PROJECT REPORT ON
RECTANGULAR DUALIZATION
AND ITS APPLICATIONS

Under the guidance of

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ACKNOWLEDGEMENT

We are greatly indebted to Dr. Krishnendra Shekhawat for giving us this opportunity to work under him and for being a constant source of support and inspiration. His valuable suggestions were of immense help throughout our project work. This topic has a lot of ongoing research and development which helped us to gain knowledge in the area of rectangular dualization and its applications. Working under him was an extremely knowledgeable experience for us.

CONTENTS

1. INTRODUCTION

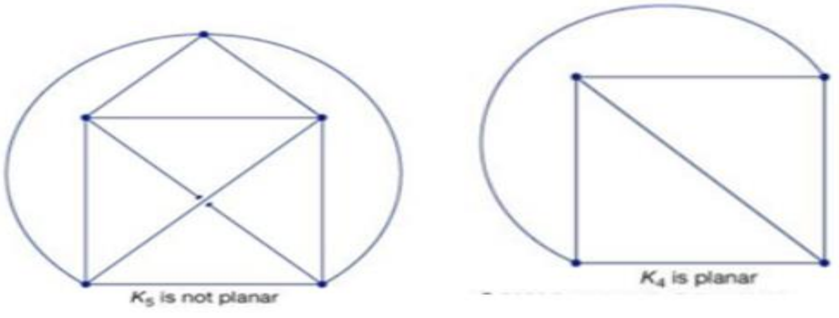
Rectangular Dualization is the computation of a rectangular dual of a planar graph. Rectangular Dualization was originally introduced to generate rectangular topologies for floor planning of integrated circuits [6]. Practical networks in the circuits includes hundreds and thousands of nodes and links. For humans, the visualization and working with such large structures become really difficult. Hence, one of the simplest ways to solve the problem could be to model the relationship between sites and links by a weighted undirected graph. The aim is to enhance the readability of the drawing, making it easy to find which nodes are connected by an edge. To reduce the complexity of visualizing circuits and to avoid technical drawbacks, the graph should not have any edge crossings. So, ***planar*** graphs are considered. The next step would be to focus on the proximity of vertices. For this problem, a concept of ***maximizing angular resolution*** was studied by the researchers. Maximum angular resolution is the smallest angle between adjacent edges, in such a way that lines representing connections are kept as separate as possible [6]. Orthogonal graph drawing solves the angular resolution by forcing all angles between adjacent edges to be $\pi/2$ [6]. This motivates the construction of Rectangular Duals of planar graphs.

Addition of networks to a large communication network would increase its complexity. Construction of ***Automated Best Connected Rectangular Duals*** could be considered to solve part of the problem. Automated Best Connected Rectangular Duals ensures that the Rectangular Dual of the planar graph remains best connected at each stage.

In our work, we first studied the concepts related to Rectangular Dual of a planar graph. We then started with construction of Automated Best Connected Rectangular Duals for planar graphs on ***n*** = 4,5,6,7,8,9. We then focussed on the problems that arose. The first problem was, how are the obtained Automated Best Connected Rectangular Duals for any given number of sub rectangles, distinct. Secondly, how can we conclude that the obtained number of Automated Best Connected Rectangular Duals, is the maximum possible. Lastly, we tried to construct an algorithm for the generation of Automated Best Connected Rectangular Duals for a given number of sub rectangles.

2. PLANAR GRAPHS

A graph *G* is planar if it can be drawn in the plane in such a way that no pair of edges cross each other.



2.1 RECTANGULAR DUAL OF A PLANAR GRAPH

A Rectangular Graph is a plane graph where all regions are four-sided and all edges are oriented in either the vertical or the horizontal direction. In addition the graph enclosure must also be rectangular.

A rectangular subdivision system of a rectangle R is a partition of R into a set F= R_1 ...$R_n$of non-overlapping rectangles such that no four rectangles meet at the same point.

A rectangular dual of a planar graph $G = (V, E)$ is a rectangular subdivision system Γ and a one-to-one correspondence f: $V \to \Gamma$ such that two vertices *u* and *v* are adjacent in *G* if and only if their corresponding rectangles f (*u*) and f (*v*) share a common boundary [1].

For further understanding of rectangular dual we have a planar graph in Figure 1 and its corresponding three rectangular dual in figure 2 shown below:

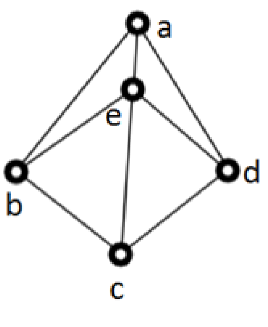


Figure 1

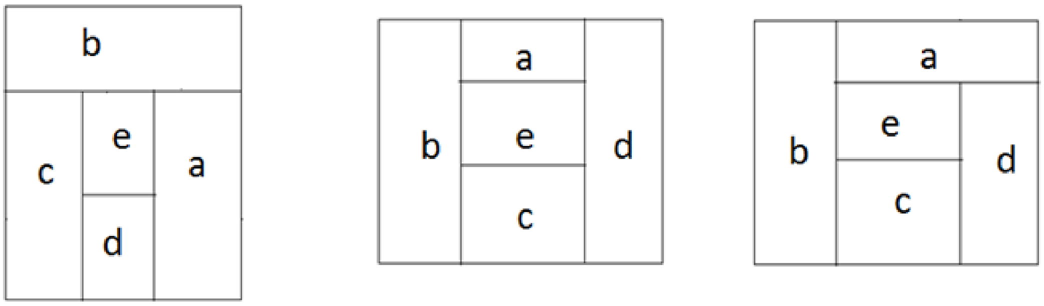


Figure 2

2.2 DUAL GRAPH OF A RECTANGULAR DUAL

The dual graph of a rectangular dual is a simple undirected graph, obtained by representing each of its sub-rectangles as a vertex and then drawing an edge between any two vertices if the corresponding sub-rectangles are adjacent [2].

