Equations

Mass Balance:

1D:

Gas:
$$\frac{\partial}{\partial t}(\varphi n) + \frac{\partial}{\partial x}(\varphi n u_g) = 0$$

Water:
$$\frac{\partial}{\partial t}(\varphi m) + \frac{\partial}{\partial x}(\varphi m u_w) = 0$$

2D:

Gas:
$$\frac{\partial}{\partial t}(\varphi n) + \frac{\partial}{\partial x}(\varphi n u_g) + \frac{\partial}{\partial y}(\varphi n v_g) = 0$$

Water:
$$\frac{\partial}{\partial t}(\varphi m) + \frac{\partial}{\partial x}(\varphi m u_w) + \frac{\partial}{\partial y}(\varphi m v_w) = 0$$

Pressure Evolution:

1D:
$$\frac{\partial}{\partial t}(P_w) + \tilde{\eta}\rho_w \frac{\partial}{\partial x}(\varphi n u_g) + \tilde{\eta}\tilde{\rho}_g \frac{\partial}{\partial x}(\varphi m u_w) = 0$$

$$\tilde{\eta} = \frac{\frac{c_w c_g}{\varphi}}{s_g \rho_w c_w + s_w \tilde{\rho}_g c_g}, \qquad \tilde{\rho}_g = \rho_g - \frac{s_g P_c'}{c_g} \qquad P_c' = \frac{\partial P_c}{\partial s_w}$$

2D:
$$\frac{\partial}{\partial t}(P_w) + \tilde{\eta}\rho_w \left[\frac{\partial}{\partial x}(\varphi n u_g) + \frac{\partial}{\partial y}(\varphi n v_g)\right] + \tilde{\eta}\tilde{\rho}_g \left[\frac{\partial}{\partial x}(\varphi m u_w) + \frac{\partial}{\partial y}(\varphi m v_w)\right] = 0$$

Y-momentum Equation

1D

Gas:
$$\frac{\partial}{\partial t}(nu_g) + \frac{1}{\varphi}\frac{\partial}{\partial x}(\varphi nu_g^2) + s_g\frac{\partial}{\partial x}(P_w + P_c) = -\hat{k}_g u_g + \hat{k}(u_w - u_g) + ng + \frac{1}{\varphi}\varepsilon_g\frac{\partial}{\partial x}\left(n\frac{\partial}{\partial x}(\varphi u_g)\right)$$

Water:
$$\frac{\partial}{\partial t}(mu_w) + \frac{1}{\varphi}\frac{\partial}{\partial x}(\varphi mu_w^2) + s_w\frac{\partial}{\partial x}(P_w) = -\hat{k}_w u_w - \hat{k}(u_w - u_g) + mg + \frac{1}{\varphi}\varepsilon_w\frac{\partial}{\partial x}\left(m\frac{\partial}{\partial x}(\varphi u_w)\right)$$

2D

Gas:
$$\frac{\partial}{\partial t}(nv_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi nu_g v_g) + \frac{\partial}{\partial y}(\varphi nv_g^2)\right] + s_g\frac{\partial}{\partial y}(P_w + P_c) = -\hat{k}_g v_g + \hat{k}(v_w - v_g) + ng + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi\tau_{yx}^g) + \frac{\partial}{\partial y}(\varphi\tau_{yy}^g)\right]$$

water:
$$\frac{\partial}{\partial t}(mv_w) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi mu_w v_w) + \frac{\partial}{\partial y}(\varphi mv_w^2)\right] + s_w\frac{\partial}{\partial y}(P_w) = -\hat{k}_w v_w - \hat{k}(v_w - v_g) + mg + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi\tau_{yx}^w) + \frac{\partial}{\partial y}(\varphi\tau_{yy}^w)\right]$$

X-momentum Equation

Gas:
$$\frac{\partial}{\partial t}(nu_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial y}(\varphi nv_g u_g) + \frac{\partial}{\partial x}(\varphi nu_g^2)\right] + s_g\frac{\partial}{\partial x}(P_w + P_c) = -\hat{k}_g u_g + \hat{k}(u_w - u_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi\tau_{xx}^g) + \frac{\partial}{\partial y}(\varphi\tau_{xy}^g)\right]$$

water:
$$\frac{\partial}{\partial t}(mu_w) + \frac{1}{\varphi}\left[\frac{\partial}{\partial y}(\varphi mv_w u_w) + \frac{\partial}{\partial x}(\varphi mu_w^2)\right] + s_w\frac{\partial}{\partial x}(P_w) = -\hat{k}_w u_w - \hat{k}(u_w - u_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi\tau_{xx}^w) + \frac{\partial}{\partial y}(\varphi\tau_{xy}^w)\right]$$

Gas viscous-stress components τ^g

$$\tau_{xx}^g = 2\mu_{g,eff}\frac{\partial u_g}{\partial x} + \kappa_{g,eff}\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = \left(2\mu_{g,eff} + \kappa_{g,eff}\right)\frac{\partial u_g}{\partial x} + \kappa_{g,eff}\frac{\partial v_g}{\partial y}$$

$$\tau_{yy}^g = 2\mu_{g,eff}\frac{\partial v_g}{\partial y} + \kappa_{g,eff}\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = \left(2\mu_{g,eff} + \kappa_{g,eff}\right)\frac{\partial v_g}{\partial y} + \kappa_{g,eff}\frac{\partial u_g}{\partial x}$$

$$\tau_{xy}^g = \tau_{yx}^g = \mu_{g,eff}\left(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x}\right)$$

Water viscous-stress components τ^w

$$\tau_{xx}^w = 2\mu_{w,eff}\frac{\partial u_w}{\partial x} + \kappa_{w,eff}\left(\frac{\partial u_w}{\partial x} + \frac{\partial v_w}{\partial y}\right) = \left(2\mu_{w,eff} + \kappa_{w,eff}\right)\frac{\partial u_w}{\partial x} + \kappa_{w,eff}\frac{\partial v_w}{\partial y}$$

$$\tau_{yy}^w = 2\mu_{w,eff}\frac{\partial v_w}{\partial y} + \kappa_{w,eff}\left(\frac{\partial u_w}{\partial x} + \frac{\partial v_w}{\partial y}\right) = \left(2\mu_{w,eff} + \kappa_{w,eff}\right)\frac{\partial v_w}{\partial y} + \kappa_{w,eff}\frac{\partial u_w}{\partial x}$$

$$\tau_{xy}^w = \tau_{yx}^w = \mu_{w,eff}\left(\frac{\partial u_w}{\partial y} + \frac{\partial v_w}{\partial x}\right)$$

