{n+m})^2|\,S_n) = \mathbf{E}(S_n^2|\,S_n) + 2\mathbf{E}(S_n(X_{n+1} + \cdots + X_{n+m})|\,S_n) + \mathbf{E}((X_{n+1} + \cdots + X_{n+m})^2|\,S_n) = S_n^2 + 2S_n\mathbf{E}(X_{n+1} + \cdots + X_{n+m}) + \mathbf{E}(X_{n+1} + \cdots + X_{n+m})^2 = S_n^2 + 2S_n m\mu + m\sigma^2 + m^2\mu^2 = (S_n + m\mu)^2 + m\sigma^2$.

(iv) $\mathbf{E}(S_m|\,S_n) = \sum_{k\le m} \mathbf{E}(X_k|\,S_n) = \frac{m}{n} S_n$ from (ii).

2. (i) $\mathbf{E}(N_{t+s}|\,\mathcal{F}_t) = \mathbf{E}(N_t + N_{t+s} - N_s|\,\mathcal{F}_t) \overset{\text{CE1}}{=} \mathbf{E}(N_t|\,\mathcal{F}_t) + \mathbf{E}(N_{t+s} - N_s|\,\mathcal{F}_t) \overset{\text{CE2,CE3}}{=} N_t + \mathbf{E}(N_{t+s} - N_s) = N_t + \lambda s$.

(ii) $\mathbf{E}(N_{t+s}^2|\mathcal{F}_t) = (N_t + \lambda s)^2 + \lambda s$ (the argument is basically the same as in solution to part (iii) in the previous problem, just recall that $\mathbf{E}\,N_s = \mathrm{Var}\,(N_s) = \lambda s$).

(iii) $\mathbf{E}(N_s|\mathcal{F}_t) = N_s$, $\mathbf{E}(N_s^2|\mathcal{F}_t) = N_s^2$ by CE2, as N_s (and hence N_s^2 as well) is $\mathcal{F}_t$-measurable.

(iv) We know that $\mathbf{E}(N_s|N_t) =: \eta(N_s)$ is a function of N_t, with $\eta(n) = \mathbf{E}(N_s|N_t = n)$. From Example 2.5 we know that $(N_s|N_t = n) \sim B_{n,s/t}$ (see also the proof of Theorem 5.2). Hence $\eta(n) = sn/t$, and so $\mathbf{E}(N_s|N_t) = \frac{s}{t}N_t$. Similarly, one can find that $\mathbf{E}(N_s^2|N_t) = \frac{s(t-s)}{t^2}N_t + \frac{s^2}{t^2}N_t^2$.

3. (i) $\{\tau \le t\} = \Omega$ if $m \le t$ and $= \emptyset$ if $m > t$. In either case, the set is an element of $\mathcal{F}_t$. This is an ST.

(ii) $\{\tau_1 \wedge \tau_2 \le t\} = \{\tau_1 \le t\} \cup \{\tau_2 \le t\} \in \mathcal{F}_t$ as each of the events in the union belongs to $\mathcal{F}_t$. Likewise, $\{\tau_1 \vee \tau_2 \le t\} = \{\tau_1 \le t\} \cap \{\tau_2 \le t\} \in \mathcal{F}_t$. This is an ST.

(iii) $\{\tau > t\} = \bigcap_{k=0}^{t}\{X_{k+1}/X_k \le 1\}$, where the last event in the intersection does not need to belong to $\mathcal{F}_t$ (as X_{t+1} is, generally speaking, not $\mathcal{F}_t$-measurable). So $\{\tau \le t\} = \{\tau > t\}^c$ does not need to belong to $\mathcal{F}_t$ as well, and hence τ may not be an ST.

(iv) $\{\tau > t\} = \bigcap_{j=0}^{t}\left\{\sum_{k=0}^{t} X_k \le X_t^2\right\} \in \mathcal{F}_t$ as each of the events in the intersection $\in \mathcal{F}_t$. So $\{\tau \le t\} = \{\tau > t\}^c \in \mathcal{F}_t$ as well. This is an ST.

