

Random Numbers

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Abstract—This manual provides a simple introduction to the generation of random numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
raw/master/manual/codes/coeffs.h
gcc exrand.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution:

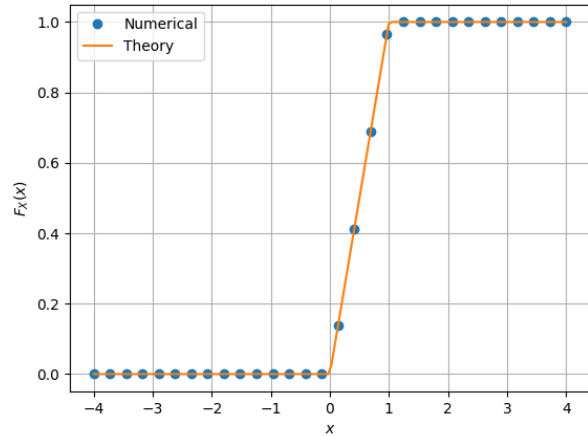


Fig. 1.2: The CDF of U

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

The following code plots Fig. 1.2

1.3 Find a theoretical expression for $F_U(x)$.

Solution: U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.2)$$

Therefore, we have:

$$F_U(x) = \int_0^x U(x) dx \quad (1.3)$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.4)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: Add the following function to coeffs.h

```
double variance(char *str)
{
    int i=0,c;
    FILE *fp;
    double x, temp=0.0;

    fp = fopen(str,"r");
    //get numbers from file
    while(fscanf(fp,"%lf",&x)!=EOF)
    {
        //Count numbers in file
        i=i+1;
        //Add all numbers in file
        temp = temp+x*x;
    }
    double mn = mean(str);
    fclose(fp);
    temp = temp/(i-1);
    return temp - mn*mn ;
}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution: Since

$$dF_U(x) = p_U(x)dx \quad (1.8)$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \quad (1.9)$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.10)$$

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x)dx \quad (1.11)$$

$$= \int_0^1 x^2 dx \quad (1.12)$$

$$= \frac{1}{3} \quad (1.13)$$

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \quad (1.14)$$

$$= \int_0^1 x dx \quad (1.15)$$

$$= \frac{1}{2} \quad (1.16)$$

Therefore, the mean is $\frac{1}{2}$, and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.17)$$

$$= \frac{1}{12} \quad (1.18)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Add the following line to **exrand.c** and execute the code:

```
gaussian("gau.dat", 1000000);
gcc exrand.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

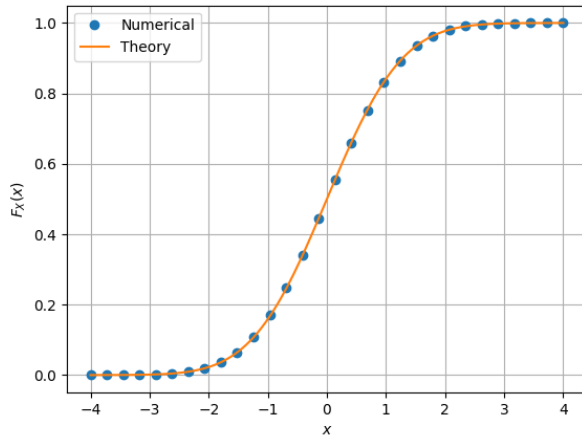


Fig. 2.2: The CDF of X

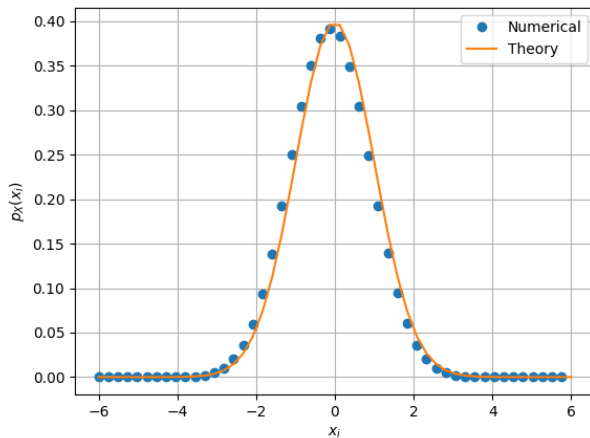


Fig. 2.3: The PDF of X

The CDF of X is plotted in Fig. 2.2

- 2.3 Load `gau.dat` in python and plot the empirical PDF of X using the samples in `gau.dat`. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/pdf_plot.py
python3 pdf_plot.py
```

To find the CDF theoretically, consider

- 2.4 Find the mean and variance of X by writing a C program.

Solution: Use the main and variance functions in `coeffs.h`, and execute the code below

