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Short Communication

Heuristic method to check the realisability of a graph into a rectangular plan

A. Recuero[a,*], O. Río[a], M. Alvarez[b]

[a]*Instituto Eduardo Torroja-CSIC, Serrano Galvache S/N, 28033 Madrid, Spain*
[b]*Facultad de Informática-UPM, Canipus de Monte Gancedo, Boadilla del Monte, Madrid, Spain*

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Abstract

The problem of mapping a graph into rectangles so that they cover a rectangular plan is considered. The graph may represent the basic scheme of a house or of a building, in which different design requirements are collected. To cope with this problem the authors have chosen a three steps approach: determining whether a graph can be mapped in this way (first step); automatically generating some of its realisations (second step); then how a given plan can be reduced to the rectangular one is analysed (third step).

A heuristic approach to the general solution of the first step is proposed. It is a problem with a practical application in areas such as architectural design or compacting of electronic circuits, to which no general solution has been found in the specialised bibliography. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Graphs; Mapping into rectangles; Architectural design

1. Introduction

Architectural design is the result of the process carried out by a designer to specify an architectural program, both formally and spatially. The architectural space is divided into cells, usually rectangular ones, within appropriate dimensions so that they cover a plan, in many cases rectangular (or that can be transformed into rectangular) as well. From the requirements previously established in the architectural program and from a deep knowledge of the building nature, hierarchic relations are settled to define the importance of each cell and the partial or global relations among cells.

The functional structure of the building, defining uses, organisation and inter-relations, and the formal structure of spaces for these uses are created this way. Architectural spaces are generated taking the program as a basis and they are functionally and formally organised, with their corresponding links or connections. This will give rise to the geometry that is the realisation of a topology represented by the basic scheme.

The work presented here can be most helpful for obtaining design alternatives that fulfil the functional requirements previously stated in the architectural program, in the initial stages of the design process (elaboration of basic schemes). The basic schemes are displayed as graphs, where functional links are defined, specifying the type and degree of the inter-cells existing relations; in these graphs, vertices represent the cells and edges represent the relations. As a result of those analyses, every pair of vertices will be either mandatory connected or disconnected, or possibly connected. The different possible combinations of the latter give raise to many possible graphs. In this article, one of these resulting graphs is considered, where existing links are the only ones taken into account.

From the primary graph a secondary one can be deduced called skeleton in which edges become cell borders (walls) and vertices are border junctions. This skeleton is a topological representation of the inter-cell structure of the spatial system from which the architectural design (plan) can be deduced.

The fact that the given graph does not represent a unique plan means that, once the designer has described his requirements in the form of a graph, there is still a great variety of possible architectural solutions. Even before starting the transformation of a graph into a plan, some elaboration of the graph may be performed, as part of the design process. The advantage of performing some of the needed operations on the topological abstract graph rather than on the plan, is that, at this stage, all decisions may be completely objective,

* Corresponding author. Tel.: + 349-1302-0440; fax: + 349-1302-0700.
E-mail address: recuero@pinarl.csic.es (A. Recuero).

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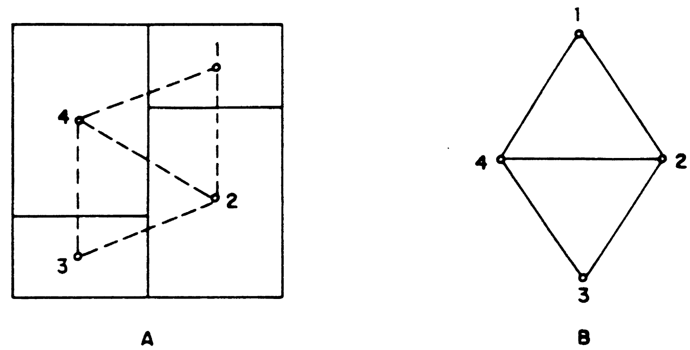

Fig. 1. Association of a graph to a plan.

since considerations regarding the shape of spaces and their look do not interfere with judgement. Also, it allows the designer to concentrate on the main issues to be solved in the different stages of the design.

The problem of mapping a graph into a given plan whose sides are parallel to two perpendicular directions, seems not having a straight forward approach. This is the reason why authors have chosen an indirect approach divided into three steps. The two aspects of the problem for a rectangular plan are considered:

- determining whether a graph can be mapped in this way (first step);
- automatically generating some of its realisations (second step);
- then how a given plan can be reduced to the rectangular one is analysed (third step).

