EXERCISES

1. If S is a nonempty subset of a vector space X, prove (as asserted in Sec. 9.1) that the span of S is a vector space.

2. Prove (as asserted in Sec. 9.6) that BA is linear if A and B are linear transformations.
Prove also that A^{-1} is linear and invertible.

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

4. Prove (as asserted in Sec. 9.30) that null spaces and ranges of linear transformations are vector spaces.

5. Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $\mathbf{y} \in R^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$.
Hint: Under certain conditions, equality holds in the Schwarz inequality.

6. If $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0),$$

prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of R^2, although f is not continuous at $(0, 0)$.

7. Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.
Hint: Proceed as in the proof of Theorem 9.21.

8. Suppose that f is a differentiable real function in an open set $E \subset R^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

9. If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset R^n$ into R^m, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in E.

10. If f is a real function defined in a convex open set $E \subset R^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, \ldots, x_n$.
Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

11. If f and g are differentiable real functions in R^n, prove that

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$

and that $\nabla(1/f) = -f^{-2}\nabla f$ wherever $f \neq 0$.

12. Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of R^2 into R^3 by

$$\begin{aligned} f_1(s, t) &= (b + a\cos s)\cos t \\ f_2(s, t) &= (b + a\cos s)\sin t \\ f_3(s, t) &= a\sin s. \end{aligned}$$