#### 1

# Random Numbers

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**Uniform Random Numbers** 

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Abstract—This manual provides a simple introduction to the generation of random numbers

#### 1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate  $10^6$  samples of U using a C program and save into a file called uni.dat .

**Solution:** Download the following files and execute the C program.

wget https://github.com/gadepall/probability/ raw/master/manual/codes/exrand.c wget https://github.com/gadepall/probability/ raw/master/manual/codes/coeffs.h gcc exrand.c ./a.out

1.2 Load the uni.dat file into python and plot the empirical CDF of *U* using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \le x\right) \tag{1.1}$$

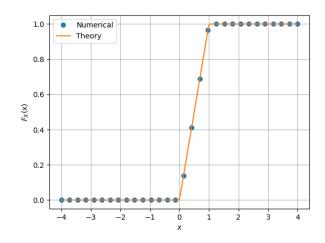


Fig. 1.2: The CDF of U

wget https://github.com/gadepall/probability/ raw/master/manual/codes/cdf\_plot.py python3 cdf\_plot.py

The following code plots Fig. 1.2

1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:** U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases}$$
 (1.2)

Therefore, we have:

$$F_U(x) = \int_0^x U(x)dx \tag{1.3}$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases}$$
 (1.4)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i$$
 (1.5)

**Solution:** 

and its variance as

$$var[U] = E[U - E[U]]^2$$
 (1.6)

Write a C program to find the mean and variance of U.

**Solution:** Add the following function to coeffs.h

```
double variance(char *str)
int i=0,c;
FILE *fp;
double x, temp=0.0;
fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp + x * x;
double mn = mean(str);
fclose(fp);
temp = temp/(i-1);
return temp - mn*mn;
}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E\left[U^{k}\right] = \int_{-\infty}^{\infty} x^{k} dF_{U}(x) \tag{1.7}$$

Solution: Since

$$dF_U(x) = p_U(x)dx (1.8)$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x) dx \tag{1.9}$$

Also,

$$p_{U}(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases}$$
 (1.10)

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x) dx \qquad (1.11)$$

$$= \int_0^1 x^2 dx$$
 (1.12)

$$=\frac{1}{3}$$
 (1.13)

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x) dx$$
 (1.14)

$$= \int_0^1 x dx \tag{1.15}$$

$$=\frac{1}{2}$$
 (1.16)

Therefore, the mean is  $\frac{1}{2}$ , and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2$$
 (1.17)

$$=\frac{1}{12}$$
 (1.18)

#### 2 Central Limit Theorem

2.1 Generate 10<sup>6</sup> samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where  $U_i$ , i = 1, 2, ..., 12 are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:** Add the following line to **exrand.c** and execute the code:

gaussian("gau.dat", 1000000); gcc exrand.c ./a.out

2.2 Load gau.dat in python and plot the empirical CDF of *X* using the samples in gau.dat. What properties does a CDF have?

**Solution:** 

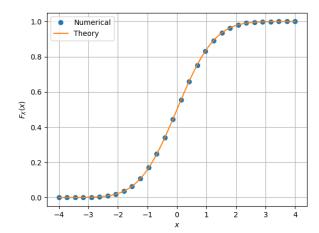


Fig. 2.2: The CDF of X

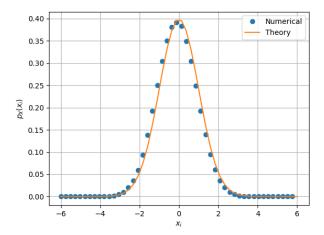


Fig. 2.3: The PDF of X

The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

**Solution:** The PDF of *X* is plotted in Fig. 2.3 using the code below

wget https://github.com/gadepall/probability/ raw/master/manual/codes/pdf plot.py python3 pdf plot.py

To find the CDF theoretically, consider

2.4 Find the mean and variance of X by writing a C program.

**Solution:** Use the main and variance functions in **coeffs.h**, and execute the code below

gcc exrand.c ./a.out

We get

mean = 0.000685variance = 1.000025

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

**Solution:** We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$
 (2.4)

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \tag{2.5}$$

$$=0 (2.6)$$

Also,

$$E[X^{2}] = \int_{-\infty}^{\infty} \frac{x^{2}}{\sqrt{2\pi}} \exp\left(-\frac{x^{2}}{2}\right)$$

$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^{2}}{2}\right)} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^{2}}{2}\right)}$$
(2.8)

$$=0+\frac{1}{\sqrt{2\pi}}\times\sqrt{2\pi}\tag{2.9}$$

$$= 1 \tag{2.10}$$

Hence,

$$var(X) = E[X^2] - E[X]^2$$
 (2.11)

$$= 1 \tag{2.12}$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in ./codes/exrancd.c, we get mean = 0.000685and variance = 1.000025, which closely matches the theoretical values.

2.6 Find the theoretical CDF of X Solution: To find the theoretical CDF, consider:

$$Q_X(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \qquad (2.13)$$

$$\operatorname{erfc}(\frac{x}{\sqrt{2}})$$

$$= \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.14}$$

The CDF is then:

$$F_X(x) = 1 - Q_X(x) (2.15)$$

$$=1-\frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \tag{2.16}$$

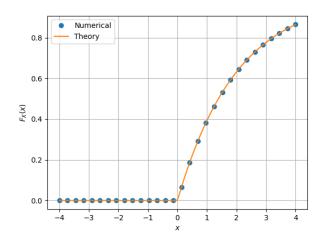


Fig. 3.2: The CDF of V

#### 3 From Uniform to Other

# 3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF. **Solution:** Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
  int i=0,c;
FILE *fp, *fp2;
  double x, temp=0.0;

fp = fopen("uni.dat","r");
  fp2 = fopen(str, "w");
//get numbers from file
  while(fscanf(fp,"%lf",&x)!=EOF)
{
    temp = -2*log(1-x);
    fprintf(fp2,"%lf\n",temp);
}

fclose(fp);
fclose(fp2);
return;
}
```