The first step only is considered in this article given a heuristic approach. In fact, only a set of necessary conditions is considered to check on the realisability of a graph. They cover all the cases the authors could think of, but no proof that it is sufficient is provided. Further details on graphs can be found in Recuero [1,2].

2. Mapping a graph into rectangles covering a rectangle

Let us consider a rectangle (overall rectangle) divided into rectangles. When a vertex is placed in every rectangle and those vertices whose rectangles share a part of a side are connected by means of edges, a graph associated to the partition is obtained as shown in Fig. 1. Reciprocally, when a graph can be associated to (mapped into) a partition of the type described, it will be said realisable, and it wi1l be send that the partition is a realisation of the graph. This is a restricted meaning of realisable, used here for simplicity. In the general case, the plan to be covered is not taken into account.

A partition is associated with only one graph, while a realisable graph may be mapped into many different partitions. Fig. 2 shows three realisations of the graph in Fig. 1(b), different from that of Fig. 1(a).

This problem has practical application in areas such as architectural design or compacting of electronic circuits. The problem was first presented by Ungar [3] and though several authors have made contributions: Levin [4], Mitchell et al. [5], Earl and March [6], Magiera [7], Roth et al.; Roth and Hashimshony [8,9], Steadman [10], Schwarz et al. [11,12 among others], no general solution has been found in the specialised bibliography.

3. Realisability of a graph

The concept of wrapping of a rectangle is introduced here in order to study the realisability of a graph. Let us consider a rectangle (base rectangle), such as that one shadowed in the different realisations of Fig. 3. Its wrapping is the set of

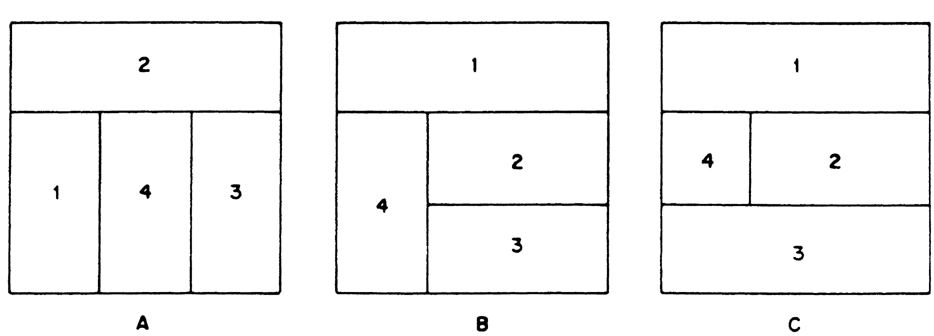

Fig. 2. Different realisations of the graph of Fig. 1(b) into plans.