Final Form of Equation

Y-momentum Equation

Gas:

$$\frac{\partial}{\partial t}(nv_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi n u_g v_g) + \frac{\partial}{\partial y}(\varphi n v_g^2)\right] + s_g \frac{\partial}{\partial y}(P_w + P_c) = -\hat{k}_g v_g + \hat{k}(v_w - v_g) + ng + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi \mu_{g,eff}(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x})\right) + \right.$$

$$\left. \frac{\partial}{\partial y}\left(\varphi\left((2\mu_{g,eff} + \kappa_{g,eff})\frac{\partial v_g}{\partial y} + \kappa_{g,eff}\frac{\partial u_g}{\partial x}\right)\right)\right]$$

Water:

$$\frac{\partial}{\partial t}(mv_w) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}(\varphi m u_w v_w) + \frac{\partial}{\partial y}(\varphi m v_w^2)\right] + s_w \frac{\partial}{\partial y}(P_w) = -\hat{k}_w v_w - \hat{k}(v_w - v_g) + mg + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi \mu_{w,eff}(\frac{\partial u_w}{\partial y} + \frac{\partial v_w}{\partial x})\right) + \right.$$

$$\left. \frac{\partial}{\partial y}\left(\varphi\left((2\mu_{w,eff} + \kappa_{w,eff})\frac{\partial v_w}{\partial y} + \kappa_{w,eff}\frac{\partial u_w}{\partial x}\right)\right)\right]$$

X-momentum Equation

Gas:

$$\frac{\partial}{\partial t}(nu_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial y}(\varphi n v_g u_g) + \frac{\partial}{\partial x}(\varphi n u_g^2)\right] + s_g \frac{\partial}{\partial x}(P_w + P_c) = -\hat{k}_g u_g + \hat{k}(u_w - u_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\left((2\mu_{g,eff} + \kappa_{g,eff})\frac{\partial u_g}{\partial x} + \right.\right.\right.$$

$$\left.\left.\left. \kappa_{g,eff}\frac{\partial v_g}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(\varphi \mu_{g,eff}(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x})\right)\right]$$

Water:

$$\frac{\partial}{\partial t}(mu_w) + \frac{1}{\varphi}\left[\frac{\partial}{\partial y}(\varphi m v_w u_w) + \frac{\partial}{\partial x}(\varphi m u_w^2)\right] + s_w \frac{\partial}{\partial x}(P_w) = -\hat{k}_w u_w - \hat{k}(u_w - u_g) + \frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi((2\mu_{w,eff} + \kappa_{w,eff})\frac{\partial u_w}{\partial x} +\right.\right.$$

$$\left.\left.\kappa_{w,eff}\frac{\partial v_w}{\partial y})\right) + \frac{\partial}{\partial y}(\varphi\mu_{w,eff}(\frac{\partial u_w}{\partial y} + \frac{\partial v_w}{\partial x}))\right]$$

The last term discretization: **(Y-momentum, Gas)**

$$\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\mu_{g,eff}(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x})\right) + \frac{\partial}{\partial y}(\varphi((2\mu_{g,eff} + \kappa_{g,eff})\frac{\partial v_g}{\partial y} + \kappa_{g,eff}\frac{\partial u_g}{\partial x}))\right] \qquad Let\ 2\mu_{g,eff} + \kappa_{g,eff} = \alpha$$

$$= \frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\mu_{g,eff}(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x})\right) + \frac{\partial}{\partial y}(\varphi(\alpha\frac{\partial v_g}{\partial y} + \kappa_{g,eff}\frac{\partial u_g}{\partial x}))\right]$$

$$C_{i+\frac{1}{2},j+\frac{1}{2}} = \mu_{g,eff_{i+\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{g_{i+\frac{1}{2},j+1}}^k - u_{g_{i+\frac{1}{2},j}}^k}{\Delta y} + \frac{v_{g_{i+1,j+\frac{1}{2}}}^{k+1} - v_{g_{i,j+\frac{1}{2}}}^{k+1}}{\Delta x}\right) \qquad C_{i-\frac{1}{2},j+\frac{1}{2}} = \mu_{g,eff_{i-\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{g_{i-\frac{1}{2},j+1}}^k - u_{g_{i-\frac{1}{2},j}}^k}{\Delta y} + \frac{v_{g_{i,j+\frac{1}{2}}}^{k+1} - v_{g_{i-1,j+\frac{1}{2}}}^{k+1}}{\Delta x}\right)$$

$$(visc)_x = \frac{1}{\varphi_{i,j+\frac{1}{2}}}\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}C_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}C_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}$$

$$D_{i,j} = \alpha_{i,j}\frac{v_{g_{i,j+\frac{1}{2}}}^{k+1} - v_{g_{i,j-\frac{1}{2}}}^{k+1}}{\Delta y} + \kappa_{g,eff_{i,j}}\frac{u_{g_{i+\frac{1}{2},j}}^k - u_{g_{i-\frac{1}{2},j}}^k}{\Delta x} \qquad D_{i,j+1} = \alpha_{i,j+1}\frac{v_{g_{i,j+\frac{3}{2}}}^{k+1} - v_{g_{i,j+\frac{1}{2}}}^{k+1}}{\Delta y} + \kappa_{g,eff_{i,j+1}}\frac{u_{g_{i+\frac{1}{2},j+1}}^k - u_{g_{i-\frac{1}{2},j+1}}^k}{\Delta x}$$

$$(visc)_y = \frac{1}{\varphi_{i,j+\frac{1}{2}}} \frac{\varphi_{i,j+1}D_{i,j+1} - \varphi_{i,j}D_{i,j}}{\Delta y} \qquad\qquad (visc)_{i,j+\frac{1}{2}} = (visc)_x + (visc)_y$$

$$(visc)_{i,j+\frac{1}{2}} = \frac{1}{\varphi_{i,j+\frac{1}{2}}} \left(\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}} C_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}} C_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{\varphi_{i,j+1}D_{i,j+1} - \varphi_{i,j}D_{i,j}}{\Delta y} \right)$$

$$(visc)_{i,j+\frac{1}{2}} = \frac{1}{\varphi_{i,j+\frac{1}{2}}} \left(\frac{1}{\Delta x} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} C_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}} C_{i-\frac{1}{2},j+\frac{1}{2}} \right) + \frac{1}{\Delta y} \left(\varphi_{i,j+1}D_{i,j+1} - \varphi_{i,j}D_{i,j} \right) \right)$$

$$(visc)_{i,j+\frac{1}{2}} = \frac{1}{\varphi_{i,j+\frac{1}{2}}} \left[\frac{1}{\Delta x} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} \left(\mu_{g,eff_{i+\frac{1}{2},j+\frac{1}{2}}} \left(\frac{u^k_{g_{i+\frac{1}{2},j+1}} - u^k_{g_{i+\frac{1}{2},j}}}{\Delta y} + \frac{v^{k+1}_{g_{i+1,j+\frac{1}{2}}} - v^{k+1}_{g_{i,j+\frac{1}{2}}}}{\Delta x} \right) \right) - \varphi_{i-\frac{1}{2},j+\frac{1}{2}} \left(\mu_{g,eff_{i-\frac{1}{2},j+\frac{1}{2}}} \left(\frac{u^k_{g_{i-\frac{1}{2},j+1}} - u^k_{g_{i-\frac{1}{2},j}}}{\Delta y} + \right. \right. \right. \right.$$

