$(\Omega, \mathcal{F}, \mathbb{F}, P)$ filtered prob. space

A r.v. τ on that space is called "stopping time" ST if one has —— ①

$$\underline{\{\tau \le t\} \in \mathcal{F}_t} \text{ for each } t = 0, 1, 2, \ldots$$

for an ST τ

$$\{\tau = t\} = \underbrace{\{\tau \le t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau \le t-1\}^c}_{\in \mathcal{F}_{t-1} \subset \mathcal{F}_t} \in \mathcal{F}_t \quad \text{—— ②}$$

$$\{\tau \le t\} = \bigcup_{s=0}^{t} \underbrace{\{\tau = s\}}_{\in \mathcal{F}_s \subset \mathcal{F}_t} \qquad \text{since ② } \Rightarrow \text{ ①}$$

<u>About name</u>. τ (random) time when we decide to stop doing something (stop gambling or to sell a block of share at a stock exchange)

$\tau = t$, you act on the basis of you already know by that time $\therefore \{\tau = t\} \in \mathcal{F}_t$

<u>Example</u> (First hitting time)

adapted process $\{X_t\}$, a (boundary) fn u_t, $t = 0, 1, 2, \ldots$

Show that the first hitting (or crossing) time

$$\tau := \inf\{t \ge 0 : X_t \ge u_t\} \text{ is a S.T.} \quad \text{—— ⊛}$$

Sol For any $t = 0, 1, 2, \ldots$

$$\{\tau \le t\} = \bigcup_{s=0}^{t} \{X_s \ge u_s\} \in \mathcal{F}_t$$

$\in \mathcal{F}_s \subset \mathcal{F}_t$

Note that use of inf rather than min in (*), when $X_t < u_t \ \forall t$ $\therefore$ the set of t-values appearing on RHS of (*) is empty, since $\inf \phi = \infty$, we set $\tau = \infty$ then.

<u>Example</u> Coin tossing

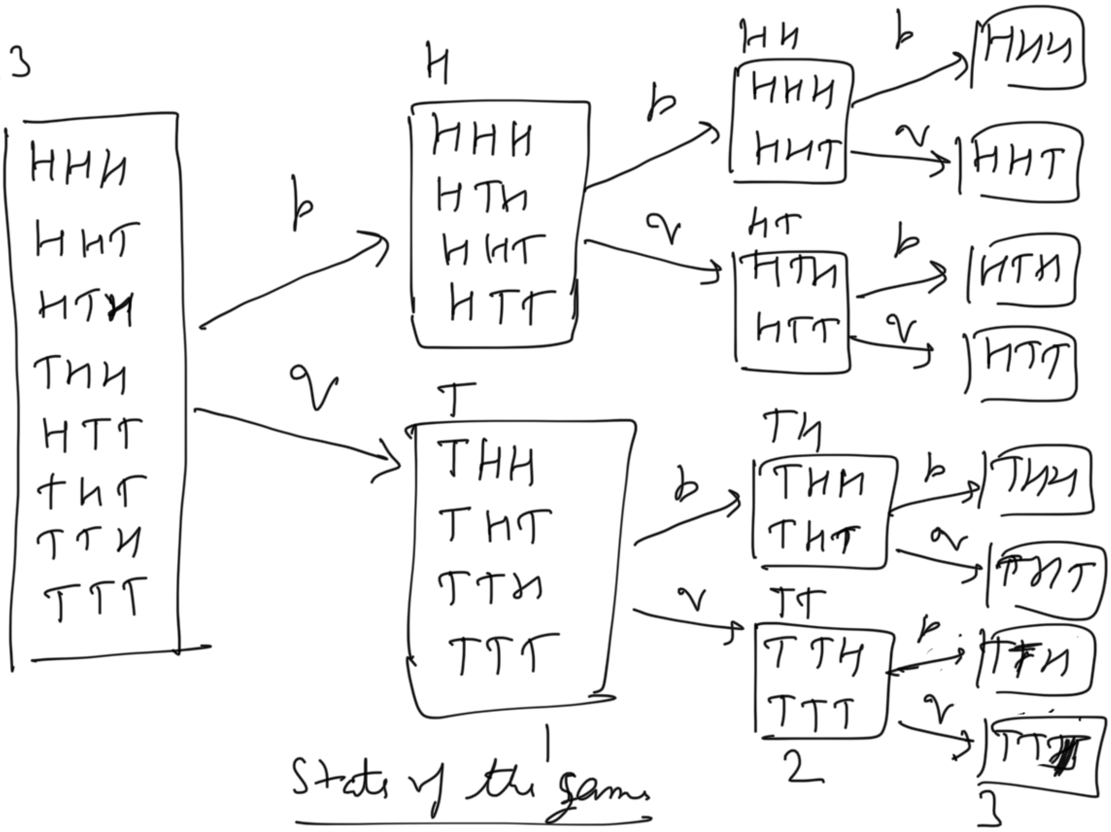
$$t_0 < t_1 < \dots < t_N$$

$$\Omega_k = \{H, T\}^k$$

$\mathcal{F}_0 = \{\phi, \Omega\}$; $\mathcal{F}_1 = \sigma\{A_H, A_T\}$; $\mathcal{F}_2 = \sigma\{A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$

......

eg $N = 3$



State of the game

Now let us suppose that for each heads, a player wins \$1 and for each tails player losses \$1.

