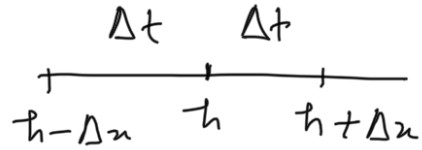
Brownian motion process (BM process):



$X(t)$ = position of particle at time t

$$= \Delta x \left(X_1 + X_2 + \cdots + X_{[\frac{t}{\Delta t}]} \right)$$

$$X_i = \begin{cases} +1 & \text{if } i^{th} \text{ step of length } \Delta x \text{ is to right} \\ -1 & \text{" " " " " " left} \end{cases}$$

where $[\cdot]$ greatest integer less than or equal to the number

eg $[4.4] = 4$,

X_i's are indep.

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$$

$$E(X_i) = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$$

$$E(X_i^2) = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$$

$$V(X_i) = E(X_i^2) - (E(X_i))^2 = 1 - 0 = 1$$

$$E(X(t)) = E\left(\Delta x \sum_{i=1}^{[t/\Delta t]} X_i \right)$$

$$= \Delta x \sum_{i=1}^{[t/\Delta t]} E(X_i) \to 0$$

$$= 0$$

$$V(X(t)) = V\left(\Delta x \sum^{[t/\Delta t]} X_i \right)$$

$$= (\Delta x)^2 \sum_{i=1}^{\left[\frac{t}{\Delta t}\right]} \underbrace{V(X_i)}_{1}$$

$$= (\Delta x)^2 \left[\frac{t}{\Delta t}\right]$$

If $\Delta x \to 0$, $\Delta t \to 0$

(i) $\Delta x = \Delta t \to 0$ $\quad E(X(t)) = 0$, $V(X(t)) \to 0$

(ii) $\Delta x = \sigma\sqrt{\Delta t}$, $\sigma > 0$

as $\Delta t \to 0$

$E(X(t)) = 0$, $\quad V(X(t)) \to \sigma^2 t$

(i) $X(t) \sim N(0, \sigma^2 t)$

(ii) $X(t)$ has independent increment i.e. $X(t_1)$, $X(t_2) - X(t_1)$, $t_2 > t_1$ are indep

(iii) $X(t)$ " stationary increment

, i.e., $X(t) \stackrel{d}{=} X(2t) - X(t)$

A S.P $\{X(t), t \geq 0\}$ is said to be BM process if

(i) $X(0) = 0$

(ii) $\{X(t), t \geq 0\}$ has stationary & indep. increment

(iii) $\forall t > 0$, $X(t) \sim N(0, \sigma^2 t)$

$\sigma^2 = 1$ $\quad W(t) \to$ Wiener process or standard BM process

$W_t = W(t)$ MG ?

$$E(W_{t+s} | W_1, \dots, W_t) = E(W_{t+s} - W_t + W_t | W_1, \dots, W_t)$$

$$= E(W_{t+s} - W_t | W_1, \dots, W_t) + E(W_t | W_1, \dots, W_t)$$

$$= \underbrace{E(W_{t+s} - W_t)}_{\text{using indep. increment}} + W_t$$

$$= 0 + W_t \qquad W_s \sim N(0, s)$$

$$= W_t$$

$W_t = W(t)$ is M.G.

→ S.P. $\{Y_t = W_t^2 - t\}_{t \geq 0}$ is a M.G.

$$E|Y_t| \leq E|W_t^2| + t = E(W_t^2) + t = t + t = 2t < \infty$$

$$E(Y_{t+s} | \mathcal{F}_t) = E(W_{t+s}^2 - (t+s) | \mathcal{F}_t)$$

$$= E((W_{t+s} - W_t + W_t)^2 | \mathcal{F}_t) - (t+s)$$

$$= E((W_{t+s} - W_t)^2 + W_t^2 + 2W_t(W_{t+s} - W_t) | \mathcal{F}_t) - (t+s)$$

$$= E((W_{t+s} - W_t)^2 | \mathcal{F}_t) + E(W_t^2 | \mathcal{F}_t) + 2E(W_t(W_{t+s} - W_t) | \mathcal{F}_t) - (t+s)$$

$$= \underbrace{E(W_{t+s} - W_t)^2}_{\text{indep.}} + W_t^2 + 2W_t E(W_{t+s} - W_t | \mathcal{F}_t) - (t+s)$$

$$= s + W_t^2 + 2W_t \times \underbrace{E(W_{t+s} - W_t)}_{\text{indep increment}} - (t+s)$$

$$= s + W_t^2 + 2W_t \times 0 - (t+s)$$

$$= W_t^2 - t = Y_t$$

Y_t is MG.