$$\left. \left. \left. \left. \frac{v^{k+1}_{g_{i,j+\frac{1}{2}}} - v^{k+1}_{g_{i-1,j+\frac{1}{2}}}}{\Delta x} \right) \right) \right) + \frac{1}{\Delta y} \left(\varphi_{i,j+1} \left(\left(2\mu_{g,eff_{i,j+1}} + \kappa_{g,eff_{i,j+1}} \right) \left(\frac{v^{k+1}_{g_{i,j+\frac{3}{2}}} - v^{k+1}_{g_{i,j+\frac{1}{2}}}}{\Delta y} \right) + \kappa_{g,eff_{i,j+1}} \left(\frac{u^k_{g_{i+\frac{1}{2},j+1}} - u^k_{g_{i-\frac{1}{2},j+1}}}{\Delta x} \right) \right) - \right.$$

$$\left. \varphi_{i,j} \left(\left(2\mu_{g,eff_{i,j}} + \kappa_{g,eff_{i,j}} \right) \left(\frac{v^{k+1}_{g_{i,j+\frac{1}{2}}} - v^{k+1}_{g_{i,j-\frac{1}{2}}}}{\Delta y} \right) + \kappa_{g,eff_{i,j}} \left(\frac{u^k_{g_{i+\frac{1}{2},j}} - u^k_{g_{i-\frac{1}{2},j}}}{\Delta x} \right) \right) \right) \right]$$

Y-momentum, Gas Equation

$$\frac{1}{\Delta t}\left(n_{i,j+\frac{1}{2}}^{k+1}\, v_{g_{i,j+\frac{1}{2}}}^{k+1} - n_{i,j+\frac{1}{2}}^{k}\, v_{g_{i,j+\frac{1}{2}}}^{k}\right) + \frac{1}{\varphi_{i,j+\frac{1}{2}}\,\Delta y}\left(\varphi_{i,j+1}\, n_{i,j+1}^{k+1}\, \frac{v_{g_{i,j+\frac{1}{2}}}^{k} + v_{g_{i,j+\frac{3}{2}}}^{k}}{2}\, \frac{v_{g_{i,j+\frac{1}{2}}}^{k+1} + v_{g_{i,j+\frac{3}{2}}}^{k+1}}{2} - \varphi_{i,j}\, n_{i,j}^{k+1}\, \frac{v_{g_{i,j-\frac{1}{2}}}^{k} + v_{g_{i,j+\frac{1}{2}}}^{k}}{2}\, \frac{v_{g_{i,j-\frac{1}{2}}}^{k+1} + v_{g_{i,j+\frac{1}{2}}}^{k+1}}{2}\right) +$$

$$\frac{1}{\varphi_{i,j+\frac{1}{2}}\,\Delta x}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\, n_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}\, \frac{u_{g_{i+\frac{1}{2},j}}^{k} + u_{g_{i+\frac{1}{2},j+1}}^{k}}{2}\, \frac{v_{g_{i,j+\frac{1}{2}}}^{k+1} + v_{g_{i+1,j+\frac{1}{2}}}^{k+1}}{2} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\, n_{i-\frac{1}{2},j+\frac{1}{2}}^{k+1}\, \frac{u_{g_{i-\frac{1}{2},j}}^{k} + u_{g_{i-\frac{1}{2},j+1}}^{k}}{2}\, \frac{v_{g_{i-1,j+\frac{1}{2}}}^{k+1} + v_{g_{i,j+\frac{1}{2}}}^{k+1}}{2}\right) +$$

$$s_{g_{i,j+\frac{1}{2}}}^{k+\frac{1}{2}}\, \frac{1}{\Delta y}\left(P_{w_{i,j+1}}^{k+1} - P_{w_{i,j}}^{k+1}\right) = -s_{g_{i,j+\frac{1}{2}}}^{k+\frac{1}{2}}\, \frac{1}{\Delta y}\left(P_{c_{i,j+1}}^{k+1} - P_{c_{i,j}}^{k+1}\right) - \hat{k}_{g_{i,j+\frac{1}{2}}}^{k+\frac{1}{2}}\, v_{g_{i,j+\frac{1}{2}}}^{k+1} + \hat{k}_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}\left(v_{w_{i,j+\frac{1}{2}}}^{k+1} - v_{g_{i,j+\frac{1}{2}}}^{k+1}\right) + n_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}\, g +$$

$$\color{red}{\frac{1}{\varphi_{i,j+\frac{1}{2}}}\left[\frac{1}{\Delta x}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\left(\mu_{g,eff_{i+\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{g_{i+\frac{1}{2},j+1}}^{k} - u_{g_{i+\frac{1}{2},j}}^{k}}{\Delta y} + \frac{v_{g_{i+1,j+\frac{1}{2}}}^{k+1} - v_{g_{i,j+\frac{1}{2}}}^{k+1}}{\Delta x}\right)\right) - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\left(\mu_{g,eff_{i-\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{g_{i-\frac{1}{2},j+1}}^{k} - u_{g_{i-\frac{1}{2},j}}^{k}}{\Delta y} + \right.\right.\right.}$$

$$\color{red}{\left.\left.\left.\frac{v_{g_{i,j+\frac{1}{2}}}^{k+1} - v_{g_{i-1,j+\frac{1}{2}}}^{k+1}}{\Delta x}\right)\right)\right) + \frac{1}{\Delta y}\left(\varphi_{i,j+1}\left(\left(2\mu_{g,eff_{i,j+1}} + \kappa_{g,eff_{i,j+1}}\right)\left(\frac{v_{g_{i,j+\frac{3}{2}}}^{k+1} - v_{g_{i,j+\frac{1}{2}}}^{k+1}}{\Delta y}\right) + \kappa_{g,eff_{i,j+1}}\left(\frac{u_{g_{i+\frac{1}{2},j+1}}^{k} - u_{g_{i-\frac{1}{2},j+1}}^{k}}{\Delta x}\right)\right) - \right.}$$

$$\color{red}{\left.\varphi_{i,j}\left(\left(2\mu_{g,eff_{i,j}} + \kappa_{g,eff_{i,j}}\right)\left(\frac{v_{g_{i,j+\frac{1}{2}}}^{k+1} - v_{g_{i,j-\frac{1}{2}}}^{k+1}}{\Delta y}\right) + \kappa_{g,eff_{i,j}}\left(\frac{u_{g_{i+\frac{1}{2},j}}^{k} - u_{g_{i-\frac{1}{2},j}}^{k}}{\Delta x}\right)\right)\right)\right]}$$

