Stochastic Calculus   Ito's integrals

White Noise

$\{W(t), t \geq 0\}$ SBM/Weiner process, f having a cont. derivative on $[a,b]$

Sto. integral

$$\int_a^b f\,dW(t) \equiv \lim_{\substack{n\to\infty \\ \max(t_i - t_{i-1}) \to 0}} \sum_{i=1}^{n} f(t_{i-1})[W(t_i) - W(t_{i-1})] \quad \longrightarrow (4)$$

where $a = t_0 < t_1 < \dots < t_n = b$ is a partition of $[a,b]$

$$\sum_{i=1}^{n} f(t_{i-1})(W(t_i) - W(t_{i-1})) = f(b)W(b) - f(a)W(a) - \sum_{i=1}^{n} W(t_i)(f(t_i) - f(t_{i-1}))$$

$$f(t_n)W(t_n) - f(t_0)W(t_0) - W(t_1)(f(t_1) - f(t_0)) - W(t_2)(f(t_2) - f(t_1)) - \dots - W(t_n)(f(t_n) - f(t_{n-1}))$$

defn

$$\boxed{\int_a^b f(t)\,dW(t) = f(b)W(b) - f(a)W(a) - \int_a^b W(t)\,df(t)}$$

$$E\left(\int_a^b f(t)\,dW(t)\right) = 0$$

$$V\left(\int_a^b f(t)\,dW(t)\right) = \underbrace{E\left(\int_a^b f(t)\,dW(t)\right)^2 = \int_a^b f^2(t)\,dt}_{\text{Ito's isometry}},$$

since

$$V\left(\sum_{i=1}^{n} f(t_{i-1})(W(t_i) - W(t_{i-1}))\right) = \sum f^2(t_{i-1})\, V(W(t_i) - W(t_{i-1}))$$

$i=1 \qquad\qquad i=1$

$t_i - t_{i-1}$

$$= \sum_{i=1}^{n} f^2(t_{i-1}) \times (t_i - t_{i-1})$$

If integrand $f_t \equiv f(t)$ is non-random

$$\int_a^b f(t)\, dW(t) \sim N\left(0, \int_a^b f^2(t)\,dt\right)$$

Example (1) Find the distribution of $X := \int_0^t s\, dW_s$

Sol Since integrand is non-random

$$X \sim N\left(0, \int_0^t s^2 ds\right) \equiv N\left(0, \frac{t^3}{3}\right)$$

(2)
$$\int_0^t W_s\, dW_s \simeq \sum_j W_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right)$$

$$\overset{*}{=} \frac{W_t^2}{2} - \frac{1}{2}\sum_j \left(W_{t_{j+1}} - W_{t_j}\right)^2 \to \frac{W_t^2}{2} - \frac{t}{2}$$

Ex. $*$
$$\sum_{k=1}^{n} a_{k-1}(a_k - a_{k-1}) = \frac{1}{2}a_n^2 - \frac{1}{2}\sum_{k=1}^{n}(a_k - a_{k-1})^2$$

$$\sqrt{h}\, Z \overset{d}{=} W_{t+h} - W_t \sim N(0,h) \quad , \quad Z \sim N(0,1)$$

$$\sum_{j=1}^{n}\left(W_{tj/n} - W_{t(j-1)/n}\right)^2 \overset{d}{=} \sum_{j=1}^{n}\left(\sqrt{\frac{t}{n}}\, Z_j\right)^2$$

$$= t \underbrace{\frac{1}{n}\sum_{j=1}^{n} Z_j^2}_{\to E[Z^2] = 1} \to t$$

$$\int_0^t W_s\,dW_s = \frac{W_t^2}{2} - \frac{t}{2}$$

$$W_t^2 = \int_0^t ds + 2\int_0^t W_s\,dW_s$$

Illy $W_t^3 = 3\int_0^t W(s)\,ds + 3\int_0^t W_s^2\,dW_s$

smooth fns

$$x_T^2 = 2\int_0^T x_s\,dx_s$$

$$x_T^3 = 3\int_0^T x_s^2\,dx_s$$

Itô formula:

$\{X_t\}_{t\in[0,T]}$ is an Itô process (on a filtered prob. space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a BM $\{W_t\}_{t\in[0,T]}$ on it) if

$$X_t = X_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s \quad , t\in[0,T] \qquad \text{—— (1)}$$

where → X_0 is $\mathcal{F}_0$ mble

→ $\{a_t = a_t(\omega)\}_{t\in[0,T]}$ is an $\mathbb{F}$-adapted process with mble trajectories s.t. $\int_0^T |a_s|\,ds < \infty$

→ $\{b_t = b_t(\omega)\}_{t\in[0,T]}$ is an $\mathbb{F}$-adapted process s.t. $\int_0^T E(b_s^2)\,ds < \infty$

—x—

(1) has on $[0,T]$ stochastic differential

$$dX_t = a_t\,dt + b_t\,dW_t \qquad \text{—— (2)}$$

multiplication table

$dt \cdot dt = (dt)^2 = 0$ $\qquad \Sigma(\Delta t_k)^2 \to 0$

$dt\,dW_t = dW_t\,dt = 0$ $\qquad \Sigma(\Delta t_k)(\Delta W_k) \to 0$

$$dW_t\, dW_t = (dW_t)^2 = dt \qquad \sum (\Delta W_k)^2 \to t$$

Let $\{X_t\}$ Itô process with sto differentials ②, f twice cont differentiable fn, let $Y_t = f(X_t)$, $t \in [0,T]$