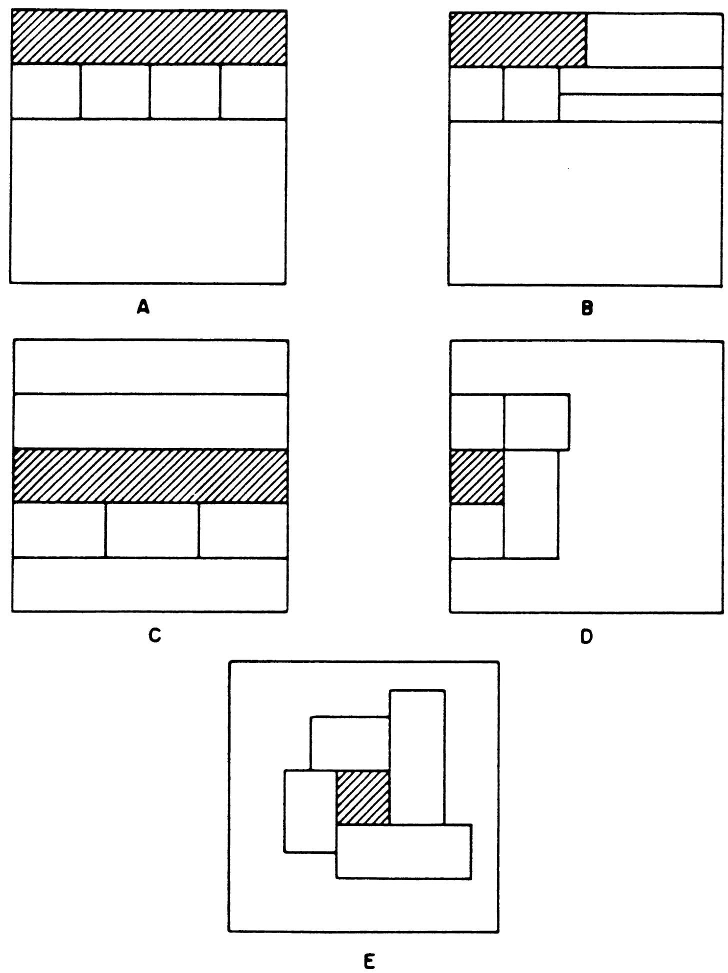

Fig. 3. Different forms of wrapping a rectangle with 4 rectangles.

rectangles that have any contact with it, sharing either a part of a side or a corner.

Two rectangles in contact only at a corner are send to be vertical to each other. When two rectangles are vertical at a corner, two other rectangles will also be vertical at the same corner, so that if vertical rectangles exist, then they are grouped in sets of four rectangles vertically by pairs.

When the rectangles of the wrapping form a cycle the wrapping is said to be closed, otherwise it is said to be open. Inner rectangles have closed wrappings while outer rectangles (with one or more sides lying on sides of the overall rectangle) have open wrappings.

In the five cases in Fig. 3, the wrapping of the shadowed rectangle is formed by 4 rectangles. The wrapping may consist of either one part (cases A, B, D or E), or of two independent parts (case C), depending on whether all the rectangles are consecutive or not. When the base rectangle has one of its sides as long as a full side of the overall rectangle, it will be said that this rectangle has a complete dimension. In this case, the wrapping may have rectangles either on two opposite sides (case C) or on one side (case A). When the wrapping has only one part, it may cover any number of contiguous sides of the base rectangle (1 in A, 2 in B, 3 in D, and 4 in E).

Let us see now some conditions necessarily accomplished by the rectangles in the wrapping:

- The rectangles in each part form an ordered sequence, that is, each rectangle may be preceded by only one rectangle and followed by only another one.
- If the wrapping has one part then it may be closed having in such a case at least as many rectangles as the number of rectangles vertical-to-the-base (VTTB) one plus four. Fig. 4 shows examples of closed wrappings with the minimum number of rectangles.
- If there are more than 2 VTTB rectangles in the wrapping, then it is closed.
- If the wrapping has two parts then the base rectangle has a complete dimension and there are no rectangle verticals to the base one.
- In the latter case, if one of the two parts includes a rectangle with a complete dimension then it is the only rectangle in this part.
- If the wrapping has two parts, then the rectangles placed

