Discounted Dynamic programming:

(inflation etc) \$ now, 5years time is not and the same thing

introduce discounted returns

$$\sum_{t=0}^{T} \alpha^t R(X_t, a_t) \quad , \quad \alpha \in (0,1)$$

discount factor

α close to 0 if do not care much about distant future

α " " 1 one can allow the whole time horizon

T (a finite horizon) or infinite (an infinite horizon which makes sense due to discounting as the series can now be convergent.

Optimal criterion

$$E_\pi\left(\sum_{t=0}^{T} \alpha^t R(X_t, a_t) \mid X_0 = i\right) \to \max_\pi =: V_n(i) \text{ or } V(i)$$

where E_π stands for the exp. under the policy π (We know that the evolution of the process $\{X_t\}$ depends on actions taken, so we will have different distn of $\{X_t\}$ for different policies)

→ Finite horizon case

$$V_n(i) = \max_a \left[R(i,a) + \alpha E_a\left(V_{n-1}(X_1) \mid X_0 = i\right)\right]$$

$$= \max \left[R(i,a) + \alpha \sum p_{ij}(a) V_{n-1}(j)\right]$$

→ Infinite horizon case

$$V(i) = \max_a \left[R(i,a) + \alpha E_a \left(V(X_1) \mid X_0 = i \right) \right]$$

$$= \max_a \left[R(i,a) + \alpha \sum_j p_{ij}(a) V(j) \right]$$

<u>Example</u> Lifetime portfolio selection (optimal consumption - saving)

(Non random environment)

X_t : wealth of particular individual at the beginning of t^{th} time period, $t = 0, 1, 2, \ldots, T$

unit X_t

consume C_t ; invest in non-riskey asset at fixed rate $r > 1$: $X_t - C_t$

$$X_{t+1} = r(X_t - C_t)$$

Consuming c units of wealth leads to the utility $u(c)$

$u(\cdot) \to$ utility fn

$$C_t = X_t - r^{-1} X_{t+1}$$

$$\max_{(X_t)} \sum_{t=0}^{T} \alpha^t u(C_t) = \max_{(X_t)} \sum_{t=0}^{T} \alpha^t u\left(X_t - \frac{1}{r} X_{t+1}\right)$$

$$= \max_{X_t} \phi(X_1, \ldots, X_{T-1})$$

let X_0, X_T fixed

$$0 = \frac{\partial \phi}{\partial X_t} = -\frac{\alpha^{t-1}}{r} u'\left(X_{t-1} - \frac{1}{r} X_t\right) + \alpha^t u'\left(X_t - \frac{1}{r} X_{t+1}\right)$$

$$, \quad t = 1, \dots, T-1$$

$$\Rightarrow u'\left(X_{t-1} - \frac{X_t}{r}\right) = \alpha r \, u'\left(X_t - \frac{1}{r} X_{t+1}\right) \quad \text{☆}$$

Special case of Bernoulli utility $u(x) = \log x$

(similar result hold for $u(x) = x^r$, $0 < r < 1$, as well)

$$u'(x) = \frac{1}{x}$$

From ☆

$$\frac{1}{X_{t-1} - \frac{X_t}{r}} = \alpha r \times \frac{1}{X_t - \frac{1}{r} X_{t+1}}$$

$$\Rightarrow X_t - \frac{1}{r} X_{t+1} = \alpha r X_{t-1} - \alpha X_t$$

$$\Rightarrow -\frac{1}{r} X_{t+1} + (1+\alpha) X_t - \alpha r X_{t-1} = 0$$

$$\Rightarrow X_{t+1} - r(1-\alpha) X_t + \alpha r^2 X_{t-1} = 0$$

Consider the partial soln of the form $X_t = \lambda^t$ for fixed λ

$$\lambda^{t+1} - r(1+\alpha)\lambda^t + \alpha r^2 \lambda^{t-1} = 0$$

$$\Rightarrow \lambda^2 - r(1+\alpha)\lambda + \alpha r^2 = 0$$

$$\Rightarrow \lambda_1 = r, \quad \lambda_2 = \alpha r$$

general soln

$$X_t = b_1 \lambda_1^t + b_2 \lambda_2^t$$

$$= \underbrace{b_1 r^t}_{\text{deposit}} + \underbrace{b_2 \alpha^t r^t}_{\text{consume \& deposit}}$$

deposit consumption

Here $b_1 + b_2 = X_0$

$b_1 r^T + b_2 \alpha^T r^T = X_T$ ✡

$\Rightarrow b_2 = \dfrac{X_0 - r^{-T} X_T}{1-\alpha^T}$; $b_1 = X_0 - b_2$

∴ Optimal policy has the form

$$C_t = X_t - \frac{1}{r} X_{t+1}$$

$$= \cancel{b_1} r^t + b_2 \alpha^t r^t - \frac{1}{r}\left(\cancel{b_1} r^{t+1} + b_2 \alpha^{t+1} r^{t+1}\right)$$

$$= b_2 (1-\alpha)(\alpha r)^t$$

For instance $T = \infty$ (individual is going to live a very long life) in the "no-bequest" situation ($X_T = 0$)

✡ $b_1 + b_2 = X_0$

$b_1 r^T + b_2 \alpha^T r^T = X_T$, $0 < \alpha < 1$

$\Rightarrow b_2 = X_0,\ b_1 = 0$

∴ $C_t = (1-\alpha) X_t$