Rectangular Duals of Planar Graphs

Krzysztof Koźmiński and Edwin Kinnen
Department of Electrical Engineering, University of Rochester, Rochester, New York 14627

Let one face of a cube be dissected into rectangles, no 4 of which meet at a single point. The dual graph of this configuration is a 4-connected triangulated plane graph. This paper shows that any 4-connected plane triangulation with at least 6 vertices and at least one vertex of degree 4 is dual to a cube with one face dissected into rectangles. The proof of this result contains an implicit algorithm for obtaining such a dissection. The paper also discusses a related problem: Given a graph G with all faces triangular except the outer face, does there exist a dissection of a rectangle into rectangles for which G describes the adjacency relations among the rectangles?

I. INTRODUCTION

The results presented here were obtained while investigating methods for solving a problem of rectangular dualization for area planning in VLSI design. The problem is to find a dissection of a rectangle into rectangles that satisfies specified adjacency relations among the component rectangles and results in the smallest area under a set of constraints on these rectangles. Earlier this problem was investigated for its application to architectural design [1, 6, 7]. A subproblem that has to be solved for these applications is to determine if the adjacency relations can be satisfied. This can be stated as follows: *given a plane graph G, find if there exists a dissection D of a rectangle into rectangles such that (1) there exists a 1-1 correspondence between the vertices of G and the rectangles in the dissection, and (2) for any edge of G the rectangles corresponding to its endpoints abut.* Any such dissection D is called a *rectangular dual* of G.

A theoretical characterization of graphs that have rectangular duals has been reported [1]; however, an efficient implementation of this theory as an algorithm does not appear possible [1, 7]. Necessary and sufficient conditions are developed below for the case when the nodes in the rectangular dual D have degree at most 3, or, equivalently, the faces of G have degree 3. This constraint on the node degrees allows for an efficient construction of a rectangular dual and is thought to have little importance in applications.

Section 2 discusses the concept of embedding a rectangular dissection on the face of a cube and presents a theorem characterizing graphs dual to such a configuration. Ap-

NETWORKS, Vol. 15 (1985) 145–157
 CCC 0028-3045/85/020145-13$04.00

plication of this theorem to the problem of finding a rectangular dual and an outline of an algorithm for rectangular dualization are discussed in Sections 3 and 4.

Before proceeding with the main theorem, we state an useful lemma characterizing 4-connected plane triangulations.

Lemma 1. A triangulated plane graph G is 4-connected if and only if each simple cycle of G that is not a face has length at least 4.

The proof of this lemma is given in the Appendix.

2. MAIN THEOREM

Let a face of a cube be dissected into rectangles, no 4 of which meet at a single point. The nondissected sides of the cube and the rectangles in the dissection are interpreted as the faces of a planar graph D embedded on the cube. Consider the plane graph G that is dual to this configuration.

As all vertices of D have degree 3, each face of G is a triangle, i.e., G is a plane triangulation. G has at least six vertices. The side of the cube opposing the side dissected into rectangles is dualized into a vertex of degree 4. We note that at least 4 rectangles are needed to fully enclose some area defined by a rectilinear polygon on a plane. A generalization of this observation for rectilinear polygons on a cube, expressed in terms of a dual graph, is that each cycle of G that is not a face has length at least 4. According to lemma 1, this is equivalent to G being 4-connected. A plane 4-connected triangulation with at least 6 vertices and at least one vertex of degree 4 is called a *4-triangulation.*

Theorem 1. For a cube D with one face dissected into rectangles, no 4 of which meet at a single point, to be dual to a plane graph G, it is necessary and sufficient that G is a 4-triangulation.

Proof. (A) Necessity: This is shown in the paragraph preceding the statement of the theorem.

(B) Sufficiency: Contrary to the theorem, suppose that for some 4-triangulation there exists no dual cube with one face dissected into rectangles. From the class of all 4-triangulations with this property, take graph G with the smallest possible number of vertices. Let t be a vertex of degree 4 in G and r, u, l, b be the four vertices adjacent to t, ordered clockwise. There are two possibilities: (a) at least one of the vertices r, u, l, b has degree 4, and (b) all r, u, l, b have degree at least 5.

Case a. Without a loss of generality assume degree 4 for the vertex r as shown in the representative sketch of G in Figure 1.

