Random Numbers

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Contents

1	**Uniform Random Numbers**	1
2	**Central Limit Theorem**	2
3	**From Uniform to Other**	4
4	**Triangular Distributions**	4
5	**Maximal Likelihood**	6
6	**Gaussian to Other**	9
7	**Conditional Probability**	10
8	**Two Dimensions**	11

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/coeffs.h
gcc exrand.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution:

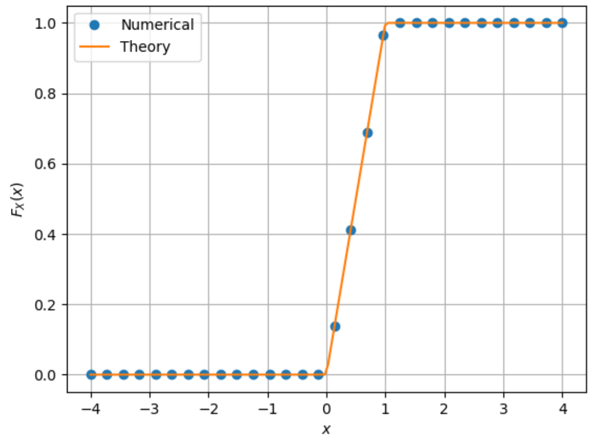

Fig. 1.2: The CDF of U

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

The following code plots Fig. 1.2

1.3 Find a theoretical expression for $F_U(x)$.
Solution: U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.2}$$

Therefore, we have:

$$F_U(x) = \int_0^x U(x)dx \tag{1.3}$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: Add the following function to coeffs.h

```
double variance(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x*x;
}
double mn = mean(str);
fclose(fp);
temp = temp/(i-1);
return temp - mn*mn ;

}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: Since

$$dF_U(x) = p_U(x)dx \tag{1.8}$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \tag{1.9}$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.10}$$

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{1.11}$$

$$= \int_0^1 x^2 dx \tag{1.12}$$

$$= \frac{1}{3} \tag{1.13}$$

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \tag{1.14}$$

$$= \int_0^1 x dx \tag{1.15}$$

$$= \frac{1}{2} \tag{1.16}$$

Therefore, the mean is $\frac{1}{2}$, and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.17}$$

$$= \frac{1}{12} \tag{1.18}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Add the following line to **exrand.c** and execute the code:

```
gaussian("gau.dat", 1000000);
gcc exrand.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution:

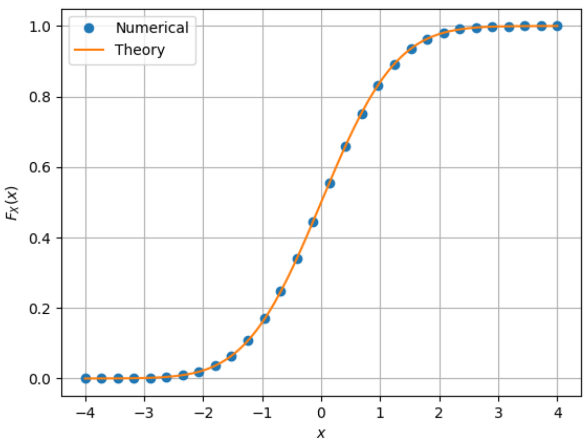

Fig. 2.2: The CDF of X

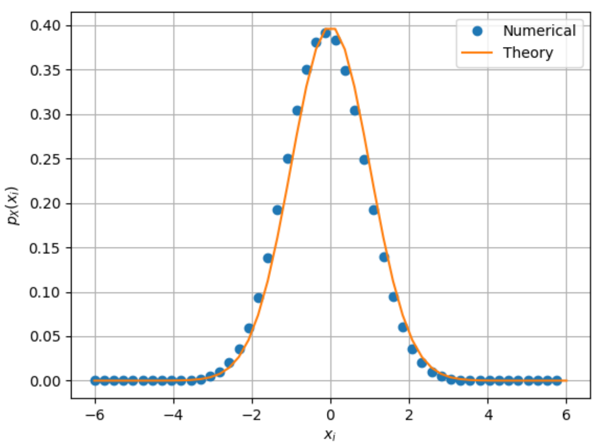

Fig. 2.3: The PDF of X

The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/pdf_plot.py
python3 pdf_plot.py
```

To find the CDF theoretically, consider

2.4 Find the mean and variance of X by writing a C program.
Solution: Use the main and variance functions in **coeffs.h**, and execute the code below

