4.5 Problems

1. Derive the optimality equation (4.2) using the total probability formula (you may consider the case of the discrete state space only).

2. A person must buy a block of land during the next three weeks. The lowest prices he can be offered on particular weeks are independent random variables $\$100,000 \times Z_j$, $j = 1, 2, 3$, distributed according to the following table:

x	$\mathbf{P}(Z_1 = x)$	$\mathbf{P}(Z_2 = x)$	$\mathbf{P}(Z_3 = x)$
2.2	0.3	0.2	0.2
2.3	0.5	0.6	0.5
2.4	0.2	0.2	0.3

Each week the person has to make a decision: either to buy or not to buy. If he does not buy for the best price (this week), the opportunity is lost (he cannot return to the offer later).

(i) Set this as a stochastic dynamic decision problem: define the decision process, possible actions, reward function *etc.*

(ii) Write down the optimality equation for the optimal value function.

(iii) Draw a decision tree for this problem. Derive the optimum policy for the buyer. What is the expected price when one follows the optimum policy?

3. *A Gambling Model.* At each play of the game, a gambler can bet any non-negative amount up to his present fortune and will either win or lose that amount with probabilities p and $q = 1-p$ respectively. The gambler is allowed to make T bets, and his objective is to maximise the expected Bernoulli utility of his final fortune. What is the optimal strategy?

Hints. [First try to solve the problem without reading the hints!] The gambler's goal is to maximise the expectation of the logarithm of his final fortune. Take the process X_t = the gambler's fortune at time t. Take actions to be the fractions of the gambler's fortune that he bets (so now the set of possible actions is $A = [0, 1]$). Given $X_{t-1} = x$, we have $X_t = x + axZ_t$, where $Z_t = \pm 1$ w.p.'s p and q, respectively. The optimality equation for $V_n(x)$—the maximal expected return if the gambler has a present fortune of x and is allowed n more gambles—takes now this form:

$$V_n(x) = \max_a \mathbf{E}_a(V_{n-1}(X_{N-n+1}) | X_{N-n} = x) = \max_a \mathbf{E}_a(V_{n-1}(X_1) | X_0 = x)$$

(as the process $\{X_t\}$ is homogeneous in time). Note that $V_0(x) = \log x$ (the Bernoulli utility of the fortune x).

(i) When $p \le 1/2$, show that $V_n(x) = \log x$ and the optimum strategy is always to bet 0. [So if a game unfavourable for you, never play it!]

(ii) When $p > 1/2$, derive a general formula for $V_n(x)$ and show that the optimal decision is to bet each time the fraction $p - q$ of one's fortune.

4. A person has to sell a block of shares during the next four days. He believes that the prices Z_j, $j = 1, \ldots, 4$, of the block on particular days are independent $U(0,1)$-RVs. The objective is to maximize the expected selling price.

(i) Set this as a stochastic dynamic problem: define the decision process $\{X_t\}$, possible actions, reward function *etc.*

(ii) Write down the optimality equation for the optimal value function. Use it to find $V_n(x)$ for $n = 1, \ldots, 4$.

(iii) What is the optimal policy? What is the maximum expected price (i.e., the value $\mathbf{E}\, V_4(X_1)$)?

Hint. (ii) You may well wish to use the formula $\mathbf{E}\max(Z_j, c) = \frac{1}{2}(1 + c^2)$, $c \in [0,1]$. Verify it.

5. For the option model from Example 4.2, show that when $\mu = \mathbf{E}\, Y_j > 0$, one has $s_n = \infty$ for $n > 1$. In other words, it is never optimal to exercise the option before maturity when $\mu > 0$.

Hints. It suffices to show that $s_2 = \infty$. (Why?) Using the optimality equation for $n = 2$, write down the explicit expression for $V_2(s) - s$ and recall that s_2 is the minimum value for which $V_2(s) - s \le -c$ (the LHS decreases in s). Does such a value exist? Recall that $\mathbf{E}\, V_1(s + Y_1) = \mathbf{E}\max\{s + Y_1 - c, 0\}$.

6. *Diversification pays, or do not put all eggs in one basket.*
(i) Show that putting a fixed total of wealth equally into independent identically distributed investments will yield the same mean gain as any other portfolio, but will minimise the variance. [Thus such an investment portfolio is, in a sense, the most "reliable" one: the uncertainty is then minimal!] In other words, if $X_1, \ldots, X_n$ are i.i.d. RVs (representing profits from investments) with finite mean $\mu = \mathbf{E}\, X_1$ and variance $\sigma^2 = \mathrm{Var}\,(X_1) < \infty$, then the mean of the random variable

$$Y := \lambda_1 X_1 + \cdots + \lambda_n X_n$$

(the total gain), where the values

$$\lambda_j \ge 0, \ j = 1, \ldots n, \quad \lambda_1 + \cdots + \lambda_n = 1$$

represent proportions of one's wealth invested into different assets, does not depend on the choice of λ_j, while the minimum of $\mathrm{Var}\,(Y)$ is attained on the portfolio $\lambda_j = 1/n$, $j = 1, \ldots, n$.
(ii) However, if you are using a strictly concave[5] utility function $u(x)$ (such as the Bernoulli utility $u(x) = \log x$), then the investment portfolio $\lambda_j = 1/n$,

[5]That u is "strictly concave" means that, for any $x_1 < x_2 \in \mathbf{R}$ and $\alpha \in (0,1)$, one has $u(\alpha x_1 + (1-\alpha)x_2) > \alpha u(x_1) + (1-\alpha)u(x_2)$. In other words, if you draw a straight line through the points $(x_1, u(x_1))$ and $(x_2, u(x_2))$, then the graph of the function $u(x)$ will be *strictly above* that straight line on the interval $x \in (x_1, x_2)$. For a smooth u, strict concavity is equivalent to the condition that $u''(x) < 0$ everywhere.

$j = 1, \ldots, n$, is the optimal choice—it maximises the expected utility $\mathbf{E}\,u(Y)$. Prove that assertion (you may prove it in the case of the Bernoulli utility only).

(iii) Moreover, in (ii) above one can relax the assumption of having i.i.d. X's and require only that the X's are *exchangeable*, which means that, for any *permutation* of the indices $i_1, \ldots, i_n$, the distribution of the random vector $(X_{i_1}, \ldots, X_{i_n})$ is the same as that of the original $(X_1, \ldots, X_n)$. [In particular, i.i.d. RVs are exchangeable, and $X_1 = X_2 = \cdots = X_n$ are also exchangeable.]

7. Prove (4.19).