(v) $\{\tau = t\} = \{X_t > 10\} \cap \left[\bigcap_{k=t+1}^{10}\{X_k \le 10\}\right]$. Here, generally speaking, $[\cdots] \notin \mathcal{F}_t$, so τ does not need to be an ST.

4. That $\{X_t\}$ is adapted follows from the definition of CE. Integrability is obvious from CE4. The latter also implies the MG property, as for $0 \le s < t$, one has $\mathbf{E}(X_t|\mathcal{F}_s) = \mathbf{E}(\mathbf{E}(Y|\mathcal{F}_t)|\mathcal{F}_s) = \mathbf{E}(Y|\mathcal{F}_s) = X_s$.

5. As X_{t_1}, X_{t_2} are $\mathcal{F}_{t_3}$-measurable, $\mathbf{E}(X_{t_2} - X_{t_1})(X_{t_4} - X_{t_3}) = \mathbf{E}((\cdots)|\mathcal{F}_{t_3}) = \mathbf{E}\left[(X_{t_2} - X_{t_1})\mathbf{E}(X_{t_4} - X_{t_3}|\mathcal{F}_{t_3})\right] = 0$ since $\mathbf{E}(X_{t_4} - X_{t_3}|\mathcal{F}_{t_3}) = \mathbf{E}(X_{t_4}|\mathcal{F}_{t_3}) - X_{t_3} = 0$, $\{X_t\}_{t\ge0}$ being an MG. Square integrability is needed to ensure that the expectation of the product exists (by Cauchy–Bunyakovskii's inequality).

6. The SP is clearly adapted. Integrability: $\mathbf{E}|X_n| = \mathbf{E}|S_n^2 - n\sigma^2| \le \mathbf{E}\,S_n^2 + n\sigma^2 = \mathrm{Var}\,(S_n) + (\mathbf{E}\,S_n)^2 + n\sigma^2 = 2n\sigma^2 < \infty$.

(i) $\mathbf{E}(X_{n+1}|\mathcal{F}_n) = \mathbf{E}\left((S_n + Y_{n+1})^2 - (n+1)\sigma^2|\mathcal{F}_n\right) = \mathbf{E}\left(S_n^2 + 2S_nY_{n+1} + Y_{n+1}^2|\mathcal{F}_n\right) - (n+1)\sigma^2$, where $\mathbf{E}(S_n^2|\mathcal{F}_n) = S_n^2$, $\mathbf{E}(S_nY_{n+1}|\mathcal{F}_n) = S_n\mathbf{E}(Y_{n+1}|\mathcal{F}_n) = S_n\mathbf{E}\,Y_{n+1} = 0$, $\mathbf{E}(Y_{n+1}^2|\mathcal{F}_n) = \mathbf{E}\,Y_{n+1}^2 = \sigma^2$. So $\mathbf{E}(X_n|\mathcal{F}_n) = S_n^2 - n\sigma^2 = X_n$, that's an MG!

(ii) As all $X_k = S_k^2 - k\sigma^2$, $k = 1, \ldots, n$, are functions of $Y_1, \ldots, Y_n$, one has $\mathcal{F}_n' \subset \mathcal{F}_n$. Hence, using CE4, $\mathbf{E}(X_n|\mathcal{F}_n') = \mathbf{E}\left[\mathbf{E}(X_n|\mathcal{F}_n)|\mathcal{F}_n'\right] = \mathbf{E}(X_n|\mathcal{F}_n') = X_n$ from (i). Done.

7. DIY. This is no different from the proof of Theorem 11.6 (use the fact the this is a process with independent increments and, for part (iii), the explicit form of the Poisson process' MGF).

8. (i) We know from Example 11.2 that $\{S_n\}$ is an MG. By Theorem 11.2, $0 = \mathbf{E}\,S_0 = \mathbf{E}\,S_\tau = a\mathbf{P}(S_\tau = a) + b\mathbf{P}(S_\tau = b) = (a-b)\mathbf{P}(S_\tau = a) + b$, so that $\mathbf{P}(S_\tau = a) = \frac{b}{b-a}$, $\mathbf{P}(S_\tau = b) = \frac{-a}{b-a}$.