```
gcc exrand.c
./a.out
```

We get

```
mean = 0.000685
variance = 1.000025
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.4}$$
$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{2.5}$$
$$= 0 \tag{2.6}$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.7}$$
$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} \tag{2.8}$$
$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.9}$$
$$= 1 \tag{2.10}$$

Hence,

$$\mathrm{var}(X) = E[X^2] - E[X]^2 \tag{2.11}$$
$$= 1 \tag{2.12}$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in **./codes/exrandc.c**, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

2.6 Find the theoretical CDF of X
Solution: To find the theoretical CDF, consider:

$$Q_X(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{2.13}$$
$$= \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.14}$$

The CDF is then:

$$F_X(x) = 1 - Q_X(x) \tag{2.15}$$
$$= 1 - \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.16}$$

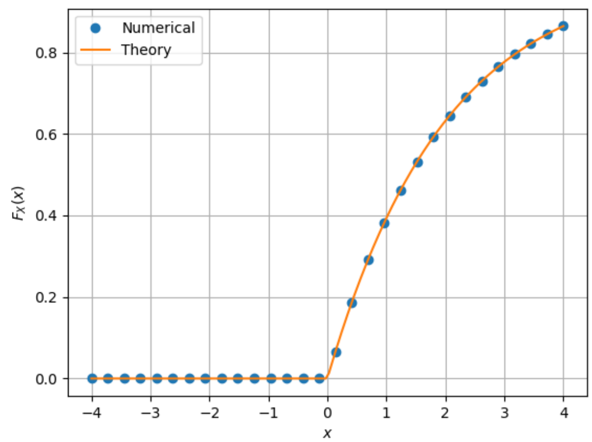

Fig. 3.2: The CDF of V

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF. **Solution:**
Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
int i=0,c;
FILE *fp, *fp2;
double x, temp=0.0;

fp = fopen("uni.dat","r");
fp2 = fopen(str, "w");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
temp = -2*log(1-x);
fprintf(fp2,"%lf\n",temp);
}

fclose(fp);
fclose(fp2);

return ;

}
```

Using this function in **exrand.c** prints the numbers in **log.dat**

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

Therefore,

$$F_V(x) = \begin{cases} 0, & 1-\exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & 1-\exp\left(-\frac{x}{2}\right) \in (0,1) \\ 1, & 1-\exp\left(-\frac{x}{2}\right) \in (1,\infty) \end{cases} \tag{3.7}$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & x \in (0,\infty) \end{cases} \tag{3.8}$$

The CDF of V is plotted in Fig. 3.2

4 Triangular Distributions

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Use the function 'triangular' in **exrand.c** and execute the following code:

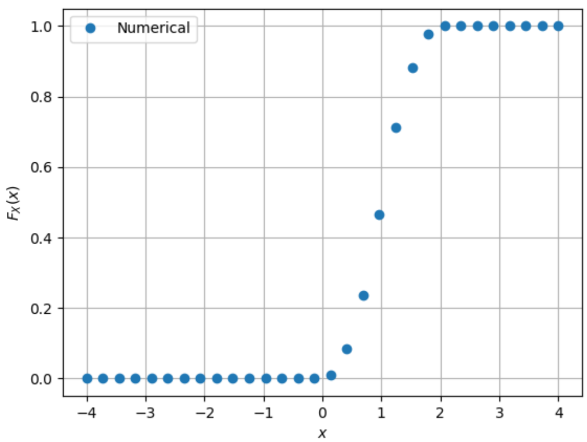

Fig. 4.2: The CDF of T

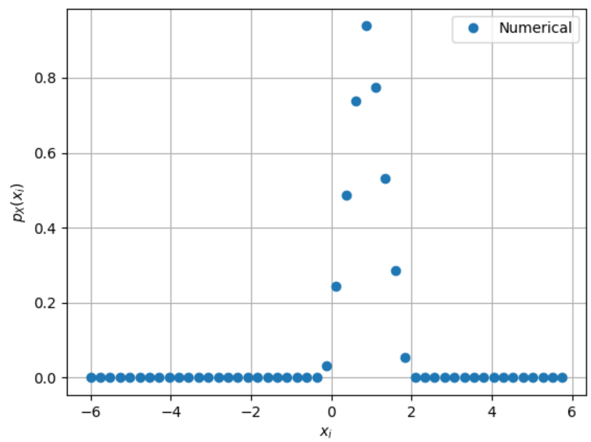

Fig. 4.3: The PDF of T

