1. $X(s) + X(t) = 2X(s) + X(t) - X(s)$.

   Now $2X(s)$ is normal with mean 0 and variance $4s$ and $X(t) - X(s)$ is normal with mean 0 and variance $t - s$. As $X(s)$ and $X(t) - X(s)$ are independent, it follows that $X(s) + X(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

2. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s - t_1)$ given that $X(0) = 0$ and $X(t_2 - t_1) = B - A$, which by Equation (10.4) is normal with mean $\frac{s - t_1}{t_2 - t_1}(B - A)$ and variance $\frac{(s - t_1)}{t_2 - t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \frac{(s - t_1)(B - A)}{t_2 - t_1}$ and variance $\frac{(s - t_1)(t_2 - s)}{t_2 - t_1}$.

3. $E[X(t_1)X(t_2)X(t_3)]$

$$
\begin{aligned}
&= E[E[X(t_1)X(t_2)X(t_3) \mid X(t_1), X(t_2)]] \\
&= E[X(t_1)X(t_2)E[X(t_3) \mid X(t_1), X(t_2)]] \\
&= E[X(t_1)X(t_2)X(t_2)] \\
&= E[E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \\
&= E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \qquad (*) \\
&= E[X(t_1)\{(t_2 - t_1) + X^2(t_1)\}] \\
&= E[X^3(t_1)] + (t_2 - t_1)E[X(t_1)] \\
&= 0
\end{aligned}
$$

   where the equality $(*)$ follows since given $X(t_1)$, $X(t_2)$ is normal with mean $X(t_1)$ and variance $t_2 - t_1$. Also, $E[X^3(t)] = 0$ since $X(t)$ is normal with mean 0.

4. (a) $P\{T_a < \infty\} = \lim_{t\to\infty} P\{T_a \le t\}$

$$
\begin{aligned}
&= \frac{2}{\sqrt{2r}} \int_0^\infty e^{-y^2/2} dy \quad \text{by (10.6)} \\
&= 2P\{N(0,1) > 0\} = 1
\end{aligned}
$$

   Part (b) can be proven by using

$$E[T_a] = \int_0^\infty P\{T_a > t\} dt$$

   in conjunction with Equation (10.7).

5. $P\{T_1 < T_{-1} < T_2\} = P\{\text{hit 1 before } -1 \text{ before 2}\}$

$$
\begin{aligned}
&= P\{\text{hit 1 before } -1\} \\
&\quad \times P\{\text{hit } -1 \text{ before 2} \mid \text{hit 1 before } -1\} \\
&= \frac{1}{2} P\{\text{down 2 before up 1}\} \\
&= \frac{1}{2}\frac{1}{3} = \frac{1}{6}
\end{aligned}
$$

   The next to last equality follows by looking at the Brownian motion when it first hits 1.

6. The probability of recovering your purchase price is the probability that a Brownian motion goes up c by time t. Hence the desired probability is

$$1 - P\{\max_{0 \le s \le t} X(s) \ge c\} = 1 - \frac{2}{\sqrt{2\pi t}} \int_{c/\sqrt{t}}^\infty e^{-y^2/2} dy$$

7. Let $M = \{\max_{t_1 \le s \le t_2} X(s) > x\}$. Condition on $X(t_1)$ to obtain

$$P(M) = \int_{-\infty}^\infty P(M \mid X(t_1) = y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} dy$$

   Now, use that

$$P(M \mid X(t_1) = y) = 1, \quad y \ge x$$

   and, for $y < x$

$$
\begin{aligned}
P(M \mid X(t_1) = y) &= P\{\max_{0 < s < t_2 - t_1} X(s) > x - y\} \\
&= 2P\{X(t_2 - t_1) > x - y\}
\end{aligned}
$$