For example: Figure 4 is dual graph of rectangular dual in Figure 3.

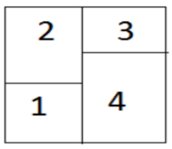


Figure 3

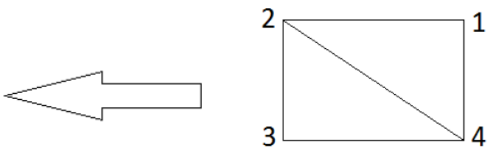


Figure 4

2.3 CONDITIONS FOR EXISTENCE OF RECTANGULAR DUAL

Any planar graph has a dual but not every planar graph has a rectangular dual. We have presented the necessary and sufficient conditions which a graph must fulfil for the existence of rectangular dual. The following theorem is one of the most important theorem of rectangular dualization on Proper triangular planar graphs.

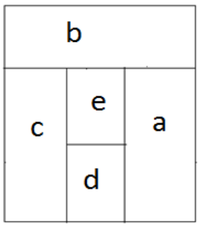
Theorem [3, 4] A Planar graph G has a rectangular dual R with four rectangles on the boundary of R if and only if

- Every interior face is a triangle and exterior face is a quadrangle.
- G has no separating triangles.

Hence graph must be Proper triangular planar graph. (PTP)

For better understanding of this theorem we have two graphs shown below where Figure 7 is the proper triangular planar graph since it is satisfying all the conditions of PTP. Hence its rectangular dual exist with four rectangles on the boundary of R shown in Figure 6.

While Figure 9 is not a PTP since its exterior face is also a triangle. Hence its rectangular dual does not exist with four rectangles on the boundary of R.



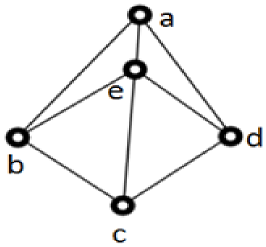
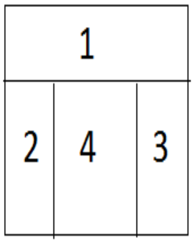
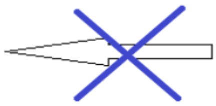


Figure 5

Figure 6





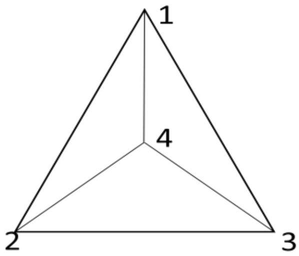


Figure 7

Figure 8

2.4 SEPARATING TRIANGLE

A triangle of a planar graph is a separating cycle if its removal separates the graph. Hence it does not form the boundary of a face.

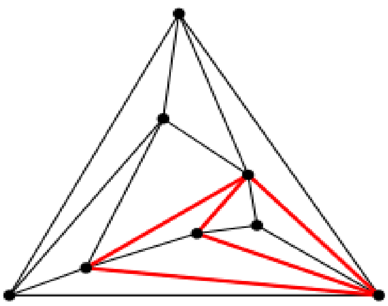
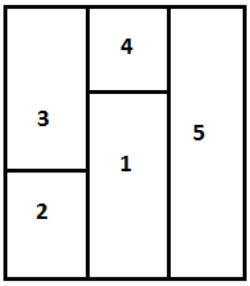


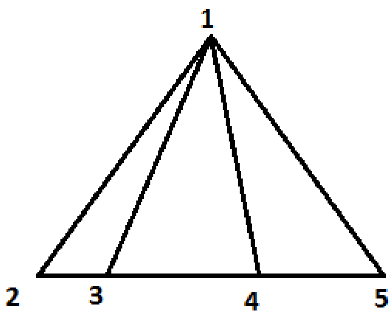
Figure 9: The two red triangles are separating triangles.

3. DEGREE OF CONNECTIVITY

The degree of connectivity of a rectangular dual is given in terms of connectivity of the corresponding dual graph [2].In our work, we have considered the number of edges in the dual graph as a measure of connectivity. So, if two dual graphs have same number of vertices, then the dual graph having more number of edges is said to be more connected [2].

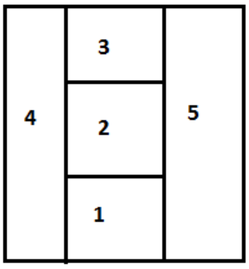


R1

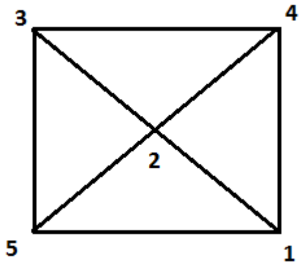


G1

Figure 10 (n=5, 3n-8=7 edges)



R2



G2

Figure 11 (n=5, 3n-7=8edges)

In Figure 10, G1 is the dual graph of rectangular dual graph R1. In Figure 11, G2 is the dual graph of rectangular dual graph R2. G1 and G2 have same number of vertices (i.e., 5) but different number of edges. G1 has 7 edges, G2 has 8 edges. Since G2 has more edges compared to G1, R2 is said to be more connected compared to R1.