```
gcc exrand.c
./a.out
```

We get

```
mean = 0.000685
variance = 1.000025
```

- 2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.4)$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (2.5)$$

$$= 0 \quad (2.6)$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.8)$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (2.9)$$

$$= 1 \quad (2.10)$$

Hence,

$$\text{var}(X) = E[X^2] - E[X]^2 \quad (2.11)$$

$$= 1 \quad (2.12)$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in `./codes/exrand.c`, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

- 2.6 Find the theoretical CDF of X

Solution: To find the theoretical CDF, consider:

$$Q_X(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (2.13)$$

$$= \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \quad (2.14)$$

The CDF is then:

$$F_X(x) = 1 - Q_X(x) \quad (2.15)$$

$$= 1 - \frac{\operatorname{erfc}(\frac{x}{\sqrt{2}})}{2} \quad (2.16)$$

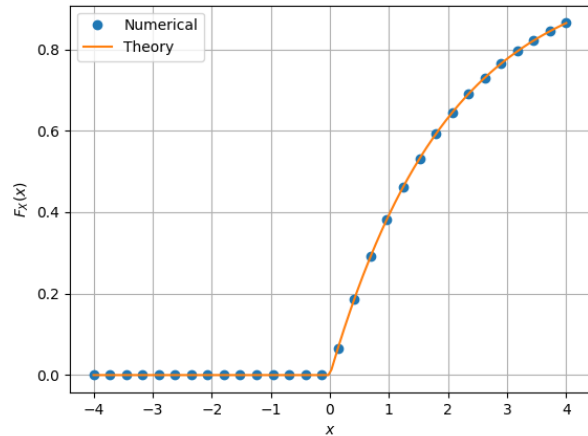


Fig. 3.2: The CDF of V

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF. **Solution:**

Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
    int i=0,c;
    FILE *fp, *fp2;
    double x, temp=0.0;

    fp = fopen("uni.dat","r");
    fp2 = fopen(str, "w");
    //get numbers from file
    while(fscanf(fp,"%lf",&x)!=EOF)
    {
        temp = -2*log(1-x);
        fprintf(fp2,"%lf\n",temp);
    }

    fclose(fp);
    fclose(fp2);

    return ;
}
```

Using this function in **exrand.c** prints the numbers in **log.dat**

3.2 Find a theoretical expression for $F_V(x)$.

Solution: We have:

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \quad (3.7)$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \quad (3.8)$$

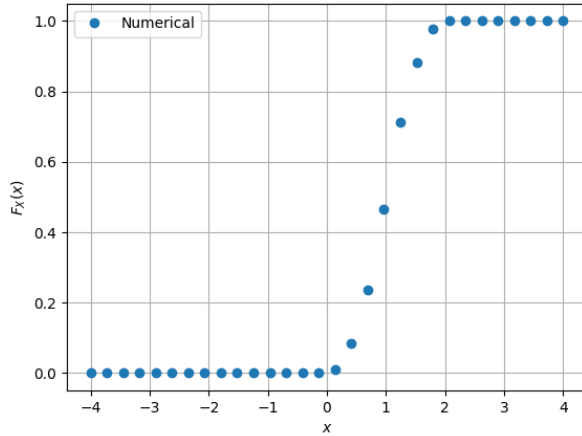
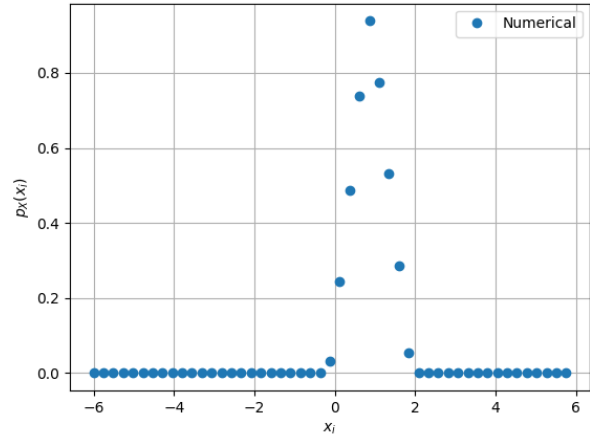
The CDF of V is plotted in Fig. 3.2