```
wget https://github.com/Rahuboy/AI1110/blob
    /main/RandomNumbers/codes/coeffs.h
wget https://github.com/Rahuboy/AI1110/blob
    /main/RandomNumbers/codes/exrand.c
gcc exrand.c
./a.out
```

4.2 Find the CDF of T.

Solution:

```
wget https://github.com/Rahuboy/AI1110/
    blob/main/RandomNumbers/codes/
    cdf_plot.py
python3 cdf_plot.py
```

The above code plots Fig. 4.2

4.3 Find the PDF of T.

```
wget https://github.com/Rahuboy/AI1110/
    blob/main/RandomNumbers/codes/
    pdf_plot.py
python3 pdf_plot.py
```

The above code plots Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution: When

$$Z = X + Y \tag{4.2}$$

where X, Y and Z are random variables, we have:

$$p_Z(t) = (p_X * p_Y)(t) \tag{4.3}$$

$$= \int_{-\infty}^{\infty} p_X(\tau)p_Y(t-\tau)d\tau \tag{4.4}$$

Here, $p_X(t) = p_Y(t) = p_U(t)$. Therefore:

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau)p_U(t-\tau)d\tau \tag{4.5}$$

$$= \int_0^1 p_U(t-\tau)d\tau \tag{4.6}$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t-\tau)d\tau \tag{4.7}$$

$$= \int_0^t p_U(t-\tau)d\tau \tag{4.8}$$

$$= \int_0^t 1 d\tau \tag{4.9}$$

$$= t \tag{4.10}$$

when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t-\tau)d\tau \tag{4.11}$$

$$= \int_{t-1}^1 p_U(t-\tau)d\tau \tag{4.12}$$

$$= \int_{t-1}^1 1 d\tau \tag{4.13}$$

$$= 2 - t \tag{4.14}$$

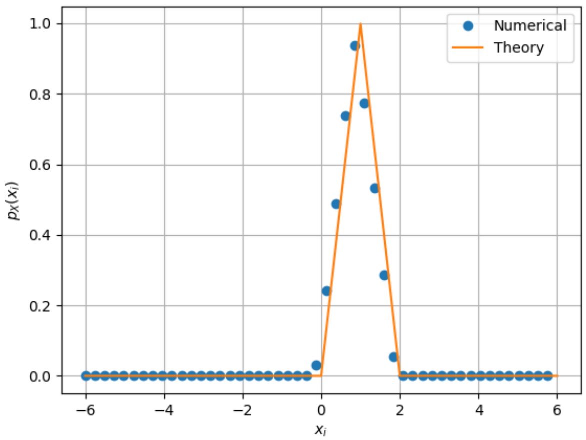

Fig. 4.5: The PDF of T

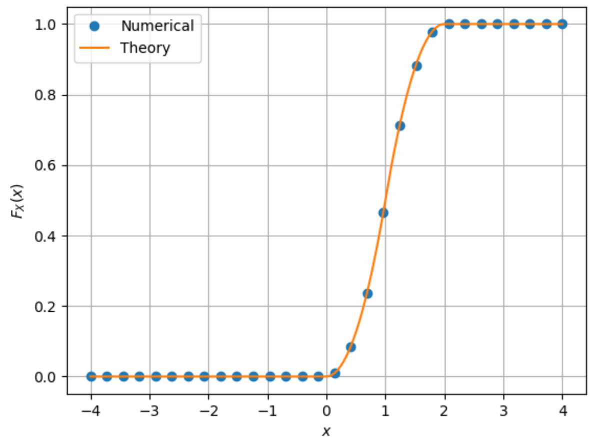

Fig. 4.6: The CDF of T

Therefore, we have:

$$p_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 2 - x, & x \in (1, 2) \\ 0, & x \in (2, \infty) \end{cases} \quad (4.15)$$

To find the CDF, we use:

$$F_T(x) = \int_{-\infty}^{x} p_T(t)dt \quad (4.16)$$

We get:

$$F_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{x^2}{2}, & x \in (0, 1) \\ -\frac{x^2}{2} + 2x - 1, & x \in (1, 2) \\ 1, & x \in (2, \infty) \end{cases} \quad (4.17)$$

4.5 Verify your result for the PDF through a plot.
Solution: Execute the following code:

```
python3 pdf_plot.py
```

The theoretical PDF is plotted in Fig. 4.5

4.6 Verify your result for the CDF through a plot.
Solution: Execute the following code:

```
python3 cdf_plot.py
```

The theoretical CDF is plotted in Fig. 4.6

5 Maximal Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Use the function "bernoulli" in **exrand.c** and execute the code below:

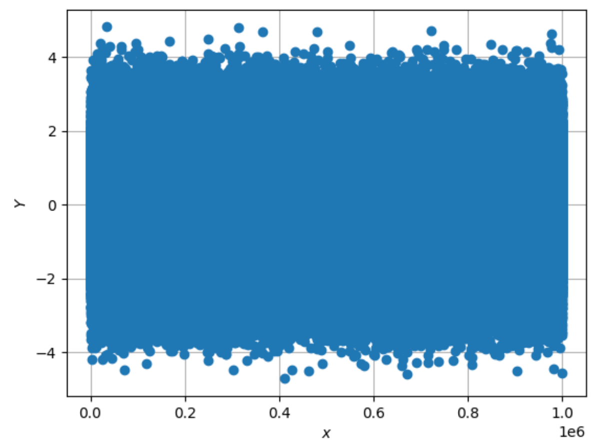

Fig. 5.3: The Plot of Y

```
gcc exrand.c
./a.out
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, $X\imath\{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution: Use the functions 'bernoulli' and 'maxlike' in **exrand.c**:

```
gcc exrand.c
./a.out
```

5.3 Plot Y.
Y is plotted in Fig. 5.3

5.4 Guess how to estimate X from Y.
Solution: To estimate X from Y, we define the

following function:

$$\operatorname{sgn}(y) = \begin{cases} -1, & y \in (-\infty, 0] \\ 1, & y \in (0, \infty) \end{cases} \tag{5.2}$$

Using $\operatorname{sgn} y$, we can operate on Y to find corresponding values of X.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: Use the function "maxlike_proberr" in **exrand.c** to find the respective probabilities:

```
gcc exrand.c
./a.out
P_(e|0) = 0.312414
P_(e|1) = 0.310985
```

5.6 Find P_e.

Solution: Assume a general value of A. Our estimation function predicts that the data points above the x axis correspond to $X = 1$, and the data points below the x-axis correspond to $X = -1$. This isn't always the case, as $Y = AX + N$, and the N causes some spill-over. We have:

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.5}$$
$$= \Pr\left(AX + N < 0 | X = 1\right) \tag{5.6}$$
$$= \Pr\left(N < -A\right) \tag{5.7}$$
$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.8}$$
$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.9}$$
$$= Q_N(A) \tag{5.10}$$

