price process $\{S_t = (S_t^1, \dots, S_t^n)\}_{t \geq 0}$ is SP on $(\Omega, \mathcal{F}, P)$

n diff risky asset

bond price $B_t = \begin{cases} (1+r)^t & \text{in discrete time } t = 0, 1, 2, \dots \\ e^{rt} & \text{" cont time } t \geq 0, \ r \geq 0 \end{cases}$

European contingent claims X:

with maturity at T

"risk-neutral" or "arbitrage free" price of the claim

is obtained as the value of the self financing portfolio that replicates the claim X at its maturity time T.

Key facts about the no-arbitrage pricing theory

→ $\exists$ an EMM P^* on $(\Omega, \mathcal{F})$ and SP

$$\underset{\sim}{S}_t^* = \begin{cases} (1+r)^{-t} \underset{\sim}{S}_t & \text{in discrete time} \\ e^{-rt} \underset{\sim}{S}_t & \text{" cont time} \end{cases}$$

is and MG on $(\Omega, \mathcal{F}, P^*)$, then the market is arbitrage free.

→ For an attainable claim X with maturity T at any time t arbitrage-free price is given by

$$P_t(X) = \begin{cases} E^*\left[(1+r)^{t-T} X \mid \mathcal{F}_t\right] & \text{discrete} \\ E^*\left(e^{-r(T-t)} X \mid \mathcal{F}_t\right) & \text{cont} \end{cases}$$

$t \in [0,T]$

→ If EMM P^* is unique, then market is complete, i.e., any claim is attainable and so can always be priced.

Black-Scholes Framework :

cont. time model

→ bond with interest rate r having price dynamics $B_t = e^{rt}$, $t \in [0,T]$
(i.e. $dB_t = r B_t \, dt$, $B_0 = 1$)

→ risky asset (a stock) of which the price S_t satisfies the Black-Scholes SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad t \in [0,T] \qquad \text{—} \star$$

$$\underbrace{\frac{dS_t}{S_t}}_{\text{short term return}} = \underbrace{\mu \, dt}_{\text{systematic return with const. rate}} + \underbrace{\sigma \, dW_t}$$

which is perturbed by the normal white noise dW_t, the volatility coef σ specifying the effect of noise on the returns.

Itô formula

... $f(t,x) = \ln x$

$$d(\ln S_t) = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2$$

$$\partial_t f = 0$$
$$\partial_x f = \frac{1}{x}$$
$$\partial_{xx} f = -\frac{1}{x^2}$$

$$= \mu\, dt + \sigma\, dW_t - \frac{1}{2S_t^2} \times \sigma^2 S_t^2\, dt$$

$$Y_t = f(t, X_t)$$

$$dY_t = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2}\partial_{xx} f(t, X_t)(dX_t)^2$$

$$= \mu\, dt + \sigma\, dW_t - \frac{\sigma^2}{2} dt$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

$$\Rightarrow \ln S_t - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

$$\Rightarrow S_t = S_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}, \quad t \in [0, T] \qquad (1)^*$$

Q1 Is the Black-Scholes market is arbitrage free? If yes then can one price attainable claims.

Q2 Is the Black-Scholes market complete (as Q1 positive). If answer to Q2 is also positive, one can price any claim on the market.

Answer to Q1, Q2 are given in terms of EMMs on $(\Omega, \mathcal{F})$ What prob. measure are equivalent to (1.) ☆

Let $Z_t := S_0 \exp\{\int_0^t a_s ds + \sigma W_t\}$, $t \in [0,T]$

$(a_t)_{t\in[0,T]}$ is an adapted process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$

Ex. An Itô process

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s, \quad t \in [0,T]$$ is

an MG iff $a_t = 0$, $t \in [0,T]$

$(e^{-rt} Z_t)$ is an Itô process it will be an MG

iff the coeff of the dt term in the

sto. differential is zero

$$d(e^{-rt} Z_t) = d\Big(\underbrace{\exp\{\textstyle\int_0^t a_s ds\}}_{=: X_t} \times \underbrace{\exp\{-rt + \sigma W_t\}}_{=: Y_t}\Big)$$

$$= Y_t \, dX_t + X_t \, dY_t$$

$$= Y_t a_t X_t \, dt + X_t \left(\left(-r + \frac{\sigma^2}{2}\right) Y_t \, dt + \sigma Y_t \, dW_t\right)$$

$$Y_t = f(t,x) = e^{-rt + \sigma x}$$

$$\partial_t f = -rf, \quad \partial_x f = \sigma f, \quad \partial_{xx} f = \sigma^2 f$$

$$dY_t = -r Y_t \, dt + \sigma Y_t \, dW_t + \frac{1}{2}\sigma^2 Y_t (dW_t)^2$$

$$= X_t Y_t \left(a_t - r + \frac{\sigma^2}{2} \right) dt + \sigma X_t Y_t \, dW_t$$

As $X_t Y_t > 0$ the coeff of dt is zero iff $a_t = r - \frac{\sigma^2}{2}$, $t \in [0, T]$

That is there is a unique solⁿ to the problem, and we have the following key result

<u>Thm?</u> $\exists$ a unique EMM in the Black-Scholes market. Under the prob. P^*, the price process has the geometric BM process dynamics specified by

$$S_t = S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma \tilde{W}_t \right\}, \, t \in [0, T] \quad — ②^*$$

or equivalently by

$$dS_t = r S_t \, dt + \sigma S_t \, d\tilde{W}_t, \, t \in [0, T]$$

when $(\hat{W}_t)$ is SBM process on $(\Omega, \mathcal{F}, P^*)$.

—x—

$\therefore$ According to previous section, the Black-Scholes market is arbitrage free and complete and so one can price any contingent claim X in the market, using the valuation formula

✓ $$P_t(X) = E^*\left(e^{-r(T-t)} X \mid \mathcal{F}_t\right)$$

Example: Pricing European Calls in the Black-Scholes market.

$$X = g(S_T) \quad \text{with } g(s) := (s-k)^+$$

Note that using ②*

$$\{S_T > k\} = \left\{ \frac{W_T}{\sqrt{T}} > -h_0 \right\} \quad \text{, when } \frac{W_T}{\sqrt{T}} \sim N(0,1)$$

$$\text{and } h_0 := \frac{\ln\left(\frac{S_0}{k}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \equiv h - \sigma\sqrt{T}$$

We have

$$P_0(X) = e^{-rT} E^* (S_T - k)^+$$

$$= e^{-rT} E^*\left((S_T - k) 1_{\{S_T > k\}}\right)$$

$$= e^{-rT} \int_{-h_0}^{\infty} \left(S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T} x} - k\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

$$= S_0 \int_{-h_0}^{\infty} \frac{e^{-(x - \sigma\sqrt{T})^2/2}}{\sqrt{2\pi}} \, dx - k e^{-rT} \int_{-h_0}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

$$= S_0 \, \Phi(h_0 + \sigma\sqrt{T}) - k e^{-rT} \, \Phi(h_0)$$

[Φ cdf of $N(0,1)$)

$$= S_0 \, \Phi(h) - k e^{-rT} \, \Phi(h - \sigma\sqrt{T}) \quad \text{——} \; \hat{\star}$$

Black-Scholes formula

—x—

How does one replicate claims in the Black-
[illegible] arket?