Application in Finance :

Pricing Stock option :

amt v at time t in the future is not worth as much as given v immediately

v at time t is $ve^{-\alpha t}$

$\alpha \to$ discount factor

$e^{-\alpha t}$ discount fn

($\equiv$ we can earn interest at a continuously compounded rate of $100\alpha\%$ per unit time)

$$\text{Stock price} \quad 100 \begin{cases} 200 \\ 50 \end{cases}$$

time 0 price   time 1 price

Suppose that for any y, at a cost cy, you can purchase at time 0, the option to buy y unit of stock at cost k per share ($k = 150$)

$$\begin{cases} \text{buy/sell } x \text{ unit of stock} \\ \text{buy/sell } y \text{ unit of option} \end{cases}$$

time 0 price

k strike price

We are interested in determining the appropriate value of C, the unit cost of an option. Specifically we will show that unless $C = \frac{50}{3}$ there will be combination of purchases that will always result in +ve gain.

value of holding at time 1

$$\text{value} = \begin{cases} 200x + 50y & \text{if price is } 200 \\ 50x & \text{if price is } 50 \end{cases}$$

let us choose y st

$$200x + 50y = 50x$$

$$\Rightarrow \quad y = -3x$$

If $y = -3x$, value of holding at time 1 is $50x$

Original cost of purchasing x unit of stock and $y = -3x$ unit of option

$$\text{Original cost} = 100x - 3xc$$

$$\text{gain} = 50x - (100x - 3xc) = x(3c - 50)$$

$$\begin{cases} 0 & \text{if } 3c = 50 \\ +ve & \text{if } 3c > 50 \\ -ve & \text{if } 3c < 50 \end{cases}$$

→ If $c = 20$ (unit cost per option)

$x = 1, \ y = -3$

purchase 1 unit of stock and simultaneously sell 3 units of options initially cost us $100 - 3 \times 20 = 40$

value of holding at time 1 is 50 whether the stock goes up to 200 or down to 50

guranteed profit $= 50 - 40 = 10$

→ If $c = 15$

$x = -1, \ y = 3$

intial gain $= 100 - 45 = 55$

value of holding at time 1 is -50

guranted profit $= 55 - 50 = 5$

A sure win betting scheme is called an arbitrage (risk free profit).

Thus, only option cost C that does not result in an arbitrage is $C = \frac{50}{3}$.

Arbitrage thm:

Expt whose set of outcomes $S = \{1, \cdots, m\}$

Wagers $1, 2, \cdots, n$

betting scheme $\underset{\sim}{x} = (x_1, x_2, \cdots, x_n)$

Outcome of expt is j, $j \in S$, then

return from $\underset{\sim}{x} = \sum_{i=1}^{n} x_i \, r_i(j)$

$r_i(\cdot)$ return fn for unit bet on wager i.

$\exists \ \underset{\sim}{p} = (p_1, \cdots, p_m)$ on the set of possible outcomes of expt under which each of wagers has expected return 0 or else there is a betting scheme that guratees a +ve win.

Thm: Exactly one of the following is true

Either

m

(i) $\exists$ a $\underset{\sim}{p} = (p_1, \rightarrow p_m)$ for which $\sum_{j=1} p_j r_i(j) = 0$

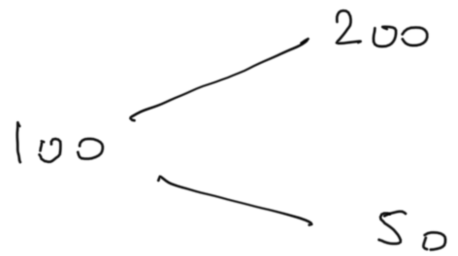
$\forall i = 1, \ldots, m$

or

(ii) $\exists$ a betting scheme $\underset{\sim}{x} = (x_1, \rightarrow x_n)$ for which

$$\sum_{i=1}^{n} x_i \, r_i(j) > 0 \quad \forall j = 1, \ldots, m.$$

Example (contd)



set. C $\qquad\qquad k = 150$

return from purchasing 1 unit of stock $= \begin{cases} 200 - 100 = 100 & \text{if price is } 200 \\ 50 - 100 = -50 & \text{if " " } 50 \end{cases}$

