How does one replicate claims in the Black-Scholes market:

Self financing discrete

trading strategy (Δ_t, b_t), $t = 1, \dots, T$ is self financing for portfolio time t value V_t

$$V_t := \Delta_t S_t + b_t (1+r)^t = \Delta_{t+1} S_t + b_{t+1} (1+r)^t, \quad t < T$$

$$\therefore V_{t+1} - V_t = \Delta_{t+1} S_{t+1} + b_{t+1} (1+r)^{t+1} - \Delta_{t+1} S_t - b_{t+1} (1+r)^t$$

$$= \Delta_{t+1} (S_{t+1} - S_t) + b_{t+1} (B_{t+1} - B_t), \quad t < T$$

Cont bivariate process (Δ_t, b_t), $t \in [0, T]$ is trading strategy if it is predictable and integrals $\int_0^u \Delta_t \, dS_t$ are well defined

time t portfolio value $V_t := \Delta_t S_t + b_t B_t$

the trading strategy (Δ_t, b_t) is self financing if

$$dV_t = \Delta_t \, dS_t + b_t \, dB_t, \quad t < T$$

So, in the cont-time case, the absence of arbitrage means that there is no self-financing trading strategy that has zero initial value and a.s. $V_T \geq 0$ at the terminal time T with prob $P(V_T > 0) > 0$.

Replicating (or hedging) a claim X means constructing a self-financing trading strategy that would generate the same cash flow as X at expiry.

→ Self financing strategy which replicates the claim $X = g(S_T)$ (so that $V_T(\omega) = X(\omega)$ a.s.)

then the claim value $P_t(X) = V_t$, $t \leq T$

or otherwise there will be an arbitrage opportunity.

$$P_t(X) = E^*\left(e^{-r(T-t)} X \mid \mathcal{F}_t\right)$$

$$= e^{-r(T-t)} E^*\left(g(S_T) \mid \mathcal{F}_t\right)$$

$$= e^{-r(T-t)} E^*\left(g(S_T) \mid S_t\right)$$

| ∵ price process is Markovian

$$=: f(t, S_t)$$

Assuming the fn $f(t,s)$ is smooth enough, one can apply Itô formula

$$\underline{\underline{\Delta_t dS_t + b_t dB_t}} = dV_t = d\,P_t(X) = d\,f(t, S_t)$$

$$= \partial_t f(t,S_t)\,dt + \partial_S f(t,S_t)\,dS_t + \frac{1}{2}\partial_{SS} f(t,S_t)\,(dS_t)^2$$

✓ | $S_t = \mu S_t\,dt + \sigma S_t\,dW_t$

$$= \left(\partial_t f(t,S_t) + \frac{1}{2}\partial_{SS} f(t,S_t)\,\sigma^2 S_t^2\right)dt + \partial_S f(t,S_t)\,dS_t$$

$\therefore$ replicating strategy must be of the form

$$\Delta_t = \partial_S f(t, S_t) \; ; \; b_t = \big(f(t, S_t) - \Delta_t S_t\big)/B_t \; , t \in [0, T]$$

Thus # of shares in the hedge is given by the partial derivative of the time t claim value

Sensitivities of claim process

delta $\Delta = \partial_{S_0} C$

Gamma $\Gamma := \partial_{SS} f(t, S_t)$

vega $\nu := \partial_\sigma P_t(X)$

theta $\Theta := \partial_T P_t(X)$

rho $\rho := \partial_r P_t(X)$

Example Sensitivities for a European call and put:

w.l.o.g. $t = 0$ European call $C = C(S_0, T, k, r, \sigma)$

Using $P_0(X) = S_0 \, \Phi(h) - k e^{-rT} \, \Phi(h - \sigma\sqrt{T})$

$$h_0 := \frac{\ln\left(\frac{S_0}{k}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \equiv h - \sigma\sqrt{T}$$

$$\partial_{S_0} h = \frac{1}{\sigma\sqrt{T} S_0} \; , \quad \Phi'(h) = \phi(h) := \frac{1}{\sqrt{2\pi}} e^{-h^2/2}$$

$$\Delta = \partial_{S_0} C = \Phi(h) + S_0 \Phi'(h) \partial_{S_0} h - k e^{-rT} \Phi'(h - \sigma\sqrt{T}) \times \partial_{S_0} h .$$

