CSC421 HW5 06月22日2019年

1. a) $\log p(x) = \log \int_z p(x,z)\,dz \geq E_q\left[\log \frac{p(x,z)}{q}\right] = F(q)$

$$E_q\left[\log \frac{p(z|x)p(x)}{q}\right] = E_q[\log p(x)] + E_q\left[\log \frac{p(z|x)}{q}\right]$$
$$= \log p(x) - E_q\left[\log \frac{q}{p(z|x)}\right]$$
$$= \log p(x) - D_{KL}(q \| p(z|x)) \quad \blacksquare$$

b) $D_{KL}(q(z)\|p(z)) = E_q\left[\log \frac{q(z)}{p(z)}\right] = E_q\left[\log \frac{\prod q(z_i)}{\prod p(z_i)}\right]$

$$= E_q\left[\sum \log \frac{q(z_i)}{p(z_i)}\right]$$
$$= \sum E_q\left[\log \frac{q(z_i)}{p(z_i)}\right]$$
$$= \sum_{i=1}^{n} D_{KL}(q(z_i)\|p(z_i))$$

$Var(x) = E[x^2] - E[x]^2$

c) Recall $q(z_i) = \mathcal{N}(z_i; \mu_i, \sigma_i)$

$p(z_i) = \mathcal{N}(z_i; 0, 1)$

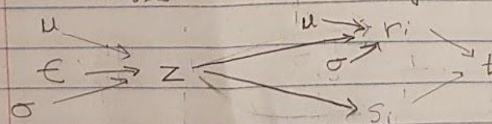
$$D_{KL}(q(z_i)\|p(z_i)) = E_q\left[\log \frac{q(z_i)}{p(z_i)}\right]$$
$$= E_q[\log q(z_i) - \log p(z_i)]$$
$$= E_q\left[-\frac{1}{2}\log 2\pi\sigma_i^2 - \frac{(z_i-\mu_i)^2}{2\sigma_i^2} + \frac{1}{2}\log 2\pi + \frac{z_i^2}{2}\right]$$
$$= -\frac{1}{2}\log 2\pi\sigma_i^2 + \frac{1}{2}\log 2\pi + \frac{1}{2}E_q[z_i^2] - \frac{1}{2\sigma_i^2}E_q[z_i^2 - 2\mu_i z_i + \mu_i^2]$$
$$= -\frac{1}{2}\log \sigma_i^2 + \frac{1}{2}(\sigma_i^2 + \mu_i^2) - \frac{1}{2\sigma_i^2}(\sigma_i^2 + \mu_i^2 - 2\mu_i^2 + \mu_i^2)$$
$$= -\frac{1}{2}\log \sigma_i^2 + \frac{1}{2}(\sigma_i^2 + \mu_i^2) - \frac{1}{2}$$
$$= -\log \sigma_i + \frac{1}{2}(\sigma_i^2 + \mu_i^2) - \frac{1}{2}$$

d) $\nabla_\theta E_q[\log q(z_i) - \log p(z_i)] = E_\epsilon[\nabla_\theta t_i] = \int_{-\infty}^{\infty} p(\epsilon)\nabla_\theta t_i \, d\epsilon \quad \epsilon \sim \mathcal{N}(0,1)$

Fact: $\int_{-\infty}^{\infty} p(\epsilon)\,d\epsilon = 1$ — $\nabla_\theta t_i$ is a fct of ϵ.

$\therefore \int_{-\infty}^{\infty} p(\epsilon)\nabla_\theta t_i \, d\epsilon \approx \frac{1}{N}\sum_{j=1}^{N} \nabla_\theta t_i(\epsilon_j)$ where $\epsilon_j \sim \mathcal{N}(0,1)$, $j = 1 \ldots N$

Monte Carlo Estimation



$\bar{s} = 1$ $\quad r = \log q_i(z_i) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{(z-\mu)^2}{2\sigma^2}$

$\bar{r} = -1$ $\quad \Rightarrow \frac{dr}{dz} = \frac{-2(z-\mu)}{2\sigma^2} = \frac{-(z-\mu)}{\sigma^2}$

$\frac{dr}{d\mu} = +\frac{(z-\mu)}{2\sigma^2} \qquad \frac{dr}{d\sigma} = -\frac{1}{2}\cdot\frac{1}{2\pi\sigma^2}\cdot 4\pi\sigma + \frac{3}{2}\frac{(z-\mu)}{\sigma^3} = \frac{-1}{\sigma} + \frac{3}{2}\frac{(z-\mu)}{\sigma^3}$

$s = \log p(z_i) = -\frac{1}{2}\log 2\pi - \frac{z^2}{2}$

$\rightarrow \frac{ds}{dz} = -z$

$\therefore \bar{z} = \bar{r}\frac{dr}{dz} + \bar{s}\frac{ds}{dz}$

Finally: $\bar{\mu} = \bar{z}(1) + \bar{r}\frac{dr}{d\mu} \qquad \bar{\sigma} = \bar{z}\epsilon + \bar{r}\frac{dr}{d\sigma}$