As there exist edges rt, ru, and rb, the remaining edge incident to r must end at some vertex v. Both faces containing the edge rv are triangles; hence edges uv and bv must also exist. By deleting vertex r together with the incident edges and introducing edge vt, a plane triangulation G' results which will be shown to be a 4-triangulation:

Note that G must contain at least 7 vertices to contradict the theorem, as there exists an unique 4-triangulation on 6 vertices; this 4-triangulation is dual to a non-dissected cube. Consequently G' has at least 6 vertices. Vertex t in G' has degree 4.

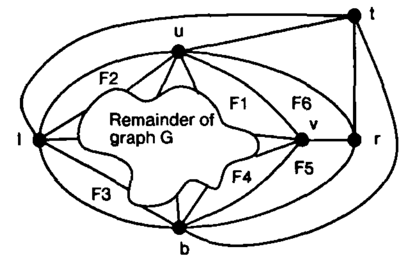

FIG. 1. Sketch of graph G with vertex r of degree 4.

Consider any two vertices x, $y \neq r$ in G. From the 4-connectedness and from Menger's theorem, there are 4 vertex-disjoint paths between x and y [5]. Take paths such that each is a shortest possible. If none of these paths uses edges ur or rb, then the same paths exist in G' with the edge vt substituted in place of the path vrt wherever it appears. Otherwise, suppose that one of the 4 paths passes through the edge ur. Being a shortest possible path, it must not use vertex t or v; therefore it must use the edge rb. Any other path using vertex t would pass through l and b, and not be a shortest possible. Thus vertex t is unused and by substituting path urb in G by utb in G' one concludes that there are 4 vertex-disjoint paths between any two vertices of G'. Menger's theorem states that such a graph is 4-connected.

As the 4-triangulation G' has fewer vertices than G and G was assumed to be a smallest contradictory graph, an embedding of a dual graph of G can be found in the form of a cube with one face dissected into rectangles. This embedding can be transformed into an embedding of the dual of G as shown in Figure 2, thus invalidating the contradictory assumption.

Case b. All vertices r, u, l, b have degree at least 5. G appears as shown in Figure 3, with at least four more vertices (labeled v_1, v_2, v_3, v_4). Initially we show that there eixsts a cycle C in G passing through r, t, l but not through b or u, such that deleting the vertices of C from G separates G into two parts each containing at least 2 vertices. C is called a cutting cycle.

Case b1. Suppose that there is an edge v_1v_3 in G (not shown in Figure 3). $C = tlv_3v_1r$ is a cutting cycle which separates G into two parts, one of which contains at least b and v_2 and another at least u and v_4.

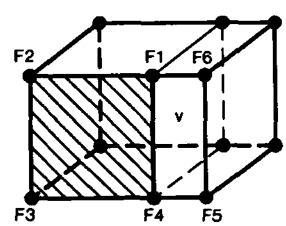

FIG. 2. Construction of an embedding of the dual of G from an embedding of a dual of G'. The dissected side is shaded.

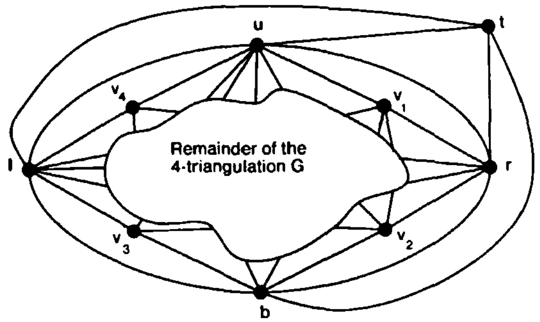

FIG. 3. Sketch of the graph G with 4 vertices of degree at least 5.

Case b2. Alternatively, suppose that there is no edge $v_1 v_3$ in G. All vertices adjacent to b and different from t, l, r lie on a path $v_2 y_1 \cdots y_n v_3$ originating in v_2 and ending in v_3, created from edges not incident with b but from the faces containing b. Analogously, all vertices adjacent to v_3 lie on a path $lx_1 \cdots x_m b$. Referring to Figure 4, these paths have a vertex in common, $y_n = x_m$; this follows from the triangularity of all faces. None of $y_1 \cdots y_n$ or $x_1 \cdots x_m$ can be equal to u or v_1 as the existence of an edge bu, bv_1, or $v_3 u$ would contradict the 4-connectedness of G, e.g., the existence of an edge bv_1 would imply a cut-set $\{r, b, v_1\}$ in G separating v_2 from t. Also $v_1 v_3$ does not exist by assumption. Then $C = tlx_1 \cdots x_m y_{n-1} \cdots y_1 v_2 r$ is a cutting cycle which separates G into two parts, one of which contains at least b and v_1 and another at least u and v_3.