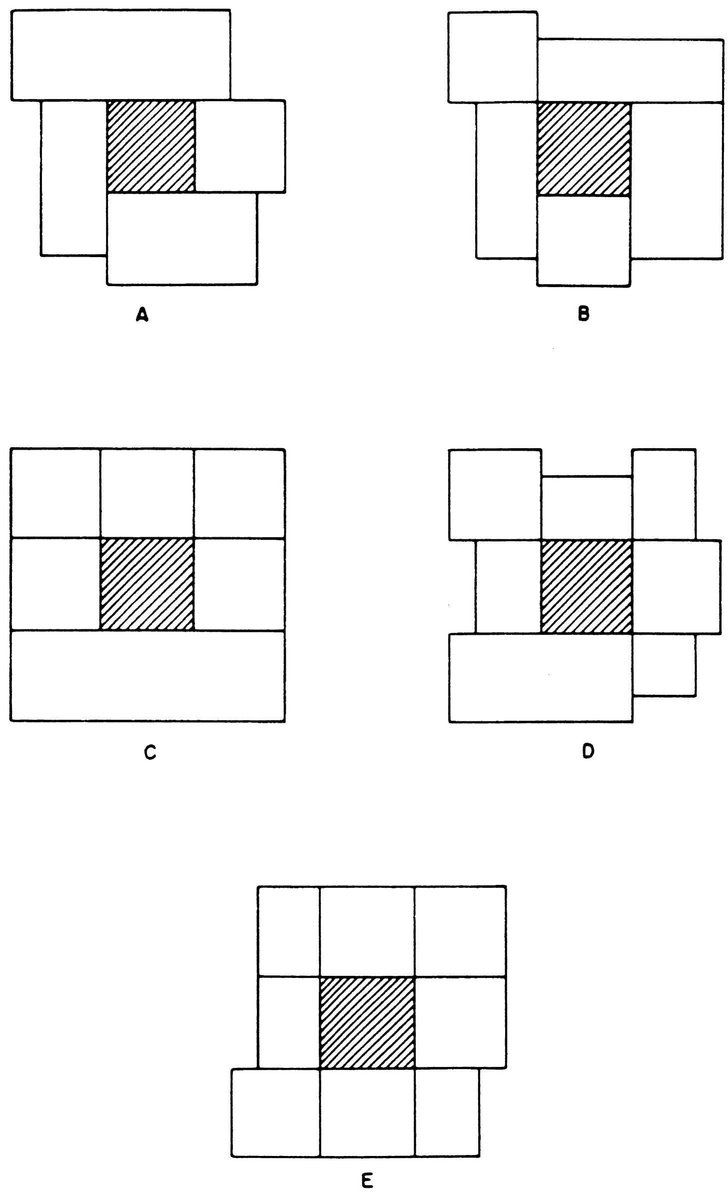

Fig. 4. Closed wrapping with the minimum number of rectangles.

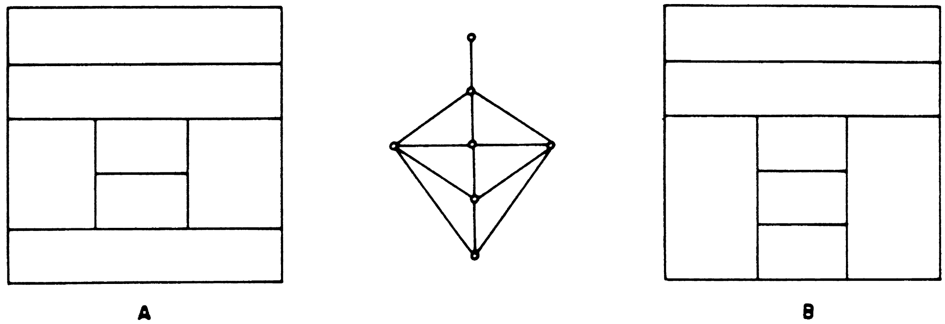

Fig. 5. Realisations of a graph with two end rectangles.

Table 1
Operations and verifications

Number	Operations (O)	Verifications (V)
1	Check whether the graph is connected or not.	(a) A disconnected graph is not realisable.
2	Count all degree 1 vertices and mark their adjacent vertices.	(a) There are no more than 2 degree 1 vertices.
3	Taking each vertex at a time as base vertex.	
3.1.	Find all the vertices ATTB one.	
3.2.	Group them in sets of connected vertices and calculate the local degree of each of the vertices, considering only connections to vertices of the same set.	(a) There are no more than 4 of these sets.
		(b) No vertex has a local degree greater than 2.
		(c) If all vertices of a set have a local degree equal to 2 then they form a cycle of length grater than 3.
		(d) If there is a cyclic set then it is the only existing set.
		(e) If the base is adjacent to a degree 1 vertex then only one other open set may exist, beside that one containing the degree 1 vertex.
		(f) In this latter case, when that second set exists, if in that set a vertex is adjacent to another of degree 1 then it is the only vertex in the set.
3.3.	Check if there are vertices, other than the base one, simultaneously adjacent to pairs of end nodes of two of the sets found, or to the end nodes of a set with more than two vertices. If so, vertical vertices are among them. If there is one vertex adjacent to the end vertices of the same set, and this is the only set and has more than 3 vertices, then the vertex can be considered either VTTB one or not, but if there are other sets or if this set has 3 or less vertices then the vertex is not VTTB one.	(a) Only one vertex can be adjacent to the same pair of vertices.
		(b) If the base is adjacent to a degree 1 vertex then no VTTB vertices exist.
3.3.1.	When a vertical node is incorporated, it increases by 1 the local degree of the vertices it connects and its local degree is 2. If the connected vertices belong to different sets then both sets are united.	
3.1.	Write down the number of parts and for each of them their number of vertices and whether they are closed, open or undefined. Write down the local degree of the vertices of the set, whether they are VTTB or not, and whether they require a complete dimension or not.	(a) There are no more than two parts.
		(b) If there are two parts then there is no VTTB rectangle.
		(c) If there are more than two VTTB vertices then there is only a cyclic part.
		(d) If there is one cyclic part (all its vertices have a local degree of 2), then it is the only part and it has at least as many vertices as the number of VTTB ones plus 4.
		(e) If the base requires a complete dimension then each part of its wrapping does not contain more than one vertex requiring a complete dimension.
		(f) No vertex in the wrapping has a local degree greater than 2.
3.5.	If the wrapping has one part then return to 3, until a all vertices have been checked.	
3.5.1.	The wrapping has two parts. Mark as "related to" all the vertices belonging to each of the two parts or connected to them by ways not passing through the base vertex.	(a) No vertex is related to both parts.
3.5.2.	Return to 3.	
4.	All vertices are already classified into inner and outer, according to their wrapping.	(a) If there are less than 5 vertices in the graph then all of them are outer vertices. If there are 5 or more vertices then at least 4 of them are outer vertices.
		(b) If all the vertices of the graph are outer vertices, then there are at least 2 degree 1 or 2 vertices.

