Thm 2. A finite single-period financial market is arbitrage-free iff $\exists$ a prob. P^* on $(\Omega, 2^{\Omega})$ s.t. $P^*(\{\omega\}) > 0$ for every $\omega \in \Omega$ and $E^*\left(\frac{S_1}{1+r}\right) = S_0$, i.e., the process $\{S_t(1+r)^{-t}\}_{t=0,1}$ is a MG under P^*

Pf: $n=2$ $\quad \underset{\sim}{S}_t = (S_t^1, S_t^2)$, $t=0,1$

two riskey asset

Set $D := \operatorname{conv}\left\{ \frac{\underset{\sim}{S}_1(\omega_1)}{1+r}, \ldots, \frac{\underset{\sim}{S}_1(\omega_N)}{1+r} \right\}$

Two possibilities: either (I) $\underset{\frown}{S}_0 \in D_0$ or (II) $\underset{\frown}{S}_0 \notin D_0$

We will prove that $NA \Leftrightarrow \exists$ EMM (equiv. martingale measures) in three steps

step 1 shows $\exists$ EMM $\Leftrightarrow$ (I), ✓

step 2 proves $\exists$ EMM $\Rightarrow$ NA ✓

and step 3 " NA $\Rightarrow$ (I) ✓

Step 1: $S_0 \in D_0$

$$\therefore S_0 = \sum_{k=1}^{N} p_k^* \frac{\underset{\sim}{S}_1(\omega_k)}{1+r} =: E^*\left(\frac{\underset{\sim}{S}_1}{1+r}\right)$$

$\therefore$ step I holds ✓

Step 2: Assume $\exists$ EMM, but NA does not hold

If NA does not hold, $\exists$ arbitrage opportunity

(Δ, h)

$(\underset{\sim}{\Delta}, b)$ s.t. $V_t = \underset{\sim}{\Delta} \cdot \underset{\sim}{S}_t + b_t$

start with $V_0 \equiv \underset{\sim}{\Delta} \cdot \underset{\sim}{S}_0 + b = 0 \Leftrightarrow b = -\underset{\sim}{\Delta} \cdot \underset{\sim}{S}_0$

$$V_1(\omega) \equiv \underset{\sim}{\Delta} \cdot \underset{\sim}{S}_1(\omega) + b(1+r)$$

$$= \underset{\sim}{\Delta} \cdot \underset{\sim}{S}_1(\omega) - \underset{\sim}{\Delta} \cdot S_0(1+r) \begin{cases} \geq 0, \forall \omega \\ > 0 \text{ for atleast one } \omega \end{cases}$$

$\therefore$ NA does not hold

Consider

$$\underset{\sim}{\Delta} \cdot \underset{\sim}{S}_0 \overset{\text{Step2}}{=} \underset{\sim}{\Delta} \cdot E^*\left(\frac{\underset{\sim}{S}_1}{1+r}\right) = E^*\left(\frac{\underset{\sim}{\Delta} \cdot \underset{\sim}{S}_1}{1+r}\right)$$

$$= \sum_{k=1}^{N} p_k^* \frac{\underset{\sim}{\Delta} \cdot \underset{\sim}{S}_1(\omega_k)}{1+r}$$

$$\overset{\text{using } (*)}{>} \sum_{k=1}^{N} p_k^* \underset{\sim}{\Delta} \cdot \underset{\sim}{S}_0$$

$= \underset{\sim}{\Delta} \cdot \underset{\sim}{S}_0$ a contradiction, So our assumption was wrong; implying $\exists$ EMM $\Rightarrow$ NA.

$\therefore$ step2 holds

Step3 Suppose NA holds, but (I) does not

i.e., (II) holds $\underset{\sim}{S}_0 \notin D_0$

Then by the separation thm $\exists$ a st. line L seperating $\underset{\sim}{S}_0$ from D_0. Denoting by $\Delta := (\Delta^1, \Delta^2)$ a normal to L vector "pointing" in the direction of D_0, we have

$$\underset{\sim}{\Delta} \cdot (\underset{\sim}{x} - S_0) > 0 \quad , x \in D_0$$

$$\underset{\sim}{\Delta} \cdot \left(\frac{S_1(\omega)}{1+r} - S_0 \right) \begin{cases} \geq 0 & , \forall \omega \\ > 0 & \text{for atleast one } \omega . \end{cases}$$

This means arbitrage opportunity.
Contradiction to NA condition.

—x—

Thm3 (The arbitrage pricing thm): In a single-period arbitrage-free market, the discounted value process $\{V_t (1+r)^{-t}\}_{t=0,1}$ of any trading strategy $(\underset{\sim}{\Delta}, b)$ is an MG under any EMM P^*

$$E^*\left(\frac{V_1}{1+r}\right) = V_0 . \quad \text{——} (1) \checkmark$$

In particular, if X is an attainable claim in the market then its time $t=0$ value is given by
$$X^* := E^*\left(\frac{X}{1+r}\right) \quad \text{——} (2)$$

Where the RHS does not depend on the choice of EMM P^*.

