13.10 Problems

1. For a single-period binomial market, sketch the graph of the call price $C = C(K)$ as a function of strike K. What features of this function do you think will be common to the call prices in more general markets as well?

2. A *straddle* is a portfolio consisting of a call and a put on the same underlying stock, with the same strike and same expiry. Plot the graph of the payoff function (of the terminal stock price) of the straddle (on one share of the stock).

3. A *bull spread* is a portfolio formed by buying a call with strike price K_1 and selling a call with strike price $K_2 > K_1$. Both calls are on the same underlying stock and have the same expiry date T. A bull spread is obviously a contingent claim.

   (i) Plot the payoff function of a bull spread.

   (ii) What is cheaper: a bull spread or a call with the strike K_1 (on the same underlying stock, with the same expiry date)? Explain.

   (iii) Consider a single-period binomial market. Assume that the current stock price is $S_0 = 5$, the possible values of S_1 are 4 and 6, $r = 0.1$, $K_1 = 3$ and $K_2 = 5$. Price the bull spread using the risk-neutral valuation ("fair price") formula (13.11).

   (iv) Under the assumptions from part (iii), construct a replicating portfolio for the bull spread and verify that the portfolio does replicate the claim.

4. Consider a single-period binomial financial market with $\Omega = \{\omega_1, \omega_2\}$, with the current (time $t = 0$) asset prices $S_0 = 5$ and $B_0 = 1$, and the terminal (time $t = 1$) stock prices $S_1(\omega_1) = 20/3$ and $S_1(\omega_2) = 40/9$. Assume the interest rate $r = 1/9$.

   (i) Show that the market is arbitrage-free.

   (ii) Consider a contingent claim X with $X(\omega_1) = 7$ and $X(\omega_2) = 2$. Find the arbitrage-free value of this claim at time $t = 0$.

   (iii) Construct a replicating portfolio for the claim X and verify that its values at times $t = 0$ and $t = 1$ coincide with the claim price you found in part (ii) and the claim value X, respectively.

5. For a single-period binomial market:

   (i) sketch the graph of the put price $P = P(K)$ as a function of strike K;

   (ii) price a put with strike $K = 3.8$ on one share of the stock with current price $S_0 = 4$, if $r = 5\%$ and the possible values of the time $t = 1$ stock price are $S_1 = 3.6$ and $S_1 = 4.6$;

   (iii) find the time $t = 0$ value of the call with the same strike.

6. Consider a single period trinomial market: $\Omega = \{d, m, u\}$ (i.e., at time $t = 1$ there are three possible states of the world now), with $S_1(d) = dS_0$, $S_1(m) = mS_0$ and $S_1(u) = uS_0$, where $0 < d < m < u$. The time $t = 0, 1$ bond prices are $B_0 = 1$ and $B_1 = 1 + r$, respectively, where $d < 1 + r < u$.

(i) Use Theorem 13.2 to verify (or disprove) that the market is arbitrage free.

(ii) Let X be a contingent claim, with some values $X_d = X(d)$, $X_m = X(m)$ and $X_u = X(u)$ for the possible states of the world in our model. Assuming that $X_d < X_m < X_u$, draw a diagram showing the set of all hedges (Δ, b) for the claim (similar to Fig. 13.3 for the binomial market). Does there exist a perfect hedge (i.e., a replicating portfolio) for X?

7. Consider a single period trinomial market with two stocks (stock 1 and stock 2) and one bond with $r = 0$. Assume that $\Omega = \{\omega_1, \omega_2, \omega_3\}$ (i.e., at time $t = 1$ there are three possible states of the world), with the initial (time $t = 0$) stock prices $S_0^1 = 4$ and $S_0^2 = 5$ for stock 1 and stock 2, respectively, and the time $t = 1$ stock prices

$$\begin{aligned} S_1^1(\omega_1) &= 6, \quad & S_1^2(\omega_1) &= 6; \\ S_1^1(\omega_2) &= 4, \quad & S_1^2(\omega_2) &= 4; \\ S_1^1(\omega_3) &= 2, \quad & S_1^2(\omega_3) &= 7. \end{aligned}$$

(i) Depict the time $t = 0$ and $t = 1$ stock prices by points on the (S_1^1, S_2^1)-plane.

(ii) Use Theorem 13.2 to prove or disprove that the market is arbitrage free.

(iii) Find the EMM $\mathbf{P}^*$ and use the Completeness Theorem 13.4 to prove or disprove that the market is complete.

(iv) Price the call on (one share of) stock 1 with expiry $T = 1$ and strike $K = 5$.

(v) Find the strategy replicating the call from part (iv). Do we really need stock 2 to replicate the call on stock 1?

8. Consider a $T = 2$ period binomial market with $u = 1.75$, $d = 0.5$ and $r = 0.25$. Assuming the time $t = 0$ stock price $S_0 = 400$,

(i) use the diagram method (see Section 13.5) to price a put option with maturity $T = 2$ and strike $K = 450$;

(ii) construct a replicating portfolio for this option;

(iii) directly verify that the replicating strategy you found in part (ii) is self-financing;

(iv) price a call with the same maturity and strike price.

9. Consider a four-period binomial financial market with $u = 4/3$, $d = 2/3$ and $r = 0$ (i.e., the prices have already been discounted). Let the current (time $t = 0$) stock price be $S_0 = 81$. Use the diagram method to price a call with maturity $T = 4$ and strike $K = 120$.

10. On 28 July 2000 ($t = 0$), the BHP[25] stock price at the ASX[26] was $S_0 = 18.50$.

(i) Assuming the (continuously compounded) interest rate $r = 0.062$ and volatility $\sigma = 0.2$, use the Black-Scholes formula to find the price of the call option with maturity one month later ($T = 1/12$ year) and strike $K = 18.50$.

(ii) The real-life buy/sell prices of the option were $0.57/0.63$, respectively. Taking the mean of these two values as the market price of the option, find the volatility value such that the Black-Scholes formula gives the closest value to the market price. This value is called the *implied volatility.*

11. Suppose that the current stock price is $S_0 = 100$. Assume volatility 40% and (continuously compounded) interest rate $r = 4\%$ (the time unit is one year). Use the Black-Scholes formula and the put-call parity to find the prices of the call and put options with strike $K = 110$ and time to maturity (i) 18 months; (ii) one month. Comment on your findings (in particular, on the prices you found in (ii)).