Using this function in **exrand.c** prints the numbers in **log.dat** 

3.2 Find a theoretical expression for  $F_V(x)$ .

**Solution:** We have:

$$F_V(x) = \Pr\left(V \le x\right) \tag{3.2}$$

$$= \Pr(-2\ln(1 - U) \le x) \tag{3.3}$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= F_U \left( 1 - \exp\left(-\frac{x}{2}\right) \right) \tag{3.6}$$

Therefore,

$$F_{V}(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases}$$

$$(3.7)$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases}$$
 (3.8)

The CDF of V is plotted in Fig. 3.2

#### 4 Triangular Distributions

#### 4.1 Generate

$$T = U_1 + U_2 (4.1)$$

**Solution:** Use the function 'triangular' in **exrand.c** and execute the following code:

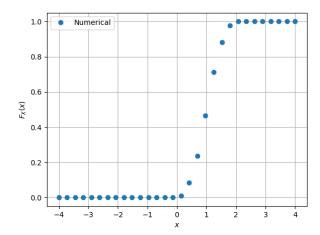


Fig. 4.2: The CDF of T

wget https://github.com/Rahuboy/AI1110/blob /main/RandomNumbers/codes/coeffs.h wget https://github.com/Rahuboy/AI1110/blob /main/RandomNumbers/codes/exrand.c gcc exrand.c ./a.out

#### 4.2 Find the CDF of T.

#### **Solution:**

wget https://github.com/Rahuboy/AI1110/blob/main/RandomNumbers/codes/cdf\_plot.py
python3 cdf\_plot.py

The above code plots Fig. 4.2

4.3 Find the PDF of T.

wget https://github.com/Rahuboy/AI1110/blob/main/RandomNumbers/codes/pdf\_plot.py
python3 pdf\_plot.py

The above code plots Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of *T*.

Solution: When

$$Z = X + Y \tag{4.2}$$

where X, Y and Z are random variables, we

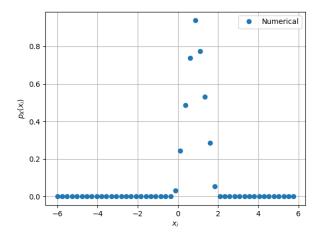


Fig. 4.3: The PDF of T

have:

$$p_Z(t) = (p_X * p_Y)(t)$$
 (4.3)

$$= \int_{-\infty}^{\infty} p_X(\tau) p_Y(t-\tau) d\tau \tag{4.4}$$

Here,  $p_X(t) = p_Y(t) = p_U(t)$ . Therefore:

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t - \tau) d\tau \tag{4.5}$$

$$= \int_0^1 p_U(t-\tau)d\tau \tag{4.6}$$

When t < 0 and t > 2, the integral evaluates to 0. When 0 < t < 1:

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau$$
 (4.7)

$$= \int_0^t p_U(t-\tau)d\tau \tag{4.8}$$

$$= \int_0^t 1d\tau \tag{4.9}$$

$$=t \tag{4.10}$$

when 1 < t < 2:

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau$$
 (4.11)

$$= \int_{t-1}^{1} p_U(t-\tau)d\tau$$
 (4.12)

$$= \int_{t-1}^{1} 1 d\tau \tag{4.13}$$

$$=2-t\tag{4.14}$$

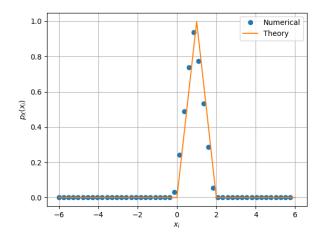
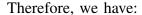


Fig. 4.5: The PDF of T



$$p_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 2 - x, & x \in (1, 2) \\ 0, & x \in (2, \infty) \end{cases}$$
(4.15)

To find the CDF, we use:

$$F_T(x) = \int_{-\infty}^{x} p_T(t)dt \tag{4.16}$$

We get:

$$F_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{x^2}{2}, & x \in (0, 1) \\ -\frac{x^2}{2} + 2x - 1, & x \in (1, 2) \\ 1, & x \in (2, \infty) \end{cases}$$
(4.17)

4.5 Verify your result for the PDF through a plot. **Solution:** Execute the following code:

The theoretical PDF is plotted in Fig. 4.5

4.6 Verify your result for the CDF through a plot. **Solution:** Execute the following code:

The theoretical CDF is plotted in Fig. 4.6

## 5 MAXIMAL LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ . Solution: Use the function "bernoulli" in **exrand.c** and execute the code below:

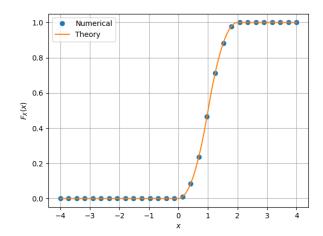


Fig. 4.6: The CDF of T

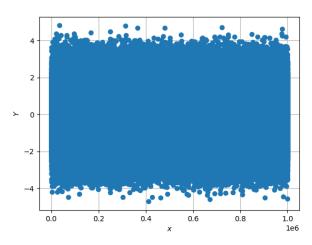


Fig. 5.3: The Plot of Y

gcc exrand.c

5.2 Generate

$$Y = AX + N, (5.1)$$

where A = 5 dB,  $X_1\{1, -1\}$ , is Bernoulli and  $N \sim \mathcal{N}(0, 1)$ .

**Solution:** Use the functions 'bernoulli' and 'maxlike' in **exrand.c**:

gcc exrand.c ./a.out

5.3 Plot *Y*.

Y is plotted in Fig. 5.3

5.4 Guess how to estimate *X* from *Y*.

**Solution:** To estimate *X* from *Y*, we define the

following function:

$$sgn(y) = \begin{cases} -1, & y \in (-\infty, 0] \\ 1, & y \in (0, \infty) \end{cases}$$
 (5.2)

Using sgn y, we can operate on Y to find corresponding values of X.

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1)$$
 (5.3)

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1)$$
 (5.4)

**Solution:** Use the function "maxlike\_proberr" in **exrand.c** to find the respective probabilities:

gcc exrand.c

./a.out

$$P(e|0) = 0.312414$$

$$P(e|1) = 0.310985$$

### 5.6 Find $P_e$ .

**Solution:** Assume a general value of A. Our estimation function predicts that the data points above the x axis correspond to X = 1, and the data points below the x-axis correspond to X = -1. This isn't always the case, as Y = AX + N, and the N causes some spill-over. We have:

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1)$$
 (5.5)

$$= \Pr(AX + N < 0 | X = 1) \tag{5.6}$$

$$= \Pr\left(N < -A\right) \tag{5.7}$$

$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.8}$$

$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.9}$$

$$=Q_N(A) \tag{5.10}$$

where  $Q_N$  is the Q-function of the normal distribution.