8. (a) Let $X(t)$ denote the position at time t. Then

$$X(t) = \sqrt{\Delta t} \sum_{i=1}^{[t/\Delta t]} X_i$$

where

$$X_i = \begin{cases} +1, & \text{if } i^{th} \text{ step is up} \\ -1, & \text{if } i^{th} \text{ step is down} \end{cases}$$

As

$$\begin{aligned} E[X_1] &= p - 1(1-p) \\ &= 2p - 1 \\ &= \mu\sqrt{\Delta t} \end{aligned}$$

and

$$\begin{aligned} Var(X_i) &= E\left[X_i^2\right] - (E[X_i])^2 \\ &= 1 - \mu^2\Delta t \quad \text{since } X_i^2 = 1 \end{aligned}$$

we obtain

$$\begin{aligned} E[X(t)] &= \sqrt{\Delta t}\left[\frac{t}{\Delta t}\right]\mu\sqrt{\Delta t} \\ &\to \mu t \text{ as } \Delta t \to 0 \end{aligned}$$

$$\begin{aligned} Var(X(t)) &= \Delta t\left[\frac{t}{\Delta t}\right](1-\mu^2\Delta t) \\ &\to t \text{ as } \Delta t \to 0. \end{aligned}$$

(b) By the gambler's ruin problem the probability of going up A before going down B is

$$\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

when each step is either up 1 or down 1 with probabilities p and $q = 1 - p$. (This is the probability that a gambler starting with B will reach his goal of $A + B$ before going broke.) Now, when $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t}), q = 1 - p = \frac{1}{2}(1 - \mu\sqrt{\Delta t})$ and so $q/p = \frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}$. Hence, in this case the probability of going up $A/\sqrt{\Delta t}$ before going down $B/\sqrt{\Delta t}$ (we divide by $\sqrt{\Delta t}$ since each step is now of this size) is

$$(*) \qquad \frac{1-\left[\dfrac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right]^{B/\sqrt{\Delta t}}}{1-\left[\dfrac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right]^{(A+B/\sqrt{\Delta t})}}$$

Now

$$\begin{aligned} \lim_{\Delta t \to 0}\left[\frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right]^{1/\sqrt{\Delta t}} &= \lim_{h\to 0}\left[\frac{1-\mu h}{1+\mu h}\right]^{1/h} \\ &= \lim_{n\to\infty}\left[\frac{1-\dfrac{\mu}{n}}{1+\dfrac{\mu}{n}}\right]^n \\ &\quad \text{by } n = 1/h \\ &= \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu} \end{aligned}$$

where the last equality follows from

$$\lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n = e^x$$

Hence the limiting value of $(*)$ as $\Delta t \to 0$ is

$$\frac{1-e^{-2\mu B}}{1-e^{-2\mu(A+B)}}$$

11. Let $X(t)$ denote the value of the process at time $t = nh$. Let $X_i = 1$ if the i^{th} change results in the state value becoming larger, and let $X_i = 0$ otherwise. Then, with $u = e^{\sigma\sqrt{h}}, d = e^{-\sigma\sqrt{h}}$

$$\begin{aligned} X(t) &= X(0)u^{\sum_{i=1}^n X_i} d^{n-\sum_{i=1}^n X_i} \\ &= X(0)d^n\left(\frac{u}{d}\right)^{\sum_{i=1}^n X_i} \end{aligned}$$

Therefore,

$$\begin{aligned} \log\left(\frac{X(t)}{X(0)}\right) &= n\log(d) + \sum_{i=1}^n X_i\log(u/d) \\ &= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\sum_{i=1}^{t/h} X_i \end{aligned}$$

By the central limit theorem, the preceding becomes a normal random variable as $h \to 0$. Moreover, because the X_i are independent, it is easy to see that the process has independent increments. Also,

$$\begin{aligned} E&\left[\log\left(\frac{X(t)}{X(0)}\right)\right] \\ &= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\frac{t}{h}\frac{1}{2}(1+\frac{\mu}{\sigma}\sqrt{h}) \\ &= \mu t \end{aligned}$$

and

$$\begin{aligned} Var\left[\log\left(\frac{X(t)}{X(0)}\right)\right] &= 4\sigma^2 h\frac{t}{h}p(1-p) \\ &\to \sigma^2 t \end{aligned}$$

where the preceding used that $p \to 1/2$ as $h \to 0$.