The last term discretization: **(Y-momentum, Water)**

$$\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\mu_{w,eff}\left(\frac{\partial u_w}{\partial y}+\frac{\partial v_w}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\varphi\left(\left(2\mu_{w,eff}+\kappa_{w,eff}\right)\frac{\partial v_w}{\partial y}+\kappa_{w,eff}\frac{\partial u_w}{\partial x}\right)\right)\right]\qquad Let\ 2\mu_{w,eff}+\kappa_{w,eff}=\alpha$$

$$=\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\mu_{w,eff}\left(\frac{\partial u_w}{\partial y}+\frac{\partial v_w}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\varphi\left(\alpha\frac{\partial v_w}{\partial y}+\kappa_{w,eff}\frac{\partial u_w}{\partial x}\right)\right)\right]$$

$$C_{i+\frac{1}{2},j+\frac{1}{2}}=\mu_{w,eff\,i+\frac{1}{2},j+\frac{1}{2}}\left(\frac{u_{w\,i+\frac{1}{2},j+1}^{k}-u_{w\,i+\frac{1}{2},j}^{k}}{\Delta y}+\frac{v_{w\,i+1,j+\frac{1}{2}}^{k+1}-v_{w\,i,j+\frac{1}{2}}^{k+1}}{\Delta x}\right)\qquad C_{i-\frac{1}{2},j+\frac{1}{2}}=\mu_{w,eff\,i-\frac{1}{2},j+\frac{1}{2}}\left(\frac{u_{w\,i-\frac{1}{2},j+1}^{k}-u_{w\,i-\frac{1}{2},j}^{k}}{\Delta y}+\frac{v_{w\,i,j+\frac{1}{2}}^{k+1}-v_{w\,i-1,j+\frac{1}{2}}^{k+1}}{\Delta x}\right)$$

$$(visc)_x=\frac{1}{\varphi_{i,j+\frac{1}{2}}}\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}C_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}C_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}$$

$$D_{i,j}=\alpha_{i,j}\frac{v_{w\,i,j+\frac{1}{2}}^{k+1}-v_{w\,i,j-\frac{1}{2}}^{k+1}}{\Delta y}+\kappa_{w,eff\,i,j}\frac{u_{w\,i+\frac{1}{2},j}^{k}-u_{w\,i-\frac{1}{2},j}^{k}}{\Delta x}\qquad D_{i,j+1}=\alpha_{i,j+1}\frac{v_{w\,i,j+\frac{3}{2}}^{k+1}-v_{w\,i,j+\frac{1}{2}}^{k+1}}{\Delta y}+\kappa_{w,eff\,i,j+1}\frac{u_{w\,i+\frac{1}{2},j+1}^{k}-u_{w\,i-\frac{1}{2},j+1}^{k}}{\Delta x}$$

$$(visc)_y=\frac{1}{\varphi_{i,j+\frac{1}{2}}}\frac{\varphi_{i,j+1}D_{i,j+1}-\varphi_{i,j}D_{i,j}}{\Delta y}$$

$$(visc)_{i,j+\frac{1}{2}}=(visc)_x+(visc)_y\qquad\qquad (visc)_{i,j+\frac{1}{2}}=\frac{1}{\varphi_{i,j+\frac{1}{2}}}\left(\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}C_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}C_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}+\frac{\varphi_{i,j+1}D_{i,j+1}-\varphi_{i,j}D_{i,j}}{\Delta y}\right)$$

$$(visc)_{i,j+\frac{1}{2}}=\frac{1}{\varphi_{i,j+\frac{1}{2}}}\left(\frac{1}{\Delta x}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}C_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i-\frac{1}{2},j+\frac{1}{2}}C_{i-\frac{1}{2},j+\frac{1}{2}}\right)+\frac{1}{\Delta y}\left(\varphi_{i,j+1}D_{i,j+1}-\varphi_{i,j}D_{i,j}\right)\right)$$

$$
\begin{aligned}
(visc)_{i,j+\frac{1}{2}} = \frac{1}{\varphi_{i,j+\frac{1}{2}}} &\left[\frac{1}{\Delta x} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} \left(\mu_{w,eff_{i+\frac{1}{2},j+\frac{1}{2}}} \left(\frac{u^k_{w_{i+\frac{1}{2},j+1}} - u^k_{w_{i+\frac{1}{2},j}}}{\Delta y} + \frac{v^{k+1}_{w_{i+1,j+\frac{1}{2}}} - v^{k+1}_{w_{i,j+\frac{1}{2}}}}{\Delta x} \right) \right) - \varphi_{i-\frac{1}{2},j+\frac{1}{2}} \left(\mu_{w,eff_{i-\frac{1}{2},j+\frac{1}{2}}} \left(\frac{u^k_{w_{i-\frac{1}{2},j+1}} - u^k_{w_{i-\frac{1}{2},j}}}{\Delta y} + \right. \right. \right.\right. \\
&\left. \left. \left. \left. \frac{v^{k+1}_{w_{i,j+\frac{1}{2}}} - v^{k+1}_{w_{i-1,j+\frac{1}{2}}}}{\Delta x} \right) \right) \right) + \frac{1}{\Delta y} \left(\varphi_{i,j+1} \left(\left(2\mu_{w,eff_{i,j+1}} + \kappa_{w,eff_{i,j+1}} \right) \left(\frac{v^{k+1}_{w_{i,j+\frac{3}{2}}} - v^{k+1}_{w_{i,j+\frac{1}{2}}}}{\Delta y} \right) + \kappa_{w,eff_{i,j+1}} \left(\frac{u^k_{w_{i+\frac{1}{2},j+1}} - u^k_{w_{i-\frac{1}{2},j+1}}}{\Delta x} \right) \right) - \right.\right. \\
&\left. \left. \varphi_{i,j} \left(\left(2\mu_{w,eff_{i,j}} + \kappa_{w,eff_{i,j}} \right) \left(\frac{v^{k+1}_{w_{i,j+\frac{1}{2}}} - v^{k+1}_{w_{i,j-\frac{1}{2}}}}{\Delta y} \right) + \kappa_{w,eff_{i,j}} \left(\frac{u^k_{w_{i+\frac{1}{2},j}} - u^k_{w_{i-\frac{1}{2},j}}}{\Delta x} \right) \right) \right) \right]
\end{aligned}
$$