Similarly,

$$P_{e|1} = Q_N(A) \tag{5.11}$$

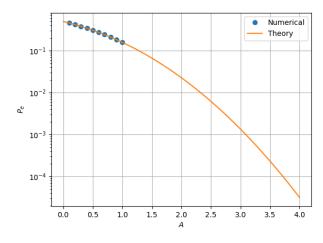


Fig. 5.7: The Plot of  $P_e$ 

Therefore,

$$P_e = P_{e|0} \times \Pr(X = 1) + P_{e|1} \times \Pr(X = -1)$$
(5.12)

$$=\frac{1}{2}P_{e|0} + \frac{1}{2}P_{e|1} \tag{5.13}$$

$$= \frac{1}{2}Q_N(A) + \frac{1}{2}Q_N(A) \tag{5.14}$$

$$=Q_N(A) \tag{5.15}$$

5.7 Verify by plotting the theoretical  $P_e$ .

**Solution:** The graph of  $P_e$  is plotted in Fig. 5.7 5.8 Now, consider a threshold  $\delta$  while estimating X from Y. Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

**Solution:** To estimate X from Y, we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases}$$
 (5.16)

Therefore,

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1)$$
 (5.17)

= 
$$\Pr(AX + N < \delta | X = 1)$$
 (5.18)

$$= \Pr\left(N < \delta - A\right) \tag{5.19}$$

$$= \int_{-\infty}^{\delta - A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
 (5.20)

$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
 (5.21)

$$=Q_N(A-\delta) \tag{5.22}$$

Where  $Q_N$  is the Q-function of the normal distribution. Similarly,

$$P_{e|1} = Q_N(A + \delta) \tag{5.23}$$

Therefore,

$$P_{e} = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1)$$

$$= \frac{1}{2} (Q_{N}(A - \delta) + Q_{N}(A + \delta)) \qquad (5.25)$$

$$(5.26)$$

To minimise  $P_e$ , we differentiate the above equation wrt  $\delta$ :

$$0 = \frac{d}{d\delta} \left( \frac{1}{2} (Q_N(A - \delta) + Q_N(A + \delta)) \right)$$

$$= \frac{1}{2} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A + \delta)^2}{2}} \right)$$
(5.28)

Therefore,

$$(\delta - A)^2 = (\delta + A)^2 \tag{5.29}$$

$$\implies \delta = 0 \tag{5.30}$$

5.9 Repeat the above exercise when

$$p_X(0) = p (5.31)$$

**Solution:** Using Eq. (5.24), we have:

$$P_e = P_{e|0}p + P_{e|1}(1-p)$$

$$= pO_N(A-\delta) + (1-p)O_N(A+\delta)$$
 (5.32)

Differentiating as before, we get:

$$0 = p \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - (1 - p) \frac{1}{\sqrt{2\pi}} e^{-\frac{(A + \delta)^2}{2}}$$
(5.34)

Taking In on both sides we have:

$$\ln p - \frac{(\delta - A)^2}{2} = \ln 1 - p + \frac{(\delta + A)^2}{2} \quad (5.35)$$

$$\implies \delta = \frac{1}{2A} \ln \frac{1 - p}{p} \quad (5.36)$$

5.10 Repeat the above exercise using the MAP criterion.

**Solution:** Assume that Pr(X = -1) = p, and Pr(X = 1) = (1 - p). Then, using the Law of Total Probability, we have:

$$p_{Y}(y) = p_{Y|X=-1}(y|-1) \Pr(X = -1)$$

$$+ p_{Y|X=1}(y|1) \Pr(X = 1) \qquad (5.37)$$

$$= p \times p_{(-A+N)}(y)$$

$$+ (1-p) \times p_{(A+N)}(y) \qquad (5.38)$$

where  $p_Y(y)$  is the pdf of Y. Now,  $p_{(-A+N)}$  is just the pdf of a shifted normal distribution, and therefore:

$$p_Y(y) = p \frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1-p) \frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}$$
 (5.39)

To use the MAP criterion, we must find  $p_{X|Y}(x|y)$ . To do this, we use the Theorem of Conditional Probability:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)}$$
 (5.40)

When X = 1, we have:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)}$$
 (5.41)

$$= \frac{(1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p^{\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}}} + (1-p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}$$
(5.42)

$$= \frac{(1-p)e^{2yA}}{p+(1-p)e^{2yA}}$$
 (5.43)

