# Financial Mathematics:

**Books** (1) Elements of Sto. Modelling by K. Borovkov

(2) S. Roman, Introduction to the Mathematics of Finance, Springer, 2004.

(3) Derivatives Valuation and Risk Management by D.A. Dubofsky and T.W. Miller

Mid → 30,
End → 50
CT → 10
surprise CT → 5
project → 5

$$E(E(X|Y)) = E(X)$$

Sol

$$E(\underbrace{E(X|Y)}) = E(\psi(Y)) = \int \psi(y) f_Y(y)\, dy$$

$$E(X|Y=y) = \underbrace{\int x f_{X|Y=y}(x)\, dx} = \psi(y)$$

$$= \int \left( \int x f_{X|Y=y}(x)\, dx \right) f_Y(y)\, dy$$

$$= \int \int x f(x,y)\, dx\, dy$$

$$= \int x \underbrace{\int f(x,y)\, dy}\, dx$$

$$= \int x f_X(x)\, dx = E(X)$$

CE1 linearity

$$E(aX+bY|\mathcal{G}) = a\,E(X|\mathcal{G}) + b\,E(Y|\mathcal{G})$$

CE2 If $Z$ is $\mathcal{G}$ mble, then $E(ZX|\mathcal{G}) = Z E(X|\mathcal{G})$

$\{Z \in B^*\} \in \mathcal{G} \ \forall B^* \in \mathcal{B}$

CE3 If $X$ is indep. of $\mathcal{G}$ (meaning: for any $B \in \mathcal{B}$ and $A \in \mathcal{G}$, the event $\{X \in B\}$ and $A$ are indep.

$$E(X|\mathcal{G}) = E(X)$$

CE4 If $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{F}$ are $\sigma$-fields, then

$$E(E(X|\mathcal{G}_1)|\mathcal{G}_0) = E(X|\mathcal{G}_0)$$

Martingale (MG):

discrete time process $\{X_t\}_{t\in \underset{\sim}{T}}$ on $(\Omega, \mathcal{F}, P)$

$\underset{\sim}{T} = \{0, 1, \dots T\}, T < \infty$

or $\{0, 1, 2, \dots\}$ or $\{\dots, -1, 0, 1, \dots\}$

→ given $t$, $\mathcal{F}_t$ collection of event "observable" by that time

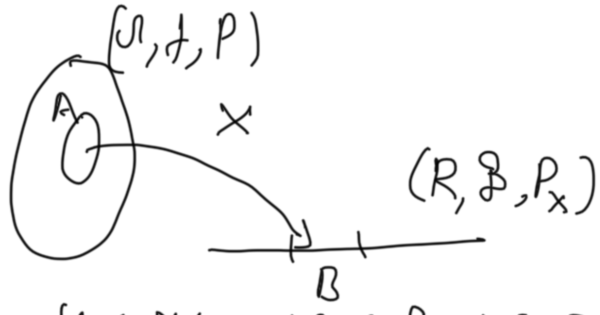
→ filtration $\underset{\sim}{F}$ is ↑ seq. of sub $\sigma$-fields

$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \dots \subset \mathcal{F}$

→ S.P. $\{X_t\}_{t\ge 0}$ is adapted to filtration $\underset{\sim}{F}$ if for any $t = 0, 1, 2, \dots$, the r.v. $X_t$ is $\mathcal{F}_t$-mble

, i.e., $\{X_t \in B\} \in \mathcal{F}_t$ for any $B \in \mathcal{B}$

$(\Omega, \mathcal{F}, \underset{\sim}{F}, P)$ filtered prob. space
or
stochastic basis




$\{\omega \mid X(\omega) \in B\} \in \mathcal{F}, \ \forall B \in \mathcal{B}$

## Binomial market model:

price $S_t$ of a given asset, $t = 0, 1, 2, \ldots$

assume $S_0$ (fixed)

for $d < u$

$$S_{t-1} \begin{cases} \nearrow S_t = u S_{t-1} \\ \searrow S_t = d S_{t-1} \end{cases}, \quad t = 1, 2, \ldots, T$$

$\mathcal{F}$ $\qquad \Omega := \{ \underset{\sim}{\nu} = \{\nu_1 \nu_2 \ldots \nu_T\} \, ; \, \nu_j = d \text{ or } u \}$

$\mathcal{F}_0 := \{\phi, \Omega\}$

$\mathcal{F}_1 := \sigma(A_d, A_u) \equiv \{\phi, A_d, A_u, \Omega\}$

where $\quad A_d := \{\nu \in \Omega \, ; \, \nu_1 = d\}$

$\qquad\quad A_u := \{\nu \in \Omega \, ; \, \nu_1 = u\}$

Once we know each of the event from $\mathcal{F}_1$, whether it occurred or not, we know $S_1$,
and vice-versa