Once the existence of a cutting cycle has been established, there also exists a shortest cutting cycle $C = tw_0 w_1, w_2, \ldots, w_n w_{n+1}$ which without a loss of generality is assumed to separate v_3 and v_1. A sketch is given in Figure 5, where $l = w_0$ and $r = w_{n+1}$.

Construct graph G_u from the subgraph contained in the interior of the cycle C by adding a vertex b' and edges $b't$, $b'w_0$, $b'w_1, \ldots, b'w_n$, $b'w_{n+1}$, as depicted in Fig-

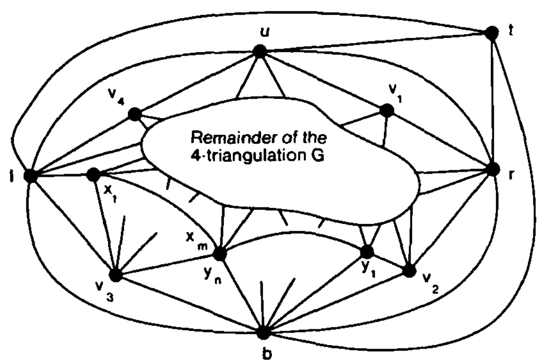

FIG. 4. A more detailed sketch of the graph G in the case when edge $v_1 v_3$ does not exist.

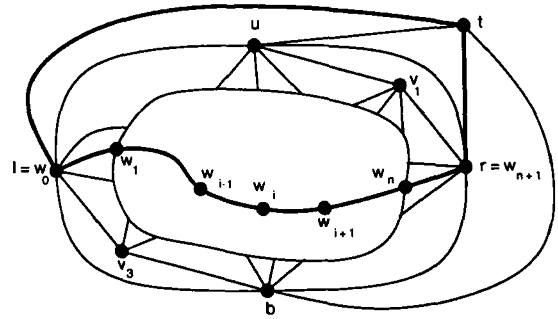

FIG. 5. A cutting cycle.

ure 6. The resulting graph G_u is a plane triangulation which will be shown to be a 4-triangulation.

Vertices t, l, u, b', r, and v_1 account for at least 6 vertices and t has degree 4. From the 4-connectedness of G, it follows that each cycle in G_u that is not a face and does not contain b' has length 4 or more. Any cycle containing b' and t that is a triangle is also a face, as there is no edge ub' by the construction and there is no edge lr by the 4-connectedness of G. Any other cycle containing b' has length at least 3 by the construction and has the form w_iw_jb', where $j = i \pm 1$ as otherwise the existence of an edge w_iw_j would imply that the cycle C is not a shortest. Therefore in G_u any cycle that is not a face has length $\geqslant 4$ and consequently G_u is 4-connected. In a similar manner, a second graph G_b can also be constructed from the subgraph contained in the exterior of the cycle C and shown to be a 4-triangulation. Both G_u and G_b have all faces of degree 3 and they have at least one vertex less than G. As G is assumed to be the smallest contradictory graph, duals of G_u and G_b can be embedded on cubes with one face dissected. These cubes can be placed one on the top of the other and merged (possibly after a homeomorphic transformation of one of them which would preserve orthogonal directions) such that the resulting cube with one face dissected is an embedding of the dual of G. See Figure 7.

This completes the proof, by invalidating the assumed nonexistence of an embedding of the dual of G in a cube with one face dissected.

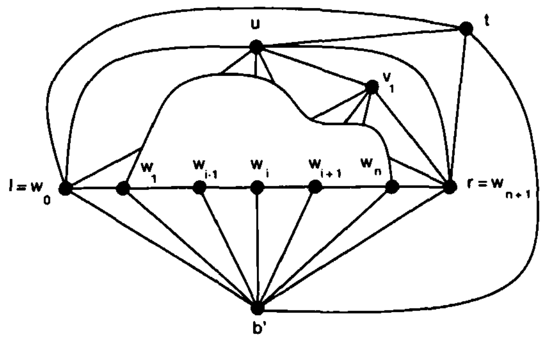

FIG. 6. The appearance of G_u.