Using Taylors formula

$$dY_t \equiv df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t) \underbrace{(dX_t)^2}_{= b_t^2\,dt}$$

From ②

$$dX_t = a_t\,dt + b_t\,dW_t$$

$$(dX_t)^2 = (dX_t)(dX_t) = (a_t\,dt + b_t\,dW_t)(a_t\,dt + b_t\,dW_t)$$

$$= a_t^2 \underbrace{(dt)^2}_{=0} + 2a_t b_t \underbrace{dW_t\,dt}_{=0} + b_t^2 \underbrace{(dW_t)^2}_{dt}$$

$$= b_t^2\,dt$$

$$dY_t \equiv df(X_t) = \left(a_t f'(X_t) + \frac{1}{2} f''(X_t)\, b_t^2\right) dt + b_t f'(X_t)\,dW_t$$

Example ① dY_t, $Y_t = \frac{1}{2} W_t^2$

$$dY_t \equiv d\left(\frac{1}{2} W_t^2\right) = f'(W_t)\,dW_t + \frac{1}{2} f''(W_t)\,(dW_t)^2$$

$$= d(f(W_t)) \qquad f(x) = \frac{1}{2}x^2,\ f'(x) = x,\ f''(x) = 1$$

$$= W_t\,dW_t + \frac{1}{2}\,dt$$

$$d\left(\frac{1}{2} W_t^2\right) = W_t\,dW_t + \frac{1}{2}\,dt$$

② $d(e^{W_t})$

$$f(x) = e^x \qquad f'(x) = e^x \qquad f''(x) = e^x$$

$$df(W_t) = f'(W_t)\,dW_t + \frac{1}{2} f''(W_t)\,dW_t^2$$

$$d(e^{W_t}) = e^{W_t}\,dW_t + \frac{1}{2} e^{W_t}\,dt$$

Statement:

$f(t,x)$ has cont. partial derivative $\partial_t f$, $\partial_x f$, and be twice cont. differentiable in x $\partial_{xx} f$;

(X_t) Itô process $dX_t = a_t\,dt + b_t\,dW_t$

Then $Y_t := f(t, X_t)$ is also an Itô process with

$$dY_t = \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \frac{1}{2}\partial_{xx} f(t,X_t)\,\underbrace{(dX_t)^2}_{= b_t^2\,dt}$$

Taylor series f for (x,y) near (a,b)

$$f(x,y) = f(a,b) + \underline{f_x(a,b)(x-a)} + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2$$
$$+ f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Example Geometric BM

$$Z_t = e^{\mu t + \sigma W_t}, \quad \mu, \sigma \in \mathbb{R}$$
$$= f(t, W_t)$$

$\mu t + \sigma x$

$$f(t,x) = e^{\cdot}$$

$$\partial_t f = \mu f(t,x)$$

$$\partial_x f = \sigma f(t,x) \quad , \quad \partial_{xx} f = \sigma^2 f(t,x)$$

$d(Z_t) = ?$

$$d(Z_t) = d(f(t, W_t))$$

$$= \partial_t f(t, W_t)\, dt + \partial_x f(t, W_t)\, dW_t + \frac{1}{2} \partial_{xx} f(t, W_t) (dW_t)^2$$

$$= \mu f(t, W_t)\, dt + \sigma f(t, W_t)\, dW_t + \frac{1}{2} \sigma^2 f(t, W_t)\, dt$$

$$= \mu Z_t\, dt + \sigma Z_t\, dW_t + \frac{1}{2} \sigma^2 Z_t\, dt$$

$$= \left(\mu + \frac{\sigma^2}{2}\right) Z_t\, dt + \sigma Z_t\, dW_t .$$

Product rule

X_t , Y_t Itô process on common filtered prob. space

$$Z_t := X_t Y_t$$

$$dZ_t \equiv d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$$

—x—

Quotient rule

$$dZ_t = d\left(X_t / Y_t\right)$$

$$= \frac{dX_t}{Y_t} - \frac{X_t}{Y_t^2} dY_t + \frac{X_t}{Y_t^3} (dY_t)^2$$

$$- \frac{1}{\ } dX_t\, dY_t$$

Y_t^2

—x—

<u>Example</u>

$X_t = W_t$

$Y_t = e^{W_t}$

$$Z_t := W_t e^{W_t} = X_t Y_t$$

$$dZ_t = X_t dY_t + Y_t dX_t + dX_t dY_t$$

$$= W_t d(e^{W_t}) + e^{W_t} dW_t + dW_t d(e^{W_t})$$

$$= W_t\left(e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt\right) + e^{W_t}dW_t + dW_t\left(e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt\right)$$

$\because d(e^{W_t})$

$$= e^{W_t}dW_t + \frac{1}{2}e^{W_t}(dW_t)^2$$

$$= e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt$$

$f(t,x) = e^x$

$\partial_t f = 0, \ \partial_x f = e^x, \ \partial_{xx} f = e^x$

$$= \left(\frac{1}{2}W_t e^{W_t} + e^{W_t}\right)dt + \left(W_t e^{W_t} + e^{W_t}\right)dW_t$$

$$= \left(1 + \frac{1}{2}W_t\right)e^{W_t}dt + (1 + W_t)e^{W_t}dW_t$$