PES1201802826 Rahil N Modi Date 6/04/2020 Page

LA Assignment

1. Find the equations of parabola $y = A + Bx + Cx^2$ that passes through 3 points $(1,1)$, $(2,-1)$, and $(3,1)$ using Gaussian Elimination.

Sol: Given: 3 Points: $(1,1)$ $(2,-1)$ $(3,1)$

Parabolic equation: $y = A + Bx + Cx^2$

Substitute 3 points in equation as they pass through it.

$$A + B + C = 1$$
$$A + 2B + 4C = -1$$
$$A + 3B + 9C = 1$$

Converting them to matrix form:

$$A \quad x = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$[A \ \ b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

By back substitution,

(1) $2C = 4$

$\boxed{C = 2}$

(2) $B + 3C = -2$

$B + 3(2) = -2$

$B = -2 - 6$

$\boxed{B = -8}$

(3) $A + B + C = 1$

$A - 8 + 2 = 1$

$A = 1 + 6$

$\boxed{A = 7}$

$\therefore$ The equation of parabola is $\boxed{y = 2x^2 - 8x + 7}$

2. Find LU decomposition for the matrix :

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}_{4\times4}$$

Sol: Initially we apply Gaussian Elimination :

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}_{4\times4}$$

Date ______ Page ______

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 + 5R_1$; $R_4 \to R_4 - 5R_1$

$$A \approx \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$l_{21} = 2$

$l_{31} = -5$

$l_{41} = 5$

$R_3 \to R_3 + 2R_2$; $R_4 \to R_4 + 2R_2$

$$A \approx \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$l_{32} = -2$

$l_{42} = -2$

$R_4 \to R_4 - 3R_3$

$l_{43} = 3$

$$A \approx \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$

$\therefore$ $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}$$

$\therefore$ $$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

(i) Find matrix T relative to standard basis of $\mathbb{R}^3$
(ii) Find the basic for 4 fundamental subspace of T.
(iii) Find Eigen Values & Eigen Vectors of T,
(iv) Decompose $T = QR$

Sol: (i) Standard Base of $\mathbb{R}^3$ $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y-z \\ y+z \\ x+y-2z \end{bmatrix}$$

$(1,0,0) \leftarrow\!\!\rightarrow (1,0,$

(i) $x+2y-z$

$$\begin{array}{ccc} x & y & z \\ 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} & -1\begin{bmatrix}0\\0\\1\end{bmatrix} \end{array} = x+2y-z$$

$$\begin{array}{ccc} x & y & z \\ 0\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} & +1\begin{bmatrix}0\\0\\1\end{bmatrix} \end{array} = y+z$$

$$1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\end{bmatrix} = x+y-2z$$

$$\therefore\ T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Date ______ Page ______

(ii) $T = \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & -2 & b_3 \end{array}\right]$

$R_3 \to R_3 - R_1$

$[T\ b] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & -1 & b_3 - b_1 \end{array}\right]$

$R_3 \to R_3 + R_2$

$[T\ b] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{array}\right] \longrightarrow \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Row Reduced form ($R_1 \to R_1 - 2R_2$

$= \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ Basis of $C(T) = \{(1,0,1), (2,1,1)\}$

Basis of $R(T)$ or $C(A^T) = \{(1,2,-1), (0,1,1)\}$

Basis of $N(A) = \{(3,-1,1)\}$

Basis of $N(A^T) = \{(-1,1,1)\}$

(iii) Eigen Vectors & Values:

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|T| = 1(-3) - 2(-1) - 1(-1)$$
$$= -\cancel{3} + \cancel{2} + \cancel{1} = 0$$

$$\lambda^3 - (0)\lambda^2 + (-3 - 1 + 2)\lambda - 0 = 0$$

$$\therefore \lambda^3 - 3\lambda = 0$$
$$\lambda(\lambda^2 - 3) = 0$$
$$\lambda = 0, \quad \lambda^2 = 3 \Rightarrow \lambda = \pm\sqrt{3}$$
$$\therefore \lambda_1 = 0; \ \lambda_2 = \sqrt{3}; \ \lambda_3 = -\sqrt{3}$$ — Eigen Values

Eigen Vectors: $(A - \lambda I)x = 0$

(1) $\lambda_1 = 0$

$$Ax = 0$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

By Gaussian Elimination:

$$R_3 \to R_3 - R_1$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$R_1 \to R_1 - 2R_2$$

Date______ Page______

$$R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad V_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