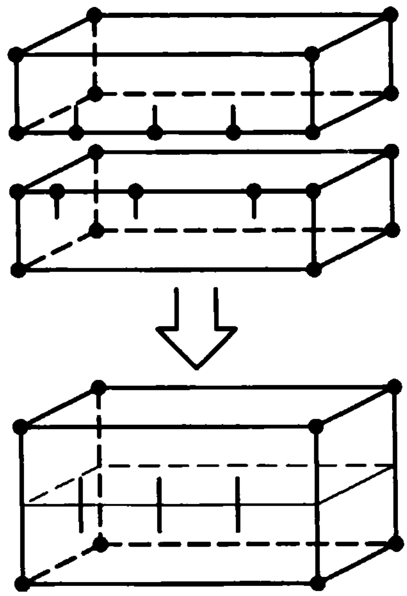

FIG. 7. Merging the embeddings of the duals of G_u and G_b into an embedding of the dual of G.

3. FINDING A RECTANGULAR DUAL OF A PLANE GRAPH

In this section, we consider the problem of finding a rectangular dual in the sense introduced by Heller et al. in [2, 4]: *Given a connected plane graph G with all faces of degree 3, find if there exists a dissection of a rectangle into rectangles for which G describes adjacencies among the rectangles.* The following is a corollary of Theorem 1:

Theorem 2. A plane graph G with all faces triangular except the outer one has a rectangular dual if and only if it can be obtained from some 4-triangulation H by deleting a vertex of H of degree 4 together with its 4 neighbor vertices.

In the following, we investigate conditions for G to be a subgraph of H as stated in the theorem. Such an H, if one exists, is called a *4-completion* of G. A method for checking the existence of H is to try to attach the missing part to G. This consists of putting a cycle on 4 additional vertices u, r, b, l containing G in its interior. These vertices are called *outer vertices.* An additional vertex t is placed outside of the cycle and connected to the vertices u, r, b, l. Finally, edges between appropriate vertices of G and u, r, b, l are added to insure that the resulting graph is a 4-connected triangulation. This latter construction is considered below for two cases, when G is a block and for the general case of a graph.

Construction of a 4-Completion of a Block

To construct a 4-completion, assign 4 labels v_{br}, v_{bl}, v_{ul}, v_{ur} to some set of $\leqslant 4$ clockwise ordered vertices of the outside face of G. If this assignment later results in a construction of a 4-completion, then, after a rectangular dual of G corresponding to this 4-completion is found, the labeled vertices correspond to the corner faces of the rectangular dual. Therefore the labeled vertices are called *corner vertices.* These corner vertices divide the cycle bounding the outer face into $\leqslant 4$ edge disjoint paths called *outer paths.* Connect each vertex on path v_{br}-v_{bl} to b, each vertex on path v_{bl}-v_{ul} to l, each vertex on path v_{ul}-v_{ur} to u and each vertex on path v_{ur}-v_{br} to r. If the resulting graph is 4-connected, then it is a 4-completion of G and its dual can be embedded on a cube with one face dissected into a rectangular dual of G. Figure 8(a) is an example of a block for which a rectangular dual is sought; the outer paths are: $v_{br}v_5 v_{bl}$, v_{bl} (a single vertex), $v_{ul}v_1 v_{ur}$, and $v_{ur}v_4 v_{br}$. The resulting 4-completion and rectangular dual are shown in Figure 8(b).

Labels need not be assigned to 4 different outer vertices of G. If some vertex is assigned two consecutive labels, then if a 4-completion is obtained, the face of a rectangular dual that corresponds to the doubly labeled vertex contains 2 corners of the rectangular embedding. For example, in Figure 8(a), $v_{ul} = v_{bl}$ and the face corresponding to $v_{ul} = v_{bl}$ contain two corners of the outermost cycle in the resulting rectangular dual. No vertex may be assigned 3 labels. In the trivial case of G consisting of only one vertex, all 4 labels are assigned to this vertex.

