→ The term "arbitrage" actually refers to the market of underlying. It means that ∃ a trading strategy with zero net investment, never making a loss and sometimes making profit.

→ The absence (or presence) of such opportunity is one of the key properties for any market model.

Q NA (no-arbitrage) condition for our model

zero net investment $0 = V_0 = \Delta S_0 + b$

$$\Rightarrow b = -\Delta S_0$$

$$\therefore V_1 = \Delta S_1 + (1+r) b = \Delta S_1 - \Delta S_0 (1+r)$$

$$= \Delta (1+r) \left[\frac{S_1}{1+r} - S_0 \right]$$

If $d < 1+r < u$, then

$$\frac{S_1}{1+r} - S_0 > 0 \quad \text{when} \quad S_1 = u S_0$$

and $\frac{S_1}{1+r} - S_0 < 0$, when $S_1 = d S_0$

which means NA.

Example (Pricing a European Put)

$$P = \frac{1}{1+r} \left[p^* (u S_0 - k)^- + (1-p^*)(d S_0 - k)^- \right]$$

$S_0 = 1$, $k = 1$, $r = 0.25$, $u = 1.75$, $d = 0.5$

$$= \frac{1}{1.25}\left[0.6 \times (1.15-1) + 0.4 \times (0.5-1)\right]$$

$$= \frac{1}{1.25}\left[0.6 \times 0 + 0.4 \times 0.5\right] = 0.16$$

Alternative

One can use the "put call parity"

C → european call P ↑ put op k common strike in NA

Portfolio
- one share of stock
- one put on one share of stock
- a short position in one call on one share of the stock

$t=0 \quad V_0 = S_0 + P - C$

$t=1 \quad V_1 = S_1 + (S_1-k)^- - (S_1-k)^+$

$= S_1 - \left[(S_1-k)^- - (S_1-k)^+\right]$

$= S_1 - (S_1-k) = k$

due to NA $V_0 = \frac{k}{1+r}$

$\therefore \; S_0 + P - C = \frac{k}{1+r}$

$\Rightarrow P = \frac{k}{1+r} - S_0 + C$

If $V_0 < \frac{k}{1+r}$ ← arbitrage.

$t=0$ borrow V_0 from bank and buy portfolio

$t=1$ sell at $t=1$

return $= k - (1+r)V_0 > 0$ RS