Date ______ Page ______

(2) $\lambda_2 = \sqrt{3}$

$$(A-\lambda I) = \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$$

$R_1 \rightarrow -R_1\left(\frac{1+\sqrt{3}}{2}\right)$

$$= \begin{bmatrix} +1 & -(1+\sqrt{3}) & +\frac{(1+\sqrt{3})}{2} \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$$

$-2-\sqrt{3} - \frac{1-\sqrt{3}}{2}$

$R_3 \rightarrow R_3 - R_1$

$$= \begin{bmatrix} +1 & -(1+\sqrt{3}) & +\frac{(1+\sqrt{3})}{2} \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & \frac{-3\sqrt{3}-5}{2} \end{bmatrix}$$

$\frac{-4-2\sqrt{3}-1-\sqrt{3}}{2}$

$R_2 \longrightarrow \frac{-(1+\sqrt{3})}{2} R_2$

$$= \begin{bmatrix} 1 & -(1+\sqrt{3}) & \left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 1 & -\left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 2+\sqrt{3} & \frac{-3\sqrt{3}-5}{2} \end{bmatrix}$$

$\frac{-3\sqrt{3}-5}{2} + \frac{(2+\sqrt{3})(1+\sqrt{3})}{2}$

$\frac{\cancel{-3\sqrt{3}}\cancel{-5}+\cancel{3\sqrt{3}}+\cancel{5}}{2}$

$R_3 \longrightarrow R_3 - (2+\sqrt{3})R_2$

$$= \begin{bmatrix} 1 & -(1+\sqrt{3}) & \left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 1 & -\left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$\frac{1+\sqrt{3}}{2} \quad \frac{-(4+2\sqrt{3})}{2}$

$R_1 \longrightarrow R_1 + (1+\sqrt{3})R_2$

$\frac{-3-\sqrt{3}}{2}$

$$= \begin{bmatrix} 1 & 0 & -\frac{(3+\sqrt{3})}{2} \\ 0 & 1 & -\left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad V_2 = \begin{bmatrix} +\left(\frac{3+\sqrt{3}}{2}\right) \\ \frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

③ $\lambda_3 = -\sqrt{3}$

$$(T - \lambda I) = \begin{bmatrix} 1+\sqrt{3} & +2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$$

$R_1 \to \frac{(\sqrt{3}-1)}{2} R_1$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$(-2+\sqrt{3}) - \frac{(1-\sqrt{3})}{2}$$

$$\frac{-4+2\sqrt{3}-1+\sqrt{3}}{2}$$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 2-\sqrt{3} & \frac{-5+3\sqrt{3}}{2} \end{bmatrix}$$

$R_2 \to \frac{(\sqrt{3}-1)}{2}$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1 & \left(\frac{\sqrt{3}-1}{2}\right) \\ 0 & 2-\sqrt{3} & \frac{-5+3\sqrt{3}}{2} \end{bmatrix}$$

$R_3 \to R_3 - (2-\sqrt{3}) R_2$

$$-[2\sqrt{3}-2+\sqrt{3}-3]$$

$$\frac{-5+3\sqrt{3}}{2} - \frac{(2-\sqrt{3})(\sqrt{3}-1)}{2}$$

$$\frac{-5+3\sqrt{3}+5-3\sqrt{3}}{2}$$

$$U = \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1 & \left(\frac{\sqrt{3}-1}{2}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - (\sqrt{3}-1) R_2$

$$\frac{1-\sqrt{3}}{2} - \frac{(3+1-2\sqrt{3})}{2}$$

$$\frac{1-\sqrt{3}-4+2\sqrt{3}}{2}$$

$$\frac{-3+\sqrt{3}}{2}$$

$$= \begin{bmatrix} 1 & 0 & -\left(\frac{3-\sqrt{3}}{2}\right) \\ 0 & 1 & \left(\frac{\sqrt{3}-1}{2}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore V_3 = \begin{bmatrix} \left(\frac{3-\sqrt{3}}{2}\right) \\ \left(\frac{1-\sqrt{3}}{2}\right) \\ 1 \end{bmatrix}$

Date______ Page______

$\therefore$ Eigen Values : $\lambda_1 = 0$; $\lambda_2 = \sqrt{3}$; $\lambda_3 = -\sqrt{3}$

Corresponding Eigen Vectors : $v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} \left(\frac{3+\sqrt{3}}{2}\right) \\ \left(\frac{1+\sqrt{3}}{2}\right) \\ 1 \end{bmatrix}$ $V_3 = \begin{bmatrix} \left(\frac{3-\sqrt{3}}{2}\right) \\ \left(\frac{1-\sqrt{3}}{2}\right) \\ 1 \end{bmatrix}$