There may be several admissible choices of a set of 4 corner vertices that yield a 4-completion, but the number of possible different choices is not greater than $n^*(n-1)^*(n^2+7^*n-6)/24$, where $n \geqslant 2$ is the number of vertices in the outer face of G. This is the number of possible assignments of 4 indistinguishable labels to at most 4 out of n vertices with no more than 2 labels assigned to a single vertex. The labels are considered indistinguishable as all cyclical permutations are admissible. It

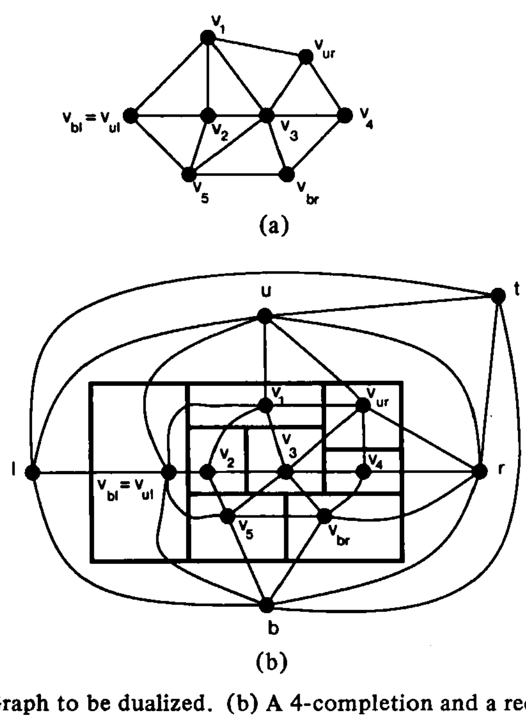

FIG. 8(a). Graph to be dualized. (b) A 4-completion and a rectangular dual.

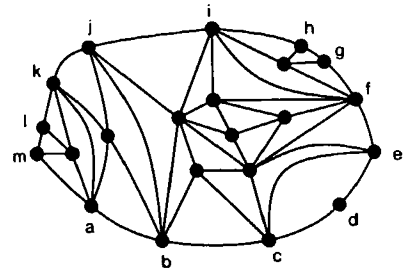

FIG. 9. ak, bj, fi, ce are shortcuts. Only jb is not a critical shortcut. Corner implying paths are *klma*, *cde*, and *fghi*.

is evident that for $n > 1$, the assignment of 3 or 4 labels to a single vertex cannot produce a 4-completion. If $n = 1$, there is only one possible labeling.

It is not necessary to examine all possible choices of corner vertices to determine if a block has a rectangular dual. The construction procedure for a 4-completion of a block fails only if a cycle of length $\leqslant 4$ is created that is not a face or if such a cycle existed in the block itself. In the latter case, no rectangular dual exists. To prevent the introduction of unwanted cycles during the construction, a check can be made on the existence of shortcuts defined below.

A *shortcut* in a plane block G is an edge that has both vertices on the outer face of G and does not belong to the outer face; see Figure 9.

Any cycle created by the addition of edges and vertices to G contains at least one of the nodes u, r, l, b and has length at least 3. All newly created cycles containing t that are not faces have length $\geqslant 4$ by the construction. Other cycles of length 3 are created only if some edge of G has both endpoints on the outer face and both of these endpoints are joined with edges to the same outer vertex. If such a cycle has a nonempty interior, the construction fails to yield a 4-completion. To avoid failures, the initial choice of vertices v_{br}, v_{bl}, v_{ul}, v_{ur} should ensure for each shortcut of G that its endpoints belong to different outer paths.

A uv path taken on the outermost cycle of a plane block G is said to *imply a corner* if vertices u, v are endpoints of a uv shortcut and if this path contains no other endpoints of a shortcut except for u and v. This uv shortcut is said to be a *critical shortcut.*

Corner implying paths may share endpoints but the paths are edge disjoint. They always contain at least 3 vertices, as a single edge would create a face of degree 2 with a shortcut.

As vertices of a shortcut uv cannot be connected to the same outer vertex of a 4-completion under construction, the path uv must contain one of the corner vertices.

Theorem 3. A plane block G with all faces of degree 3, where each cycle that is not a face has length at least 4, has a rectangular dual if and only if it has no more than 4 corner implying paths.

Proof. Necessity: If G has 5 or more corner implying paths then for any choice of v_{br}, v_{bl}, v_{ul}, v_{ur} at least one outer path will contain 3 clockwise consecutive endpoints of corner implying paths, two of which must belong to the same shortcut. Consequently the construction will not yield a 4-completion, as the shortcut with endpoints joined to the same outer vertex would create a cycle of length 3 with at least one vertex in its interior.