$$\Gamma(\ldots) \ldots \frac{1}{\ldots}\left[e^{-h^2/2} \ldots k \ldots e^{-(h - \sigma\sqrt{T})^2/2} \right]$$

$$= \Phi(h) + \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}}\left[e^{\cdots} - \frac{k}{S_0 e^{rT}} e^{\cdots/2} \right]$$

$$= \Phi(h) + \frac{e^{-h^2/2}}{\sigma\sqrt{T}\sqrt{2\pi}}\left[1 - \frac{k}{S_0 e^{rT}} e^{h\sigma\sqrt{T}} e^{-\sigma^2 T/2} \right]$$

$$e^{h\sigma\sqrt{T}} = e^{\ln\left(\frac{S_0}{k}\right) + rT + \frac{1}{2}\sigma^2 T}$$

$$= \Phi(h) + \frac{e^{-h^2/2}}{\sigma\sqrt{T}\sqrt{2\pi}}\left[1 - 1\right] = \Phi(h)$$

$$\therefore \Gamma = \partial_{S_0 S_0} C = \partial_{S_0} \Phi(h) = \Phi'(h)\, \partial_{S_0} h$$

$$= \frac{\phi(h)}{\sigma\sqrt{T} S_0}$$

$$\nu = \partial_\sigma P_t(X) = \phi(h) S_0 \sqrt{T}$$

To compute the Greeks for the put, one can use put-call parity

$$S_t + P_t - C_t = k e^{-r(T-t)}, \quad t \in [0,T]$$

$$\Delta = \partial_{S_0} P = \partial_{S_0} C - \partial_{S_0} S_0 = \Phi(h) - 1$$

Since $\Phi(h) \in (0,1)$

$\therefore$ Δ for call is always +ve

while Δ for put is " -ve,

which makes sense: the issuer of the call should be able to deliver the stock in case the

[illegible]

owner decides to exercise the option, while writer of put should be able to absorb purchasing the stock.

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Value at Risk (VAR):

VaR
- → var. – cov. approach
- → historical simulation approach
- → Monte – Carlo simulation method

Var. – cov. approach (delta – normal method)

Assumption: returns for each of the institutions assets are normally distributed

Var. of portfolio returns

$$\mathrm{Var}(\tilde{R}) = \sum_i \sum_j w_i w_j \sigma_{ij} = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where w_i : fraction of total portfolio value consisting of asset i, so $\sum_i w_i = 1$

σ_{ij} : cov. of asset i's return with asset j's return

σ_i : sd of asset i's return

ρ_{ij} : corr of asset i's return with asset j's return

→ Consider a portfolio consisting of then three assets

a. A currency swap. Because of changes in the exchange rate since the swap was first entered into, the swap now has a value of \$2 million, or 8.7% of the portfolio's total value

b. A bond. The market value of the bond is \$17 million, which is 73.9% of the portfolio total value

c. A stock. The 10,000 shares are worth \$4 million or 17.4% of the portfolio's total value.

Assume the var-cov matrix of the assets daily returns is

	swap	bond	stock
swap	0.0090	−0.0008	0.00007
bond		0.00040	−0.00010
stock			0.00300

Var of portfolio's daily return distr is given by

$$Var(\tilde{R}) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23}$$

$$= (0.087)^2 \times 0.009 + (0.739)^2 \times 0.004 + (0.174)^2 \times 0.003$$
$$+ 2\times 0.087 \times 0.739 \times (-0.0008) + 2 \times 0.087 \times 0.174 \times 0.0007 + 2 \times 0.739 \times 0.174 \times (-0.0001)$$

$$= 0.0002822$$

s.d. of daily returns dist^n = $\sqrt{0.0002822}$

$= 0.0168$ or 1.68%

1 s.d of \$ loss from the portfolio value of \$23 million is \$386375 (1.68% of \$23 million is \$386375)

VaR = $z_\alpha \times$ 1 sd of \$ loss from portfolio value

Thus there is 5% prob. that a one-day loss of $1.645 \times \$386375 = \$635587$ will be realized

There is a 1% prob. that a one-day loss of

$2.327 \times \$386375 = \$899095$ will be realized.

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