(ii) By Theorem 11.2 applied to the MG $X_n := S_n^2 - n\sigma^2 \equiv S_n^2 - n$, and the result of (i), one has $0 = \mathbf{E}\,X_0 = \mathbf{E}\,X_\tau = \mathbf{E}\,(S_\tau^2 - \tau) = \frac{a^2 b}{b-a} + \frac{b^2(-a)}{b-a} - \mathbf{E}\,\tau = -ab - \mathbf{E}\,\tau$. Therefore, $\mathbf{E}\,\tau = -ab$.

9. (i) The SP $\{Z_n\}$ is clearly adapted to $\mathbf{F}$. Integrability follows from the bound $|Z_n| \le \max_{j\le n}|X_j - X_{j-1}|\sum_{k\le n}|Y_k| \le \sum_{k\le n} C_n$. The MG property: using CE2, $\mathbf{E}\,(Z_{n+1} - Z_n|\,\mathcal{F}_n) = \mathbf{E}\,(Y_{n+1}(X_{n+1} - X_n)|\,\mathcal{F}_n) = Y_{n+1}\mathbf{E}\,(X_{n+1} - X_n|\,\mathcal{F}_n) = 0$ as $\{X_n\}$ is an MG.

(ii) $Y_1 = 1$, and for $n > 1$, one has $Y_n = 2Y_{n-1}$ if $X_{n-1} - X_{n-2} = -1$ (loss) and $Y_n = 1$ if $X_{n-1} - X_{n-2} = 1$ (win).

(iii) The RVs $\xi_n := X_n - X_{n-1}$ are i.i.d., $\mathbf{P}\,(\xi_1 = \pm 1) = \frac{1}{2}$. Hence one has $\mathbf{P}\,(\tau = n) = \mathbf{P}\,(\xi_1 = \xi_2 = \cdots = \xi_{n-1} = -1, \xi_n = 1) = 2^{-n}$, $n = 1, 2, \ldots$ (the geometric distribution). To find the expectation, either compute $\sum_{n=1}^\infty n 2^{-n}$ or, equivalently, first compute the GF $g_\tau(z) = z/(2-z)$ of τ (it's just the sum of a geometric series) and then take $\mathbf{E}\,\tau = \gamma_\tau'(1) = 2$. The statement of Theorem 11.2 does not hold as $0 = Z_0$, but as your fortune at the time of your first win is always $Z_\tau = 1$ (verify that!), one has $\mathbf{E}\,Z_\tau = 1$.

(iv) $\mathbf{E}\,(Z_n; \tau > n) = \mathbf{E}\,(Z_n; \xi_1 = \xi_2 = \cdots = \xi_n = -1) = -\sum_{j=1}^{n-1} 2^j \times 2^{-n} = -\frac{2^n - 1}{2-1}2^{-n} = 2^{-n} - 1 \not\to 0$ as $n \to \infty$. Condition (11.11) is not satisfied.

10. (i) As no filtration is specified, we consider the natural one. Integrability follows from boundedness ($0 < X_n \le 2^{2n}$). Setting $Y_n := S_n - S_{n-1}$, $n \ge 1$, we have $\mathbf{E}\,(X_{n+1}|\,\mathcal{F}_n) = \mathbf{E}\,(2^{-S_n - Y_{n+1}}|\,\mathcal{F}_n) = 2^{-S_n}\mathbf{E}\,(2^{-Y_{n+1}}|\,\mathcal{F}_n) = X_n \mathbf{E}\,2^{-Y_{n+1}} = X_n$ since $\mathbf{E}\,2^{-Y_{n+1}} = 2^{-1} \times \frac{6}{7} + 2^2 \times \frac{1}{7} = 1$.