—x—

$W(t) \sim N(0,t)$ density $f_t(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, -\infty < x < \infty$

$W(t_1)=x_1, \dots, W(t_n)=x_n \equiv W(t_1)=x_1, W(t_2)-W(t_1)=x_2-x_1, \dots$
$\dots, W(t_n)-W(t_{n-1})=x_n-x_{n-1}$

Joint density of $W(t_1), \dots, W(t_n)$ is

$$f(x_1,\dots,x_n) = f_{t_1}(x_1)\, f_{t_2-t_1}(x_2-x_1) \cdots f_{t_n-t_{n-1}}(x_n-x_{n-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \dots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right]\right\}}{(2\pi)^{n/2}\left[t_1(t_2-t_1)\cdots(t_n-t_{n-1})\right]^{1/2}}$$

Find the conditional pdf of
$[W(s) \mid W(t)=B]$; $s<t$

density is

$$f_{s|t}(x|B) = \frac{f_{s,t}(x,B)}{f_t(B)} = \frac{f_s(x)\, f_{t-s}(B-x)}{f_t(B)}$$

$$= k_1 \exp\left\{-\frac{x^2}{2s} - \frac{(B-x)^2}{2(t-s)}\right\}$$

$$= k_2 \exp\left\{ - x^2\left(\frac{1}{2s} + \frac{1}{2(t-s)}\right) + \frac{\cancel{2} Bx}{\cancel{2}(t-s)} \right\}$$

$$= k_2 \exp\left\{ - \frac{x^2 t}{2s(t-s)} + \frac{Bx}{t-s} \right\}$$

$$= k_2 \exp\left\{ - \frac{t}{2s(t-s)} \left[x^2 - \frac{2s}{t}Bx\right] \right\}$$

$$= k_3 \exp\left\{ - \frac{t}{2s(t-s)} \left(x - \frac{sB}{t}\right)^2 \right\}$$

$$= k_3 \exp\left\{ - \frac{\left(x - \frac{sB}{t}\right)^2}{\frac{2s(t-s)}{t}} \right\}$$

k_1, k_2, k_3 are indep. of x

$s<t$

$$[W(s) \mid W(t) = B] \sim N\left(\frac{sB}{t}, \frac{s(t-s)}{t}\right), \quad s<t$$

$E(W(s)|W(t)=B)$ $\qquad$ $V(W(s)|W(t)=B)$

<u>Example</u>: In a bicycle race btw two competitors, let $Y(t)$ denote the amt of time (in sec's) by which the racer that started in the inside position is ahead when $100t\%$ of the race has been completed, and suppose that $\{Y(t), 0 \le t \le 1\}$ modeled by BM process with variance parameter σ^2.

(i) If the inside racer is leading by σ sec's at the midpoint of race, what is the prob. that she is the winner?

(ii) If the inside racer wins the race by a margin of σ secs, what is the prob. that she was ahead at the midpoint?

Sol $P\left(Y(1) > 0 \mid Y\left(\frac{1}{2}\right) = \sigma \right)$

$Y(t)$ BM process, stat. & indep. increment, $Y(t) \sim N(0, \sigma^2 t)$

$= P\left(Y(1) - Y\left(\frac{1}{2}\right) > -\sigma \mid Y\left(\frac{1}{2}\right) = \sigma \right)$

$= P\left(Y(1) - Y\left(\frac{1}{2}\right) > -\sigma \right)$ | by indep. increment

$= P\left(Y\left(\frac{1}{2}\right) > -\sigma \right)$ | by stationary increment

$= P\left(Z > \dfrac{-\sigma - 0}{\sigma \frac{1}{\sqrt{2}}} \right)$ | $Z = \dfrac{Y\left(\frac{1}{2}\right) - 0}{\sigma \sqrt{\frac{1}{2}}} \sim N(0,1)$

$= P\left(Z > -\sqrt{2} \right) = 1 - P(Z \leq -\sqrt{2})$

$= 1 - \Phi(-\sqrt{2})$

$= \Phi(\sqrt{2}) \simeq 0.9213$

| $\Phi(3) + \Phi(-3) = 1$
| $\Phi(3) = P(Z \leq 3)$

(ii) $P\left(Y\left(\frac{1}{2}\right) > 0 \mid Y(1) = \sigma \right) = P(U > 0)$ —— $*$

$s < t$

$[W(s) \mid W(t) = \frac{c}{\sigma}] \sim N\left(\frac{s}{t} \frac{c}{\sigma}, \frac{s}{t}(t-s) \right)$

