Stochastic Differential Equation (SDE):

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t\,,\quad t \in [0,T]\,,\quad X_0 = x_0 \qquad \text{— (4)}$$

here $a(t,x)$, $b(t,x)$, $t \geq 0$, $x \in \mathbb{R}$ are non-random,
(W_t) BM on $(\Omega, \mathcal{F}, \mathbb{F}, P)$

Itô process $\{X_t\}_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is said to be soln of SDE (4) if

$$X_t = x_0 + \int_0^t a(t, X_s)\,ds + \int_0^t b(s, X_s)\,dW_s\,;\quad t \in [0,T] \qquad \text{— (I)}$$

Equation (I) will have unique soln provided fns a and b are regular enough, eg if they are mble (measurable) and $\exists\ C < \infty$

s.t. $|a(t,x)| + |b(t,x)| \leq C\,(1 + |x|)$

$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq C\,|x - y|$

for $t \in [0,T)$, $x, y \in \mathbb{R}$.

<u>Example</u> (Ornstein–Uhlenbeck process) OU process

$$dX_t = -r X_t\,dt + \sigma\,dW_t\,;\quad X_t\big|_{t=0} = X_0 \qquad \text{— (1)}$$

$$e^{rt}\,dX_t = -r X_t e^{rt}\,dt + \sigma e^{rt}\,dW_t\,,\quad r, \sigma > 0 \qquad \text{— (2)}$$

$$d\left(e^{rt} X_t\right) = d\left(f(t, X_t)\right)$$

$$f(t,x) = e^{rt} x$$

$$\partial_t f = rf \quad , \partial_x f = e^{rt} \quad , \partial_{xx} f = 0$$

$$d(e^{rt} X_t) = r e^{rt} X_t \, dt + e^{rt} dX_t$$

$$\xrightarrow{\text{Using (2)}} \quad d(e^{rt} X_t) = \sigma e^{rt} dW_t$$

$$e^{rt} X_t - X_0 = \sigma \int_0^t e^{rs} dW_s$$

(1)$^{++}$

$$X_t = e^{-rt} X_0 + \sigma \underbrace{\int_0^t e^{-r(t-s)} dW_s}_{Q_t} \quad \text{is a sol}^{n} \text{ of (1)}$$

sol^n (1)$^{++}$ to (1) is unique.

Let $Q_t = \int_0^t e^{-r(t-s)} dW_s$ is Gaussian process

with $E(Q_t) = 0$

$$E(Q_t^2) = E\left(\int_0^t e^{-r(t-s)} dW_s\right) \underset{\text{Itô's isometry}}{=} \int_0^t e^{-2r(t-s)} ds$$

$$= e^{-2rt} \left(\frac{e^{+2rs}}{2r}\right)_{s=0}^{t}$$

$$= \frac{e^{-2rt}}{2r} \left(e^{2rt} - 1\right) = \frac{1}{2r}\left(1 - e^{-2rt}\right)$$

If $X_0 = x_0$

$$X_t \sim N\left(e^{-rt} x_0 , \sigma^2_{r,t}\right) \quad , \text{ when } \sigma^2_{r,t} = \frac{\sigma^2}{2r}\left(1 - e^{-2rt}\right)$$

X_t is a Gaussian process if X_0 is Gaussian and

$$X_t \to N\left(0, \frac{\sigma^2}{2r}\right) \quad \text{as } t \to \infty$$

Vasicek Interest rate model :

Spot interest rate r_t is assumed to satisfy SDE

$$dr_t = a(b - r_t)\,dt + \sigma\, dW_t \,, \quad t > 0, \; a, b, \sigma, > 0, \; r_0 > 0$$

let $X_t = r_t - b$, $dX_t = dr_t$

$$dX_t = -a X_t\, dt + \sigma\, dW_t$$

$r_t - b$ is OU process

$$r_t = X_t + b \sim N\left(b + e^{-at}(r_0 - b), \sigma^2_{a,t}\right)$$

using ✱

This model has an obvious deficiency, with a +ve prob, the interest rate r_t can assume -ve values, which is undesirable.

Cox - Ingersoll - Ross Interest rate model (CIR model)

assume

$$dr_t = a(b - r_t)\,dt + \sigma \sqrt{r_t}\, dW_t \,, \quad t > 0$$

$$a, b, \sigma, r_0 > 0$$

The effect of having the factor $\sqrt{r_t}$ in the diffusion coefficient is that it "freezes" the random oscillation as $r_t \to 0$ and so the +ve drift term becomes dominating. Hence the model will never produce negative interest rate values. Moreover, r_t will never turn into zero provided that $2ab \geq \sigma^2$.

Geometric BM

GBM is a solⁿ of SDE with multiplicative noise N_t : price of asset ; r : interest rate ①+
α : volatility.

$$dN_t = r N_t \, dt + \alpha N_t \, dW_t \quad , \quad N_t\big|_{t=0} = N_0$$

$$\frac{dN_t}{N_t} = r\, dt + \alpha \, dW_t$$

$$d(\log N_t) = d(f(t, N_t)) \, ,$$

$$f(t,x) = \log x$$

$$\partial_t f = 0 \, , \quad \partial_x f = \frac{1}{x} \, , \quad \partial_{xx} f = -\frac{1}{x^2}$$

$$d(\log N_t) = \underbrace{\frac{1}{N_t} dN_t}_{r\,dt + \alpha\, dW_t} - \frac{1}{2N_t^2} \underbrace{(dN_t)^2}_{= \alpha^2 N_t^2 dt}$$

$$\Rightarrow d(\log N_t) = \left(r - \frac{\alpha^2}{2}\right) dt + \alpha \, dW_t$$

$$\int_0^t d \log N_t = \left(r - \frac{\alpha^2}{2}\right) t + \alpha W_t$$

$$\log N_t - \log N_0 = \left(r - \frac{\alpha^2}{2}\right) t + \alpha W_t$$

$$\Rightarrow \quad N_t = N_0 \exp\left\{ \left(r - \frac{\alpha^2}{2}\right)t + \alpha W_t \right\}$$

GBM. is unique solⁿ to sde (1)+

—x—