(ii) Note that $\{X_n \le 0.1\} = \{S_n \ge \log_2 10\} = \{S_n \ge 4\}$ (as S_n is integer-valued). Hence $\tau = \min\{n \ge 0 : S_n \ge 4\} = \min\{n \ge 0 : S_n = 4\}$ as the RW $\{S_n\}$ is skip-free, and so $\mathbf{P}\,(S_\tau = 4) = 1$ (the RW $\{S_n\}$ crosses any positive level w.p. 1 as it has positive trend: $\mathbf{E}\,Y_1 = 1 \times \frac{6}{7} - 2 \times \frac{1}{7} = \frac{4}{7}$.) By Theorem 11.2, $0 = \mathbf{E}\,Z_0 = \mathbf{E}\,Z_\tau = \mathbf{E}\,(S_\tau - \frac{4}{7}\tau) = 4 - \frac{4}{7}\mathbf{E}\,\tau$. Hence $\mathbf{E}\,\tau = 7$.

11. (i) Use the natural filtration. Integrability is obvious from boundedness. Set $Y_n := S_n - S_{n-1}$, $n \ge 1$. We have $\mathbf{E}\,(X_{n+1}|\,\mathcal{F}_n) = \mathbf{E}\,((q/p)^{S_n + Y_{n+1}}|\,\mathcal{F}_n) = (q/p)^{S_n}\mathbf{E}\,((q/p)^{Y_{n+1}}|\,\mathcal{F}_n) = X_n \mathbf{E}\,(q/p)^{Y_{n+1}} = X_n((q/p)^1 p + (q/p)^{-1} q) = X_n$.

(ii) Clearly, either $S_\tau = a$ or $S_\tau = b$. Set $r := q/p$. By Theorem 11.2, $1 = \mathbf{E}\,X_0 = \mathbf{E}\,X_\tau = r^a\mathbf{P}\,(S_\tau = a) + r^b\mathbf{P}\,(S_\tau = b) = (r^a - r^b)\mathbf{P}\,(S_\tau = a) + r^b$, so that $\mathbf{P}\,(S_\tau = a) = 1 - \mathbf{P}\,(S_\tau = b) = (r^b - 1)/(r^b - r^a)$.

Apply Theorem 11.2 to $\{Z_n\}$: $0 = \mathbf{E}\,Z_0 = \mathbf{E}\,Z_\tau = \mathbf{E}\,(S_\tau - (p-q)\tau) = \mathbf{E}\,S_\tau - (2p-1)\mathbf{E}\,\tau = a\mathbf{P}\,(S_\tau = a) + b(1 - \mathbf{P}\,(S_\tau = a)) - (2p-1)\mathbf{E}\,\tau$. Hence $\mathbf{E}\,\tau = (b(1-r^a) + a(r^b - 1))/[(2p-1)(r^b - r^a)]$.

12. Suppose $\omega \in \Omega$ is such that $\{U_1(\omega), U_2(\omega), \ldots\}$ is not everywhere dense in $[0,1]$. That means that there exist points $t_1 < t_2$ in $[0,1]$ s.t. $U_j(\omega) \notin (t_1, t_2)$, $j = 1, 2, \ldots$ Hence, for the EDFs F_n^* corresponding to the samples consisting of the first n RVs U_j, one has $F_n^*(t_1) = F_n^*(t_2)$, $n = 1, 2, \ldots$ But that clearly contradicts (2.87) as the limiting DF is $F(t) = t$, $t \in [0,1]$. Therefore the set of such ω's must have zero probability.

13. The RW X is normal as a linear transformation of a Gaussian vector, so one just needs to compute its mean and variance. Clearly, $\mathbf{E}\,(2W_{t_1} - W_{t_2}) =$

$2\mathbf{E}\,W_{t_1} - \mathbf{E}\,W_{t_2} = 0$, $\operatorname{Var}(X) = \mathbf{E}\,X^2 = \mathbf{E}\,(2W_{t_1} - W_{t_2})^2 = 4\mathbf{E}\,W_{t_1}^2 - 4\mathbf{E}\,W_{t_1}W_{t_2} + \mathbf{E}\,W_{t_2}^2 = 4t_1 - 4(t_1 \wedge t_2) + t_2 = t_2$. So $2W_{t_1} - W_{t_2} \sim N(0, t_2)$.