4. BEST CONNECTED RECTANGULAR DUALS

Theorem [2] Any rectangular dual graph of order n has at most $3n$ - 7 edges in its dual graph provided that n> 3.

This implies that the rectangular dual is best connected if its dual graph has $3n$-7 edges.

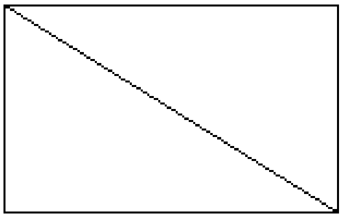


Figure 12: Dual Graph having 4 vertices, $3n$-7=5 edges.

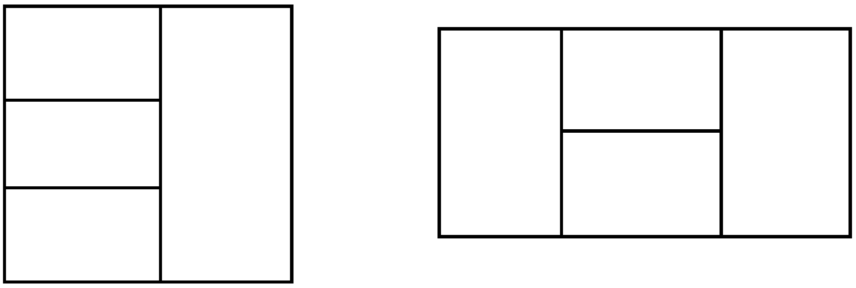


Figure 13: Best Connected Rectangular Duals on n=4 vertices. They have 3n-7=5 edges.

4.1. AUTOMATED BEST CONNECTED RECTANGULAR DUALS

An Automated Best Connected is a rectangular dual graph that remains best connected for any n [5].

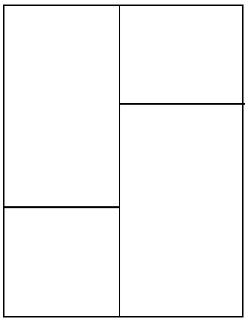


Figure 14

Figure 15

Consider a best connected rectangular dual graph. This would imply that its dual graph has 3n-7 edges. If it is possible to add a rectangle to the graph in such a way that it remains connected to three other existing rectangles, then its dual graph would have 3n-7+3 i.e., 3(n+1)-7 edges, which would imply that the rectangular dual is still best connected. Such a graph is said to be automated best connected rectangular dual.

Referring to Figure 15, it is possible to add a rectangle such that it is adjacent to three other existing rectangles. Hence, the graph is Automated Best Connected Rectangular Dual. In Figure 14, if we attempt to add a rectangle to either of the four sides, the rectangle would remain adjacent to at most two other existing rectangles. This implies that the graph would not remain best connected. Hence, the graph is not Automated Best Connected Rectangular Dual.

4.1.1 CONSTRUCTION OF AUTOMATED BEST CONNECTED RECTANGULAR DUALS

We begin constructing Automated Best Connected Rectangular Duals starting from $n=4$. For $n=4$, we get three graphs (Figure 16). Comparing the three graphs in Figure 16, we can say that the three graphs are distinct. Proceeding for five vertices, we add a rectangle to an automated best connected rectangular dual on four vertices. So, for $n=5$, we get three graphs, shown in Figure 17.

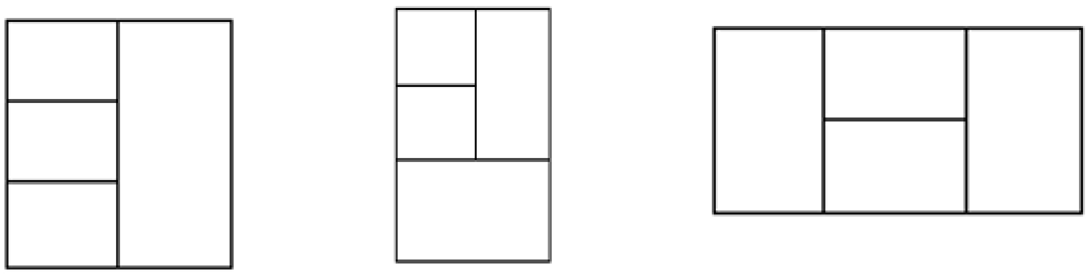


Figure 16

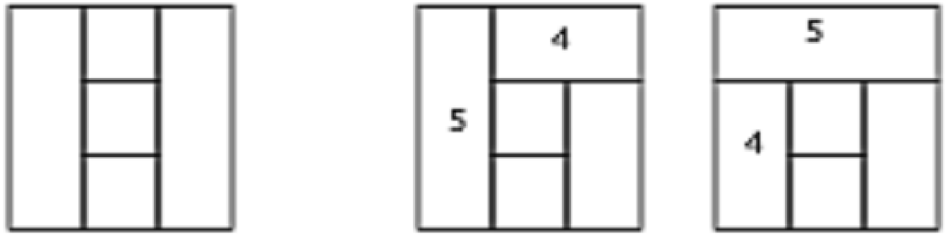


Figure 17

Similarly, proceeding for *n*=6, we get three graphs. More graphs on six vertices are possible. To construct them, we follow the following procedure:

For an Automated Best Connected Rectangular Dual on six vertices (Figure 18), we consider two adjacent rectangles in the graph. Let rectangle 1 be above rectangle 2. We change their arrangement such that rectangle1 is now at the side of rectangle 2 (Figure 19).Comparing the two graphs, we could say that they are distinct. In this manner, we obtain three graphs for six vertices. In total, we obtain six Automated Best Connected Rectangular Dual graphs on six vertices. Similarly we proceed for seven and eight vertices and nine vertices. Table 1 shows the obtained number of Automated Best Connected Rectangular Duals for a given number of sub rectangles.