on one side of the base rectangle cannot be in contact with those placed on the other side. Consequently, if the base vertex is deleted (as well as all edges having this vertex as an end node) then the graph becomes disconnected.

- If a rectangle covers an end of the overall rectangle and it is wrapped by only one rectangle then its associated vertex is of degree 1. Degree of a vertex is the number of vertices to which it is connected. Since this is the only possible realisation of such a vertex, a realisable graph cannot have more than 2 vertices of degree 1. Fig. 5, shows a realisation of a graph with 2 rectangles covering two ends of the overall rectangle, one of them corresponding to a vertex of degree 1.
- The rectangle adjacent to one of degree 1 has a complete dimension.
- Only groups of four rectangles may share a common vertex.

Let us see now how the wrapping of a rectangle can be built from the description of the graph. All vertices adjacent to vertex 1 will be included.

- Group these vertices in sets of connected vertices. If any vertex is adjacent to more than two other vertices of the same set then the graph is not realisable. The graph is also non-realisable if more than 4 of those sets exist.
- If there is only one set and all its vertices are adjacent to other two then the wrapping must have at least 4 vertices.

To find out rectangles VTTB one, the existence of vertices other than the base one simultaneously adjacent to end vertices of two of the send sets must be checked. These end vertices, ends of the open chain of connected vertices, are adjacent to one or none other vertices of the same set, depending on whether the set contains either more than one vertex or only one.

- If there is a vertex, other than the base one, simultaneously adjacent to the end nodes of a set of more than 3 vertices, and there is only one set of adjacent vertices, then this vertex may be mapped into a rectangle VTTB one or not. In such a case, the wrapping may be either open or closed. This indeterminacy may be solved by some other criteria. If such a vertex exists, but the set has 3 or less vertices, or there are other sets of adjacent-to-the-base (ATTB) vertices, then the vertex is not VTTB one.
- If a VTTB vertex is adjacent to the end nodes of two different sets then both sets are united. In any case, the VTTB, vertex, is included in the set.

When this process is completed either one open or closed set of vertices adjacent and may be VTTB or one undefined set of ATTB vertices plus 1 vertex that can be considered VTTB or not, or two open sets of vertices ATTB may result. Thus, the wrapping is completed.

Some observations can be made on the set of all vertices when all of them have a proper wrapping. The first one is that there must be a minimum number of outer vertices. If the graph has 4 or fewer vertices, then all of them must be outer ones. If the graph has more than 4 vertices, then at least 4 of then must be outer vertices.

The second one is made in the case when all the vertices are outer ones. At least two vertices of degree 1 or 2 must exist. In the realisation of the graph, the rectangles associated to these vertices must be placed at corners of the overall rectangle.

Sometimes, the indeterminacy on the type of wrapping of a vertex is solved when the wrapping of an adjacent vertex is considered. In other cases the global checks solve these indeterminacies. Nevertheless, if the indeterminacy stands then the wrapping may be alternatively considered as open or closed.

4. Algorithm for checking the realisability of a graph

The algorithm described in this section allows the above mentioned conditions to be checked. Firstly, it makes some global checks on connectivity and on the number of degree 1 vertices. Afterwards, it performs local checks on every vertex. Lastly, the final global checks are made. If all tile checks are past then the graph is considered to be realisable. In Table 1 the operations and the verifications to be made after each operation are shown.

