Example Pricing European call in binomial setting

$t=0$, price of European call with payoff $h=$

$$C = \frac{1}{(1+r)^T} \sum_{j=0}^{T} \binom{T}{j} (p^*)^j (1-p^*)^{T-j} \underbrace{(u^j d^{T-j} S_0 - k)^+}$$

$$= \frac{1}{(1+r)^T} \sum_{j=m}^{T} \binom{T}{j} (p^*)^j (1-p^*)^{T-j} (u^j d^{T-j} S_0 - k)$$

$$m := \left\lfloor \frac{\ln k - \ln S_0 - T\ln d}{\ln u - \ln d} \right\rfloor + 1$$

$u^j d^{T-j} S_0 > k$

$\ln S_0 + j\ln u + (T-j)\ln d > \ln k$

$j(\ln u - \ln d) > \ln k - \ln S_0 - T\ln d$

$$= S_0 \sum_{j=m}^{T} \binom{T}{j} \Big(\underbrace{\frac{up^*}{1+r}}_{=p^\wedge}\Big)^j \Big(\underbrace{\frac{d(1-p^*)}{1+r}}_{=1-p^\wedge}\Big)^{T-j}$$

$$- \frac{k}{(1+r)^T} \sum_{j=m}^{T} \binom{T}{j} (p^*)^j (1-p^*)^{T-j}$$

$$= S_0 P(U^\wedge \geq m) - \frac{k}{(1+r)^T} P(U^* \geq m),$$

where $U^\wedge \sim B(n, p^\wedge)$, $U^* \sim B(n, p^*)$

$n = 0, 1, \ldots, T$

—x—

$[0,T]$ split into large n time period of lengths $\delta = \frac{T}{n}$

$\delta_n := \frac{1}{n}$

trading can occur at discrete time epochs $\to$ $t = 0, \delta_n, 2\delta_n, \ldots, n\delta_n = T$

binomial model choose $u = u_n$, $d = d_n$, $r = r_n$

as $n \to \infty$

fixed maturity time $T > 0$

$[(j-1)\delta_n, j\delta_n]$, $j = 1, \ldots, n$

$$S_{(j-1)\delta_n} \nearrow S_{j\delta_n} = u_n S_{(j-1)\delta_n}, \quad \searrow S_{j\delta_n} = d_n S_{(j-1)\delta_n} \qquad j = 1, \ldots, n$$

$u_n = e^{\sigma\sqrt{\delta_n}}$

$d_n = e^{-\sigma\sqrt{\delta_n}}$

$$u_n = 1 + \sigma\sqrt{\delta_n} + O(n^{-1})$$

$$d_n = 1 - \sigma\sqrt{\delta_n} + O(n^{-1})$$

$$r_n = e^{r\delta_n} - 1 = r\delta_n + O(n^{-2})$$

$$S_n = S_0 e^{\sum_{i=1}^{n} Y_i} = S_0 e^{Z_n}$$

$Z_n = \sum_{i=1}^{n} Y_i$ is a random walk with i.i.d jumps

$$Y_{n,k} = \begin{cases} \ln u_n = \sigma\sqrt{\frac{T}{n}} & \text{w.p. } p_n^* \\ \ln d_n = -\sigma\sqrt{\frac{T}{n}} & \text{w.p. } 1 - p_n^* \end{cases}$$

with $p^* = \dfrac{1 + r_n - d_n}{u_n - d_n} = \dfrac{e^{r\delta_n} - e^{-\sigma\sqrt{\delta_n}}}{e^{\sigma\sqrt{\delta_n}} - e^{-\sigma\sqrt{\delta_n}}}$

$n \to \infty$, time $t = 0$, price of European call with payoff is

$$C = S_0 \, P(U^\wedge \geq m) - \frac{k}{(1+r)^T} P(U^* \geq m)$$

| Using ✡

$$= S_0 \left(1 - \Phi\left(\frac{m - n\hat{p}_n}{\sqrt{n\hat{p}_n(1-\hat{p}_n)}}\right)\right)$$

$$- \frac{k}{(1+r)^T} \left(1 - \Phi\left(\frac{m - np_n^*}{\sqrt{np_n^*(1-p_n^*)}}\right)\right)$$

$$= S_0 \, \Phi\left(\frac{n\hat{p}_n - m}{\sqrt{n\hat{p}_n(1-\hat{p}_n)}}\right) - \frac{k}{(1+r)^T} \Phi\left(\frac{np_n^* - m}{\sqrt{np_n^*(1-p_n^*)}}\right)$$

$$p_n^* = \frac{1+r_n-d_n}{u_n-d_n} = \frac{e^{r\delta_n} - e^{-\sigma\sqrt{\delta_n}}}{e^{\sigma\sqrt{\delta_n}} - e^{-\sigma\sqrt{\delta_n}}}$$

$$= \frac{\cancel{1} + r\frac{T}{n} - \cancel{1} + \sigma\sqrt{\frac{T}{n}}}{\cancel{1} + \sigma\sqrt{\frac{T}{n}} - \cancel{1} + \sigma\sqrt{\frac{T}{n}}} = \frac{r\sqrt{\frac{T}{n}} + \sigma}{2\sigma}$$

$$= \frac{1}{2} + \frac{r}{\sigma}\sqrt{\frac{T}{n}}$$

$$\hat{p}_n = \frac{u_n p_n^*}{1+r_n} =$$

$$C = S_0 \, \Phi(h) - k e^{-rT} \, \Phi(h - \sigma\sqrt{T}),$$

when $h := \dfrac{\ln(S_0/k) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

This is Black-Scholes formula giving the price of the European call with maturity T and strike K under the assumption of a cont. time financial market with the stock modelled by geometric BM $S_t = S_0 e^{\mu t + \sigma W_t}$

bond price $B_t = e^{rt}, t \in [0,T]$