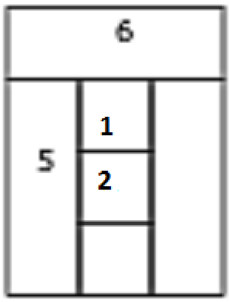


Figure 18

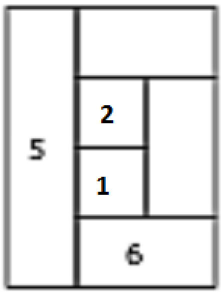


Figure 19

NUMBER OF VERTICES	NUMBER OF AUTOMATED BEST CONNECTED RECTANGULAR DUALS
5	3
6	6
7	15
8	42
9	123

Table 1

4.1.2. PROBLEMS THAT ARISE WHILE CONSTRUCTING THE AUTOMATED BEST CONNECTED RECTANGULAR DUALS

1. For a given number of sub rectangles, why is the obtained number of Automated Best Connected Rectangular Duals, the maximum possible?

For a given n, we obtain the number of Automated Best Connected Rectangular Duals for each k (where $1<k<=n-2$).k denotes the number of sub rectangles arranged successively one above the other. The values are given in Table 2.

	k=7	*k*=6	*k*=5	*k*=4	*k*=3	*k*=2
***n*=4**					1	2
***n*=5**					1	2
***n*=6**				1	2	3
***n*=7**			1	2	3	9
***n*=8**		1	2	3	9	27
***n*=9**	1	2	3	9	27	81

Table 2

`

$$S = 3 + \sum_{i=1}^{n-5} 3^i \qquad 4 < n \leq 9 \qquad (i, n \in N)$$

Figure 28

While constructing the Automated Best Connected Rectangular Duals, we obtained the formula given in Figure 28. S denotes the maximum number of Automated Best Connected Rectangular Dual for a given number of sub rectangles i.e. *n*. We have verified the formula for *n*=5, 6,7,8,9. But we haven't proves the formula mathematically for *n*>9.

2. For a given number of sub rectangles, how are the obtained Automated Best Connected Rectangular Duals distinct?

By looking at the graphs constructed for a given *n*, we say that the graphs are distinct if one graph cannot be obtained from any of the other graphs by either rotation, reflection or scaling. We then approach the problem in a graph-theoretical manner by using the concept of graph labelling.

We have two rectangular duals here in Figure 20 and Figure 21 on *n* =8. We can see these two rectangular duals are distinct as their internal arrangement of sub rectangles are different.

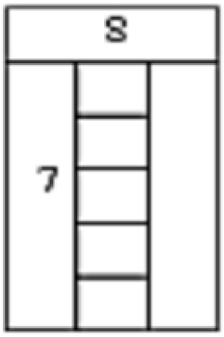


Figure 20

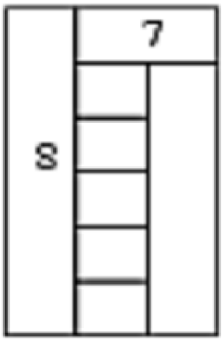


Figure 21

But how to prove mathematically that these two rectangular duals are distinct? For this we have defined a labelling series which will be explained in next sections.

4.1.3. LABELLING

In order to prove mathematically that any two rectangular duals are distinct we have defined a labelling using a set of few rules explained in later sections. Through this labelling, we can differentiate between any two rectangular duals.

4.1.3.1. LABELLING RULES

RULE 1:

Assign a number one to the first sub rectangle of any rectangular dual.

For better understanding of this rule we have an example as illustrated below.

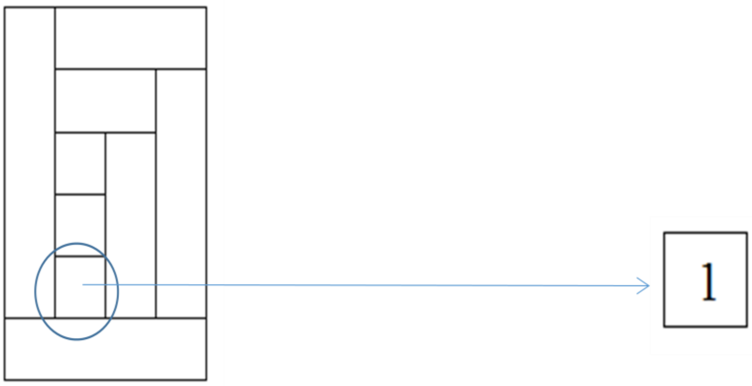


Figure 22: n=9

RULE 2:

Assign a number equal to sum of numbers assigned to its adjacent sub rectangles.

In order to clearly demonstrate the second rule we have an example illustrated below as how we labelled each sub rectangle using Rule 2.