5. Examples of non-realisable graphs

A set of non-realisable graphs is shown here. In each case, the reason why they are non-realisable is indicated, and which of the algorithm check detects the violation. In the first place, since no more than two degree 1 vertices can exist, trees (graphs without cycles) with more than one branch (a vertex of degree greater than 2 exists) are not realisable, as that shown in Fig. 6(a). Graph in Fig. 6(b) is not realisable for the same reason. Both cases are detected in V2a. Graph in Fig. 6(c) has more than 4 sets of connected ATTB vertices, which is detected by V3.2a. in the graph in Fig. 6(d) a vertex with a local degree of 3 is detected by V3.2b.

The complete graph (graphs in which every vertex is connected to every other vertices) of order 4 (K4) is not realisable because all its vertices are surrounded by cycles of length 3 (Fig. 6(e)). Higher order complete graphs will not be realisable because they do not pass V3.2b. Graph of Fig. 6(f) has two sets of ATTB vertices, one of them cyclic, which is detected by V3.2d. Graphs of Fig. 6(g) and (h) when a vertex adjacent to another of degree 1 is considered as the base one, there are more than 2 sets of ATTB vertices, which is detected by V3.2e. In the graph of Fig. 6(i) two vertices are adjacent to another adjacent to one of degree l. And at the same time they are adjacent to each other and