$Y(t) = \sigma W(s) \sim N(0, \sigma^2 t)$ $\quad W(t) \sim N(0, t)$

$$\left[\sigma W(s) \mid W(t) = \frac{c}{\sigma}\right] \sim N\left(\cancel{\sigma}\frac{s}{t}\frac{c}{\cancel{\sigma}},\ \sigma^2\frac{s(t-s)}{t}\right)$$

$$\left[Y(s) \mid Y(t) = c\right] \sim N\left(\frac{s}{t}c,\ \sigma^2\frac{s(t-s)}{t}\right), \quad s<t$$

$$s = \frac{1}{2},\ t = 1,\ c = \sigma$$

$$U = \left[Y\left(\tfrac{1}{2}\right) \mid Y(1) = \sigma\right] \sim N\left(\frac{\sigma}{2}, \frac{\sigma^2}{4}\right)$$

$$Z = \frac{U - \sigma/2}{\sigma/2} \sim N(0,1)$$

$$* = P(U>0) = P\left(Z > \frac{-\sigma/2}{\sigma/2}\right) = P(Z>-1)$$

$$= 1 - P(Z \leq -1) = 1 - \Phi(-1)$$

$$= \Phi(1) \simeq 0.8413 \qquad \Big|\ \Phi(x) + \Phi(-x) = 1$$

—x—

I $\{X(t), t \geq 0\}$ BM with drift coefficient μ is

(i) $X(0) = 0$ (ii) $\{X(t), t \geq 0\}$ stationary & indep increment

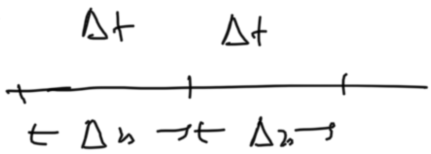
(iii) $X(t) \sim N(\mu t, t)$

$W(t) \sim N(0, t)$

$\therefore X(t) = \mu t + W(t) \sim N(\mu t, t)$, $W(t)$ weiner process

indep $\leftarrow$ $X_i = \begin{cases} +1 & \text{if } i^{th} \text{ step is in +ve direction wp } p \\ -1 & \text{'' '' '' '' '' -ve '' wp } 1-p \end{cases}$

$X(t)$ = position of particle at time t



$X(t) = \Delta x(X_1 + \cdots + X_{t_n})$

$\left[\frac{t}{\Delta t}\right]$ $\quad$ $1-p$ $\quad$ $+\; p$

$$E(X(t)) = \Delta x\left[\frac{t}{\Delta t}\right](2p-1)$$

$$V(X(t)) = (\Delta x)^2\left[\frac{t}{\Delta t}\right]\left(1-(2p-1)^2\right)$$

$$E(X_i) = 1\times p - 1\times(1-p) = 2p-1$$

$$V(X_i) = E(X_i^2) - (E(X_i))^2 = 1-(2p-1)^2$$

$\Delta x = \sqrt{\Delta t}$, $p = \frac{1}{2}(1+\mu\sqrt{\Delta t})$ and let $\Delta t \to 0$

$$2p-1 = \mu\sqrt{\Delta t}$$

$$E(X(t)) = \sqrt{\Delta t}\left[\frac{t}{\Delta t}\right]\times\mu\sqrt{\Delta t} \longrightarrow \mu t$$

$$V(X(t)) = \Delta t\left[\frac{t}{\Delta t}\right]\times\left[1-\mu^2\Delta t\right] \longrightarrow t$$

$(X(t))$ converges to BM with drift coeff. μ.

Probability that the process will hit A before $-B$; $A, B > 0$

Let $P(x) = P(X(t)$ hits A before $-B \mid X(0) = x)$,

$$-B < x < A$$

where

$P(x)$: prob. that the process will hit A before B given that we are now at x

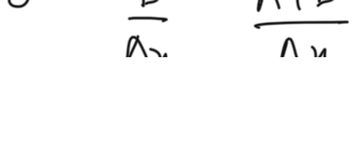
Suppose $x = 0$; $P(0)$

Boundary condition

$$P(A) = 1 \;;\; P(-B) = 0$$

$A, B > 0$

$$P(\text{up } A \text{ before down } B) = \frac{1-\left(\frac{1-p}{p}\right)^{B/\Delta x}}{1-\left(\frac{1-p}{p}\right)^{\frac{A+B}{\Delta x}}}$$



(p)