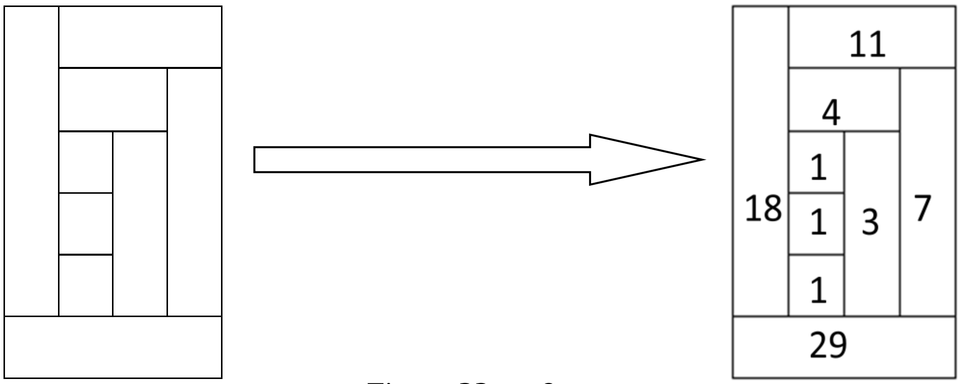


Figure 22: n=9

So the final labelling series of Figure 22 is 1,1,1,3,4,7,11,18,29.

RULE 3:

If a sub rectangle R_1 is placed adjacent to another sub rectangle R_2 whose number is k (k>1) then R_1 is assigned a number that is one plus the number of R_2 i.e. k+1.

The next example shows how we labelled the Figure 23 using third rule.

Figure 23: n=9

So the final labelling series of Figure 23 is 1,1,1,3,3,4,11,11,26.

4.1.3.2. HOW TO PROVE TWO RECTANGULAR DUALS ARE DISTINCT

Till now we defined the labelling rules to label any rectangular dual. Next step is to use this labelling series to prove any two rectangular duals are distinct. For this we will consider the following two cases:

Case 1: Two rectangular duals are distinct if their series defined by labelling is different.

For example we have two rectangular duals in Figure 24 and Figure 25 whose labelling series are different hence we can say that these two rectangular duals are distinct.

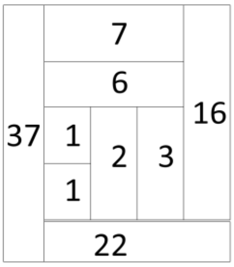


Figure 24

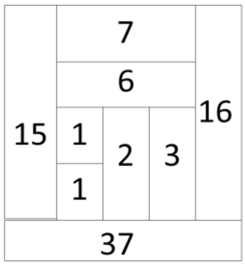


Figure 25

The labelling series of Figure 24 is 1, 1,2,3,6,7,16, 22, 37.

The labelling series of Figure 25 is 1, 1,2,3,6,7,15, 16, 37.

Case 2:

If two rectangular duals have same series defined by labelling then they are distinct provided their series have different subscript.

For example in Figure 26 and Figure 27 the labelling series is same but in Figure 26 the number of sub rectangles adjacent to sub rectangle with number 29 is 4 while in Figure 27 no of sub rectangles adjacent to sub rectangle with number 29 is 3. So the subscript is the number of adjacent sub rectangles.

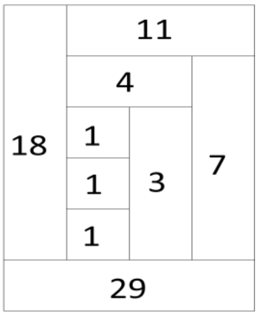


Figure 26

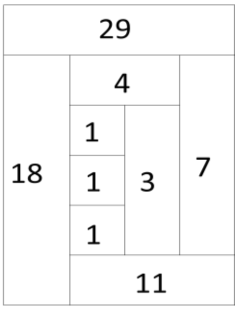


Figure 27

The labelling series of Figure 26 is 1, 1,1,3,4,7,11,18,29.

The labelling series of Figure 27 is 1, 1,1,3,4,7,11,29.

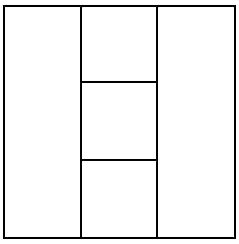
Similarly, we proceeded with the above stated rules to label all the constructed rectangular duals.

5. CONSTRUCTED AUTOMATED BEST CONNECTED RECTANGULAR DUALS

The constructed Automated Best Connected Rectangular Duals for *n*=5, 6,7,8,9 with labelling are given below. *k* is the number of sub rectangles successively arranged one above the other.