Alternatively, one may wish to use independence of the increments of the BM: $X = 2W_{t_1} - W_{t_2} = W_{t_1} - (W_{t_2} - W_{t_1})$, the summands on the right-hand side being independent with distributions $N(0, t_2)$ and $N(0, t_2 - t_1)$, resp. Hence the same answer as above.

14. As in Problem 13, X will be normal with zero mean. One can compute $\operatorname{Var}(X)$ from the covariance function of the BM (cf. solution to Problem 13; DIY!) or using independence of increments and the following alternative representation for the RV: $X = 2W_2 + (W_3 - W_2) + 2(W_4 - W_3) \sim N(0, 13)$.

15. In the notation of (2.33), we have $\boldsymbol{X} = (W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}})$, $\boldsymbol{Y} = (W_{t_1}, \ldots, W_{t_n})$, and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}A$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Hence $\boldsymbol{g}^{-1}$ is also a linear transform, $|J(\boldsymbol{g}^{-1}(\boldsymbol{y}))| = |\det A^{-1}| = |\det A|^{-1} = 1$, and so

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(\boldsymbol{g}^{-1}(\boldsymbol{y})) = \prod_{j=1}^{n} f_{W_{t_j} - W_{t_{j-1}}}(x_j - x_{j-1}),$$

which is clearly the same as $f_{t_1,\ldots,t_k}(x_1, \ldots, x_n)$ (see (11.22)).

16. Cf. solution to the previous problem. The transformation matrix is

$$A = \begin{pmatrix} \sqrt{t_1 - t_0} & \sqrt{t_1 - t_0} & \sqrt{t_1 - t_0} & \cdots & \sqrt{t_1 - t_0} \\ 0 & \sqrt{t_2 - t_1} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_2 - t_1} \\ 0 & 0 & \sqrt{t_3 - t_2} & \cdots & \sqrt{t_3 - t_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \sqrt{t_n - t_{n-1}} \end{pmatrix}.$$

For the arithmetic BM, multiply the matrix by σ (and don't forget that there can be a non-zero drift!).

17. (i) Observing that $2W_3 \sim N(0, 12)$ and $(W_5 | 2W_3 = x) \sim N(x/2, 2)$ by independent increments, by the formula (2.80) for conditional densities, $f_{(2W_3, W_5)}(x, y) = f_{W_5|2W_3}(y|x) f_{2W_3}(x) = \frac{1}{\sqrt{2\pi \cdot 2}} e^{-(y - x/2)^2/4} \frac{1}{\sqrt{2\pi \cdot 12}} e^{-x^2/24} = \frac{1}{4\pi\sqrt{6}} \exp\{-\frac{y^2}{4} + \frac{xy}{4} - \frac{5x^2}{48}\}$.

(ii) Arguing as in (i), compute $f_{(W_2, 2W_3)}(x, y) = \frac{1}{4\pi\sqrt{2}} \exp\{-\frac{y^2}{8} + \frac{xy}{2} - \frac{3x^2}{4}\}$. Next note that, due to independent increments, $(W_5 | (W_2, 2W_3)) = (x, y)) \overset{d}{=} (W_5 | 2W_3 = y) \sim N(y/2, 2)$, as in (i). Now use $f_{(W_2, 2W_3, W_5)}(x, y, z) = f_{W_5|(W_2, 2W_3)}(z | (x, y)) f_{(W_2, 2W_3)}(x, y) = \frac{1}{\sqrt{2\pi \cdot 2}} e^{-(z - y/2)^2/4} f_{(W_2, 2W_3)}(x, y) =$

$\frac{1}{s\pi\sqrt{2\pi}} \exp\{-\frac{3x^2}{4} + \frac{xy}{2} - \frac{3y^2}{16} + \frac{yz}{4} - \frac{z^2}{4}\}.$

18. Clearly, $\mathbf{E}\,\widetilde{W}_t = 0$. For $0 < s < t$, one has $\mathbf{E}\,\widetilde{W}_s\widetilde{W}_t = st\mathbf{E}\,W_{1/s}W_{1/t} = s$ since $\frac{1}{s} \wedge \frac{1}{t} = \frac{1}{t}$. Since $\{\widetilde{W}_t\}$ is a Gaussian process (indeed, $(\widetilde{W}_{t_1}, \ldots, \widetilde{W}_{t_n}) = (t_1 W_{1/t_1}, \ldots, t_n W_{1/t_n})$ is just a linear transform of the Gaussian vector $(W_{1/t_1}, \ldots, W_{1/t_n})$, and so is Gaussian itself), Theorem 11.4 yields the desired assertion.

