Question 3 (25 marks)

MAST90053 Experimental Mathematics

Due 3pm AEST on Thursday 2 July 2020

Note: This question has 3 parts and goes over 2 pages. Make sure you read everything carefully.

Background: A function $t(n,k)$ is said to be a *proper hypergeometric term* if it can be written in the form

$$t(n,k) = P(n,k)\,\frac{\Gamma(a_1 n + b_1 k + c_1)\ldots\Gamma(a_r n + b_r k + c_r)}{\Gamma(d_1 n + e_1 k + f_1)\ldots\Gamma(d_s n + e_s k + f_s)}\,x^k \tag{1}$$

where P is a polynomial in n and k, with $r, s \in \mathbb{Z}_{\geqslant 0}$, $x \in \mathbb{C}$, and $a_i, b_i, d_i, e_i \in \mathbb{Z}$.

Fix a point (n,k). A term t of the form (1) is said to be *well-defined* if none of the numbers $a_i n + b_i k + c_i$ occurring in the numerator of (1) is a negative integer. If t is well-defined at (n,k), we declare $t(n,k) = 0$ if $P(n,k) = 0$ or at least one of the numbers $d_i n + e_i k + f_i$ occurring in the denominator of (1) is a negative integer.

If t is a proper hypergeometric term of the form (1), then it satisfies a k-free recurrence relation (as found by Fasenmyer's algorithm) with

$$J = \sum_{j=1}^{r} |b_j| + \sum_{j=1}^{s} |e_j|$$

$$I = 1 + \deg(P) + J\left(-1 + \sum_{j=1}^{r} |a_j| + \sum_{j=1}^{s} |d_j|\right)$$

What you are asked to do:

(a) Write a function

```
def term_from_data(P, alst, blst, clst, dlst, elst, flst, x, n, k):
    """
    Return the proper hypergeometric term defined by Equation (1).

    You may assume that the data is valid (P is a polynomial, the three
    lists alst, blst, clst of coefficients have the same length r, the three
    lists dlst, elst, flst of coefficients have the same length s, etc.)

    EXAMPLE:

    sage: n, k = var("n, k")
    sage: P = n^2 + k
    sage: alst = [1]; blst = [0]; clst = [1]
    sage: dlst = [0, 1]; elst = [1, -1]; flst = [1, 1]
    sage: t = term_from_data(P, alst, blst, clst, dlst, elst, flst, 2, n, k)
```

```
    sage: t.simplify_full().factor()
    (n^2 + k)*2^k*factorial(n)/(k*factorial(k - 1)*factorial(-k + n))
    """
```

that takes the proper hypergeometric data appearing on the right hand side of (1) and returns the function t of n and k defined by (1).

(b) Write a function

```
def fasenmyer(tdata, n, k):
    """
    Return the recurrence relation for the summation of the proper
    hypergeometric term t defined by tdata, using Celine Fasenmyer's algorithm.

    The input is of the form

    tdata = (P, alst, blst, clst, dlst, elst, flst, x)

    The output is a list of coefficients a(n), a(n+1), ..., a(n+r) of a
    recurrence relation

    a(n)*t(n) + a(n+1)*t(n+1) + ... + a(n+r)*t(n+r) = 0

    EXAMPLE:

    sage: tdata = (SR(1), [1], [0], [1], [0, 1], [1, -1], [0, 1], 1)
    sage: fasenmyer(tdata, n, k)
    [2, -3*n/(n + 1), n/(n + 2)]
    """
```

that takes the data defining a proper hypergeometric term t and returns a recurrence relation (as a function of n) for

$$s_n = \sum_{k=-\infty}^{\infty} t(n,k)$$

(c) Use your function from (b) to obtain a recurrence for

$$s_n = \sum_{k=-\infty}^{\infty} k \binom{n}{k}$$

Verify that $s_n = 2^{n-1}n$ satisfies the recurrence you find.