where Q_N is the $Q-$function of the normal distribution.
Similarly,

$$P_{e|1} = Q_N(A) \tag{5.11}$$

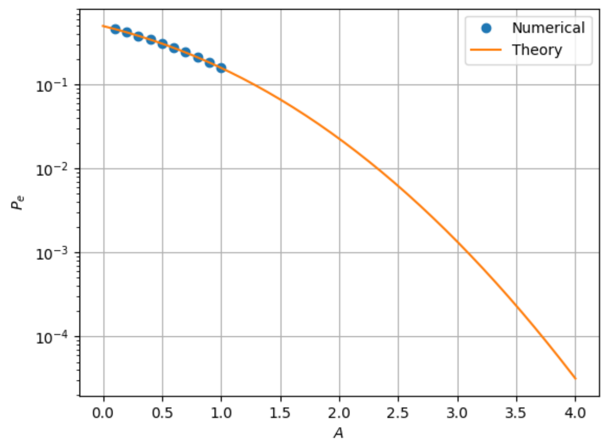

Fig. 5.7: The Plot of P_e

Therefore,

$$P_e = P_{e|0} \times \Pr\left(X = 1\right) + P_{e|1} \times \Pr\left(X = -1\right) \tag{5.12}$$
$$= \frac{1}{2} P_{e|0} + \frac{1}{2} P_{e|1} \tag{5.13}$$
$$= \frac{1}{2} Q_N(A) + \frac{1}{2} Q_N(A) \tag{5.14}$$
$$= Q_N(A) \tag{5.15}$$

5.7 Verify by plotting the theoretical P_e.

Solution: The graph of P_e is plotted in Fig. 5.7

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution: To estimate X from Y, we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \tag{5.16}$$

Therefore,

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.17}$$
$$= \Pr\left(AX + N < \delta | X = 1\right) \tag{5.18}$$
$$= \Pr\left(N < \delta - A\right) \tag{5.19}$$
$$= \int_{-\infty}^{\delta - A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{5.20}$$
$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{5.21}$$
$$= Q_N(A - \delta) \tag{5.22}$$

Where Q_N is the Q-function of the normal distribution. Similarly,

$$P_{e|1} = Q_N(A+\delta) \tag{5.23}$$

Therefore,

$$\begin{aligned} P_e &= P_{e|0}\Pr(X=1) + P_{e|1}\Pr(X=-1) && (5.24)\\ &= \frac{1}{2}(Q_N(A-\delta)+Q_N(A+\delta)) && (5.25)\\ & && (5.26)\end{aligned}$$

To minimise P_e, we differentiate the above equation wrt δ:

$$\begin{aligned} 0 &= \frac{d}{d\delta}\left(\frac{1}{2}(Q_N(A-\delta)+Q_N(A+\delta))\right) && (5.27)\\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}}\right) && (5.28)\end{aligned}$$

Therefore,

$$\begin{aligned} (\delta-A)^2 &= (\delta+A)^2 && (5.29)\\ \implies \delta &= 0 && (5.30)\end{aligned}$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.31}$$

Solution: Using Eq. (5.24), we have:

$$\begin{aligned} P_e &= P_{e|0}p + P_{e|1}(1-p) && (5.32)\\ &= pQ_N(A-\delta)+(1-p)Q_N(A+\delta) && (5.33)\end{aligned}$$

Differentiating as before, we get:

$$0 = p\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - (1-p)\frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}} \tag{5.34}$$

Taking ln on both sides we have:

$$\begin{aligned} \ln p - \frac{(\delta-A)^2}{2} &= \ln 1 - p + \frac{(\delta+A)^2}{2} && (5.35)\\ \implies \delta &= \frac{1}{2A}\ln\frac{1-p}{p} && (5.36)\end{aligned}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: Assume that $\Pr(X=-1) = p$, and $\Pr(X=1) = (1-p)$. Then, using the Law of Total Probability, we have:

$$\begin{aligned} p_Y(y) &= p_{Y|X=-1}(y|-1)\Pr(X=-1)\\ &\quad + p_{Y|X=1}(y|1)\Pr(X=1) && (5.37)\\ &= p\times p_{(-A+N)}(y)\\ &\quad + (1-p)\times p_{(A+N)}(y) && (5.38)\end{aligned}$$

where $p_Y(y)$ is the pdf of Y. Now, $p_{(-A+N)}$ is just the pdf of a shifted normal distribution, and therefore:

$$p_Y(y) = p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \tag{5.39}$$

To use the MAP criterion, we must find $p_{X|Y}(x|y)$. To do this, we use the Theorem of Conditional Probability:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\times p_X(x)}{p_Y(y)} \tag{5.40}$$

