Assignment additional : BM, SBM, Sto. Cal., White noise, etc.

Exercises

In the following exercises $\{B(t),\ t \geqslant 0\}$ is a standard Brownian motion process and T_a denotes the time it takes this process to hit a.

*1. What is the distribution of $B(s) + B(t)$, $s \leqslant t$?

2. Compute the conditional distribution of $B(s)$ given that $B(t_1) = A$ and $B(t_2) = B$, where $0 < t_1 < s < t_2$.

*3. Compute $E[B(t_1)B(t_2)B(t_3)]$ for $t_1 < t_2 < t_3$.

4. Show that

$$P\{T_a < \infty\} = 1,$$
$$E[T_a] = \infty, \qquad a \neq 0$$

*5. What is $P\{T_1 < T_{-1} < T_2\}$?

6. Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b + c$,

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$c > 0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b + c$ or when an additional time t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

7. Compute an expression for

$$P\left\{\max_{t_1 \leqslant s \leqslant t_2} B(s) > x\right\}$$

8. Consider the random walk that in each Δt time unit either goes up or down the amount $\sqrt{\Delta t}$ with respective probabilities p and $1 - p$, where $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t})$.
   (a) Argue that as $\Delta t \to 0$ the resulting limiting process is a Brownian motion process with drift rate μ.
   (b) Using part (a) and the results of the gambler's ruin problem (Section 4.5.1), compute the probability that a Brownian motion process with drift rate μ goes up A before going down B, $A > 0$, $B > 0$.

9. Let $\{X(t), t \geqslant 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the joint density function of $X(s)$ and $X(t)$, $s < t$?

*10. Let $\{X(t), t \geqslant 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(s) = c$ when
   (a) $s < t$?
   (b) $t < s$?

11. Consider a process whose value changes every h time units; its new value being its old value multiplied either by the factor $e^{\sigma\sqrt{h}}$ with probability $p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})$, or by the factor $e^{-\sigma\sqrt{h}}$ with probability $1-p$. As h goes to zero, show that this process converges to geometric Brownian motion with drift coefficient μ and variance parameter σ^2.

12. A stock is presently selling at a price of \$50 per share. After one time period, its selling price will (in present value dollars) be either \$150 or \$25. An option to purchase y units of the stock at time 1 can be purchased at cost cy.
   (a) What should c be in order for there to be no sure win?
   (b) If $c = 4$, explain how you could guarantee a sure win.
   (c) If $c = 10$, explain how you could guarantee a sure win.
   (d) Use the arbitrage theorem to verify your answer to part (a).

13. Verify the statement made in the remark following Example 10.2.

14. The present price of a stock is 100. The price at time 1 will be either 50, 100, or 200. An option to purchase y shares of the stock at time 1 for the (present value) price ky costs cy.
   (a) If $k = 120$, show that an arbitrage opportunity occurs if and only if $c > 80/3$.
   (b) If $k = 80$, show that there is not an arbitrage opportunity if and only if $20 \leqslant c \leqslant 40$.

15. The current price of a stock is 100. Suppose that the logarithm of the price of the stock changes according to a Brownian motion process with drift coefficient $\mu = 2$ and variance parameter $\sigma^2 = 1$. Give the Black-Scholes cost of an option to buy the stock at time 10 for a cost of

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(a) 100 per unit.
(b) 120 per unit.
(c) 80 per unit.
Assume that the continuously compounded interest rate is 5 percent.

A stochastic process $\{Y(t),\ t \geqslant 0\}$ is said to be a *Martingale* process if, for $s < t$,

$$E[Y(t)|Y(u),\ 0 \leqslant u \leqslant s] = Y(s)$$

16. If $\{Y(t),\ t \geqslant 0\}$ is a Martingale, show that

$$E[Y(t)] = E[Y(0)]$$

17. Show that standard Brownian motion is a Martingale.
18. Show that $\{Y(t),\ t \geqslant 0\}$ is a Martingale when

$$Y(t) = B^2(t) - t$$

What is $E[Y(t)]$?

Hint: First compute $E[Y(t)|B(u),\ 0 \leqslant u \leqslant s]$.

*19. Show that $\{Y(t), t \geqslant 0\}$ is a Martingale when

$$Y(t) = \exp\{cB(t) - c^2t/2\}$$

where c is an arbitrary constant. What is $E[Y(t)]$?

An important property of a Martingale is that if you continually observe the process and then stop at some time T, then, subject to some technical conditions (which will hold in the problems to be considered),

$$E[Y(T)] = E[Y(0)]$$

The time T usually depends on the values of the process and is known as a *stopping time* for the Martingale. This result, that the expected value of the stopped Martingale is equal to its fixed time expectation, is known as the *Martingale stopping theorem*.

*20. Let

$$T = \text{Min}\{t\colon B(t) = 2 - 4t\}$$

That is, T is the first time that standard Brownian motion hits the line $2 - 4t$. Use the Martingale stopping theorem to find $E[T]$.