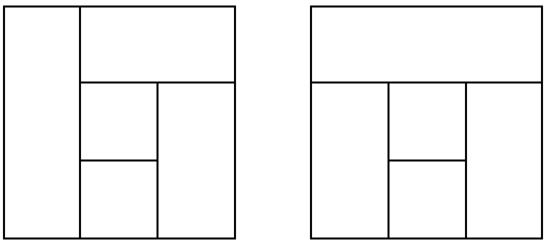
***n* = 5**

k = 3



3,3

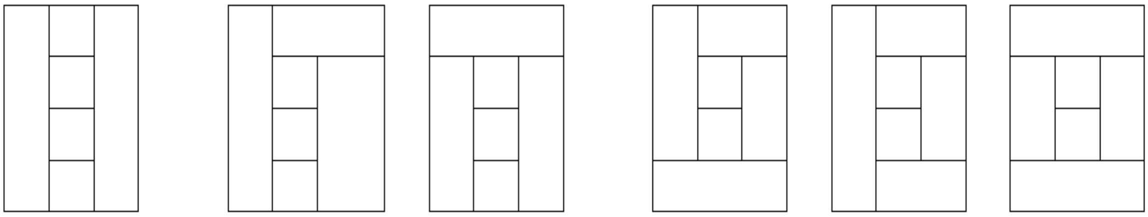
k = 2



2,3,5 2,2,5

$n = 6$

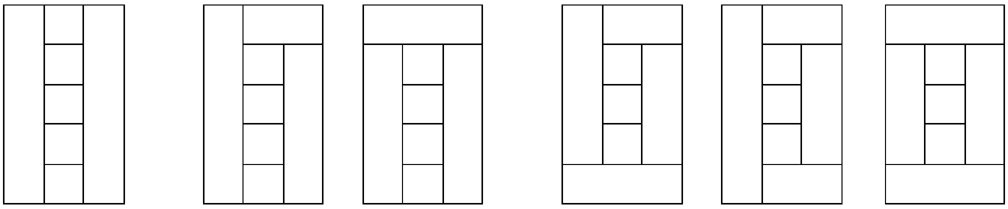
k = 4 k = 3 k = 2



4,4 3,4,7 3,3,7 2,3,5,8 2,3,3,8 2,2,5,5

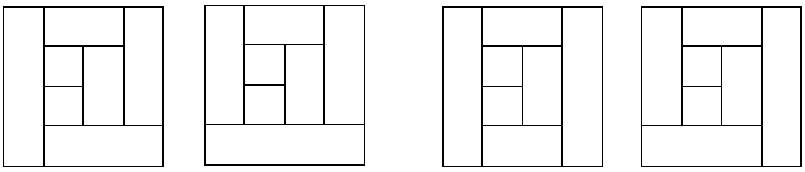
$n = 7$

k = 5 k = 4 k = 3

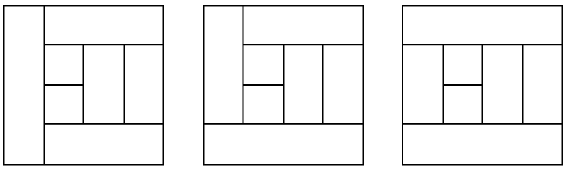


5,5 4,5,9 4,4,9 3,4,7,11 3,4,4,11 3,3,7,7

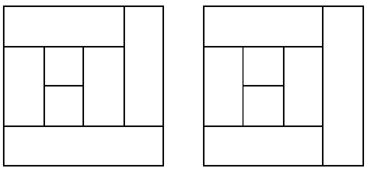
k = 2



2,3,5_3,8,13 2,3,5,5,13 2,3,3,8,8 2,3,5$_{,4}$,8,13



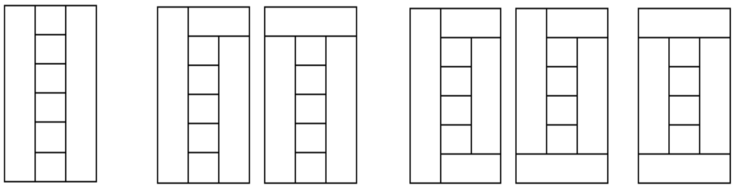
2,3,6,6,11 2,3,6,8,14 2,2,3,8,8



2,2,5,7,12 2,2,5,5,12

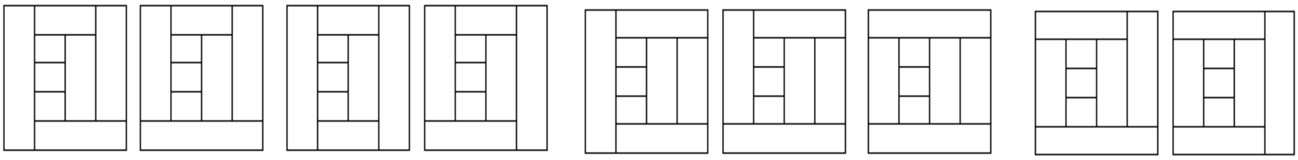
$n = 8$

k = 6 k = 5 k = 4



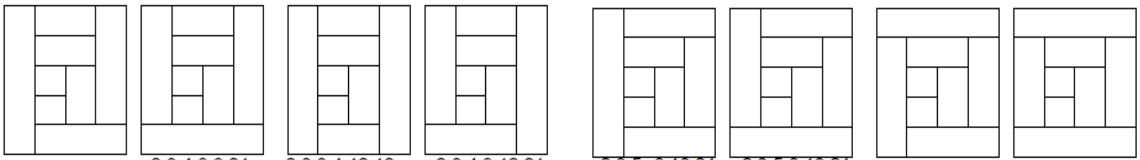
6,6 5,6,11 5,5,11 4,5,5,14 4,5,9,14 4,4,9,9

k = 3



3,4,7_2,11,18 3,4,7,7,18 3,4,4,11,11 3,4,7_5,11,18 3,4,8,8,19 3,4,8,11,19 3,3,4,4,11,11 3,3,7,10,17 3,3,7,7,17

k = 2



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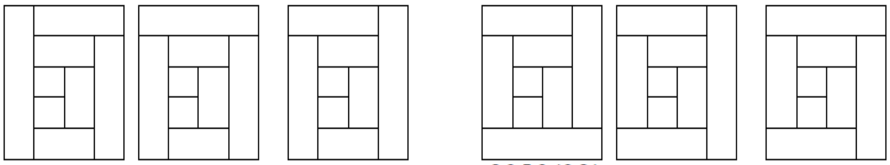

