

LA Assignment

10. Find the equations of parabola $y = A + Bx + Cx^2$ that passes through 3 points $(1, 1)$, $(2, -1)$, and $(3, 1)$ using Gaussian Elimination.

Sol: Given: 3 Points : $(1, 1)$ $(2, -1)$ $(3, 1)$

Parabolic equation: $y = A + Bx + Cx^2$

Substitute 3 points in equation as they pass through it.

$$A + B + C = 1$$

$$A + 2B + 4C = -1$$

$$A + 3B + 9C = 1$$

Converting them to matrix form:

$$A \ x = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$[A \ b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1 ; R_3 \rightarrow R_3 - R_1$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

By back substitution,

$$\textcircled{1} \quad 2C = 4$$

$$C = 2$$

$$\textcircled{2} \quad B + 3C = -2$$

$$B + 3(2) = -2$$

$$B = -2 - 6$$

$$B = -8$$

$$\textcircled{3} \quad A + B + C = 1$$

$$A - 8 + 2 = 1$$

$$A = 1 + 6$$

$$A = 7$$

\therefore The equation of parabola is $y = \underline{\underline{2x^2 - 8x + 7}}$

Q. Find LU decomposition for the matrix:

$$A = \left[\begin{array}{cccc} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{array} \right] \quad 4 \times 4$$

Sol. Initially we apply Gaussian Elimination:

$$A = \left[\begin{array}{cccc} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{array} \right] \quad 4 \times 4$$

$$R_2 \rightarrow R_2 - 2R_1 ; R_3 \rightarrow R_3 + 5R_1 ; R_4 \rightarrow R_4 - 5R_1$$

$$A \sim \left[\begin{array}{cccc} \textcircled{2} & 5 & 2 & -5 \\ 0 & \textcircled{2} & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{array} \right] \quad l_{21} = 2 \\ l_{31} = -5 \\ l_{41} = 5$$

$$R_3 \rightarrow R_3 + 2R_2 ; R_4 \rightarrow R_4 + 2R_2$$

$$A \sim \left[\begin{array}{cccc} \textcircled{2} & 5 & 2 & -5 \\ 0 & \textcircled{2} & -1 & -4 \\ 0 & 0 & \textcircled{3} & 5 \\ 0 & 0 & 9 & 11 \end{array} \right] \quad l_{32} = -2 \\ l_{42} = -2$$

$$R_4 \rightarrow R_4 - 3R_3 \quad l_{43} = 3$$

$$A \sim \left[\begin{array}{cccc} \textcircled{2} & 5 & 2 & -5 \\ 0 & \textcircled{2} & -1 & -4 \\ 0 & 0 & \textcircled{3} & 5 \\ 0 & 0 & 0 & \textcircled{-4} \end{array} \right] = U$$

$$\therefore L = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{array} \right]$$

$$\therefore A = LU = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{array} \right] \left[\begin{array}{cccc} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{array} \right]$$

3. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$$

i)

Find matrix T relative to standard basis of \mathbb{R}^3

ii)

Find the basis for 4 fundamental subspaces of T.

iii)

Find Eigen Values & Eigen Vectors of T,

iv)

Decompose $T = QR$

Sol:

i) Standard Basis of \mathbb{R}^3 $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y-z \\ y+z \\ x+y-2z \end{bmatrix}$$

$(1, 0, 0) \longleftrightarrow (1, 0, 0)$

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$$x+2y-z$$

$$\textcircled{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \textcircled{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \textcircled{3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x+2y-z$$

$$x \\ 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = y+z$$

$$z \\ 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x+y-2z$$

$$\therefore T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(ii)

$$T = \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & -2 & b_3 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_1$$

$$[T \ b] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & -1 & b_3 - b_1 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$[T \ b] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array} \right]$$