12. If we purchase x units of the stock and y of the option then the value of our holdings at time 1 is

$$\text{value} = \begin{cases} 150x + 25y & \text{if price is 150} \\ 25x & \text{if price is 25} \end{cases}$$

So if

$$150x + 25y = 25x, \text{ or } y = -5x$$

then the value of our holdings is $25x$ no matter what the price is at time 1. Since the cost of purchasing x units of the stock and $-5x$ units of options is $50x - 5xc$ it follows that our profit from such a purchase is

$$25x - 50x + 5xc = x(5c - 25)$$

(a) If $c = 5$ then there is no sure win.

(b) Selling $|x|$ units of the stock and buying $-5|x|$ units of options will realize a profit of $5|x|$ no matter what the price of the stock is at time 1. (That is, buy x units of the stock and $-5x$ units of the options for $x < 0$.)

(c) Buying x units of the stock and $-5x$ units of options will realize a positive profit of $25x$ when $x > 0$.

(d) Any probability vector $(p, 1-p)$ on $(150, 25)$, the possible prices at time 1, under which buying the stock is a fair bet satisfies the following:

$$50 = p(150) + (1-p)(25)$$

or

$$p = 1/5$$

That is, $(1/5, 4/5)$ is the only probability vector that makes buying the stock a fair bet. Thus, in order for there to be no arbitrage possibility, the price of an option must be a fair bet under this probability vector. This means that the cost c must satisfy

$$c = 25(1/5) = 5$$

13. If the outcome is i then our total winnings are

$$\begin{aligned} x_i o_i - \sum_{j \neq i} x_j &= \frac{o_i(1+o_i)^{-1} - \sum_{j \neq i}(1+o_j)^{-1}}{1 - \sum_k (1+o_k)^{-1}} \\ &= \frac{(1+o_i)(1+o_i)^{-1} - \sum_j (1+o_j)^{-1}}{1 - \sum_k (1+o_k)^{-1}} \\ &= 1 \end{aligned}$$

14. Purchasing the stock will be a fair bet under probabilities $(p_1, p_2, 1-p_1-p_2)$ on $(50, 100, 200)$, the set of possible prices at time 1, if

$$100 = 50p_1 + 100p_2 + 200(1 - p_1 - p_2)$$

or equivalently, if

$$3p_1 + 2p_2 = 2$$

(a) The option bet is also fair if the probabilities also satisfy

$$c = 80(1 - p_1 - p_2)$$

Solving this and the equation $3p_1 + 2p_2 = 2$ for p_1 and p_2 gives the solution

$$p_1 = c/40, \; p_2 = (80 - 3c)/80$$

$$1 - p_1 - p_2 = c/80$$

Hence, no arbitrage is possible as long as these p_i all lie between 0 and 1. However, this will be the case if and only if

$$80 \geq 3c$$

(b) In this case, the option bet is also fair if

$$c = 20p_2 + 120(1 - p_1 - p_2)$$

Solving in conjunction with the equation

$3p_1 + 2p_2 = 2$ gives the solution

$$p_1 = (c - 20)/30, \; p_2 = (40 - c)/20$$

$$1 - p_1 - p_2 = (c - 20)/60$$

These will all be between 0 and 1 if and only if

$$20 \leq c \leq 40.$$

15. The parameters of this problem are

$$\sigma = .05, \quad \sigma = 1, \quad x_o = 100, \quad t = 10.$$

(a) If $K = 100$ then from Equation (4.4)

$$\begin{aligned} b &= [.5 - 5 - \log(100/100)]/\sqrt{10} \\ &= -4.5\sqrt{10} = -1.423 \end{aligned}$$

and

$$\begin{aligned} c &= 100\phi(\sqrt{10} - 1.423) - 100e^{-.5}\phi(-1.423) \\ &= 100\phi(1.739) - 100e^{-.5}[1 - \phi(1.423)] \\ &= 91.2 \end{aligned}$$

The other parts follow similarly.