Sufficiency: If G has $n \leqslant 4$ corner implying paths, pick n of the vertices v_{br}, v_{bl}, v_{ul}, v_{ur} on each corner implying path, but not as the endpoints of these paths. The remaining $4 - n$ vertices can be chosen anywhere on the outermost cycle. By construction, any newly created cycle is either a face or has length $\geqslant 4$.

Construction of a 4-Completion of a Plane Connected Graph *G*

In general, G need not be a block but it may contain several maximal blocks. It is known that two maximal blocks can have at most one vertex in common.

A *block neighborhood graph (BNG)* for a given graph G is a graph where each vertex corresponds to each maximal block of G, and two vertices are connected with an edge if and only if the corresponding blocks have a vertex in common.

Theorem 4. A plane graph G with all faces of degree 3 has a rectangular dual only if no maximal block B_i of G lies inside a face of some other maximal block B_j and the block neighborhood graph of G is a path, i.e., has no vertices of degree 3 or more and no cycles.

Proof outline. As cycles of length 2 or 1 are never created in a BNG by the definition of the block neighborhood graph, suppose that there exists a cycle of length 3 or more; refer to Figure 10. Existence of this cycle implies that some cut vertex v of G belongs to 3 or more blocks. Therefore, consider the clockwise ordered set A of at least 6 edges in v that belong to the outer face of maximal blocks containing v. All eges in A belong to faces adjacent to some outer vertex of a 4-completion if it exists. As these faces have 3 edges each, where two edges are incident with an outer vertex of a 4-completion, each angle created by clockwise consecutive edges belonging to different maximal blocks must be split by an edge incident to some outer vertex. Therefore the construction of a 4-completion inserts at least 3 edges incident with the cut vertex under consideration. As cycles of length 2 are not admitted in a 4-completion, every possibility for joining these edges to outer nodes will produce a cycle of length 3 that is not a face; hence no 4-completion exists.

A conceptually similar argument holds for the BNG nodes of degree 3 or more which

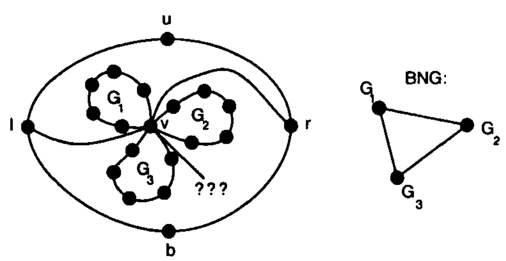

FIG. 10. No 4-completion results for any termination of the edge in question.

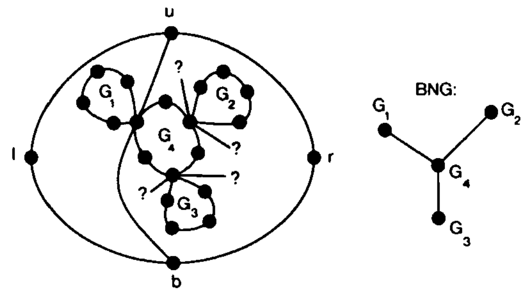

FIG. 11. No 4-completion results for any termination of the edges in question.

correspond to blocks having vertices in common with at least 3 other blocks. An illustration is given in Figure 11.

A *critical corner implying path* in a maximal block G_i of a plane graph G is a corner implying path of G_i that does not contain cut-vertices of G.

Theorem 5. A plane graph G with all faces of degree 3 has a rectangular dual if and only if (1) each cycle that is not a face has length at least 4, (2) its block neighborhood graph is a path, (3) both maximal blocks corresponding to the endpoints of the block neighborhood graph each contain at most 2 critical corner implying paths, and (4) no other maximal block contains a critical corner implying path.

This theorem is a consequence of the fact that cut-vertices of G correspond to faces having an edge in common with two opposite sides of the rectangular dual and can be proved by noting that each maximal block must have a rectangular dual.