When $X=1$, we have:

$$\begin{aligned} p_{X|Y}(1|y) &= \frac{p_{Y|X}(y|1)\times p_X(1)}{p_Y(y)} && (5.41)\\ &= \frac{(1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} && (5.42)\\ &= \frac{(1-p)\,e^{2yA}}{p+(1-p)\,e^{2yA}} && (5.43)\end{aligned}$$

Similarly, when $X=-1$, we get:

$$p_{X|Y}(-1|y) = \frac{p}{p+(1-p)\,e^{2yA}} \tag{5.44}$$

Therefore, when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$, we have:

$$\begin{aligned} \frac{(1-p)\,e^{2yA}}{p+(1-p)\,e^{2yA}} &> \frac{p}{p+(1-p)\,e^{2yA}} && (5.45)\\ e^{2yA} &> \frac{p}{(1-p)} && (5.46)\\ y &> \frac{1}{2A}\ln\frac{p}{(1-p)} && (5.47)\end{aligned}$$

Therefore, when Eq. (5.47), we can assert that

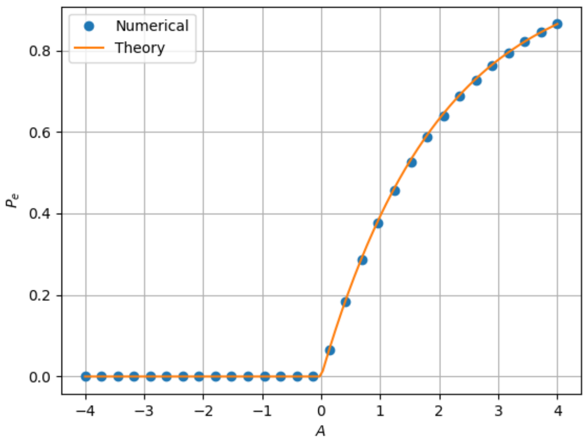

Fig. 6.1: The CDF of V

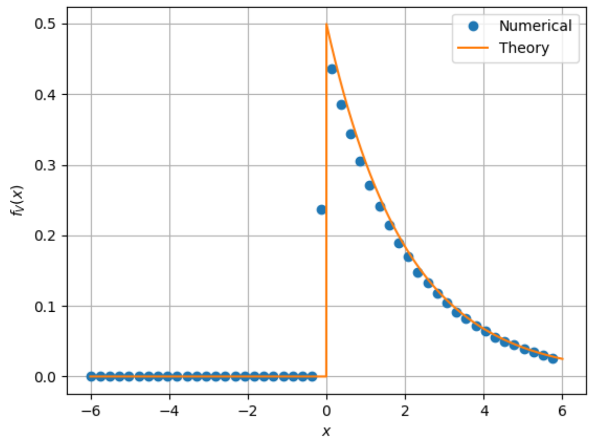

Fig. 6.1: The PDF of V

$X = 1$, and $X = -1$ otherwise. Now, consider when $p = \frac{1}{2}$. We have:

$$y > \frac{1}{2A} \ln \frac{p}{(1-p)} \tag{5.48}$$
$$= \frac{1}{2A} \ln 1 \tag{5.49}$$
$$= 0 \tag{5.50}$$

Therefore, when $y > 0$, we choose $X = 1$, and we choose $X = -1$ otherwise.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Use the function "chi" in exrand.c and execute:

```
gcc exrand.c
./a.out
```

Define the functions "chi_pdf" and "chi_cdf" in **functions.py** and execute:

```
python3 cdf_plot.py
python3 pdf_plot.py
```

The graphs are plotted in Fig. 6.1

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

Solution: We will assume that X_1 and X_2 are i.i.d. Let

$$X_1 = r\cos\theta \tag{6.3}$$
$$X_2 = r\sin\theta \tag{6.4}$$
$$\tag{6.5}$$

The Jacobian Matrix is then defined as:

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \tag{6.6}$$
$$J = \begin{pmatrix} \frac{\delta r\cos\theta}{\delta r} & \frac{\delta r\cos\theta}{\delta \theta} \\ \frac{\delta r\sin\theta}{\delta r} & \frac{\delta r\sin\theta}{\delta \theta} \end{pmatrix} \tag{6.7}$$
$$J = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix} \tag{6.8}$$
$$\implies |J| = R \tag{6.9}$$

Then,

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2) \tag{6.10}$$
$$= \frac{1}{2\pi} e^{\frac{-(x_1^2+x_2^2)}{2}} \tag{6.11}$$
$$= \frac{1}{2\pi} e^{\frac{-r^2}{2}} \tag{6.12}$$

Now, since

$$p_{r,\theta}(r,\theta) = |J| p_{X_1,X_2}(x_1,x_2) \tag{6.13}$$

we have:

$$p_{R,\theta}(r,\theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \tag{6.14}$$

Therefore,

$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r,\theta) \tag{6.15}$$
$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta \tag{6.16}$$
$$= re^{-\frac{r^2}{2}} \tag{6.17}$$
$$\tag{6.18}$$

We then have:

$$F_R(r) = \Pr(R \le r) \tag{6.19}$$
$$= \int_0^r f_R(r)dr \quad = 1 - e^{-\frac{r^2}{2}} \tag{6.20}$$

$F_V(x)$ is given by:

$$F_V(x) = F_{X_1^2+X_2^2}(x) \tag{6.21}$$
$$= F_{R^2} x \tag{6.22}$$
$$= \Pr\left(R^2 \le x\right) \tag{6.23}$$
$$= \Pr\left(R \le \sqrt{x}\right) \tag{6.24}$$

Therefore,

$$F_V(x) = \begin{cases} 0, & x \in x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \ge 0 \end{cases} \tag{6.25}$$

Comparing with Eq. 6.1 we get:

$$\alpha = \frac{1}{2} \tag{6.26}$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.27}$$

Solution: Use the function "ray" in **exrand.c** and execute:

```
gcc exrand.c
./a.out
```

Add the functions "ray_pdf" and "ray_cdf" to **functions.py** and execute the below files:

```
python3 pdf_plot.py
python3 cdf_plot.py
```

The graphs are plotted in Fig. 6.3

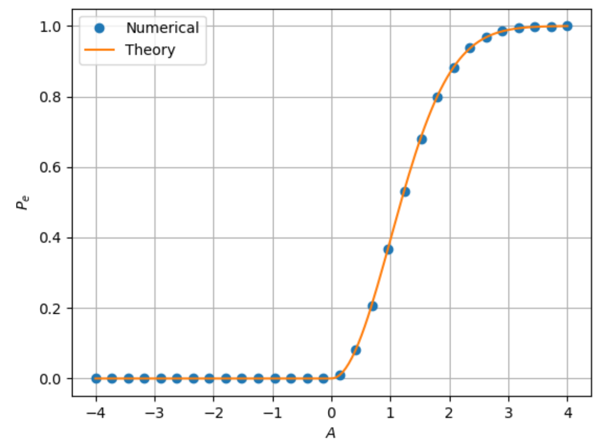

Fig. 6.3: The CDF of A

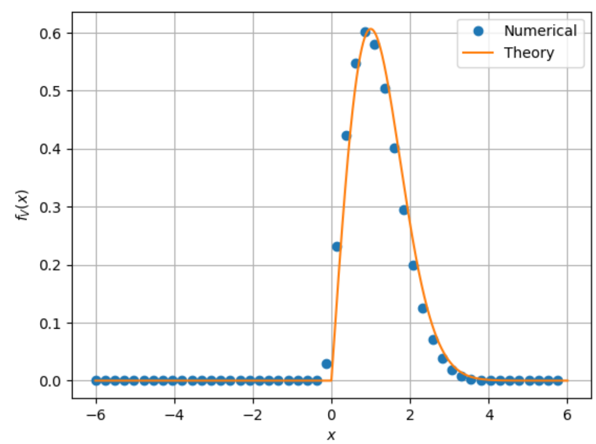

Fig. 6.3: The PDF of A

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.