Y-Momentum, Water Equation

$$\frac{1}{\Delta t}\left(m_{i,j+\frac{1}{2}}^{k+1} v_{w_{i,j+\frac{1}{2}}}^{k+1} - m_{i,j+\frac{1}{2}}^{k} v_{w_{i,j+\frac{1}{2}}}^{k}\right) + \frac{1}{\varphi_{i,j+\frac{1}{2}}\Delta y}\left(\varphi_{i,j+1}\, m_{i,j+1}^{k+1}\,\frac{v_{w_{i,j+\frac{1}{2}}}^{k} + v_{w_{i,j+\frac{3}{2}}}^{k}}{2}\,\frac{v_{w_{i,j+\frac{1}{2}}}^{k+1} + v_{w_{i,j+\frac{3}{2}}}^{k+1}}{2} - \varphi_{i,j}\, m_{i,j}^{k+1}\,\frac{v_{w_{i,j-\frac{1}{2}}}^{k} + v_{w_{i,j+\frac{1}{2}}}^{k}}{2}\,\frac{v_{w_{i,j-\frac{1}{2}}}^{k+1} + v_{w_{i,j+\frac{1}{2}}}^{k+1}}{2}\right) +$$

$$\frac{1}{\varphi_{i,j+\frac{1}{2}}\Delta x}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\, m_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}\,\frac{u_{w_{i+\frac{1}{2},j}}^{k} + u_{w_{i+\frac{1}{2},j+1}}^{k}}{2}\,\frac{v_{w_{i,j+\frac{1}{2}}}^{k+1} + v_{w_{i+1,j+\frac{1}{2}}}^{k+1}}{2} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\, m_{i-\frac{1}{2},j+\frac{1}{2}}^{k+1}\,\frac{u_{w_{i-\frac{1}{2},j}}^{k} + u_{w_{i-\frac{1}{2},j+1}}^{k}}{2}\,\frac{v_{w_{i-1,j+\frac{1}{2}}}^{k+1} + v_{w_{i,j+\frac{1}{2}}}^{k+1}}{2}\right) +$$

$$s_{w_{i,j+\frac{1}{2}}}^{k+\frac{1}{2}}\,\frac{1}{\Delta y}\left(P_{w_{i,j+1}}^{k+1} - P_{w_{i,j}}^{k+1}\right) = -\hat{k}_{w_{i,j+\frac{1}{2}}}^{k+\frac{1}{2}} v_{w_{i,j+\frac{1}{2}}}^{k+1} - \hat{k}_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}\left(v_{w_{i,j+\frac{1}{2}}}^{k+1} - v_{g_{i,j+\frac{1}{2}}}^{k+1}\right) + m_{i,j+\frac{1}{2}}^{k+1}\, g +$$

$$\frac{1}{\varphi_{i,j+\frac{1}{2}}}\left[\frac{1}{\Delta x}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\left(\mu_{w,eff_{i+\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{w_{i+\frac{1}{2},j+1}}^{k} - u_{w_{i+\frac{1}{2},j}}^{k}}{\Delta y} + \frac{v_{w_{i+1,j+\frac{1}{2}}}^{k+1} - v_{w_{i,j+\frac{1}{2}}}^{k+1}}{\Delta x}\right)\right) - \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\left(\mu_{w,eff_{i-\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{w_{i-\frac{1}{2},j+1}}^{k} - u_{w_{i-\frac{1}{2},j}}^{k}}{\Delta y} + \right.\right.\right.$$

$$\left.\left.\left.\frac{v_{w_{i,j+\frac{1}{2}}}^{k+1} - v_{w_{i-1,j+\frac{1}{2}}}^{k+1}}{\Delta x}\right)\right)\right) + \frac{1}{\Delta y}\left(\varphi_{i,j+1}\left(\left(2\mu_{w,eff_{i,j+1}} + \kappa_{w,eff_{i,j+1}}\right)\left(\frac{v_{w_{i,j+\frac{3}{2}}}^{k+1} - v_{w_{i,j+\frac{1}{2}}}^{k+1}}{\Delta y}\right) + \kappa_{w,eff_{i,j+1}}\left(\frac{u_{w_{i+\frac{1}{2},j+1}}^{k} - u_{w_{i-\frac{1}{2},j+1}}^{k}}{\Delta x}\right)\right)\right) -$$

$$\varphi_{i,j}\left(\left(2\mu_{w,eff_{i,j}} + \kappa_{w,eff_{i,j}}\right)\left(\frac{v_{w_{i,j+\frac{1}{2}}}^{k+1} - v_{w_{i,j-\frac{1}{2}}}^{k+1}}{\Delta y}\right) + \kappa_{w,eff_{i,j}}\left(\frac{u_{w_{i+\frac{1}{2},j}}^{k} - u_{w_{i-\frac{1}{2},j}}^{k}}{\Delta x}\right)\right)\right)\right]$$

The last term discretization: **(X-momentum, Gas)**

$$\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\left((2\mu_{g,eff}+\kappa_{g,eff})\frac{\partial u_g}{\partial x}+\kappa_{g,eff}\frac{\partial v_g}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\varphi\mu_{g,eff}\left(\frac{\partial u_g}{\partial y}+\frac{\partial v_g}{\partial x}\right)\right)\right]$$

Let $\quad 2\mu_{g,eff}+\kappa_{g,eff}=\alpha$

$$=\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\left(\alpha\frac{\partial u_g}{\partial x}+\kappa_{g,eff}\frac{\partial v_g}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\varphi\mu_{g,eff}\left(\frac{\partial u_g}{\partial y}+\frac{\partial v_g}{\partial x}\right)\right)\right]$$

$$A_{i,j}=\alpha_{i,j}\frac{u_{g_{i+\frac{1}{2},j}}^{k+1}-u_{g_{i-\frac{1}{2},j}}^{k+1}}{\Delta x}+\kappa_{g,eff_{i,j}}\frac{v_{g_{i,j+\frac{1}{2}}}^{k}-v_{g_{i,j-\frac{1}{2}}}^{k}}{\Delta y}$$

$$A_{i+1,j}=\alpha_{i+1,j}\frac{u_{g_{i+\frac{3}{2},j}}^{k+1}-u_{g_{i+\frac{1}{2},j}}^{k+1}}{\Delta x}+\kappa_{g,eff_{i+1,j}}\frac{v_{g_{i+1,j+\frac{1}{2}}}^{k}-v_{g_{i+1,j-\frac{1}{2}}}^{k}}{\Delta y}$$

$$(visc)_x=\frac{1}{\varphi_{i+\frac{1}{2},j}}\frac{\varphi_{i+1,j}A_{i+1,j}-\varphi_{i,j}A_{i,j}}{\Delta x}$$

$$B_{i+\frac{1}{2},j+\frac{1}{2}}=\mu_{g,eff_{i+\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{g_{i+\frac{1}{2},j+1}}^{k+1}-u_{g_{i+\frac{1}{2},j}}^{k+1}}{\Delta y}+\frac{v_{g_{i+1,j+\frac{1}{2}}}^{k}-v_{g_{i,j+\frac{1}{2}}}^{k}}{\Delta x}\right)$$