16. Taking expectations of the defining equation of a Martingale yields

$$E[Y(s)] = E[E[Y(t)/Y(u),\ 0 \le u \le s]] = E[Y(t)]$$

That is, $E[Y(t)]$ is constant and so is equal to $E[Y(0)]$.

17. $E[B(t)|B(u),\ 0 \le u \le s]$

$$= E[B(s) + B(t) - B(s)|B(u),\ 0 \le u \le s]$$

$$= E[B(s)|B(u),\ 0 \le u \le s]$$

$$+\ E[B(t) - B(s)|B(u), 0 \le u \le s]$$

$= B(s) + E[B(t) - B(s)]$ by independent increments

$$= B(s)$$

18. $E[B^2(t)|B(u),\ 0 \le u \le s] = E[B^2(t)|B(s)]$

where the above follows by using independent increments as was done in Problem 17. Since the conditional distribution of $B(t)$ given $B(s)$ is normal with mean $B(s)$ and variance $t - s$ it follows that

$$E[B^2(t)|B(s)] = B^2(s) + t - s$$

Hence,

$$E[B^2(t) - t|B(u),\ 0 \le u \le s] = B^2(s) - s$$

Therefore, the conditional expected value of $B^2(t) - t$, given all the values of $B(u)$, $0 \le u \le s$, depends only on the value of $B^2(s)$. From this it intuitively follows that the conditional expectation given the squares of the values up to time s is also $B^2(s) - s$. A formal argument is obtained by conditioning on the values $B(u)$, $0 \le u \le s$ and using the above. This gives

$$E[B^2(t) - t|B^2(u),\ 0 \le u \le s]$$

$$= E\ [E[B^2(t) - t|B(u),\ 0 \le u \le s]|B^2(u),$$

$$0 \le u \le s]$$

$$= E[B^2(s) - s|B^2(u),\ 0 \le u \le s]$$

$$= B^2(s) - s$$

which proves that $\{B^2(t) - t,\ t \ge 0\}$ is a Martingale. By letting $t = 0$, we see that

$$E[B^2(t) - t] = E[B^2(0)] = 0$$

19. Since knowing the value of $Y(t)$ is equivalent to knowing $B(t)$ we have

$$E[Y(t)|Y(u),\ 0 \le u \le s]$$

$$= e^{-c^2t/2}E[e^{cB(t)}|B(u),\ 0 \le u \le s]$$

$$= e^{-c^2t/2}E[e^{cB(t)}|B(s)]$$

Now, given $B(s)$, the conditional distribution of $B(t)$ is normal with mean $B(s)$ and variance $t - s$. Using the formula for the moment generating function of a normal random variable we see that

$$e^{-c^2t/2}E[e^{cB(t)}|B(s)]$$

$$= e^{-c^2t/2}e^{cB(s)+(t-s)c^2/2}$$

$$= e^{-c^2s/2}e^{cB(s)}$$

$$= Y(s)$$

Thus, $\{Y(t)\}$ is a Martingale.

$$E[Y(t)] = E[Y(0)] = 1$$

20. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

However, $B(T) = 2 - 4T$ and so

$$2 - 4E[T] = 0$$

or, $E[T] = 1/2$

21. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

But, $B(T) = (x - \mu T)/\sigma$ and so

$$E[(x - \mu T)/\sigma] = 0$$

or

$$E[T] = x/\mu$$

22. (a) It follows from the results of Problem 19 and the Martingale stopping theorem that

$$E[\exp\{cB(T) - c^2T/2\}]$$

$$= E[\exp\{cB(0)\}] = 1$$

Since $B(T) = [X(T) - \mu T]/\sigma$ part (a) follows.