19. Using the MG $\{Y_t := W_t^2 - t\}_{t\geq 0} : 0 = \mathbf{E}\,Y_0 = \mathbf{E}\,Y_\tau = \mathbf{E}\,(W_\tau^2 - \tau) = \mathbf{E}\,(a + b\tau - \tau) = a - (1-b)\mathbf{E}\,\tau$. Hence $\mathbf{E}\,\tau = \frac{a}{1-b}$.

20. (i) Using the first MG: $0 = \mathbf{E}\,W_0 = \mathbf{E}\,W_\tau = \mathbf{E}\,v_\tau = \mathbf{E}\,(2\tau - 4) = 2\mathbf{E}\,\tau - 4$, so that $\mathbf{E}\,\tau = 2$.

(ii) Using the second MG $\{Y_t := W_t^2 - t\}_{t\geq 0} : 0 = \mathbf{E}\,Y_0 = \mathbf{E}\,Y_\tau = \mathbf{E}\,(W_\tau^2 - \tau) = \mathbf{E}\,((2\tau - 4)^2 - \tau) = 4\mathbf{E}\,\tau^2 - 17\mathbf{E}\,\tau + 16 = 4\mathbf{E}\,\tau^2 - 18$, so that $\mathbf{E}\,\tau^2 = 4.5$.

(iii) Using the third MG $\{Z_t := e^{uW_t - u^2 t/2}\}_{t\geq 0} : 1 = \mathbf{E}\,Z_0 = \mathbf{E}\,Z_\tau = \mathbf{E}\,e^{uW_\tau - u^2\tau/2} = \mathbf{E}\,e^{u(2\tau-4) - u^2\tau/2} = \mathbf{E}\,e^{-4u-(u^2/2-2u)\tau} = e^{-4u}\mathbf{E}\,e^{-s\tau}$, so that $\mathbf{E}\,e^{-s\tau} = e^{4u}$ for $s := u^2/2 - 2u$. Solving for u: $u = 2 \pm \sqrt{2(s+2)}$; choosing "−" (why?), we obtain $l_\tau(s) = \exp\{8 - 4\sqrt{2(s+2)}\}$.

(iv) By Theorem 11.2, $0 = \mathbf{E}\,W_0 = \mathbf{E}\,W_\tau$, $0 = \mathbf{E}\,Y_0 = \mathbf{E}\,Y_\tau = \mathbf{E}\,W_\tau^2 - \mathbf{E}\,\tau$, so that $\mathbf{E}\,W_\tau = 0$, $\mathbf{E}\,W_\tau^2 = \mathbf{E}\,\tau = 2$.

21. (i) Clearly, $\mathbf{P}\,(W_\tau = a) = 1 - \mathbf{P}\,(W_\tau = b)$, and, by Theorem 11.2 for the MG $\{W_t\}$, one has $0 = \mathbf{E}\,W_0 = \mathbf{E}\,W_\tau = a\mathbf{P}\,(W_\tau = a) + b\mathbf{P}\,(W_\tau = b) = (a-b)\mathbf{P}\,(W_\tau = a) + b$, so that $\mathbf{P}\,(W_\tau = a) = \frac{b}{b-a}$.

(ii) $\mathbf{E}\,\tau = -ab$ (see (12.31)).