Using Gambler's ruin problem — ⊛ $\quad \overset{\Delta x}{\overset{\shortparallel}{i}} \quad \overset{\Delta x}{\overset{\shortparallel}{N}}$

Let $p = \frac{1}{2}(1+\mu \Delta x)$

$$\lim_{\Delta x \to 0} \left(\frac{1-p}{p}\right)^{1/\Delta x} = \lim_{\Delta x \to 0} \left(\frac{1-\mu\Delta x}{1+\mu\Delta x}\right)^{\frac{1}{\Delta x}}$$

$$= \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu}$$

Letting $\Delta x \to 0$, we have

$$P(\text{up } A \text{ before down } B) = \frac{1-e^{-2B\mu}}{1-e^{-2(A+B)\mu}} \quad \text{— (1)}$$

$$= e^{2A\mu} \frac{e^{2B\mu}-1}{e^{2(A+B)\mu}-1}$$

(i) If $\mu < 0$, $B \to \infty$

$P(\text{process ever goes to } A) = e^{2A\mu}$

(ii) $\mu \to 0$ in (1)

$$P(\text{BM goes to } A \text{ before down } B) = \frac{B}{A+B}$$

In general

$$P(x) = \frac{1-e^{-2\mu(x+B)}}{1-e^{-2\mu(A+B)}}$$

$-B \quad 0 \quad x \quad A$

<u>Example</u>: (Exercising a Stock option)

Suppose we have the option of buying, at some time in the future, one unit of a stock at a fixed price A, indep. of current market price. The current market price of the stock is taken to be 0, and we suppose that it changes in accordance with a BM process having a negative drift coeff $-d$, where $d > 0$.

The question is, when, if ever, should we exercise our option.

$$\text{Gain} = \begin{cases} x - A & \text{wp } P(A) \\ 0 & \text{wp } 1 - P(A) \end{cases}$$

Sol market price is x

Our expected gain in such a policy is $= P(x) \times (x - A)$

$P(x)$: prob. that the process will ever reach x.

$$\mu = -d < 0 \quad ; \quad P(x) = e^{-2dx}.$$

optimal value of x is one maximizing gain

$$\max \; e^{-2dx}(x - A)$$

$$f(x) = e^{-2dx}(x - A)$$

$f(x)$ is max ~~at~~ $x = A + \dfrac{1}{2d}$

II <u>Geometric BM</u>: (GBM)

Let $Y(t)$ BM drift coeff μ and var. parameter σ^2, i.e., (i) $Y(t)$ has stationary and indep increments

(ii) $Y(t) \sim N(\mu t, \sigma^2 t)$

defin. $X(t) = e^{Y(t)}$ GBM.

$s < t$ $\quad X(t)$ GBM

$$E(X(t) \mid X(u),\ 0 \le u \le s)$$

$$= E(e^{Y(t)} \mid Y(u),\ 0 \le u \le s)$$

$$= E(e^{Y(t) - Y(s) + Y(s)} \mid Y(u),\ 0 \le u \le s)$$

$$= e^{Y(s)} E(e^{Y(t) - Y(s)} \mid Y(u),\ 0 \le u \le s)$$

$$= e^{Y(s)} E(e^{Y(t) - Y(s)})$$

$$Y(t) - Y(s) \overset{d}{=} Y(t-s)$$

$$\sim N(\mu(t-s), \sigma^2(t-s))$$

$$E(e^{uY(t-s)}) = M(u) = e^{\mu(t-s)u + \frac{1}{2}\sigma^2(t-s)u}$$

$$= e^{Y(s)} e^{\mu(t-s) + \frac{1}{2}\sigma^2(t-s)}$$

$$= X(s)\, e^{(t-s)(\mu + \sigma^2/2)}$$

On taking expectation on both sides, we have

$$\Rightarrow E(X(t)) = E(E(X(t) \mid X(u),\ 0 \le u \le s))$$

$$= E(X(s))\, e^{(t-s)(\mu + \sigma^2/2)}$$

X_n price of stock at time n

assume

We may [write] $Y_n = \frac{X_n}{X_{n-1}}$, $n \geq 1$, are IID

$$X_n = Y_n X_{n-1} = Y_n Y_{n-1} X_{n-2}$$
$$\cdots = Y_n Y_{n-1} \cdots Y_1 X_0$$

$$\log X_n = \sum_{i=1}^{n} \log Y_i + \log X_0$$

Since $\{\log Y_i\}_{i \geq 1}$ are indep

Using suitable normalized, approx, $\log(X_n)$ is BM with a drift. Hence $X_n = e^{\log X_n}$ GBM