Similarly, when X = -1, we get:

$$p_{X|Y}(-1|y) = \frac{p}{p + (1-p)e^{2yA}}$$
 (5.44)

Therefore, when  $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$ , we have:

$$\frac{(1-p)e^{2yA}}{p+(1-p)e^{2yA}} > \frac{p}{p+(1-p)e^{2yA}}$$
 (5.45)

$$e^{2yA} > \frac{p}{(1-p)} \tag{5.46}$$

$$y > \frac{1}{2A} \ln \frac{p}{(1-p)}$$
 (5.47)

Therefore, when Eq. (5.47), we can assert that

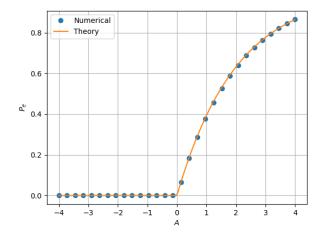


Fig. 6.1: The CDF of V

X = 1, and X = -1 otherwise. Now, consider when  $p = \frac{1}{2}$ . We have:

$$y > \frac{1}{2A} \ln \frac{p}{(1-p)}$$
 (5.48)

$$= \frac{1}{2A} \ln 1 \tag{5.49}$$

$$=0 (5.50)$$

Therefore, when y > 0, we choose X = 1, and we choose X = -1 otherwise.

#### 6 Gaussian to Other

6.1 Let  $X_1 \sim \mathcal{N}(0,1)$  and  $X_2 \sim \mathcal{N}(0,1)$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

**Solution:** Use the function "chi" in exrand.c and execute:

gcc exrand.c ./a.out

Define the functions "chi pdf" and "chi cdf" in functions.py and execute:

The graphs are plotted in Fig. 6.1

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0\\ 0 & x < 0, \end{cases}$$
 (6.2)

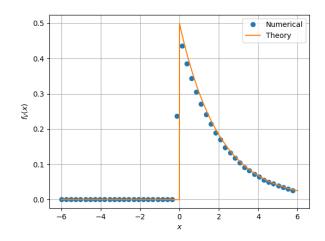


Fig. 6.1: The PDF of V

find  $\alpha$ .

**Solution:** We will assume that  $X_1$  and  $X_2$  are i.i.d. Let

$$X_1 = r\cos\theta \tag{6.3}$$

$$X_2 = r \sin \theta \tag{6.4}$$

(6.5)

The Jacobian Matrix is then defined as:

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta x} & \frac{\delta x_2}{\delta \theta} \end{pmatrix}$$
 (6.6)

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\delta r \cos \theta}{\delta r} & \frac{\delta r \cos \theta}{\delta \theta} \\ \frac{\delta r \sin \theta}{\delta r} & \frac{\delta r \sin \theta}{\delta \theta} \end{pmatrix}$$

$$J = \begin{pmatrix} \cos \theta & -R \sin \theta \\ \sin \theta & R \cos \theta \end{pmatrix}$$
(6.8)

$$J = \begin{pmatrix} \cos \theta & -R \sin \theta \\ \sin \theta & R \cos \theta \end{pmatrix} \tag{6.8}$$

$$\implies |J| = R \tag{6.9}$$

Then,

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$
 (6.10)

$$=\frac{1}{2\pi}e^{\frac{-(x_1^2+x_2^2)}{2}}\tag{6.11}$$

$$=\frac{1}{2\pi}e^{\frac{-r^2}{2}}\tag{6.12}$$

Now, since

$$p_{r,\theta}(r,\theta) = |J| p_{X_1,X_2}(x_1, x_2)$$
 (6.13)

we have:

$$p_{R,\theta}(r,\theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}}$$
 (6.14)

Therefore,

$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r,\theta)$$
 (6.15)

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta \qquad (6.16)$$

$$= re^{-\frac{r^2}{2}} \tag{6.17}$$

(6.18)

We then have:

$$F_R(r) = \Pr\left(R \le r\right) \tag{6.19}$$

$$= \int_0^r f_R(r)dr = 1 - e^{-\frac{r^2}{2}}$$
 (6.20)

 $F_V(x)$  is given by:

$$F_V(x) = F_{X_1^2 + X_2^2}(x) \tag{6.21}$$

$$=F_{R^2}x\tag{6.22}$$

$$= \Pr\left(R^2 \le x\right) \tag{6.23}$$

$$= \Pr\left(R \le \sqrt{x}\right) \tag{6.24}$$

Therefore,

$$F_V(x) = \begin{cases} 0, & x \in x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \ge 0 \end{cases}$$
 (6.25)