21. Let $\{X(t),\ t \geqslant 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. That is,

$$X(t) = \sigma B(t) + \mu t$$

Let $\mu > 0$, and for a positive constant x let

$$\begin{aligned} T &= \text{Min}\{t\colon X(t) = x\} \\ &= \text{Min}\left\{t\colon B(t) = \frac{x - \mu t}{\sigma}\right\} \end{aligned}$$

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That is, T is the first time the process $\{X(t),\ t \geqslant 0\}$ hits x. Use the Martingale stopping theorem to show that

$$E[T] = x/\mu$$

22. Let $X(t) = \sigma B(t) + \mu t$, and for given positive constants A and B, let p denote the probability that $\{X(t),\ t \geqslant 0\}$ hits A before it hits $-B$.
    (a) Define the stopping time T to be the first time the process hits either A or $-B$. Use this stopping time and the Martingale defined in Exercise 19 to show that

    $$E[\exp\{c(X(T) - \mu T)/\sigma - c^2 T/2\}] = 1$$

    (b) Let $c = -2\mu/\sigma$, and show that

    $$E[\exp\{-2\mu X(T)/\sigma\}] = 1$$

    (c) Use part (b) and the definition of T to find p.

    Hint: What are the possible values of $\exp\{-2\mu X(T)/\sigma^2\}$?
23. Let $X(t) = \sigma B(t) + \mu t$, and define T to be the first time the process $\{X(t),\ t \geqslant 0\}$ hits either A or $-B$, where A and B are given positive numbers. Use the Martingale stopping theorem and part (c) of Exercise 22 to find $E[T]$.

*24. Let $\{X(t),\ t \geqslant 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. Suppose that $\mu > 0$. Let $x > 0$ and define the stopping time T (as in Exercise 21) by

$$T = \text{Min}\{t\colon X(t) = x\}$$

Use the Martingale defined in Exercise 18, along with the result of Exercise 21, to show that

$$\text{Var}(T) = x\sigma^2/\mu^3$$

25. Compute the mean and variance of
    (a) $\int_0^1 t\, dB(t)$
    (b) $\int_0^1 t^2\, dB(t)$
26. Let $Y(t) = tB(1/t)$, $t > 0$ and $Y(0) = 0$.
    (a) What is the distribution of $Y(t)$?
    (b) Compare $\text{Cov}(Y(s), Y(t))$.
    (c) Argue that $\{Y(t),\ t \geqslant 0\}$ is a standard Brownian motion process.

*27. Let $Y(t) = B(a^2 t)/a$ for $a > 0$. Argue that $\{Y(t)\}$ is a standard Brownian motion process.

28. For $s < t$, argue that $B(s) - \frac{s}{t}B(t)$ and $B(t)$ are independent.
29. Let $\{Z(t),\ t \geqslant 0\}$ denote a Brownian bridge process. Show that if

$$Y(t) = (t + 1)Z(t/(t + 1))$$

then $\{Y(t),\ t \geqslant 0\}$ is a standard Brownian motion process.

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30. Let $X(t) = N(t+1) - N(t)$ where $\{N(t),\ t \geqslant 0\}$ is a Poisson process with rate λ. Compute

$$\mathrm{Cov}[X(t), X(t+s)]$$

*31. Let $\{N(t), t \geqslant 0\}$ denote a Poisson process with rate λ and define $Y(t)$ to be the time from t until the next Poisson event.
(a) Argue that $\{Y(t),\ t \geqslant 0\}$ is a stationary process.
(b) Compute $\mathrm{Cov}[Y(t), Y(t+s)]$.

32. Let $\{X(t), -\infty < t < \infty\}$ be a weakly stationary process having covariance function $R_X(s) = \mathrm{Cov}[X(t), X(t+s)]$.
(a) Show that

$$\mathrm{Var}(X(t+s) - X(t)) = 2R_X(0) - 2R_X(t)$$

(b) If $Y(t) = X(t+1) - X(t)$ show that $\{Y(t),\ -\infty < t < \infty\}$ is also weakly stationary having a covariance function $R_Y(s) = \mathrm{Cov}[Y(t), Y(t+s)]$ that satisfies

$$R_Y(s) = 2R_X(s) - R_X(s-1) - R_X(s+1)$$

33. Let Y_1 and Y_2 be independent unit normal random variables and for some constant w set

$$X(t) = Y_1 \cos wt + Y_2 \sin wt, \qquad -\infty < t < \infty$$

(a) Show that $\{X(t)\}$ is a weakly stationary process.
(b) Argue that $\{X(t)\}$ is a stationary process.

34. Let $\{X(t),\ -\infty < t < \infty\}$ be weakly stationary with covariance function $R(s) = \mathrm{Cov}(X(t), X(t+s))$ and let $\tilde{R}(w)$ denote the power spectral density of the process.
(i) Show that $\tilde{R}(w) = \tilde{R}(-w)$. It can be shown that

$$R(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(w)e^{iws}\,dw$$

(ii) Use the preceding to show that

$$\int_{-\infty}^{\infty} \tilde{R}(w)\,dw = 2\pi E[X^2(t)]$$