$\mathcal{F}_2 := \sigma\{A_{dd}, A_{du}, A_{ud}, A_{uu}\}$,

when $A_{du} = \{\underset{\sim}{v} \in \Omega : v_1 = d, v_2 = u\}$ etc

Note that

$$A_d = A_{dd} \cup A_{du}$$

$$A_u = A_{ud} \cup A_{uu}$$

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$$

the process $(S_t)$ is adapted to filtration

$$\underset{\sim}{F} := \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T\}$$

$(S_t^2)$ , — - - - -

$Y_t = S_{t+1} - S_t \xrightarrow{?}$ Not adapted to filtration

— x —

→ $(\Omega, \mathcal{F}, F, P)$ discrete time SP $(X_t)_{t \geq 0}$ adapted to filtration $\underset{-}{F}$, $(X_t)$ is MG for any $t = 0, 1, 2, \ldots$

$E|X_t| < \infty$

and $E(X_{t+1} | \mathcal{F}_t) = X_t$

— x —

let $X_t$ is MG, for any $s \geq 1$,

$$E(X_{t+s} | \mathcal{F}_t) = E\Big(\underbrace{E(X_{t+s} | \mathcal{F}_{t+s-1})}_{X_{t+s-1}} \Big| \mathcal{F}_t\Big) \quad \because (X_t) \text{ is MG}$$

$$= E(X_{t+s-1} | \mathcal{F}_t)$$

$$\vdots$$

$$= E(X_{t+1} | \mathcal{F}_1)$$

$= X_t$

—x—

$\Rightarrow \quad E(E(X_{t+s}|\mathcal{F}_t)) = E(X_t)$

$E(X_{t+s}) = E(X_t) = E(X_0)$

→ A cont time MG is defined as an adapted (in that case, to a cont time filtration, i.e., a family of σ-fields $(\mathcal{F}_t)_{t\geq 0}$ st for any $s, t \geq 0$ one has $\mathcal{F}_t \subset \mathcal{F}_{t+s} \subset \mathcal{F}$) process $(X_t)_{t\geq 0}$ is MG

$E|X_t| < \infty$

$E(X_{t+s}|\mathcal{F}_t) = X_t$

—x—

→ Submartingale, adapted process st. $E(X_{t+1}|\mathcal{F}_t) \geq X_t$

→ Supermartingale, " , st $E(X_{t+1}|\mathcal{F}_t) \leq X_t$

Example (1) Random walk

$X_0 := 0$, $X_n := Y_1 + \cdots + Y_n$, $n \geq 1$, $Y_j$ IID rvs with $E|Y_1| < \infty$. When is the SP $\{X_n\}_{n\geq 0}$ is MG?

Sol $E|X_n| \leq E|Y_1| + \cdots + E|Y_n| < \infty$

$E(X_{n+1}|\mathcal{F}_n) = E(X_n + Y_{n+1}|\mathcal{F}_n)$

$$= E(X_n | \mathcal{F}_n) + E(Y_{n+1} | \mathcal{F}_n)$$

$$= X_n + E(Y_1)$$

$\{X_n\}$ is MG iff $E(Y_1) = 0$

(2) Geometric random walk

$$X_n := \underline{X_0 \, e^{Y_1 + \cdots + Y_n}}, \quad n \geq 1$$

where $X_0 :=$ const $\geq 0$, $Y_j$'s IID r.v.'s

When $\{X_n\}_{n \geq 0}$ is an MG wrt filtration $\mathcal{F}_n := \sigma(Y_1, \ldots, Y_n)$

$$E(X_n) = X_0 \, E(e^{Y_1 + \cdots + Y_n})$$

$$= X_0 \, [E(e^{Y_1})]^n$$

$< \infty$ iff $E(e^{Y_1}) = \phi_Y(1) < \infty$

$$E(X_{n+1} | \mathcal{F}_n) = E(X_0 \, e^{Y_1 + Y_2 + \cdots + Y_n + Y_{n+1}} | \mathcal{F}_n)$$

$$= E(X_0 \, e^{Y_1 + \cdots + Y_n} \, e^{Y_{n+1}} | \mathcal{F}_n)$$

$$= X_0 \, e^{Y_1 + \cdots + Y_n} \, E(e^{Y_{n+1}} | \mathcal{F}_n)$$

$$= X_n \, E(e^{Y_1})$$

$X_n$ is MG iff $E(e^{Y_1}) = \phi_Y(1) = 1$

—— x ——