(iv) $T = QR$

Sol: Given $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\quad\quad\quad\quad x_1 \quad x_2 \quad x_3$

From Gram Schmidt Process :

$$q_1 = \frac{x_1}{\|x_1\|} = \frac{(1,0,1)}{\sqrt{1^2+1^2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$e = b - p$

$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} -$

$$e = x_2 - (q_1^T x_2) q_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{3}{\sqrt{2}}\right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} +1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$\hat{x} a$

$\frac{a^T b}{a^T a}$

$\frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$

$$\therefore q_2 = \frac{e}{\|e\|} = \frac{(+\frac{1}{2}, 1, -\frac{1}{2})}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} = \begin{bmatrix} +1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$E = x_3 - \left[(q_1^T x_3) q_1 + (q_2^T x_3) q_3 \right]$$

$-1/\sqrt{6} + \frac{2}{\sqrt{6}} + 2/\sqrt{6}$

$$= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \left[-3/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + \frac{3}{\sqrt{6}} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right]$$

dependent

$$= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \left[\begin{bmatrix} -3/2 \\ 0 \\ -3/2 \end{bmatrix} + \begin{bmatrix} 3/6 \, 2 \\ 1 \\ -1/2 \end{bmatrix} \right] = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_{3\times3} = Q_{3\times2}\, R_{2\times3}$$

left inverse exist (Q^T)

$$\underset{3\times3}{T} = \underset{3\times2}{Q}\ \underset{2\times3}{R}$$

$$Q^T T = \underbrace{Q^T Q}_{I} R_{2\times2}$$

$$R = Q^T T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \end{bmatrix}_{2\times3}$$

$$\therefore\ T = QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}_{3\times2} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \end{bmatrix}_{2\times3}$$

4. Find a best fit line $y = C + Dx$ for the following data

x	-4	1	2	3
y	4	6	10	8

Sol: In the form of matrix

$$Ax = b$$

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}_{4\times2} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$(4 > 2)$ System is inconsistent, Hence we use least square method.
$m > n$

Date ____ Page ____

Normal equation

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{116} \begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 193/29 \\ 20/29 \end{bmatrix}$$

$\therefore$ Best fit line $y = C + Dx$

$$y = \frac{193}{29} + \frac{20}{29} x$$

5. Find the projection matrices P and Q onto the plane $x_1 + x_2 + 3x_3 + 4x_5 = 0$ and its orthogonal complement respectively.

Sol: Plane Equation in the form of matrix $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}}_{1\times 5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times 1} = \begin{bmatrix} 0 \end{bmatrix}$$

pivot   free

To find a matrix with columns as span of plane
we find $N(A) = \{ (-1,1,0,0,0),\ (-3,0,1,0,0),\ (0,0,0,1,0),\ (-4,0,0,0,1) \}$

Plane :

$$B = \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{5\times4}$$

Projection : $P = B(B^TB)^{-1}B^T$

$$= \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} B^T$$

$$= B \left(\begin{bmatrix} 2 & 3 & 0 & 4 \\ 3 & 10 & 0 & 12 \\ 0 & 0 & 1 & 0 \\ 4 & 12 & 0 & 17 \end{bmatrix} \right)^{-1} B^T$$

$$= \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 4} \left(\begin{bmatrix} \frac{26}{27} & \frac{-3}{27} & 0 & \frac{-4}{27} \\ -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 1 & 0 \\ \frac{-4}{27} & \frac{-12}{27} & 0 & \frac{11}{27} \end{bmatrix}_{4\times 4} \right) \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/27 & -3/27 & 0 & -4/27 \\ 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 1 & 0 \\ -4/27 & -12/27 & 0 & 11/27 \end{bmatrix}_{5\times 4} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 26/27 & -1/27 & -3/27 & 0 & -4/27 \\ -1/27 & 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 0 & 1 & 0 \\ -4/27 & -4/27 & -12/27 & 0 & 11/27 \end{bmatrix}_{5\times 5}$$

Orthogonal complement of $N(A)$ is $C(A^T)$

$$C(A^T) = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} = C$$

$$C = A(A^TA)^{-1}A^T$$

$$= \begin{bmatrix}1\\1\\3\\0\\4\end{bmatrix}_{5\times1} \left([1\ 1\ 3\ 0\ 4] \begin{bmatrix}1\\1\\3\\0\\4\end{bmatrix} \right)^{-1} [1\ 1\ 3\ 0\ 4]_{1\times5}$$

$$= \begin{bmatrix}1\\1\\3\\0\\4\end{bmatrix} (27) [1\ 1\ 3\ 0\ 4]$$

$$Q = 27 \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \\ 1 & 1 & 3 & 0 & 4 \\ 3 & 3 & 9 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 12 & 0 & 16 \end{bmatrix}$$