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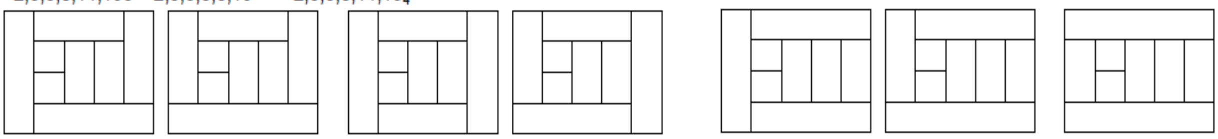

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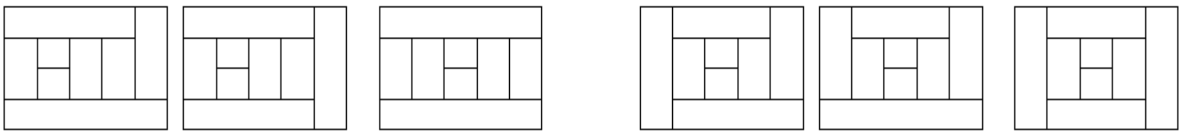

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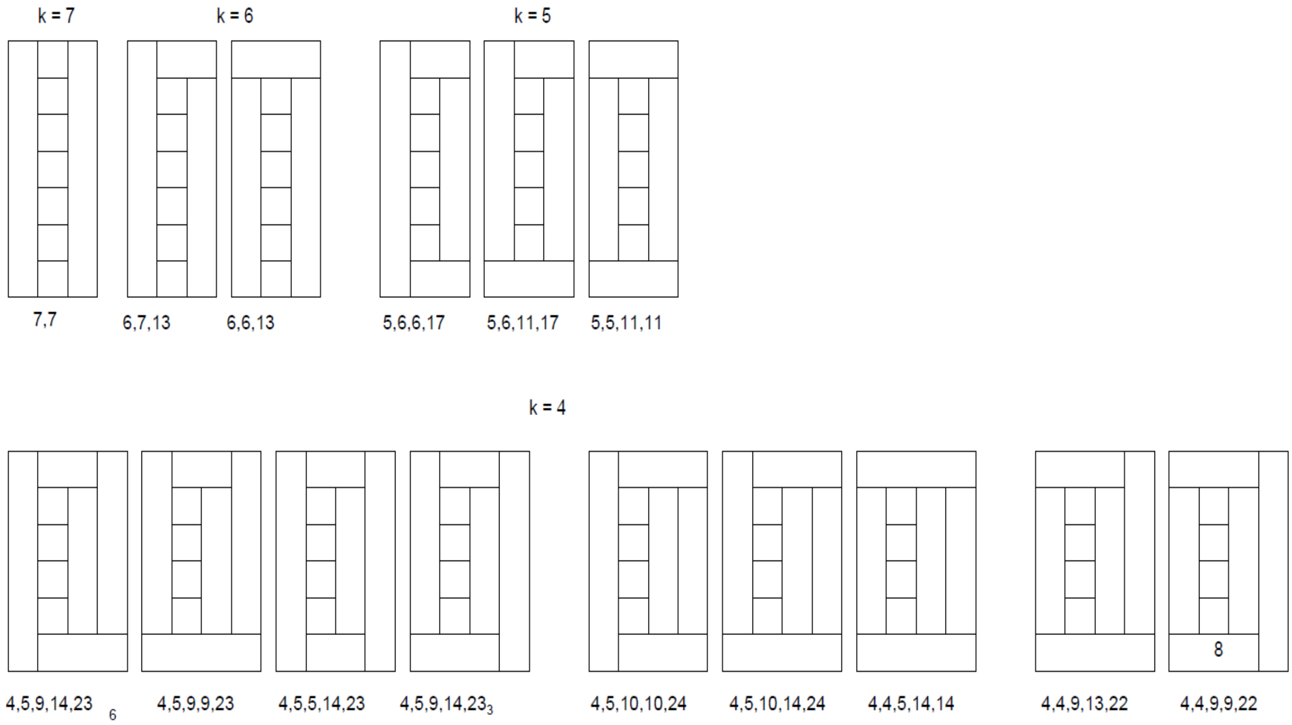
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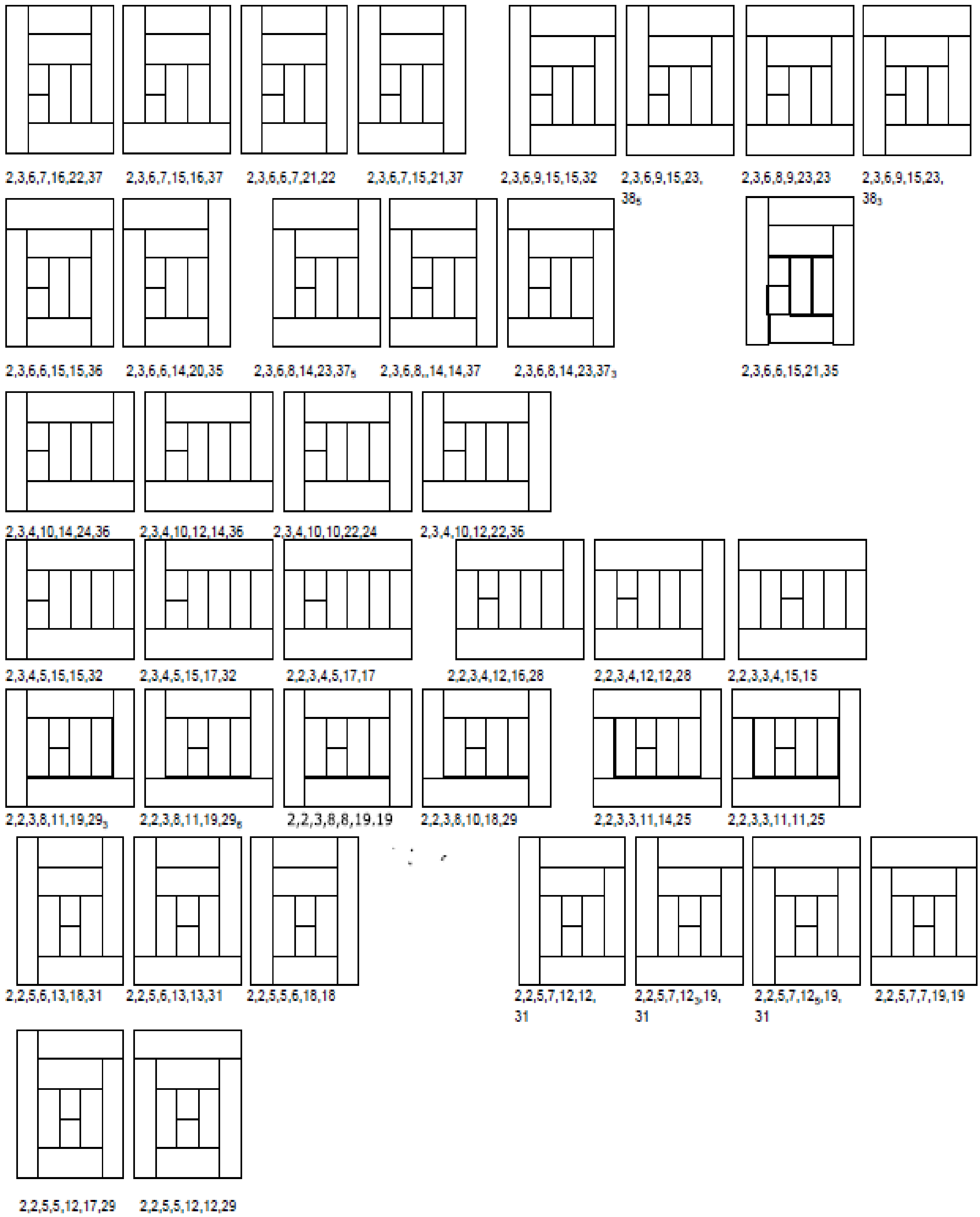