Row Reduced form: $R_2 \rightarrow R_1 - 2R_2$

$$= \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{c} 3 \\ -1 \\ 1 \end{array} \right]$$

$$\therefore \text{Basis of } C(T) = \{(1, 0, 1), (2, 1, 1)\}$$

$$\text{Basis of } R(T) \text{ or } C(A^T) = \{(1, 2, -1), (0, 1, 1)\}$$

$$\text{Basis of } N(A) = \{(3, -1, 1)\}$$

$$\text{Basis of } N(A^T) = \{(-1, 1, 1)\}$$

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Eigen Vectors & Values:

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\begin{aligned}|T| &= 1(-3) - 2(-1) - 1(-1) \\ &= -3 + 2 + 1 = 0\end{aligned}$$

$$\lambda^3 - (0)\lambda^2 + (-3+2)\lambda - 0 = 0$$

$$\therefore \lambda^3 - 3\lambda = 0$$

$$\lambda(\lambda^2 - 3) = 0$$

$$\lambda = 0, \quad \lambda^2 = 3 \Rightarrow \lambda = \pm\sqrt{3}$$

$$\therefore \lambda_1 = 0 ; \underline{\lambda_2 = \sqrt{3}} ; \underline{\lambda_3 = -\sqrt{3}} \quad \text{— Eigen Values}$$

$$\text{Eigen Vectors : } (A - \lambda I)x = 0$$

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$$\lambda_1 = 0$$

$$Ax = 0$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

By Gaussian Elimination:

$$R_3 \rightarrow R_3 - R_1$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$R^2 \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore V_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$$\textcircled{2} \quad \lambda_2 = \sqrt{3}$$

$$(A - \lambda I) = \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$$

$$R_1 \rightarrow -R_1 \left(\frac{1+\sqrt{3}}{2} \right)$$

$$= \begin{bmatrix} 1 & -(1+\sqrt{3}) & +\frac{(1+\sqrt{3})}{2} \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$$

$$-2-\sqrt{3} \quad -1-\sqrt{3} \\ \hline 2$$

$$R_3 \rightarrow R_3 - R_1$$

$$= \begin{bmatrix} 1 & -(1+\sqrt{3}) & +\frac{(1+\sqrt{3})}{2} \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -\frac{3\sqrt{3}-5}{2} \end{bmatrix}$$

$$-4 \quad -2\sqrt{3}-1-\sqrt{3} \\ \hline 2$$

$$R_2 \rightarrow -\frac{(1+\sqrt{3})}{2} R_2$$

$$= \begin{bmatrix} 1 & -(1+\sqrt{3}) & \left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 1 & -\left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 2+\sqrt{3} & -\frac{3\sqrt{3}-5}{2} \end{bmatrix}$$

$$-\frac{3\sqrt{3}-5}{2} + \frac{(2+\sqrt{3})(1+\sqrt{3})}{2}$$

$$-\underline{3\sqrt{3}-5} + \underline{3\sqrt{3}+8}$$

$$R_3 \rightarrow R_3 - (2+\sqrt{3})R_2$$

$$2$$

$$= \begin{bmatrix} 1 & -(1+\sqrt{3}) & \left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 1 & -\left(\frac{1+\sqrt{3}}{2}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1+\sqrt{3}}{2} \quad -\frac{(4+2\sqrt{3})}{2}$$

$$R_1 \rightarrow R_1 + (1+\sqrt{3})R_2$$

$$= \begin{bmatrix} 1 & 0 & \frac{-(3+\sqrt{3})}{2} \\ 0 & 1 & -\frac{(1+\sqrt{3})}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{-3-\sqrt{3}}{2}$$

$$\therefore v_2 = \begin{bmatrix} \frac{-(3+\sqrt{3})}{2} \\ \frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix} //$$