$$B_{i+\frac{1}{2},j-\frac{1}{2}}=\mu_{g,eff_{i+\frac{1}{2},j-\frac{1}{2}}}\left(\frac{u_{g_{i+\frac{1}{2},j}}^{k+1}-u_{g_{i+\frac{1}{2},j-1}}^{k+1}}{\Delta y}+\frac{v_{g_{i+1,j-\frac{1}{2}}}^{k}-v_{g_{i,j-\frac{1}{2}}}^{k}}{\Delta x}\right)$$

$$(visc)_y=\frac{1}{\varphi_{i+\frac{1}{2},j}}\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}B_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}B_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}$$

$$(visc)_{i+\frac{1}{2},j}=(visc)_x+(visc)_y$$

$$(visc)_{i+\frac{1}{2},j}=\frac{1}{\varphi_{i+\frac{1}{2},j}}\left(\frac{\varphi_{i+1,j}A_{i+1,j}-\varphi_{i,j}A_{i,j}}{\Delta x}+\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}B_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}B_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}\right)$$

$$(visc)_{i+\frac{1}{2},j} = \frac{1}{\varphi_{i+\frac{1}{2},j}} \left(\frac{1}{\Delta x} \left(\varphi_{i+1,j}\, A_{i+1,j} - \varphi_{i,j}\, A_{i,j} \right) + \frac{1}{\Delta y} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\, B_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i+\frac{1}{2},j-\frac{1}{2}}\, B_{i+\frac{1}{2},j-\frac{1}{2}} \right) \right)$$

$$(visc)_{i+\frac{1}{2},j} = \frac{1}{\varphi_{i+\frac{1}{2},j}} \left[\frac{1}{\Delta x} \left(\varphi_{i+1,j} \left(\left(2\mu_{g,eff_{i+1,j}} + \kappa_{g,eff_{i+1,j}} \right) \left(\frac{u_{g_{i+\frac{3}{2},j}}^{k+1} - u_{g_{i+\frac{1}{2},j}}^{k+1}}{\Delta x} \right) + \kappa_{g,eff_{i+1,j}} \left(\frac{v_{g_{i+1,j+\frac{1}{2}}}^{k} - v_{g_{i+1,j-\frac{1}{2}}}^{k}}{\Delta y} \right) \right) \right) - $$

$$\varphi_{i,j} \left(\left(2\mu_{g,eff_{i,j}} + \kappa_{g,eff_{i,j}} \right) \left(\frac{u_{g_{i+\frac{1}{2},j}}^{k+1} - u_{g_{i-\frac{1}{2},j}}^{k+1}}{\Delta x} \right) + \kappa_{g,eff_{i,j}} \left(\frac{v_{g_{i,j+\frac{1}{2}}}^{k} - v_{g_{i,j-\frac{1}{2}}}^{k}}{\Delta y} \right) \right) + \frac{1}{\Delta y} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} \left(\mu_{g,eff_{i+\frac{1}{2},j+\frac{1}{2}}} \left(\frac{u_{g_{i+\frac{1}{2},j+1}}^{k+1} - u_{g_{i+\frac{1}{2},j}}^{k+1}}{\Delta y} + \right. \right. \right.$$

$$\left. \frac{v_{g_{i+1,j+\frac{1}{2}}}^{k} - v_{g_{i,j+\frac{1}{2}}}^{k}}{\Delta x} \right) \right) - \varphi_{i+\frac{1}{2},j-\frac{1}{2}} \left(\mu_{g,eff_{i+\frac{1}{2},j-\frac{1}{2}}} \left(\frac{u_{g_{i+\frac{1}{2},j}}^{k+1} - u_{g_{i+\frac{1}{2},j-1}}^{k+1}}{\Delta y} + \frac{v_{g_{i+1,j-\frac{1}{2}}}^{k} - v_{g_{i,j-\frac{1}{2}}}^{k}}{\Delta x} \right) \right) \right) \right]$$

X-momentum, Gas Equation

$$\frac{1}{\Delta t}\left(n_{i+\frac{1}{2},j}^{k+1}\,u_{g_{i+\frac{1}{2},j}}^{k+1} - n_{i+\frac{1}{2},j}^{k}\,u_{g_{i+\frac{1}{2},j}}^{k}\right) + \frac{1}{\varphi_{i+\frac{1}{2},j}\,\Delta x}\left(\varphi_{i+1,j}\,n_{i+1,j}^{k+1}\,\frac{u_{g_{i+\frac{1}{2},j}}^{k}+u_{g_{i+\frac{3}{2},j}}^{k}}{2}\,\frac{u_{g_{i+\frac{1}{2},j}}^{k+1}+u_{g_{i+\frac{3}{2},j}}^{k+1}}{2} - \varphi_{i,j}\,n_{i,j}^{k+1}\,\frac{u_{g_{i-\frac{1}{2},j}}^{k}+u_{g_{i+\frac{1}{2},j}}^{k}}{2}\,\frac{u_{g_{i-\frac{1}{2},j}}^{k+1}+u_{g_{i+\frac{1}{2},j}}^{k+1}}{2}\right) +$$

$$\frac{1}{\varphi_{i+\frac{1}{2},j}\,\Delta y}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\,n_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}\,\frac{v_{g_{i,j+\frac{1}{2}}}^{k}+v_{g_{i+1,j+\frac{1}{2}}}^{k}}{2}\,\frac{u_{g_{i+\frac{1}{2},j}}^{k+1}+u_{g_{i+\frac{1}{2},j+1}}^{k+1}}{2} - \varphi_{i+\frac{1}{2},j-\frac{1}{2}}\,n_{i+\frac{1}{2},j-\frac{1}{2}}^{k+1}\,\frac{v_{g_{i,j-\frac{1}{2}}}^{k}+v_{g_{i+1,j-\frac{1}{2}}}^{k}}{2}\,\frac{u_{g_{i+\frac{1}{2},j-1}}^{k+1}+u_{g_{i+\frac{1}{2},j}}^{k+1}}{2}\right) +$$

$$s_{g_{i+\frac{1}{2},j}}^{k+\frac{1}{2}}\,\frac{1}{\Delta x}\left(P_{w_{i+1,j}}^{k+1} - P_{w_{i,j}}^{k+1}\right) = -s_{g_{i+\frac{1}{2},j}}^{k+\frac{1}{2}}\,\frac{1}{\Delta x}\left(P_{c_{i+1,j}}^{k+1} - P_{c_{i,j}}^{k+1}\right) - \hat{k}_{g_{i+\frac{1}{2},j}}^{k+\frac{1}{2}}\,u_{g_{i+\frac{1}{2},j}}^{k+1} + \hat{k}_{i+\frac{1}{2},j}^{k+\frac{1}{2}}\left(u_{w_{i+\frac{1}{2},j}}^{k+1} - u_{g_{i+\frac{1}{2},j}}^{k+1}\right) + {\color{red}\frac{1}{\varphi_{i+\frac{1}{2},j}}\left[\frac{1}{\Delta x}\left(\varphi_{i+1,j}\left(\left(2\mu_{g,eff_{i+1,j}} + \right.\right.\right.}$$