4. AN ALGORITHM OUTLINE

An algorithm for finding a rectangular dual of a plane graph G has been developed [3]; however, a detailed exposition of this algorithm is beyond the scope of this paper. The first stage of the algorithm checks the degrees of vertices and faces for a given plane block G. Then Theorems 3, 4 and 5 are implemented to produce a plane graph H which is a 4-completion of G, provided that G does not contain non-face cycles of length <4. In the second stage, H is checked for 4-connectedness by application of the 4-completion splitting procedures described in the proof of Theorem 1. This stage proceeds in a "divide and conquer" manner by splitting H along the shortest cutting cycle until either the trivial 4-completion on 6 vertices results or a cutting cycle of length $\leqslant 3$ is found. The latter case is reported as an existence of a cycle of length <4 that is not a face in the dualized graph. The third and final stage of the algorithm uses information about cutting paths from stage 2 to create a rectangular dual of G by combining the rectangular subgraphs in a bottom-up fashion.

The computational complexity of the first stage is linear in the number of edges, as the only activity required is to check the degrees of the faces and vertices of G and to scan the list of edges to determine shortcuts. A clockwise scan of the vertices on the outermost cycle of each maximal block of G uncovers the critical shortcuts and corner implying paths.

In the second stage, the number of times the 4-dual splitting algorithm is applied is linear in the number of vertices of the graph G. This follows as each application of a cutting cycle to split H results in at least one vertex of G to be adjacent to at least one more outer vertex of the corresponding part of H. Each vertex may be adjacent to at most 4 outer vertices. Consequently, the splitting procedure is applied at most $4n$ times, where n is the number of vertices in G. A Lee algorithm is used to find shortest paths from l to r not going through t, b, or u. The complexity of this algorithm is $O(n)$, where n is the number of vertices of the graph. Thus the total complexity of the second stage is $O(n^2)$.

The computational complexity of the third stage can also be shown to be linear in the number of vertices in G.

5. CONCLUSION

A theory of dual rectangularization for plane graphs with all faces of degree 3 is given. An algorithm based on this theory is characterized by time complexity $O(n^2)$.

As a final remark, note that the property of a cutting cycle being a shortest one is not necessary for the algorithm. It is only required that any edge of a 4-completion with both endpoints on the cutting cycle belongs to this cycle. The final appearance of the rectangular dual is determined by the choice of the cutting cycles that split the 4-completions.

APPENDIX: PROOF OF LEMMA 1

Suppose that a plane triangulation contains a cycle of length $\leqslant 3$ that is not a face. Vertices of this cycle are a cut-set of cardinality less than 4, separating the vertices in the interior of this cycle from those outside of the cycle and clearly the triangulation is not 4-connected.

Conversely, suppose that in a triangulation G all cycles that are not faces have length 4. The lemma can be easily verified for all triangulations with no more than 6 vertices. Suppose it holds for triangulations with $n \geqslant 6$ vertices. For a triangulation with $n + 1$ vertices, all vertices should have degree at least 4, as neighbors of a vertex with smaller degree would create a cycle of length < 4 and not a face. From the planarity it follows that there is at least one vertex of degree 4 or 5 in G.

Case 1. There is a vertex of degree 4, say, v. The proof resembles part (a) from Theorem 1, but is done in the opposite direction. Let the neighbors of v be v_1, v_2, v_3, v_4. Delete v together with the adjacent edges. If there is vertex $w \neq v_2, v_4$ adjacent simultaneously to v_1 and v_3, add an edge $v_2 v_4$ (see Figure A1 and note that edges wv_4 and wv_2 cannot exist simultaneously with wv_1 and wv_3 in a plane triangulation with more than 6 vertices). If there is no such vertex w, add edge $v_1 v_3$. Without loss of generality, let the edge $v_2 v_4$ be added. No cycles of length < 4 that are not faces are created in this way and the resulting triangulation G' has n vertices; hence it is 4-connected by the induction hypothesis. Accordingly to Menger's theorem, there

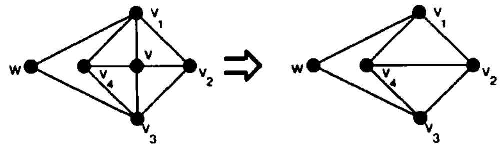

FIG. A1

exist 4 vertex-disjoint paths between any two vertices of G', the same paths exist in G with path v_2vv_4 substituted in place of edge v_2v_4 wherever this last occurred. Deleting less than 4 vertices from G leaves at least one of the vertices v_1, v_2, v_3, v_4 adjacent to v and at least one path from the leftover vertex to any other vertex of G that is different from v. Hence G is 4-connected.

