Calculate Heisenberg uncertainty for Gaussian wave packets

$\|\psi_\alpha(x)\|^2 = \frac{1}{d\sqrt{\pi}} \exp\left(-\frac{x^2}{d^2}\right)$ let's consider $\psi_\alpha(x)$ as: $\psi_\alpha(x) = \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx - \frac{x^2}{2d^2}}$

Now you want to calculate moments of $\|\psi_\alpha(x)\|^2$. so let's remember charectristic function.

$$\langle e^{\lambda x} \rangle = \int dx \frac{1}{\pi^{1/2} d} e^{\left(\lambda x - \frac{x^2}{d^2}\right)} = \frac{1}{\pi^{1/2} d} \int dx\, e^{-\left(\frac{d\lambda}{2} + \frac{x}{d}\right)^2} e^{\frac{d^2}{4}\lambda^2}$$

$$= \frac{e^{d^2/4\,\lambda^2}}{\pi^{1/2}} \int e^{-\vartheta^2} d\vartheta = e^{d^2/4\,\lambda^2}$$

$$\left\langle \sum_n \frac{1}{n!} (\lambda x)^n \right\rangle = \sum_m \frac{1}{m!} \left(\frac{d^2}{4}\lambda^2\right)^m \Longrightarrow \langle x^{2n+1} \rangle = 0$$

$$\langle x^{2n} \rangle = ?$$

$$\left\langle \frac{1}{(2n)!} (\lambda x)^{2n} \right\rangle = \frac{1}{n!} \left(\frac{d^2}{4}\lambda^2\right)^n$$

$$\frac{1}{(2n)!} \lambda^{2n} \langle x^{2n} \rangle = \frac{1}{n!} \frac{d^{2n}}{4^n} \lambda^{2n} \Longrightarrow \langle x^{2n} \rangle = \frac{(2n)!}{n!} \left(\frac{d^2}{4}\right)^n$$

therefore, $\langle x \rangle = 0$, $\langle x^2 \rangle = 2\frac{d^2}{4} = \frac{d^2}{2}$, $\Delta x = \langle x^2 \rangle - \langle x \rangle^2 = d^2/2$

as we know from general charetristics function how obtain moments in 3D space we can do it for here but I don't want discuss about them now.

Now it's turn of momentum.

$$\langle p|\psi\rangle = \int dx \langle p|x\rangle \langle x|\psi\rangle = \int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}xp} \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx - \frac{x^2}{2d^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\pi^{1/4}\sqrt{d}} \int e^{-\frac{i}{\hbar}x(p - \hbar k) - \frac{x^2}{2d^2}} dx$$

$$\frac{x^2}{2d^2}=a^2,\quad 2ab=\frac{i}{\hbar}x(p-\hbar k) \Rightarrow \sqrt{2}\,\frac{1}{d}\,b=\frac{i}{\hbar}\,(p-\hbar k) \Rightarrow b=\frac{d}{\sqrt{2}}\,\frac{i}{\hbar}(p-\hbar k)$$

$$\frac{1}{\sqrt{2\pi\hbar}}\,\frac{1}{\pi^{1/4}\sqrt{d}}\int \exp\left(-\left[\left[\frac{x}{\sqrt{2}d}+\frac{d}{\sqrt{2}}\,\frac{i}{\hbar}(p-\hbar k)\right]^2-\frac{d^2}{2}\,\frac{1}{\hbar^2}(p-\hbar k)^2\right]\right)dx$$

$$=\frac{1}{\sqrt{2\pi\hbar}}\,\frac{1}{\pi^{1/4}\sqrt{d}}\exp\left(-\frac{d^2}{2}\,\frac{1}{\hbar^2}(p-\hbar k)^2\right)\int \exp(-v^2)\,dv\,(\sqrt{2}d)$$

$$=\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left(-\frac{d^2}{2}\,\frac{1}{\hbar^2}(p-\hbar k)^2\right)$$

$$\|\tilde{\psi}(p)\|^2=\frac{d}{\hbar\sqrt{\pi}}\exp\left(-\frac{d^2}{\hbar^2}(p-\hbar k)^2\right) \longrightarrow \langle p\rangle=\hbar K,\quad \langle p^2\rangle-\langle p\rangle^2=\frac{\hbar^2}{2d^2}$$

$$\sigma_p\,\sigma_x=\frac{\hbar}{\sqrt{2}d}\,\frac{d}{\sqrt{2}}=\hbar/2 \quad \square$$