$${\color{red}\kappa_{g,eff_{i+1,j}}\right)\left(\frac{u_{g_{i+\frac{3}{2},j}}^{k+1}-u_{g_{i+\frac{1}{2},j}}^{k+1}}{\Delta x}\right) + \kappa_{g,eff_{i+1,j}}\left(\frac{v_{g_{i+1,j+\frac{1}{2}}}^{k}-v_{g_{i+1,j-\frac{1}{2}}}^{k}}{\Delta y}\right)\right) - \varphi_{i,j}\left(\left(2\mu_{g,eff_{i,j}} + \kappa_{g,eff_{i,j}}\right)\left(\frac{u_{g_{i+\frac{1}{2},j}}^{k+1}-u_{g_{i-\frac{1}{2},j}}^{k+1}}{\Delta x}\right) + }$$

$${\color{red}\kappa_{g,eff_{i,j}}\left(\frac{v_{g_{i,j+\frac{1}{2}}}^{k}-v_{g_{i,j-\frac{1}{2}}}^{k}}{\Delta y}\right)\right)\right) + \frac{1}{\Delta y}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\left(\mu_{g,eff_{i+\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{g_{i+\frac{1}{2},j+1}}^{k+1}-u_{g_{i+\frac{1}{2},j}}^{k+1}}{\Delta y} + \frac{v_{g_{i+1,j+\frac{1}{2}}}^{k}-v_{g_{i,j+\frac{1}{2}}}^{k}}{\Delta x}\right)\right) - }$$

$${\color{red}\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\left(\mu_{g,eff_{i+\frac{1}{2},j-\frac{1}{2}}}\left(\frac{u_{g_{i+\frac{1}{2},j}}^{k+1}-u_{g_{i+\frac{1}{2},j-1}}^{k+1}}{\Delta y} + \frac{v_{g_{i+1,j-\frac{1}{2}}}^{k}-v_{g_{i,j-\frac{1}{2}}}^{k}}{\Delta x}\right)\right)\right)\right]}$$

The last term discretization: **(Water, X-momentum)**

$$\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\left(\left(2\mu_{w,eff}+\kappa_{w,eff}\right)\frac{\partial u_w}{\partial x}+\kappa_{w,eff}\frac{\partial v_w}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\varphi\mu_{w,eff}\left(\frac{\partial u_w}{\partial y}+\frac{\partial v_w}{\partial x}\right)\right)\right]$$

Let $2\mu_{w,eff}+\kappa_{w,eff}=\alpha$

$$=\frac{1}{\varphi}\left[\frac{\partial}{\partial x}\left(\varphi\left(\alpha\frac{\partial u_w}{\partial x}+\kappa_{w,eff}\frac{\partial v_w}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(\varphi\mu_{w,eff}\left(\frac{\partial u_w}{\partial y}+\frac{\partial v_w}{\partial x}\right)\right)\right]$$

$$A_{i,j}=\alpha_{i,j}\frac{u_{w\,i+\frac{1}{2},j}^{k+1}-u_{w\,i-\frac{1}{2},j}^{k+1}}{\Delta x}+\kappa_{w,eff\,i,j}\frac{v_{w\,i,j+\frac{1}{2}}^{k}-v_{w\,i,j-\frac{1}{2}}^{k}}{\Delta y}$$

$$A_{i+1,j}=\alpha_{i+1,j}\frac{u_{w\,i+\frac{3}{2},j}^{k+1}-u_{w\,i+\frac{1}{2},j}^{k+1}}{\Delta x}+\kappa_{w,eff\,i+1,j}\frac{v_{w\,i+1,j+\frac{1}{2}}^{k}-v_{w\,i+1,j-\frac{1}{2}}^{k}}{\Delta y}$$

$$(visc)_x=\frac{1}{\varphi_{i+\frac{1}{2},j}}\frac{\varphi_{i+1,j}\,A_{i+1,j}-\varphi_{i,j}\,A_{i,j}}{\Delta x}$$

$$B_{i+\frac{1}{2},j+\frac{1}{2}}=\mu_{w,eff\,i+\frac{1}{2},j+\frac{1}{2}}\left(\frac{u_{w\,i+\frac{1}{2},j+1}^{k+1}-u_{w\,i+\frac{1}{2},j}^{k+1}}{\Delta y}+\frac{v_{w\,i+1,j+\frac{1}{2}}^{k}-v_{w\,i,j+\frac{1}{2}}^{k}}{\Delta x}\right)$$

$$B_{i+\frac{1}{2},j-\frac{1}{2}}=\mu_{w,eff\,i+\frac{1}{2},j-\frac{1}{2}}\left(\frac{u_{w\,i+\frac{1}{2},j}^{k+1}-u_{w\,i+\frac{1}{2},j-1}^{k+1}}{\Delta y}+\frac{v_{w\,i+1,j-\frac{1}{2}}^{k}-v_{w\,i,j-\frac{1}{2}}^{k}}{\Delta x}\right)$$

$$(visc)_y=\frac{1}{\varphi_{i+\frac{1}{2},j}}\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\,B_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\,B_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}$$

$$(visc)_{i+\frac{1}{2},j}=(visc)_x+(visc)_y$$

$$(visc)_{i+\frac{1}{2},j}=\frac{1}{\varphi_{i+\frac{1}{2},j}}\left(\frac{\varphi_{i+1,j}\,A_{i+1,j}-\varphi_{i,j}\,A_{i,j}}{\Delta x}+\frac{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\,B_{i+\frac{1}{2},j+\frac{1}{2}}-\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\,B_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}\right)$$

$$(visc)_{i+\frac{1}{2},j} = \frac{1}{\varphi_{i+\frac{1}{2},j}} \left(\frac{1}{\Delta x} \left(\varphi_{i+1,j} \, A_{i+1,j} - \varphi_{i,j} \, A_{i,j} \right) + \frac{1}{\Delta y} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} \, B_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i+\frac{1}{2},j-\frac{1}{2}} \, B_{i+\frac{1}{2},j-\frac{1}{2}} \right) \right)$$

$$(visc)_{i+\frac{1}{2},j} = \frac{1}{\varphi_{i+\frac{1}{2},j}} \left[\frac{1}{\Delta x} \left(\varphi_{i+1,j} \left(\left(2\mu_{w,eff_{i+1,j}} + \kappa_{w,eff_{i+1,j}} \right) \left(\frac{u_{w_{i+\frac{3}{2},j}}^{k+1} - u_{w_{i+\frac{1}{2},j}}^{k+1}}{\Delta x} \right) + \kappa_{w,eff_{i+1,j}} \left(\frac{v_{w_{i+1,j+\frac{1}{2}}}^{k} - v_{w_{i+1,j-\frac{1}{2}}}^{k}}{\Delta y} \right) \right) \right. -$$