(3)

$$\lambda = -\sqrt{3}$$

$$(\tilde{A} - \lambda I)^3 = \begin{bmatrix} 1+\sqrt{3} & +2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$$

$$R_1 \rightarrow \frac{(\sqrt{3}-1)}{2} R_1$$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1$$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 2-\sqrt{3} & \frac{-5+3\sqrt{3}}{2} \end{bmatrix} \quad \frac{(-2+\sqrt{3}) - (-\sqrt{3})}{2}$$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 2-\sqrt{3} & \frac{-4+2\sqrt{3}-1+\sqrt{3}}{2} \end{bmatrix} \quad \frac{-4+2\sqrt{3}-1+\sqrt{3}}{2}$$

$$R_2 \rightarrow \frac{(\sqrt{3}-1)}{2}$$

$$= \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1 & \left(\frac{\sqrt{3}-1}{2}\right) \\ 0 & 2-\sqrt{3} & \frac{-5+3\sqrt{3}}{2} \end{bmatrix}$$

$$R_3 \rightarrow R_3 - (2-\sqrt{3})R_2$$

$$= \begin{bmatrix} 2\sqrt{3}-2+\sqrt{3}-3 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & \sqrt{3}-1 & \left(\frac{1-\sqrt{3}}{2}\right) \\ 0 & 1 & \left(\frac{\sqrt{3}-1}{2}\right) \\ 0 & 0 & 0 \end{bmatrix} \quad \frac{-5+3\sqrt{3}-(2-\sqrt{3})(\sqrt{3}-1)}{2} \quad \frac{-5+3\sqrt{3}+5-3\sqrt{3}}{2}$$

$$V_3 = \begin{bmatrix} \left(\frac{3-\sqrt{3}}{2}\right) \\ \left(\frac{1-\sqrt{3}}{2}\right) \\ 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - (\sqrt{3}-1)R_2$$

$$= \begin{bmatrix} 1 & 0 & \left(\frac{3-\sqrt{3}}{2}\right) \\ 0 & 1 & \left(\frac{\sqrt{3}-1}{2}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1-\sqrt{3}}{2} - \frac{(3+1-2\sqrt{3})}{2}$$

$$\frac{1-\sqrt{3}-4+2\sqrt{3}}{2}$$

$$\frac{-3+\sqrt{3}}{2}$$

Eigen Values : $\lambda_1 = 0$; $\lambda_2 = \sqrt{3}$; $\lambda_3 = -\sqrt{3}$

Corresponding Eigen Vectors : $v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} (\frac{3+\sqrt{3}}{2}) \\ (\frac{1+\sqrt{3}}{2}) \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} (\frac{3-\sqrt{3}}{2}) \\ (\frac{1-\sqrt{3}}{2}) \\ 1 \end{bmatrix}$

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Soli:

$$T = QR$$

Given $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$

$$\begin{matrix} x_1 & x_2 & x_3 \end{matrix}$$

From Gram Schmidt Process :

$$q_1 = \frac{x_1}{\|x_1\|} = \frac{(1, 0, 1)}{\sqrt{1^2 + 1^2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$e = b - q_1$$

$$e = x_2 - (q_1^T x_2) q_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{3}{\sqrt{2}}\right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

\hat{e}_2

$$\therefore q_2 = \frac{e}{\|e\|} = \frac{(+\frac{1}{2}, 1, -\frac{1}{2})}{\sqrt{\frac{1}{4} + 1 + \frac{1}{4}}} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\frac{3}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$$E = x_3 - [(q_1^T x_3) q_1 + (q_2^T x_3) q_2]$$

$$= \frac{-1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \left[-\frac{3}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{3}{\sqrt{6}} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \right]$$

dependent

$$= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \left[\begin{bmatrix} -\frac{3}{2} \\ 0 \\ -\frac{3}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} \right] = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T_{3 \times 3} = Q_{3 \times 2} R_{2 \times 3}$$

left inverse exist (R^T)

$$T = Q_{3 \times 3} R_{3 \times 2} I_{2 \times 3}$$

$$Q^T T = Q^T Q I_{2 \times 2}$$

$$R = Q^T T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}_{2 \times 3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}_{3 \times 3}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}} \end{bmatrix}_{2 \times 3}$$

$$\therefore T = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}_{3 \times 2} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}} \end{bmatrix}_{2 \times 3}$$

4. Find a best fit line $y = C + Dx$ for the following data

x	-4	1	2	3
y	4	6	10	8

Sol: In the form of matrix

$$Ax = b$$

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}_{4 \times 2} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

(4x2) System is inconsistent. Hence we use least square method.

m \times n

Normal equation

$$\hat{A}^T \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} C \\ P \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} C \\ P \end{bmatrix} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} C \\ P \end{bmatrix} = \frac{1}{116} \begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} \frac{193}{29} \\ \frac{20}{29} \end{bmatrix}$$

\therefore Best fit line $y = C + Dx$

$$y = \frac{193}{29} + \frac{20}{29} x$$

5. Find the projection matrices P and Q onto the plane
 $x_1 + x_2 + 3x_3 + 4x_4 = 0$ and its orthogonal complement

respectively.

