Problems – Martingales

2.5.1 Use the law of total probability for conditional expectations $E[E\{X|Y,Z\}|Z] = E[X|Z]$ to show

$$E[X_{n+2}|X_0,\ldots,X_n] = E[E\{X_{n+2}|X_0,\ldots,X_{n+1}\}|X_0,\ldots,X_n].$$

Conclude that when X_n is a martingale,

$$E[X_{n+2}|X_0,\ldots,X_n] = X_n.$$

2.5.2 Let $U_1, U_2, \ldots$ be independent random variables each uniformly distributed over the interval $(0, 1]$. Show that $X_0 = 1$ and $X_n = 2^n U_1 \cdots U_n$ for $n = 1, 2, \ldots$ defines a martingale.

2.5.3 Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ be the sum of n independent random variables, each exponentially distributed with mean $E[\varepsilon] = 1$. Show that

$$X_n = 2^n \exp(-S_n), \quad n \geq 0$$

defines a martingale.

2.5.4 Let $\xi_1, \xi_2, \ldots$ be independent Bernoulli random variables with parameter p, $0 < p < 1$. Show that $X_0 = 1$ and $X_n = p^{-n}\xi_1 \cdots \xi_n$, $n = 1, 2, \ldots$, defines a nonnegative martingale. What is the limit of X_n as $n \to \infty$?

2.5.5 Consider a stochastic process that evolves according to the following laws: If $X_n = 0$, then $X_{n+1} = 0$, whereas if $X_n > 0$, then

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{1}{2} \\ X_n - 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

(a) Show that X_n is a nonnegative martingale.

(b) Suppose that $X_0 = i > 0$. Use the maximal inequality to bound

$$\Pr\{X_n \geq N \quad \text{for some } n \geq 0 | X_0 = i\}.$$

Note: X_n represents the fortune of a player of a fair game who wagers \$1 at each bet and who is forced to quit if all money is lost ($X_n = 0$). This *gambler's ruin* problem is discussed fully in Chapter 3, Section 3.5.3.