Let X_k be a r.v. on Ω that denote players winning for each ...

for each $\omega \in \Omega$,

$$X_k(\omega) = N_H([\omega_k]) - N_T([\omega_k])$$

$N_H([\omega_k])$ denotes # of heads in $[\omega_k]$

$N_T([\omega_k])$ " " tails in $[\omega_k]$

$X_0, X_1, \ldots, X_N$ is SP

$\{X_t\}_{t=0}^{N}$ is SP. when $X_0 = 0$

X_k is $\mathcal{F}_k$ mble, since X_t is adapted to filtration.

$$P(\{\omega\}) = p^{N_H(\omega)} q^{N_T(\omega)}$$

for each $\omega \in \Omega$

$p + q = 1$

$$\mathcal{F}(S) \to \mathcal{F}(SH)$$
$$\mathcal{F}(S) \to \mathcal{F}(ST)$$

State

$$P(\mathcal{F}_{k+1}(SH) \mid \mathcal{F}_k(S)) = p; \quad P(\mathcal{F}_{k+1}(ST) \mid \mathcal{F}_k(S)) = q$$

$$P(\mathcal{F}_k(S)) = p^{N_H(S)} q^{N_T(S)}$$

SP $\{X_t\}$ is <u>M.C.</u>

$$\left[\begin{aligned} & E(X_{k+1} \mid \mathcal{F}_k(S)) = X_k(\mathcal{F}_k(S)) \\ \equiv\ & \frac{E(X_{k+1} 1_{\mathcal{F}_k(S)})}{P(\mathcal{F}_k(S))} = X_k(\mathcal{F}_k(S)) = N_H(S) - N_T(S) \\ & (= W(S) \text{ say}) \end{aligned}\right.$$

$$X_{k+1}(\mathcal{F}_{k+1}(SH)) = W(S) + 1$$

$$X_{k+1}(A_{k+1}(\delta T)) = W(\delta) - 1$$

$$P(A_{k+1}(\delta H)) = p\,P(A_k(\delta)) \;;\; P(A_{k+1}(\delta T)) = q\,P(A_k(\delta))$$

$$E(X_{k+1} 1_{A_k(\delta)}) = \underbrace{E(X_{k+1} 1_{A_k(\delta)} \mid A_{k+1}(\delta H))}\,\underline{P(A_{k+1}(\delta H))}$$

$$+ E(X_{k+1} 1_{A_k(\delta)} \mid A_{k+1}(\delta T))\,\underline{P(A_{k+1}(\delta T))}$$

$$= \underbrace{(W(\delta)+1)}\; p\,P(A_k(\delta)) + \underbrace{(W(\delta)-1)}\; q\,P(A_k(\delta))$$

$$\frac{E(X_{k+1} 1_{A_k(\delta)})}{P(A_k(\delta))} = W(\delta) + p - q$$

$$E(X_{k+1} \mid A_k(\delta)) = N_H(\delta) - N_T(\delta) + p - q$$
$$= X_k(\delta) + p - q$$

(X_t) is MG iff $p = q = \frac{1}{2}$

i.e. coin is fair

<u>Thm</u> $(X_t)_{t\geq 0}$ MG, τ S.T. on common filtered prob. space, then

$Z_t := X_{t\wedge\tau}$, $t = 0, 1, 2, \ldots$ is MG on that space

| $a \wedge b = \min(a,b)$

Sol

$$Z_0 = X_0$$

For $t \geq 0$

$$Z_{t+1} = \sum_{k=0}^{t} X_k 1_{\{\tau = k\}} + X_{t+1} 1_{\{\tau > t\}}$$

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$E|Z_{t+1}| \le E\left(\sum_{k=0}^{t+1} |X_k|\right) = \sum_{k=0}^{t+1} E|X_k| < \infty$$

$\Big|\ \because X_t$ MG

$$E(Z_{t+1} | \mathcal{F}_t) = E\left(\sum_{k=0}^{t} X_k \mathbb{1}_{\{\tau = k\}} + X_{t+1} \mathbb{1}_{\{\tau > t\}} \Big| \mathcal{F}_t\right)$$

$$= \sum_{k=0}^{t} X_k \mathbb{1}_{\{\tau = k\}} + \underbrace{E\left(X_{t+1} \mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t\right)}$$

$$\| $$

$$\underbrace{\mathbb{1}_{\{\tau > t\}} E(X_{t+1} | \mathcal{F}_t)}$$

$$\|$$

$$\mathbb{1}_{\{\tau > t\}} X_t$$

$$= \sum_{k=0}^{t} X_k \mathbb{1}_{\{\tau = k\}} + X_t \mathbb{1}_{\{\tau > t\}} \quad \Big|\ \because X_t \text{ is MG}$$

$$= Z_t$$

$\therefore \{Z_t = X_{t \wedge \tau}\}$ is M.G.

<u>Thm:</u> Optimal Stopping thm (martingale stopping thm)

$\{X_t\}$ MG, τ bdded S.T. (i.e. for a constt $c < \infty$ one has $\tau < c$ a.s.)

Then $E(X_\tau) = E(X_0)$ ①

then $E(X_\tau) = E(X_0)$ —— (1)

(Thus, in a fair game, one cannot invent a rule for quitting the game that would "beat the system": the game will remain fair)

Sol $\quad Z_t = X_{t\wedge\tau}$ is MG.

$\because$ MG has const mean $\quad E(X_{t\wedge\tau}) = E(X_0)$

setting $t := c$ yields (1).