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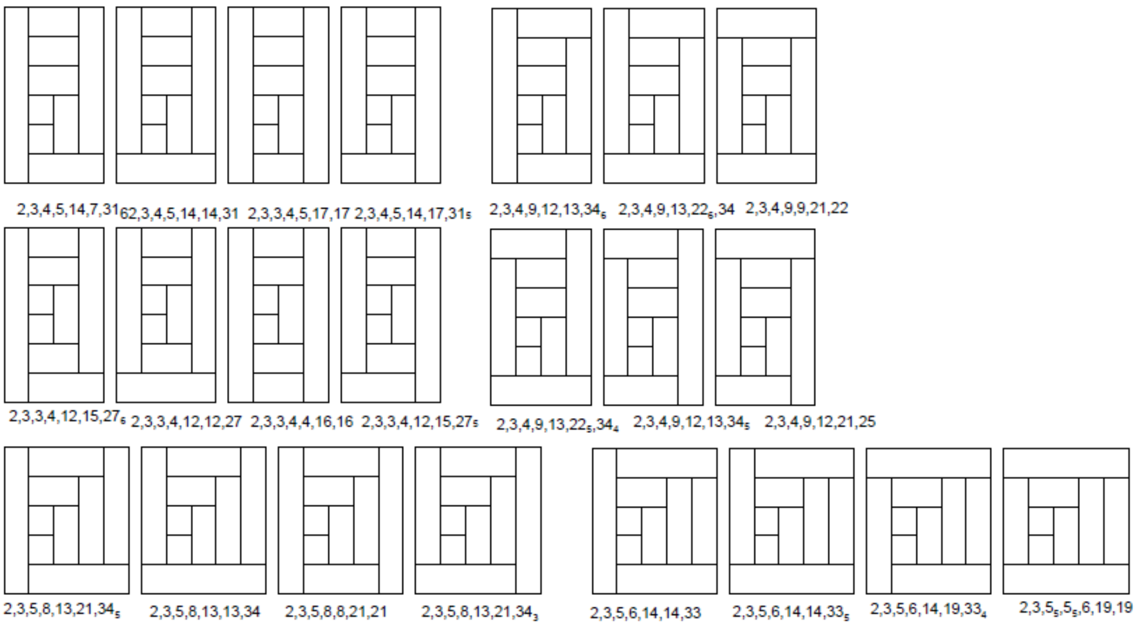

6. CONCLUSION

We have constructed Automated Best Connected Rectangular Duals on n=4,5,6,7,8,9.We were able to define a labelling rule to mathematically prove that the constructed rectangular duals were distinct. We verified the maximum number of Automated Best Connected Rectangular Dual for a given number of sub rectangles using a formula which is applicable for 5<n<=9. There are two problems that we were not able to address due to time constraint and we could work on them in future. Firstly, we haven't proved the formula mathematically, for maximum number of Automated Best Connected Rectangular Dual for a given number of sub rectangles. Secondly, we tried but did not manage to develop a general algorithm for construction of Automated Best Connected Rectangular Duals.

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