Sol: Plane in the form of matrix
Equation $Ax = b$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 0 & 4 & x_1 \\ 1 & 1 & 3 & 0 & 4 & x_2 \\ 1 & 1 & 3 & 0 & 4 & x_3 \\ 1 & 1 & 3 & 0 & 4 & x_4 \\ 1 & 1 & 3 & 0 & 4 & x_5 \end{array} \right]_{5 \times 5} = [0]$$

To find a matrix with columns as span of plane
 we find $N(A) = \{ (-1, 1, 0, 0, 0), (-3, 0, 1, 0, 0), (0, 0, 0, 1, 0), (-4, 0, 0, 0, 1) \}$

Plane :

$$B = \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5×4

Projection : $P = B(B^T B)^{-1} B^T$

$$= B \left(\begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= B \left(\begin{bmatrix} 9 & 3 & 0 & 4 \\ 3 & 10 & 0 & 12 \\ 0 & 0 & 1 & 0 \\ -4 & 12 & 0 & 17 \end{bmatrix} \right)^{-1} B^T$$

$$= \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \\ \\ \text{5x4} \\ \end{array} \begin{bmatrix} \frac{26}{27} & -\frac{3}{27} & 0 & -\frac{4}{27} \\ -\frac{3}{27} & \frac{18}{27} & 0 & -\frac{12}{27} \\ 0 & 0 & 1 & 0 \\ -\frac{4}{27} & -\frac{12}{27} & 0 & \frac{11}{27} \end{bmatrix} \begin{array}{c} \\ \\ \\ \text{4x4} \\ \end{array}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{27} & -\frac{3}{27} & 0 & -\frac{4}{27} \\ \frac{26}{27} & -\frac{3}{27} & 0 & -\frac{4}{27} \\ -\frac{3}{27} & \frac{18}{27} & 0 & -\frac{12}{27} \\ 0 & 0 & 1 & 0 \\ -\frac{4}{27} & -\frac{12}{27} & 0 & \frac{11}{27} \end{bmatrix} \begin{array}{c} \\ \\ \\ \text{5x4} \\ \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{26}{27} & -\frac{1}{27} & -\frac{3}{27} & 0 & -\frac{4}{27} \\ -\frac{1}{27} & \frac{26}{27} & -\frac{3}{27} & 0 & -\frac{4}{27} \\ -\frac{3}{27} & -\frac{1}{27} & \frac{18}{27} & 0 & -\frac{12}{27} \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{4}{27} & -\frac{1}{27} & -\frac{12}{27} & 0 & \frac{11}{27} \end{bmatrix} \begin{array}{c} \\ \\ \\ \text{5x5} \\ \end{array}$$

Orthogonal complement of $N(A)$ is $C(A^T)$

$$C(A^T) = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} = C$$

$$C = A(A^T A)^{-1} A^T$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}_{5 \times 1} \left(\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}_{1 \times 5}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} (27) \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$$Q = 27 \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \\ 1 & 1 & 3 & 0 & 4 \\ 3 & 3 & 9 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 12 & 0 & 16 \end{bmatrix}$$