(b) This follows from part (a) since

$$-2\mu[X(T) - \mu T]/\sigma^2 - (2\mu/\sigma)^2T/2$$

$$= -2\mu X(T)/\sigma^2$$

(c) Since T is the first time the process hits A or $-B$ it follows that

$$X(T) = \begin{cases} A, & \text{with probability } p \\ -B, & \text{with probability } 1-p \end{cases}$$

Hence, we see that

$$1 = E[e^{-2\mu X(T)/\sigma^2}] = pe^{-2\mu A/\sigma^2} + (1-p)e^{2\mu B/\sigma^2}$$

and so

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$$

23. By the Martingale stopping theorem we have

$$E[B(T)] = E[B(0)] = 0$$

Since $B(T) = [X(T) - \mu T]/\sigma$ this gives the equality

$$E[X(T) - \mu T] = 0$$

or

$$E[X(T)] = \mu E[T]$$

Now

$$E[X(T)] = pA - (1-p)B$$

where, from part (c) of Problem 22,

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$$

Hence,

$$E[T] = \frac{A(1 - e^{2\mu B/\sigma^2}) - B(e^{-2\mu A/\sigma^2} - 1)}{\mu(e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2})}$$

24. It follows from the Martingale stopping theorem and the result of Problem 18 that

$$E[B^2(T) - T] = 0$$

where T is the stopping time given in this problem and $B(t) = [X(t) - \mu t]/\sigma$. Therefore,

$$E[(X(T) - \mu T)^2/\sigma^2 - T] = 0$$

However, $X(T) = x$ and so the above gives that

$$E[(x - \mu T)^2] = \sigma^2 E[T]$$

But, from Problem 21, $E[T] = x/\mu$ and so the above is equivalent to

$$Var(\mu T) = \sigma^2 x/\mu$$

or

$$Var(T) = \sigma^2 x/\mu^3$$

25. The means equal 0.

$$Var\left[\int_0^1 t\,dX(t)\right] = \int_0^1 t^2 dt = \frac{1}{3}$$

$$Var\left[\int_0^1 t^2 dX(t)\right] = \int_0^1 t^4 dt = \frac{1}{5}$$

26. (a) Normal with mean and variance given by

$$E[Y(t)] = tE[X(1/t)] = 0$$

$$Var(Y(t)) = t^2 Var[X(1/t)] = t^2/t = t$$

(b)
$$\begin{aligned} Cov(Y(s), Y(t)) &= Cov(sX(1/s), tX(1/t)) \\ &= st\, Cov(X(1/s), X(1/t)) \\ &= st\frac{1}{t}, \quad \text{when } s \le t \\ &= s, \quad \text{when } s \le t \end{aligned}$$

(c) Clearly $\{Y(t)\}$ is Gaussian. As it has the same mean and covariance function as the Brownian motion process (which is also Gaussian) it follows that it is also Brownian motion.

27. $E[X(a^2t)/a] = \frac{1}{a}E[X(a^2t)] = 0$

For $s < t$,

$$\begin{aligned} Cov(Y(s), Y(t)) &= \frac{1}{a^2} Cov(X(a^2s), X(a^2t)) \\ &= \frac{1}{a^2} a^2 s = s \end{aligned}$$

As $\{Y(t)\}$ is clearly Gaussian, the result follows.