Comparing with Eq. 6.1 we get:

$$\alpha = \frac{1}{2} \tag{6.26}$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.27}$$

**Solution:** Use the function "ray" in **exrand.c** and execute:

Add the functions "ray\_pdf" and "ray\_cdf" to **functions.py** and execute the below files:

The graphs are plotted in Fig. 6.3

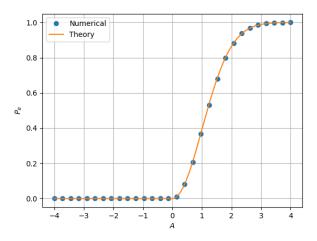


Fig. 6.3: The CDF of A

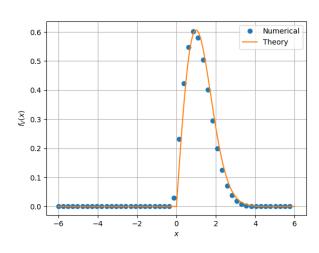


Fig. 6.3: The PDF of A

#### 7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, (7.2)$$

where A is Rayleigh with  $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0,1\right), X \in (-1,1)$  for  $0 \le \gamma \le 10$  dB.

#### **Solution:**

Add the function "cond\_prob" to coeffs.h . Also modify the function "proberr\_graph", and execute the code below:

gcc exrand.c ./a.out

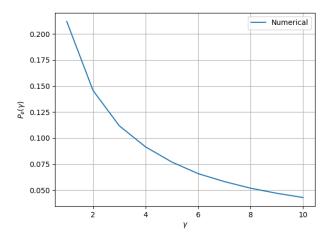


Fig. 7.1:  $P_e$  vs  $\gamma$ 

This updates **proberr\_graph.dat**. Now execute the code below:

wget https://github.com/Rahuboy/AI1110/blob/main/RandomNumbers/codes/7.py

The graph is plotted in Fig. 7.1

7.2 Assuming that *N* is a constant, find an expression for  $P_e$ . Call this  $P_e(N)$ 

# **Solution:**

We use the signum function for estimation. We need to find:

$$P_e = \Pr(\hat{X} = -1|X = 1)$$
 (7.3)

$$= \Pr(AX + N < 0 | X = 1) \tag{7.4}$$

$$= \Pr\left(A < -N\right) \tag{7.5}$$

Assuming N is a constant, we have:

$$P_e = \Pr\left(A < -N\right) \tag{7.6}$$

$$= F_A(-N) \tag{7.7}$$

$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}}, & N \le 0\\ 0, & N > 0 \end{cases}$$
 (7.8)

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x) dx \qquad (7.9)$$

Find  $P_e = E[P_e(N)]$ .

**Solution:** 

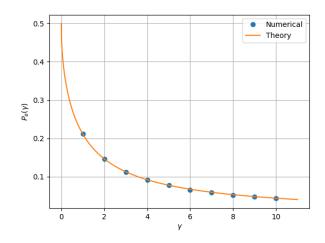


Fig. 7.4:  $P_e$  vs  $\gamma$ 

We have:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x) dx \qquad (7.10)$$

$$\implies E[P_E(N)] = \int_{-\infty}^{\infty} P_E(x) p_N(x) dx \quad (7.11)$$
$$= \int_{-\infty}^{0} (1 - e^{-\frac{x^2}{\gamma}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$=\frac{1}{2}\left(1-\sqrt{\frac{\gamma}{\gamma+2}}\right) \qquad (7.13)$$

7.4 Plot  $P_e$  in problems 7.1 and 7.3 on the same graph w.r.t  $\gamma$ . Comment.

**Solution:** Add the function "proberr2" to **functions.py**, and execute the code below:

The graph is plotted in Fig. 7.4

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n},\tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
 (8.2)

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0$$
 and  $\mathbf{y}|\mathbf{s}_1$  (8.4)

on the same graph using a scatter plot.

- 8.2 For the above problem, find a decision rule for detecting the symbols  $\mathbf{s}_0$  and  $\mathbf{s}_1$ .
- 8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for  $P_e$ . Verify this by comparing the theory and simulation plots on the same graph.