Sol $(\underset{\sim}{\Delta}, b)$

$$E^*\left(\frac{V_1}{1+r}\right) = E^*\left(\frac{\underset{\sim}{\Delta} \cdot \underset{\sim}{S}_1 + b(1+r)}{1+r}\right)$$

$$= E^* \frac{\underline{\Delta} \cdot \underset{\sim}{S_1}}{1+r} + b$$

$$= \underline{\Delta} \cdot E^*\left(\frac{S_1}{1+r}\right) + b$$

$$= \underline{\Delta} \cdot \underset{\sim}{S_0} + b \qquad \text{| using thm 2}$$

$$= V_0 \qquad \text{—— (3)} \qquad \therefore \text{(1) holds}$$

Now suppose X is an attainable claim, i.e., for some portfolio $(\underline{\Delta}, b)$ one has

$$V_1(\omega) = X(\omega) \quad , \omega \in \Omega$$

$t=0$ (Δ_1, b_1) $\quad t=1 \quad \Delta_1 S_1 + b_1(1+r) = \Delta_2 S_1 + b_2(1+r)$

A strategy (Δ_t, S_t) is called self financing

$t=0, (\Delta_1, b_1)$ $\quad \Delta_t \cdot S_t + b_t(1+r)^t = \Delta_{t+1} S_t + b_{t+1}(1+r)^t$

$t = 1, 2, \dots, T-1$

(1) & (2) implies

$$V_0 = E^*\left(\frac{V_1}{1+r}\right) = E^*\left(\frac{X}{1+r}\right) = X^*$$

—x—

Thm 3 provides one with the arbitrage-free price for any attainable claim. It turns out that, in the general case, not all claims are attainable in a financial market. A market with property that any claim is attainable then it is said to be complete.

Single-period NA binomial market is complete.

<u>Thm 4</u> (The completeness thm) An arbitrage-free market is complete iff $\exists$ a unique EMM.

pf: <u>Observation 1</u> using thm 2 $\exists P^*$ st $E^*\left(\dfrac{\underset{\sim}{S_1}}{1+r}\right) = \underset{\sim}{S_0}$

$$p_1^* S_1^1(\omega_1) + \cdots + p_H^* S_1^1(\omega_H) = (1+r) S_0^1$$

$$\cdots$$

$$p_1^* S_1^n(\omega_1) + \cdots + p_H^* S_1^n(\omega_H) = (1+r) S_0^1$$

$$p_1^* + \cdots + p_H^* = 1$$

So if the EMM is unique, the above system has unique solⁿ.

$n+1 = N$; A = system matrix
$$A = \begin{pmatrix} S_1^1(\omega_1) & S_1^1(\omega_2) & \cdots & S_1^1(\omega_H) \\ S_1^2(\omega_1) & S_1^2(\omega_2) & \cdots & S_1^2(\omega_H) \\ \cdots & \cdots & & \\ S_1^n(\omega_1) & S_1^n(\omega_2) & \cdots & S_1^n(\omega_H) \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

A is non-singular ; $\det A \neq 0$ and hence $\exists A^{-1}$.

<u>Observation 2</u> Claim $X = X(\omega)$ can be replicated when $\exists$ portfolio $(\underset{\sim}{\Delta}, b)$, $\underset{\sim}{\Delta} = (\Delta^1, \ldots, \Delta^n)$ st

$$\Delta^1 S_1^1(\omega_1) + \cdots + \Delta^n S_1^n(\omega_H) + b(1+r) = X(\omega_1) \quad \Big\} \star$$

$$\Delta^1 S_1^1(\omega_N) + \cdots + \Delta^n S_1^n(\omega_N) + b(1+r) = X(\omega_N)$$

system N linear equation for n+1 unknown $\Delta^1, \to \Delta^n, b(1+r)$

system matrix is A^T

($\Leftarrow$) Suppose $\exists$ unique EMM ; from observation 1

A is nonsingular $\det A \neq 0$ $\quad \det A = \det A^T \neq 0$

$\therefore$ For any claim X , there is unique replicating strategy in the market, which proves that market is complete

($\Rightarrow$) we know Thm2 for an NA market $\exists$ an EMM.

So we just have to show that if market is complete then EMM is unique.

Let $\exists$ two EMM $\underset{\sim}{p}^*, \underset{\sim}{\tilde{p}}^*$.

Due to completeness, by thm3, we have for any claim

$$E^* \frac{X}{1+r} = \tilde{E}^* \frac{X}{1+r}$$

$$\Leftarrow \sum_{k=1}^{N} p_k^* X(\omega_k) = \sum_{k=1}^{N} \tilde{p}_k^* X(\omega_k) \quad — \; ✡_1$$

Consider a particular case $X_i(\omega_k) := \delta_{ik} = \begin{cases} 1 & \text{if } k=i \\ 0 & \text{o.w.} \end{cases}, \; i=1,\ldots,N$

$$p_i^* = \sum_{k=1}^{N} p_k^* \delta_{ik} = \sum_{k=1}^{N} p_k^* X_i(\omega_k)$$

$$= \sum_{k=1}^{N} \tilde{p}_k^* X_i(\omega_k) \quad \Big| \text{Using } \star$$

$$= \sum_{k=1}^{N} \tilde{p}_k^* \delta_{ik} = \tilde{p}_i^*$$

$\therefore \ \underset{\sim}{p}^* = \underset{\sim}{\tilde{p}}^*$, i.e. Emm is unique