Case 2. There are no vertices of degree 4 but there is a vertex of degree 5, say, v. Let the set of its neighbors be $N = \{v_1, v_2, v_3, v_4, v_5\}$. Suppose there is a vertex $w \notin N$ adjacent simultaneously to a pair of nonadjacent vertices from N. Without a loss of generality let these vertices be v_1, v_3. Note that there cannot be an edge wv_2, as this would imply degree 4 for v_2 or a triangle being not a face in G. Producing a plane triangulation G' by deleting vertex v together with the adjacent edges and adding edges v_2v_4 and v_2v_5 does not create any cycle of length <4 that is not a face. (See Figure A2.) The same construction holds when there is no vertex w as described above. G' contains n vertices, therefore it is 4-connected by the induction hypothesis. Again from Menger's theorem, there exist 4 vertex-disjoint paths between any two vertices of G'; in particular there are paths from v_1, v_3, v_4, v_5 to any other vertex except possibly v_2 that use at most one of the added edges v_2v_4 and v_2v_5. Also, note that the degree of v_2 in G is at least 5; therefore it has at least 2 neighbors different from v, v_1 and v_3, say, u_1 and u_2, that are connected to each of v_1, v_3, v_4, v_5 with at least 4 vertex-disjoint paths using at most one of the added edges. The same paths exist in G with path v_2vv_4 substituted in place of edge v_2v_4 and path v_2vv_5 substituted in place of edge v_2v_5 wherever these occurred. Deleting less than 4 vertices from G leaves at least one of the vertices v_1, v_3, v_4, v_5 that are neighbors of v and at least one path from the left vertex to any other vertex in G that is different from v and v_2. Also, at least one of the neighbors of v_2 that is different from v must be left. Hence deleting less than 4 vertices from G will yield a connected graph. Thus the inductive proof of Lemma 1 is completed. ∎

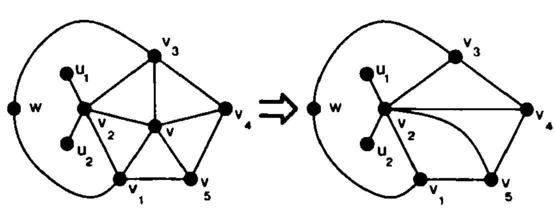

FIG. A2

The authors thank Professor Arthur H. Stone for his helpful comments and discussion. Thanks are also due to the anonymous referee for pointing out an elegant approach for embedding a rectangular dissection on the face of a cube, simplifying our initial notation.

This work has been supported in part by the Semiconductor Research Corporation, Research Triangle Park, North Carolina and by Scientific Calculations Inc., Fishers, NY.

References

[1] J. Grason, A dual linear graph representation for space filling location problems of the floor plan type. *Emerging Methods in Environmental Design and Planning* (G. T. Moore, Ed.). *Proceedings of The Design Methods Group, 1st International Conference*, Cambridge, MA (1968), pp. 170–178.

[2] W. R. Heller, G. Sorkin, and K. Maling, The planar package planner for system designers. *Proceedings of the 19th Design Automation Conference, IEEE and ACM*, Las Vegas, (1982), pp. 253–260.

[3] K. Koźmiński and E. Kinnen, An algorithm for finding a rectangular dual of a planar graph for use in area planning for VLSI integrated circuits. *Proceedings of the 21th Design Automation Conference, IEEE and ACM*, Albuquerque, June (1984).

[4] K. Maling, S. H. Mueller, and W. H. Heller, On finding most optimal rectangular package plans. *Proceedings of the 19th Design Automation Conference, IEEE and ACM*, Las Vegas, June (1982), pp. 663-670.

[5] K. Menger, Zur allgemeinen Kurventheorie. *Fund. Math.* **10** (1927) 96–115.

[6] W. J. Mitchell, J. P. Steadman and R. S. Liggett, Synthesis and optimization of small rectangular floorplans. *Environ. Planning B* 3 (1976) 37-70.

[7] P. Steadman, Graph-theoretic representation of architectural arrangement. *The Architecture of Form* (L. March, Ed). Cambridge Univ. Press, London, New York, Melborune (1976), pp. 94–115.

Received October 18, 1983
Accepted May 29, 1984