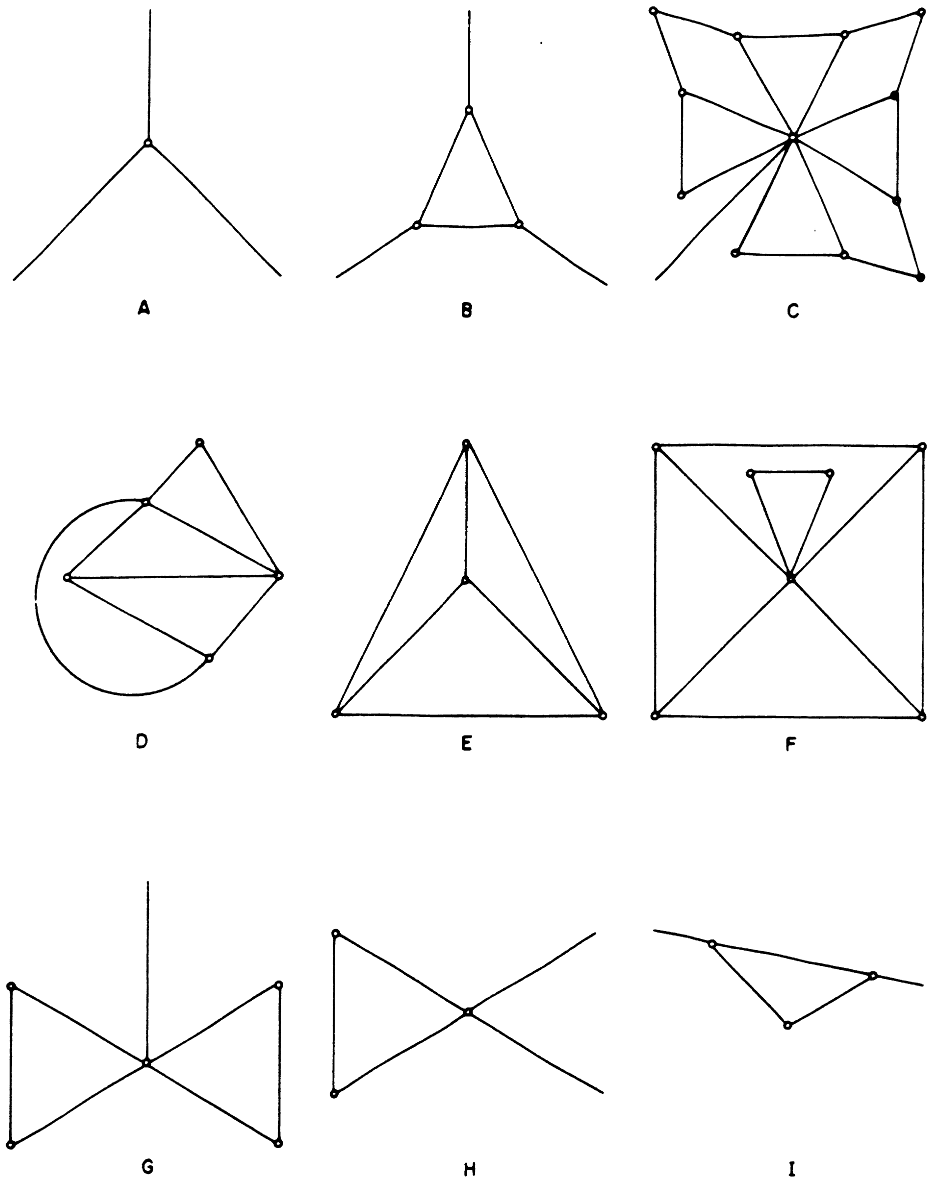

Fig. 6. Non-realisable graphs.

also adjacent to another of degree 1, which is detected by V3.2f.

In the next examples their non-realisability is detected When vertices VTTB one are considered.

- In the graph of Fig. 7(a) two vertices are simultaneously adjacent to the same pair of end nodes, condition detected by V3.3a.
- In the graph of Fig. 7(b) a VTTB rectangle exists in the wrapping of a vertex adjacent to other of degree 1, which is detected by V3.3b.
- In the graph of Fig. 7(c), 4 sets of VTTB vertices become 3 after being united by vertical vertices, which is detected by V3.4a.
- Graph of Fig. 7(d) has a vertex whose wrapping has 2 parts, one of which includes a VTTB vertex condition detected by V3.4b.
- Graph of Fig. 7(e) has a vertex with open wrapping that has 3 VTTB vertices, which is detected by V3.4c.
- In the graph of Fig. 7(f), a vertex has a closed wrapping with in sufficient vertices condition detected by V3.4d.
- In the graph of Fig. 7(g), the vertex adjacent to that of degree 1 has, in the other part of its wrapping, one vertex requiring a complete dimension together with another one, which is detected by V3.4e.
- In the graph of Fig. 7(h), When the initial sets of ATTB vertices are united by VTTB ones one vertex has a degree greater than 2, condition detected by V3.4f.