$\underset{\sim}{P} = (p, 1-p)$ Stock

p price is 200 at time 1

$1-p$ " " 50 " " 1

$E_{\underset{\sim}{p}}(\text{return}_{\text{Stock}}) = 0 \Rightarrow 100 \times p - 50(1-p) = 0$

$\Rightarrow 100 = 200p + 50(1-p)$

$\Rightarrow p = \frac{1}{3}$

$\underset{\sim}{P} = \left(\frac{1}{3}, \frac{2}{3}\right)$

return from purchasing one share of option $= \begin{cases} 50 - C & \text{if price is } 200 \\ -C & \text{if price is } 50 \end{cases}$

$E_{\underset{\sim}{p}}(\text{return}_{\text{option}}) = 0$

$\Rightarrow (50-c) \times \frac{1}{3} + (-c) \times \frac{2}{3} = 0$

$\Rightarrow C = (200-150) \times \frac{1}{3}$

$\Rightarrow C = \frac{50}{3}$

$C = E(X(t)-k)^+$

$(x-a)^+ = \begin{cases} x-a, & x > a \\ 0 & x < a \end{cases}$

Option pricing in discrete time :

$(S_t)_{t \geq 0}$ price of a security (stock) that is traded in the market

a risk free, fixed interest investment $(B_t)_{t \geq 0}$ (bond) with interest rate $r \geq 0$.

value of bond at time $t=0$ is B_0

$B_t = (1+r)^t B_0$

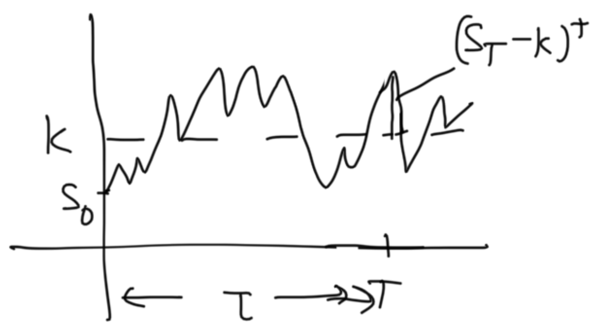
$(x-a)^+ = \begin{cases} x-a \text{ if } x > a \\ 0 \text{ if } x < a \end{cases}$

calls option buy

C

European call

$Y = (S_T - k)^+$



American call

r.v. $\tau \in [0,T]$

$Y = (S_\tau - k)^+$

put sell $Y = (k - S_T)^+$

at time T

ϵ

↳ What is the correct premium of the call Y?

Classical answer $E(Y) = E(S_T - k)^+$

Black and Scholes (1973) introduced a compelling new method to calculate the premium called fair price or Black-Scholes price

Simple pricing model

$$
\begin{tikzcd}
 & 130 = S_1^a \\
S_0 = 100 \arrow[ur, "p=0.4"] \arrow[dr, "1-p=0.6"'] & \\
 & 80 = S_1^b
\end{tikzcd}
$$

$t=0$ $\qquad$ $T=1$ $\qquad$ $k=110$

$r = 0.05$

European call at $T=1$

$$Y = (S_1 - k)^+$$

Classical way of valuation at $t=1$ is $E_p(Y) = 20 \times 0.4 = 8$

∴ value of call at $t=0$ is $C_0 = \dfrac{E_p(Y)}{1+r}$

$$= \frac{8}{1.05} = 7.62$$

$\Pi = (\Delta, B)$ — portfolio; Δ stock share, B bond share

price of call $Y \stackrel{!}{=}$ price of duplicating portfolio

choose $\Delta = 0.4$, $B = -30.48$

price of portfolio at $t=0$

$$V_0(\Pi) = 0.4 \times 100 - 30.48 = 9.52$$

price of portfolio at $t=1$

price of duplicating portfolio

$$V_1(\Pi) = \Delta S_1 + B(1+r)$$

$$= \begin{cases} 0.4 \times 130 + (-30.48) \times 1.05 & \text{if } S_1 = 130 \\ 0.4 \times 80 + (-30.48) \times 1.05 & \text{if } S_1 = 80 \end{cases}$$

$$= \begin{cases} 20 & \text{if } S_1 = 130 \\ 0 & \text{if } S_1 = 80 \end{cases}$$

$$\overset{!}{=} Y \quad \text{price of call}$$

Fair price of call at $t=0$ is the price of portfolio

$$C_0^{BS} = 100 \times 0.4 + (-30.48) = 9.52$$