(iii) Using the third MG and Theorem 11.2, for any $u \in \mathbf{R}$, one has $1 = \mathbf{E}\,Z_0 = \mathbf{E}\,Z_\tau = \mathbf{E}\,e^{uW_\tau - u^2\tau/2} = \mathbf{E}\,\mathbf{E}\,\big(e^{uW_\tau - u^2\tau/2}\big|\,\tau\big) = \mathbf{E}\,\big(e^{-u^2\tau/2}\mathbf{E}\,\big(e^{uW_\tau}\big|\,\tau\big)\big)$. By symmetry, $W_\tau = \pm a$ independently of τ, with equal probabilities, so we obtain $1 = \frac{1}{2}(e^{au} + e^{-au})\mathbf{E}\,e^{-u^2\tau/2}$. Therefore $l_\tau(s) = \mathbf{E}\,e^{-s\tau} = 1/\cosh(\sqrt{2s}a)$, $s \geq 0$.

(iv) By the chain rule, $l_\tau(s)' = \frac{\sinh(\sqrt{2s}a)}{\cosh^2(\sqrt{2s}a)}\frac{a}{\sqrt{2s}} \to a^2$ as $s \to 0$, which is the same answer as in (ii) in the case $a = -b$.

(v) DIY. Just keep differentiating.

22. One can assume w.l.o.g. that the simple processes have a common time partition $0 = t_0 < t_1 < \cdots < t_n = T$: $f_t = X_k$ and $g_t = Y_k$ for $t \in [t_{k-1}, t_k)$, for some $\mathcal{F}_{t_{k-1}}$-measurable RVs X_k and Y_k, $k = 1, 2, \ldots, n$. Then

$$\mathbf{E}\,I_t(f)I_t(g) = \mathbf{E}\sum_{k=1}^n X_k(W_{t_k} - W_{t_{k-1}})\sum_{j=1}^n Y_j(W_{t_j} - W_{t_{j-1}})$$

$$= \sum_{k=1}^n \underbrace{\mathbf{E}\,\big(X_kY_k(W_{t_k} - W_{t_{k-1}})^2\big)}_{=:A_k} + \sum_{j\neq k}\underbrace{\mathbf{E}\,X_kY_j(W_{t_k} - W_{t_{k-1}})(W_{t_j} - W_{t_{j-1}})}_{=:B_{jk}},$$

where, by CE4, $A_k = \mathbf{E}\,\mathbf{E}\,\big(X_kY_k(W_{t_k} - W_{t_{k-1}})^2\big|\,\mathcal{F}_{t_{k-1}}\big) = \mathbf{E}\,\big(X_kY_k\mathbf{E}\,\big((W_{t_k} - W_{t_{k-1}})^2\big|\,\mathcal{F}_{t_{k-1}}\big)\big) = \mathbf{E}\,X_kY_k\mathbf{E}\,(W_{t_k} - W_{t_{k-1}})^2 = (t_k - t_{k-1})\mathbf{E}\,X_kY_k$, whereas, for $j < k$, $B_{jk} = \mathbf{E}\,\mathbf{E}\,\big(X_kY_j(W_{t_k} - W_{t_{k-1}})(W_{t_j} - W_{t_{j-1}})\big|\,\mathcal{F}_{t_{k-1}}\big) = \mathbf{E}\,\big(X_kY_j(W_{t_j} -$

$W_{t_{j-1}})\mathbf{E}\,(W_{t_k} - W_{t_{k-1}} | \mathcal{F}_{t_{k-1}})) = 0$ as the last CE is zero. Thus, $\mathbf{E}\,I_t(f)I_t(g) = \sum_{k=1}^{n}(t_k - t_{k-1})\mathbf{E}\,X_k Y_k = \int_0^t \mathbf{E}\,f_s g_s\,ds$.

The identity is actually an immediate consequence of Itô's isometry (Theorem 11.7(iii)) and the following simple general fact: suppose $L : H_1 \to H_2$ is a linear mapping of one inner product space H_1 into another (H_2) such that, for any $u \in H_1$ one has $\|Lu\| = \|u\|$. Then the inner products are also preserved: for any $u, v \in H_1$, one has $(Lu, Lv) = (u, v)$. To verify that, just compare the first expression with the last one in the following chain of equalities: $\|u\|^2 + \|v\|^2 + 2(u, v) = \|u + v\|^2 = \|L(u + v)\|^2 = \|Lu\|^2 + \|Lv\|^2 + 2(Lu, Lv) = \|u\|^2 + \|v\|^2 + 2(Lu, Lv)$.