Let us see now another cause of non-realisability: in the

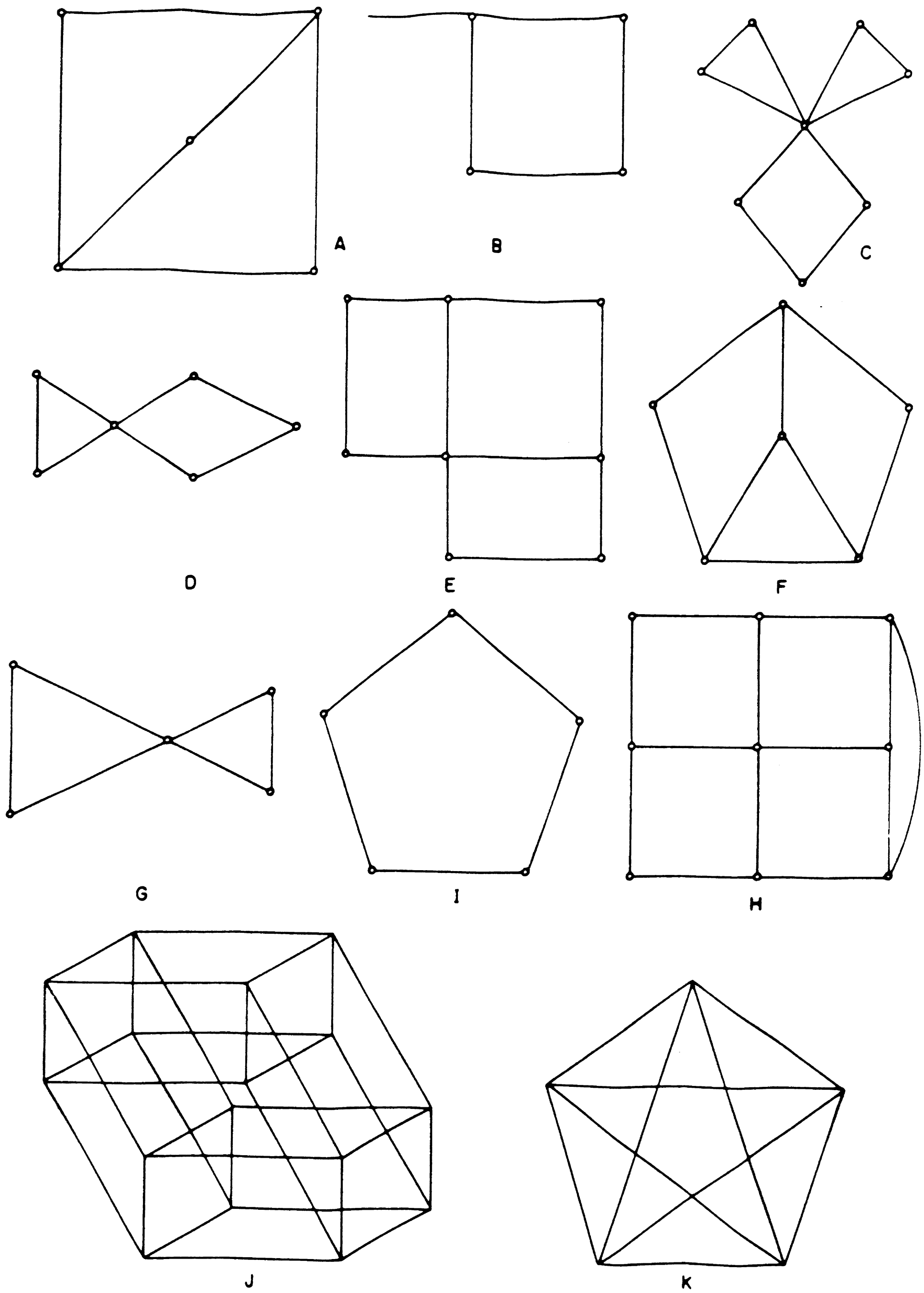

Fig. 7. Non-realisable graphs detected when VTTB vertices are considered.

graph of Fig. 7(i) the wrapping of each of the vertices has two parts but if one vertex is suppressed the resulting graph is not disconnected. This is detected by V3.5a. Let us see now examples of violation of the final global checks:

- In the graph of Fig. 7(j) shows a graph (16 vertices, each of them with a closed wrapping formed by 4 adjacent and 4 VTTB vertices) in which all vertices are inner ones, what is detected by V4a.
- In the graph of Fig. 7(k) shows a graph (10 vertices, each of them with an open wrapping formed by a set of 4 VTTB vertices) whose vertices are all of them outer and all of them of degree 4, what is detected by V4b.

Evidently, a non-planar graph is not realisable Kuratowski's Theorem says that the necessary and sufficient condition for a graph to be non-planar is that it contains either K3,3 (Fig. 8(a)) or K5 (Fig. 8(b)) graphs. The algorithm detects both graphs. K3,3 or any graph containing it can be detected by V3.2d or by V3.4f, depending on the order

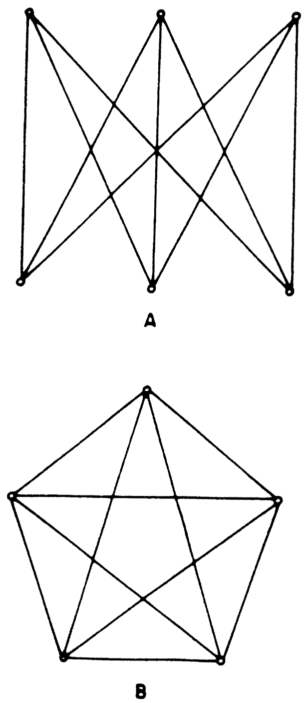

Fig. 8. K3, 3 and K5 graphs.

in which edges are considered. K5, or any graph containing it, 15 detected by V3.2b.

6. Conclusions

The problem of mapping a graph into a given plan of general shape, with its sides parallel to two perpendicular directions, does not seem to have not having a straightforward approach. This is the reason why the authors have chosen an indirect approach, considering first the case of a rectangular plan.

Of the two aspects of the problem, determining whether a graph admits this kind of realisation or not, and if so, automatically generating these realisations, only the first one is considered in this article.

The problem of deciding whether a graph can be mapped into a set of rectangles covering a rectangle is of practical interest in areas such as architectural design or compacting of electronic circuits, among others. The authors have not found references to a general solution to this problem in the bibliography they have found available.

A method based on a set of necessary conditions is presented here which can be implemented by means of an algorithm that allows one to determine whether a graph can be considered realisable or not. This method is mainly based on the analysis of each vertex wrapping.

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