6. For which range of numbers 'a' the matrix A is positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

Which 3×3 symmetric matrix B produces the function f to do

$f = x^T A x$ when

$$f = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3)$$

Sol: Consider matrix, $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

By det semi-matrices :

$$a_1 \Rightarrow a > 0 \quad \text{--- (1)}$$

$$a_2 \Rightarrow a^2 - 4 > 0$$

$$a^2 > 2 \Rightarrow \boxed{a > 2} \quad \text{--- (2)}$$

$a < 2$ — not a case due to (1)

$$\begin{aligned} a_3 \Rightarrow |A| &= a(a^2 - 4) - 2(2a - 4) + 2(4 - 2a) > 0 \\ &= a^3 - 4a - 4a + 8 + 8 - 4a > 0 \\ &= a^3 - 12a + 16 \end{aligned}$$

$$a > -4 ; \boxed{a > 2} ; a > 2$$

\therefore Range of a for A to be positive definite
 $\Rightarrow \underline{\boxed{a > 2}}$ ✘ ✘

$$B = \begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 2 \end{bmatrix}$$

3x3

f. Find the SVD decomposition of A:

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3 \times 2}$$

Sol:- SVD decomposition :

$$A = U \Sigma V^T$$

$$\text{Find } A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix}_{3 \times 2} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}_{2 \times 2} =$$

$$\text{Eigen Values: } \lambda^2 - 90\lambda + 0 = 0$$

$$\lambda_1 = 90, \lambda_2 = 0$$

$$(\lambda_1 > \lambda_2)$$

Eigen Vectors

$$\textcircled{i} \quad (A^T A - \lambda_1 I) = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$= \begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{-9}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\text{Vector } X_1 = \begin{bmatrix} +3 \\ -1 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} +3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$$

$$\textcircled{ii} \quad (A^T A - \lambda_2 I) = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \quad R_2 \rightarrow R_2 + \frac{1}{3}R_1$$

$$= \begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix} \quad R_1 \rightarrow \frac{R_1}{81} \quad \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \quad \sqrt{\frac{1}{9}} = \sqrt{\frac{10}{9}}$$

$$\therefore \text{Vector } \underline{x}_2 = \begin{bmatrix} -1/3 \\ -1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

$$\underline{y}^T = \begin{bmatrix} +3/\sqrt{10} & -1/\sqrt{10} \\ -1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$$

Singular Values: $\sigma_1 = \sqrt{90}$
 $\sigma_2 = \sqrt{10}$

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$$\text{Eigen Values: } \lambda^3 - 90\lambda^2 + 0\lambda - 0 = 0$$

$$\lambda^2(\lambda - 90) = 0 \Rightarrow \lambda_1 = 90, \lambda_2 = 0, \lambda_3 = 0$$

$$(AA^T - \lambda_1 I) = \begin{bmatrix} -80 & -20 & -20 \\ -20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{1}{4}R_1, \quad R_3 \rightarrow R_3 - \frac{1}{4}R_1$$

$$= \begin{bmatrix} -80 & -20 & -20 \\ 0 & -45 & 45 \\ 0 & 45 & -45 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2 \quad R_1 \rightarrow \frac{R_1}{-80}$$

$$= \begin{bmatrix} +1 & 1/4 & 1/4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \rightarrow \frac{R_2}{-45} \quad R_1 \rightarrow R_1 - \frac{1}{4}R_2$$

$$= \begin{vmatrix} 1 & 0 & y_2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} \quad \therefore \quad x_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \quad v = \underline{x_1} = \frac{(-\frac{1}{2}, 1, 1)}{\sqrt{\frac{1}{4} + 1 + 1}} = \underline{\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right)}$$

$$(AA^T - \lambda_2 I) = \begin{vmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{vmatrix} \quad R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 + 2R_1$$

$$= \begin{vmatrix} 10 & -20 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad R_1 \rightarrow \frac{R_1}{10}$$

$$x_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad x = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x - \text{pivot var} \\ x - 2y - 2z = 0 \\ y, z - \text{free var} \\ x = 2y + 2z$$

$$v_2 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \quad v_3 = \begin{bmatrix} -2/\sqrt{45} \\ -5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ \frac{2}{3} & 0 & -\frac{5}{\sqrt{45}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \end{bmatrix}$$

$$\therefore A = U \sum_{3 \times 3}^{3 \times 2} V^T$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ \frac{2}{3} & 0 & -\frac{5}{\sqrt{45}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$