Random Numbers

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/coeffs.h
gcc exrand.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution:

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

The following code plots Fig. 1.2

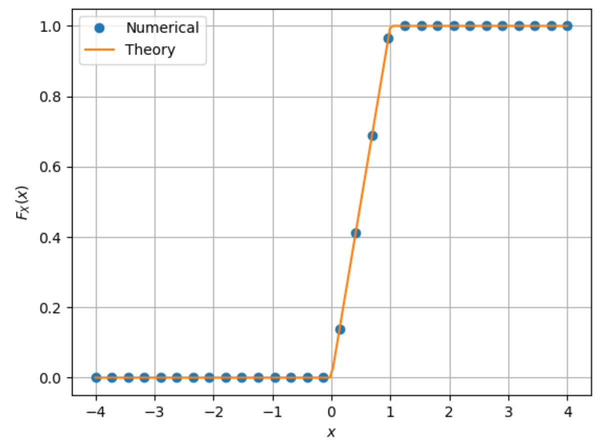

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.2}$$

Therefore, we have:

$$F_U(x) = \int_0^x U(x)dx \tag{1.3}$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and

variance of U.

Solution: Add the following function to coeffs.h

```
double variance(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x*x;
}
double mn = mean(str);
fclose(fp);
temp = temp/(i−1);
return temp − mn*mn ;

}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: Since

$$dF_U(x) = p_U(x)dx \tag{1.8}$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \tag{1.9}$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.10}$$

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{1.11}$$

$$= \int_0^1 x^2 dx \tag{1.12}$$

$$= \frac{1}{3} \tag{1.13}$$

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \tag{1.14}$$

$$= \int_0^1 x dx \tag{1.15}$$

$$= \frac{1}{2} \tag{1.16}$$

Therefore, the mean is $\frac{1}{2}$, and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.17}$$

$$= \frac{1}{12} \tag{1.18}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Add the following line to **exrand.c** and execute the code:

```
gaussian("gau.dat", 1000000);
gcc exrand.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

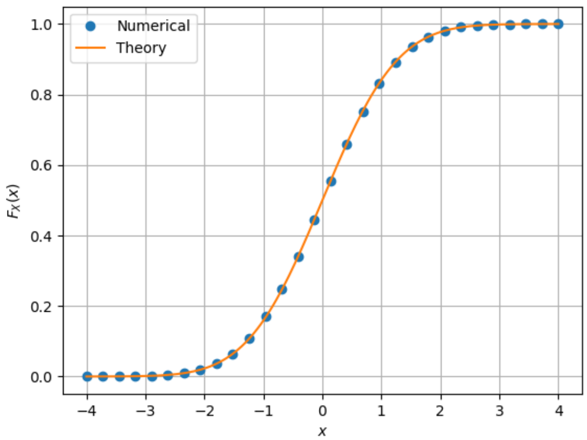

Fig. 2.2: The CDF of X

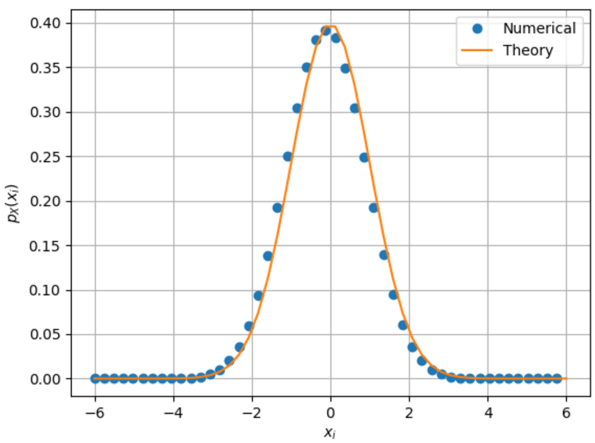

Fig. 2.3: The PDF of X

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/pdf_plot.py
python3 pdf_plot.py
```

To find the CDF theoretically, consider

2.4 Find the mean and variance of X by writing a C program.
Solution: Use the main and variance functions in **coeffs.h**, and execute the code below

```
gcc exrand.c
./a.out
```

We get

```
mean = 0.000685
variance = 1.000025
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: We have:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) && (2.4)\\ &= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Big|_{-\infty}^{\infty} && (2.5)\\ &= 0 && (2.6) \end{aligned}$$