28.
$$\begin{aligned} Cov\left(B(s) - \frac{s}{t}B(t),\ B(t)\right) &= Cov(B(s),\ B(t)) \\ &\quad - \frac{s}{t}Cov(B(t),\ B(t)) \\ &= s - \frac{s}{t}t = 0 \end{aligned}$$

29. $\{Y(t)\}$ is Gaussian with

$$E[Y(t)] = (t+1)E(Z[t/(t+1)]) = 0$$

and for $s \le t$

$$\begin{aligned} &Cov(Y(s), Y(t)) \\ &= (s+1)(t+1)\,Cov\left[Z\left[\frac{s}{s+1}\right],\ Z\left[\frac{t}{t+1}\right]\right] \\ &= (s+1)(t+1)\frac{s}{s+1}\left[1 - \frac{t}{t+1}\right] \quad (*) \\ &= s \end{aligned}$$

where $(*)$ follows since $Cov(Z(s), Z(t)) = s(1-t)$. Hence, $\{Y(t)\}$ is Brownian motion since it is also Gaussian and has the same mean and covariance function (which uniquely determines the distribution of a Gaussian process).

30. For $s < 1$

$Cov[X(t), X(t+s)]$

$$= Cov[N(t+1) - N(t), N(t+s+1) - N(t+s)]$$

$$= Cov(N(t+1), N(t+s+1) - N(t+s))$$

$$-Cov(N(t), N(t+s+1) - N(t+s))$$

$$= Cov(N(t+1), N(t+s+1) - N(t+s)) \quad (*)$$

where the equality $(*)$ follows since $N(t)$ is independent of $N(t+s+1) - N(t+s)$. Now, for $s \leq t$,

$$Cov(N(s), N(t)) = Cov(N(s), N(s) + N(t) - N(s))$$

$$= Cov(N(s), N(s))$$

$$= \lambda s$$

Hence, from $(*)$ we obtain that, when $s < 1$,

$$Cov(X(t), X(t+s)) = Cov(N(t+1), N(t+s+1))$$

$$-Cov(N(t+1), N(t+s))$$

$$= \lambda(t+1) - \lambda(t+s)$$

$$= \lambda(1-s)$$

When $s \geq 1$, $N(t+1) - N(t)$ and $N(t+s+1) - N(t+s)$ are, by the independent increments property, independent and so their covariance is 0.

31. (a) Starting at any time t the continuation of the Poisson process remains a Poisson process with rate λ.

(b) $E[Y(t)Y(t+s)]$

$$= \int_0^{\infty} E[Y(t)Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y} dy$$

$$= \int_0^{\infty} yE[Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y} dy$$

$$+ \int_s^{\infty} y(y-s)\lambda e^{-\lambda y} dy$$

$$= \int_0^{s} y\frac{1}{\lambda}\lambda e^{-\lambda y} dy + \int_s^{\infty} y(y-s)\lambda e^{-\lambda y} dy$$

where the above used that

$E[Y(t)Y(t+s)|Y(t) = y]$

$$= \begin{cases} yE(Y(t+s)) = \frac{y}{\lambda}, & \text{if } y < s \\ y(y-s), & \text{if } y > s \end{cases}$$

Hence,

$Cov(Y(t), Y(t+s))$

$$= \int_0^{s} ye^{-y\lambda} dy + \int_s^{\infty} y(y-s)\lambda e^{-\lambda y} dy - \frac{1}{\lambda^2}$$

32. (a) $Var(X(t+s) - X(t))$

$$= Cov(X(t+s) - X(t), X(t+s) - X(t))$$

$$= R(0) - R(s) - R(s) + R(0)$$

$$= 2R(0) - 2R(s)$$

(b) $Cov(Y(t), Y(t+s))$

$$= Cov(X(t+1) - X(t), X(t+s+1)$$

$$- X(t+s))$$

$$= R_x(s) - R_x(s-1) - R_x(s+1) + R_x(s)$$

$$= 2R_x(s) - R_x(s-1) - R_x(s+1), \quad s \geq 1$$

33. $Cov(X(t), X(t+s))$

$$= Cov(Y_1 \cos wt + Y_2 \sin wt,$$

$$Y_1 \cos w(t+s) + Y_2 \sin w(t+s))$$

$$= \cos wt \cos w(t+s) + \sin wt \sin w(t+s)$$

$$= \cos(w(t+s) - wt)$$

$$= \cos ws$$