6. For which range of numbers 'a' the matrix A is positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

Which 3x3 symmetric matrix B produces the function $f = x^TAx$ when

$$f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$$

Sol: Consider matrix, $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

Date________ Page________

By det semi-matrices :

$a_1 \Rightarrow \quad a > 0 \quad - (1)$

$a_2 \Rightarrow \quad a^2 - 4 > 0$

$a^2 > 2 \quad \Rightarrow \boxed{a > 2} - (2)$

$\llcorner \; a < 2$ — not a case due to (1)

$a_3 \Rightarrow \quad |A| = a(a^2-4) - 2(2a-4) + 2(4-2a) > 0$

$= a^3 - 4a - 4a + 8 + 8 - 4a > 0$

$= a^3 - 12a + 16$

$a > -4 \;;\; a > 2 \;;\; a > 2$

$\therefore$ Range of a for A to be positive definite

$\Rightarrow \boxed{a > 2}$

$$B = \begin{bmatrix} 2 & -1/2 & 0 \\ -1/2 & 2 & -1/2 \\ 0 & -1/2 & 2 \end{bmatrix}_{3\times 3}$$

7. Find the SVD decomposition of A:

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times 2}$$

Sol:- SVD decomposition :

$$\underset{3\times2}{A} = \underset{3\times3}{U} \; \underset{3\times2}{\Sigma} \; \underset{2\times2}{V^T}$$

Find $A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} =$

Eigen Values: $\lambda^2 - 90\lambda + 0 = 0$

$$\lambda(\lambda - 90) = 0$$

$$\lambda_1 = 90 \quad \lambda_2 = 0 \qquad (\lambda_1 > \lambda_2)$$

Eigen Vectors

(i) $(A^TA - \lambda_1 I) = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$$= \begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{-9}$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

Vector $x_1 = \begin{bmatrix} +3 \\ -1 \end{bmatrix}$ $\qquad v_1 = \begin{bmatrix} +3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$

(ii) $(A^TA - \lambda_2 I) = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$ $\qquad R_2 \to R_2 + \frac{1}{3} R_1$

Date ____ Page ____

$$= \begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{81} \quad \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

$\sqrt{1/9 + 1}$

$\sqrt{\frac{10}{9}}$

$\therefore$ Vector $X_2 = \begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} -1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

$$V^T = \begin{bmatrix} +3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$$

Singular Values : $\sigma_1 = \sqrt{90}$

$\sigma_2 = \sqrt{0}$

$$\underset{3\times 2}{\Sigma} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

Eigen Values: $\lambda^3 - 90\lambda^2 + 0\lambda - 0 = 0$

$\lambda^2(\lambda - 90) = 0 \implies \lambda_1 = 90,\ \lambda_2 = 0,\ \lambda_3 = 0$

$$(AA^T - \lambda_1 I) = \begin{bmatrix} -80 & -20 & -20 \\ -20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{4}R_1 \qquad R_3 \to R_3 - \frac{1}{4}R_1$

$$= \begin{bmatrix} -80 & -20 & -20 \\ 0 & -45 & 45 \\ 0 & 45 & -45 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$R_1 \to \frac{R_1}{-80}$

$R_2 \to \frac{R_2}{-45}$

$$= \begin{bmatrix} +1 & 1/4 & 1/4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{4}R_2$

$$= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \therefore X_1 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \qquad V = \frac{X_1}{\|X_1\|} = \frac{(-1/2, 1, 1)}{\sqrt{1/4 + 1 + 1}}$$

$$= \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$(AA^T - \lambda_2 I) = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$= \begin{bmatrix} 10 & -20 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 \to \frac{R_1}{10}$$

$$X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad X_3 = \begin{bmatrix} -1/2 \\ -5/4 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - 2y - 2z = 0$

x – pivot var

y, z - free var

$x = 2y + 2z$

$$V_2 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \qquad V_3 = \begin{bmatrix} -2/\sqrt{45} \\ -5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}$$

$$\therefore \underset{3\times3}{U} = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix}$$

$$\therefore A = \underset{3\times3}{U} \; \underset{3\times2}{\Sigma} \; \underset{2\times2}{V^T}$$

$$= \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 0 & -5/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} +3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$$