Also,

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) && (2.7)\\ &= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} && (2.8)\\ &= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} && (2.9)\\ &= 1 && (2.10) \end{aligned}$$

Hence,

$$\begin{aligned} \text{var}(X) &= E[X^2] - E[X]^2 && (2.11)\\ &= 1 && (2.12) \end{aligned}$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in **./codes/exrancd.c**, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

2.6 Find the theoretical CDF of X
Solution: To find the theoretical CDF, consider:

$$\begin{aligned} Q_X(x) &= \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx && (2.13)\\ &= \frac{\text{erfc}(\frac{x}{\sqrt{2}})}{2} && (2.14) \end{aligned}$$

The CDF is then:

$$F_X(x) = 1 - Q_X(x) \tag{2.15}$$
$$= 1 - \frac{\text{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.16}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF. **Solution:**
Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
int i=0,c;
FILE *fp, *fp2;
double x, temp=0.0;

fp = fopen("uni.dat","r");
fp2 = fopen(str, "w");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
temp = -2*log(1-x);
fprintf(fp2,"%lf\n",temp);
}

fclose(fp);
fclose(fp2);

return ;

}
```

Using this function in **exrand.c** prints the numbers in **log.dat**

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

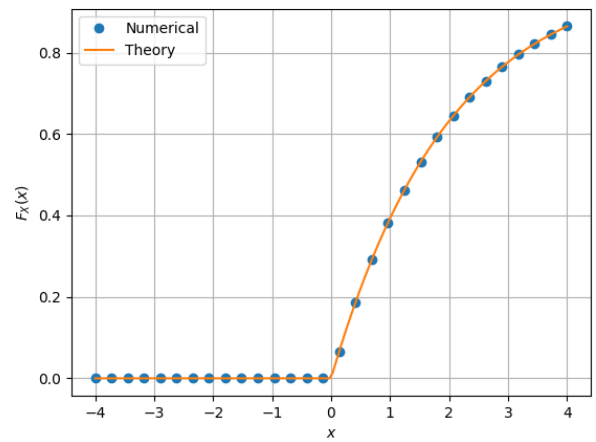

Fig. 3.2: The CDF of V

Therefore,

$$F_V(x) = \begin{cases} 0, & 1-\exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & 1-\exp\left(-\frac{x}{2}\right) \in (0,1) \\ 1, & 1-\exp\left(-\frac{x}{2}\right) \in (1,\infty) \end{cases} \tag{3.7}$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & x \in (0,\infty) \end{cases} \tag{3.8}$$

The CDF of V is plotted in Fig. 3.2

4 Triangular Distributions

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Use the function 'triangular' in **exrand.c** and execute the following code:

```
wget https://github.com/Rahuboy/AI1110/blob
    /main/RandomNumbers/codes/coeffs.h
wget https://github.com/Rahuboy/AI1110/blob
    /main/RandomNumbers/codes/exrand.c
gcc exrand.c
./a.out
```

4.2 Find the CDF of T.

Solution:

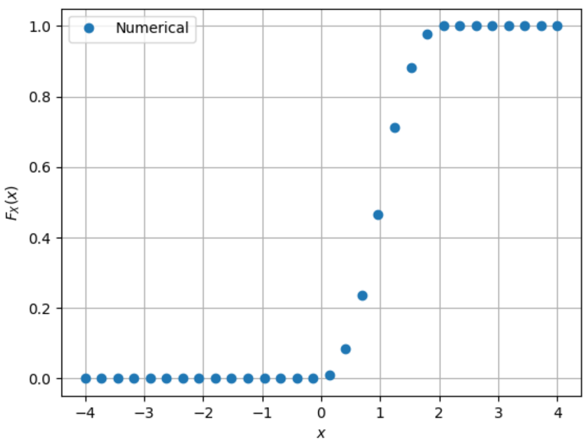

Fig. 4.2: The CDF of T

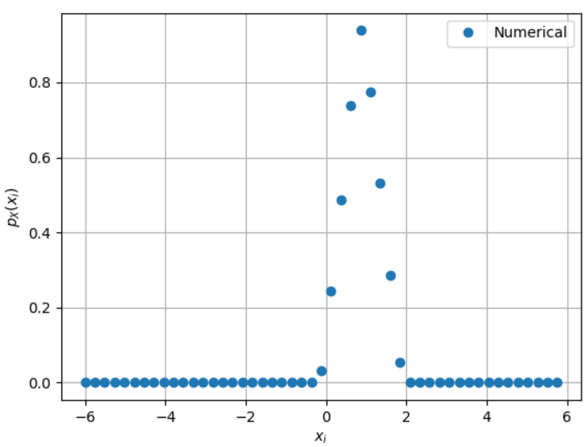

Fig. 4.3: The PDF of T