23. One has $Y_t = f(X_t)$ with $f(x) = e^x$ (so that $f'(x) = f''(x) = f(x)$) and $dX_t = g_t dW_t - \frac{1}{2}g_t^2 dt$. By Itô's formula, $dY_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 = f(X_t)(g_t dW_t - \frac{1}{2}g_t^2 dt) + \frac{1}{2}f(X_t)g_t^2 dt = f(X_t)g_t dW_t$, showing that $\{Y_t\}$ is an MG (Remark 11.3).

24. Here $X_t = f(W_t)$ with $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, and so Itô's formula yields $dX_t = -\sin(W_t)dW_t - \frac{1}{2}\cos(W_t)dt$.

25. (i) For $f(x) = e^{-2x}$ one has $f'(x) = -2f(x)$, $f''(x) = 4f(x)$, so by Itô's formula $df(X_t) = -2f(X_t)(dt + dW_t) + \frac{1}{2}\,4f(X_t)(dt + dW_t)^2 = -2f(X_t)dt - 2f(X_t)dW_t + 2f(X_t)dt = -2f(X_t)dW_t$.

(ii) It is an MG indeed, using Remark 11.3.

26. (i) Clearly, $c = S_0 = 5$ and $S_t = f(t, W_t)$ with $f(t, x) = 5e^{at+bx}$, so that $\partial_t f = af$, $\partial_x f = bf$, $\partial_{xx} f = b^2 f$. So, by Itô's formula, $dS_t = af(t, W_t)dt + \partial_x f(t, W_t)dW_t + \frac{1}{2}b^2 f(t, W_t)(dW_t)^2 = (a + \frac{b^2}{2})S_t dt + bS_t dW_t$. Comparing that with the assumed SDE, we obtain that $a + \frac{b^2}{2} = 0.2$, $b = 1$. So yes, the suspected form of S_t is correct, with $a = -0.3$, $b = 1$, $c = 5$.

(ii) Either you apply Itô's formula to the verified representation $X_t := 1/S_t = (5e^{-0.3t+W_t})^{-1} = 0.5e^{0.3t-W_t}$, or you apply it directly to $f(S_t) = 1/S_t$ using the assumed SDE. Using the latter approach, one has $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, so that $df(S_t) = -S_t^{-2}(0.2S_t dt + S_t dW_t) + \frac{1}{2}\,2S_t^{-3}(dS_t)^2 = -0.2S_t^{-1}dt - S_t^{-1}dW_t + S_t^{-3}S_t^2 dt = 0.8X_t dt - X_t dW_t$, QED.

27. (i) Here $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}\,x^{-1/2}$, $f''(x) = -\frac{1}{4}\,x^{-3/2}$, $x > 0$. So, by Itô's formula, $dZ_t = \frac{1}{2}\,V_t^{-1/2}((1 - V_t)dt + 2\sqrt{V_t}\,dW_t) + \frac{1}{2}(-\frac{1}{4}\,V_t^{-3/2})(2\sqrt{V_t})^2 dt = -\frac{1}{2}\,V^{1/2}(t)dt + dW_t = -\frac{1}{2}\,Zdt + dW_t$, which is a special case of (11.46) (with $\alpha = \sigma = 1$). Clearly, $Z_0 = \sqrt{V_0} = 1$.

(ii) Use the product rule of Itô's calculus (Theorem 11.10) with $X_t = e^{-t/2}$, $Y_t = 1 + \int_0^t e^{s/2}dW_s$. As one the factors is smooth ($dX_t = -\frac{1}{2}\,e^{-t/2}dt = -\frac{1}{2}\,X_t dt$), by Remark 11.2 this will be just the usual product rule: $dZ_t = dX_t \cdot Y_t + X_t \cdot dY_t = -\frac{1}{2}\,X_t Y_t dt + X_t e^{-t/2}dW_t = -\frac{1}{2}\,Z_t dt + dW_t$. Obviously, $Z_0 = e^0\left(1 + \int_0^0\right) = 1$.