$$\varphi_{i,j} \left(\left(2\mu_{w,eff_{i,j}} + \kappa_{w,eff_{i,j}} \right) \left(\frac{u_{w_{i+\frac{1}{2},j}}^{k+1} - u_{w_{i-\frac{1}{2},j}}^{k+1}}{\Delta x} \right) + \kappa_{w,eff_{i,j}} \left(\frac{v_{w_{i,j+\frac{1}{2}}}^{k} - v_{w_{i,j-\frac{1}{2}}}^{k}}{\Delta y} \right) \right) \right) + \frac{1}{\Delta y} \left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} \left(\mu_{w,eff_{i+\frac{1}{2},j+\frac{1}{2}}} \left(\frac{u_{w_{i+\frac{1}{2},j+1}}^{k+1} - u_{w_{i+\frac{1}{2},j}}^{k+1}}{\Delta y} + \right. \right. \right.$$

$$\left. \frac{v_{w_{i+1,j+\frac{1}{2}}}^{k} - v_{w_{i,j+\frac{1}{2}}}^{k}}{\Delta x} \right) \right) - \varphi_{i+\frac{1}{2},j-\frac{1}{2}} \left(\mu_{w,eff_{i+\frac{1}{2},j-\frac{1}{2}}} \left(\frac{u_{w_{i+\frac{1}{2},j}}^{k+1} - u_{w_{i+\frac{1}{2},j-1}}^{k+1}}{\Delta y} + \frac{v_{w_{i+1,j-\frac{1}{2}}}^{k} - v_{w_{i,j-\frac{1}{2}}}^{k}}{\Delta x} \right) \right) \right) \right]$$

X-momentum, Water Equation

$$\frac{1}{\Delta t}\left(m_{i+\frac{1}{2},j}^{k+1} u_{w_{i+\frac{1}{2},j}}^{k+1} - m_{i+\frac{1}{2},j}^{k} u_{w_{i+\frac{1}{2},j}}^{k}\right) + \frac{1}{\varphi_{i+\frac{1}{2},j}\Delta x}\left(\varphi_{i+1,j}\, m_{i+1,j}^{k+1}\, \frac{u_{w_{i+\frac{1}{2},j}}^{k} + u_{w_{i+\frac{3}{2},j}}^{k}}{2}\, \frac{u_{w_{i+\frac{1}{2},j}}^{k+1} + u_{w_{i+\frac{3}{2},j}}^{k+1}}{2} - \right.$$

$$\left. \varphi_{i,j}\, m_{i,j}^{k+1}\, \frac{u_{w_{i-\frac{1}{2},j}}^{k} + u_{w_{i+\frac{1}{2},j}}^{k}}{2}\, \frac{u_{w_{i-\frac{1}{2},j}}^{k+1} + u_{w_{i+\frac{1}{2},j}}^{k+1}}{2}\right) + \frac{1}{\varphi_{i+\frac{1}{2},j}\Delta y}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\, m_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}\, \frac{v_{w_{i,j+\frac{1}{2}}}^{k} + v_{w_{i+1,j+\frac{1}{2}}}^{k}}{2}\, \frac{u_{w_{i+\frac{1}{2},j}}^{k+1} + u_{w_{i+\frac{1}{2},j+1}}^{k+1}}{2} - \right.$$

$$\left. \varphi_{i+\frac{1}{2},j-\frac{1}{2}}\, m_{i+\frac{1}{2},j-\frac{1}{2}}^{k+1}\, \frac{v_{w_{i,j-\frac{1}{2}}}^{k} + v_{w_{i+1,j-\frac{1}{2}}}^{k}}{2}\, \frac{u_{w_{i+\frac{1}{2},j-1}}^{k+1} + u_{w_{i+\frac{1}{2},j}}^{k+1}}{2}\right) + s_{w_{i+\frac{1}{2},j}}^{k+\frac{1}{2}}\, \frac{1}{\Delta x}\left(P_{w_{i+1,j}}^{k+1} - P_{w_{i,j}}^{k+1}\right) = -\hat{k}_{w_{i+\frac{1}{2},j}}^{k+\frac{1}{2}} u_{w_{i+\frac{1}{2},j}}^{k+1} - \hat{k}_{i+\frac{1}{2},j}^{k+\frac{1}{2}}\left(u_{w_{i+\frac{1}{2},j}}^{k+1} - u_{g_{i+\frac{1}{2},j}}^{k+1}\right) +$$

$$\frac{1}{\varphi_{i+\frac{1}{2},j}}\left[\frac{1}{\Delta x}\left(\varphi_{i+1,j}\left(\left(2\mu_{w,eff_{i+1,j}} + \kappa_{w,eff_{i+1,j}}\right)\left(\frac{u_{w_{i+\frac{3}{2},j}}^{k+1} - u_{w_{i+\frac{1}{2},j}}^{k+1}}{\Delta x}\right) + \kappa_{w,eff_{i+1,j}}\left(\frac{v_{w_{i+1,j+\frac{1}{2}}}^{k} - v_{w_{i+1,j-\frac{1}{2}}}^{k}}{\Delta y}\right)\right) - \varphi_{i,j}\left(\left(2\mu_{w,eff_{i,j}} + \right.\right.\right.$$

$$\left.\left.\kappa_{w,eff_{i,j}}\right)\left(\frac{u_{w_{i+\frac{1}{2},j}}^{k+1} - u_{w_{i-\frac{1}{2},j}}^{k+1}}{\Delta x}\right) + \kappa_{w,eff_{i,j}}\left(\frac{v_{w_{i,j+\frac{1}{2}}}^{k} - v_{w_{i,j-\frac{1}{2}}}^{k}}{\Delta y}\right)\right)\right) + \frac{1}{\Delta y}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\left(\mu_{w,eff_{i+\frac{1}{2},j+\frac{1}{2}}}\left(\frac{u_{w_{i+\frac{1}{2},j+1}}^{k+1} - u_{w_{i+\frac{1}{2},j}}^{k+1}}{\Delta y} + \frac{v_{w_{i+1,j+\frac{1}{2}}}^{k} - v_{w_{i,j+\frac{1}{2}}}^{k}}{\Delta x}\right)\right) - \right.$$

$$\left.\left.\varphi_{i+\frac{1}{2},j-\frac{1}{2}}\left(\mu_{w,eff_{i+\frac{1}{2},j-\frac{1}{2}}}\left(\frac{u_{w_{i+\frac{1}{2},j}}^{k+1} - u_{w_{i+\frac{1}{2},j-1}}^{k+1}}{\Delta y} + \frac{v_{w_{i+1,j-\frac{1}{2}}}^{k} - v_{w_{i,j-\frac{1}{2}}}^{k}}{\Delta x}\right)\right)\right)\right]$$