```
wget https://github.com/Rahuboy/AI1110/
    blob/main/RandomNumbers/codes/
    cdf_plot.py
python3 cdf_plot.py
```

The above code plots Fig. 4.2

4.3 Find the PDF of T.

```
wget https://github.com/Rahuboy/AI1110/
    blob/main/RandomNumbers/codes/
    pdf_plot.py
python3 pdf_plot.py
```

The above code plots Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution: When

$$Z = X + Y \tag{4.2}$$

where X, Y and Z are random variables, we have:

$$p_Z(t) = (p_X * p_Y)(t) \tag{4.3}$$
$$= \int_{-\infty}^{\infty} p_X(\tau)p_Y(t-\tau)d\tau \tag{4.4}$$

Here, $p_X(t) = p_Y(t) = p_U(t)$. Therefore:

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau)p_U(t-\tau)d\tau \tag{4.5}$$
$$= \int_0^1 p_U(t-\tau)d\tau \tag{4.6}$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t-\tau)d\tau \tag{4.7}$$
$$= \int_0^t p_U(t-\tau)d\tau \tag{4.8}$$
$$= \int_0^t 1d\tau \tag{4.9}$$
$$= t \tag{4.10}$$

when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t-\tau)d\tau \tag{4.11}$$
$$= \int_{t-1}^1 p_U(t-\tau)d\tau \tag{4.12}$$
$$= \int_{t-1}^1 1d\tau \tag{4.13}$$
$$= 2 - t \tag{4.14}$$

Therefore, we have:

$$p_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 2 - x, & x \in (1, 2) \\ 0, & x \in (2, \infty) \end{cases} \tag{4.15}$$

To find the CDF, we use:

$$F_T(x) = \int_{-\infty}^{x} p_T(t)dt \tag{4.16}$$

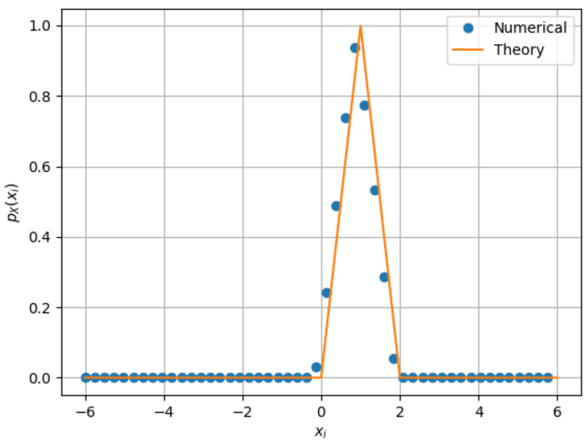

Fig. 4.5: The PDF of T

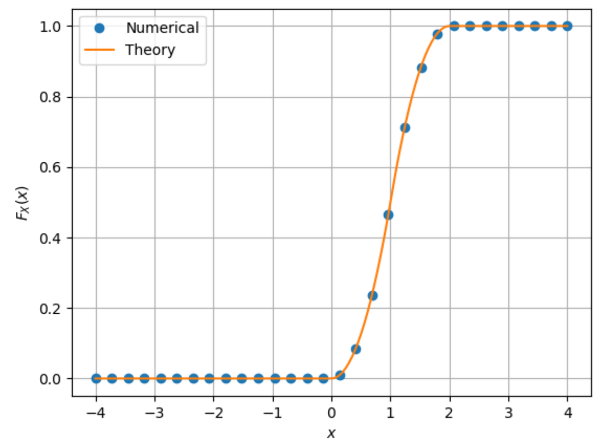

Fig. 4.6: The CDF of T

We get:

$$F_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{x^2}{2}, & x \in (0, 1) \\ -\frac{x^2}{2} + 2x - 1, & x \in (1, 2) \\ 1, & x \in (2, \infty) \end{cases} \tag{4.17}$$

4.5 Verify your result for the PDF through a plot.
Solution: Execute the following code:

```
python3 pdf_plot.py
```

The theoretical PDF is plotted in Fig. 4.5

4.6 Verify your result for the CDF through a plot.
Solution: Execute the following code:

```
python3 cdf_plot.py
```

The theoretical CDF is plotted in Fig. 4.6

5 Maximal Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X \imath \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution: Use the functions 'bernoulli' and 'maxlike' in **exrand.c**:

```
gcc exrand.c
./a.out
```

5.2 Plot Y.
Y is plotted in Fig. 5.2

5.3 Guess how to estimate X from Y.
Solution: To estimate X from Y, we define the following function:

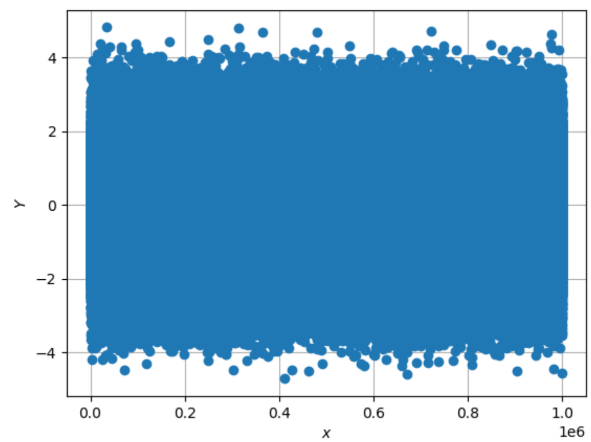

Fig. 5.2: The Plot of Y

$$\text{sgn}(y) = \begin{cases} -1, & y \in (-\infty, 0] \\ 1, & y \in (0, \infty) \end{cases} \tag{5.2}$$

Using $\text{sgn}\, y$, we can operate on Y to find corresponding values of X.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: Use the function "maxlike_proberr" in **exrand.c** to find the respective probabilities:

```
gcc exrand.c
./a.out
P_(e|0) = 0.312414
```

```
P_(e|1) = 0.310985
```

5.5 Find P_e.
Solution: Assume a general value of A. Our estimation function predicts that the data points above the x axis correspond to $X = 1$, and the data points below the x-axis correspond to $X = -1$. This isn't always the case, as $Y = AX + N$, and the N causes some spill-over. We have:

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.5}$$
$$= \Pr\left(AX + N < 0 | X = 1\right) \tag{5.6}$$
$$= \Pr\left(N < -A\right) \tag{5.7}$$
$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.8}$$
$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.9}$$
$$= Q_N(A) \tag{5.10}$$

where Q_N is the Q−function of the normal distribution.
Similarly,

$$P_{e|1} = Q_N(A) \tag{5.11}$$

Therefore,

$$P_e = P_{e|0} \times \Pr\left(X = 1\right) + P_{e|1} \times \Pr\left(X = -1\right) \tag{5.12}$$
$$= \frac{1}{2} P_{e|0} + \frac{1}{2} P_{e|1} \tag{5.13}$$
$$= \frac{1}{2} Q_N(A) + \frac{1}{2} Q_N(A) \tag{5.14}$$
$$= Q_N(A) \tag{5.15}$$

5.6 Verify by plotting the theoretical P_e.
The graph of P_e is plotted in Fig. 5.6

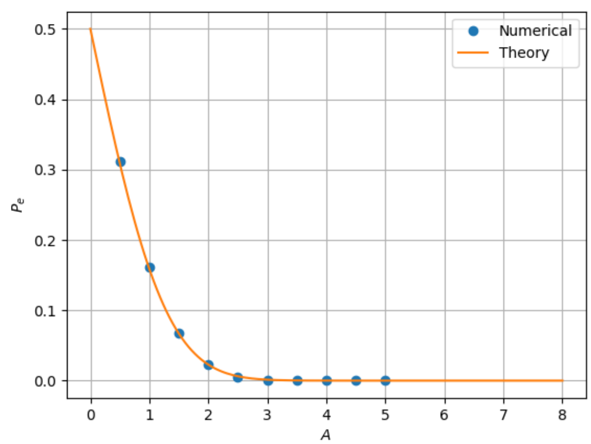

Fig. 5.6: The Plot of P_e