4 TRIANGULAR DISTRIBUTIONS

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Use the function 'triangular' in **exrand.c** and execute the following code:

Fig. 4.2: The CDF of T Fig. 4.3: The PDF of T

```
wget https://github.com/Rahuboy/AI1110/blob
/main/RandomNumbers/codes/coeffs.h
wget https://github.com/Rahuboy/AI1110/blob
/main/RandomNumbers/codes/exrand.c
gcc exrand.c
./a.out
```

4.2 Find the CDF of T .

Solution:

```
wget https://github.com/Rahuboy/AI1110/
blob/main/RandomNumbers/codes/
cdf_plot.py
python3 cdf_plot.py
```

The above code plots Fig. 4.2

4.3 Find the PDF of T .

```
wget https://github.com/Rahuboy/AI1110/
blob/main/RandomNumbers/codes/
pdf_plot.py
python3 pdf_plot.py
```

The above code plots Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: When

$$Z = X + Y \quad (4.2)$$

where X , Y and Z are random variables, we

have:

$$p_Z(t) = (p_X * p_Y)(t) \quad (4.3)$$

$$= \int_{-\infty}^{\infty} p_X(\tau) p_Y(t - \tau) d\tau \quad (4.4)$$

Here, $p_X(t) = p_Y(t) = p_U(t)$. Therefore:

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t - \tau) d\tau \quad (4.5)$$

$$= \int_0^1 p_U(t - \tau) d\tau \quad (4.6)$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau \quad (4.7)$$

$$= \int_0^t p_U(t - \tau) d\tau \quad (4.8)$$

$$= \int_0^t 1 d\tau \quad (4.9)$$

$$= t \quad (4.10)$$

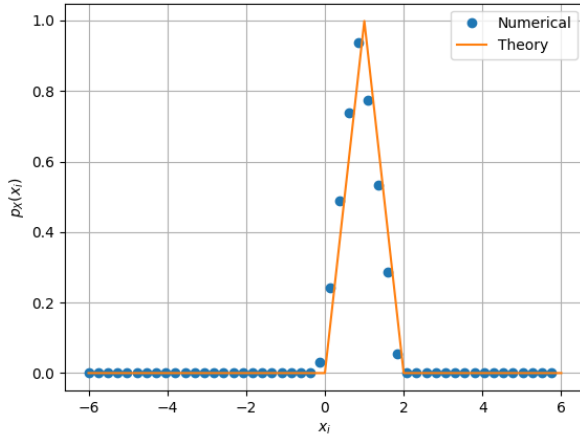
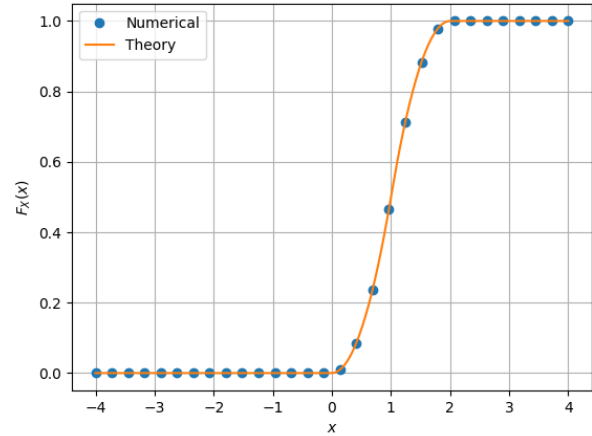
when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau \quad (4.11)$$

$$= \int_{t-1}^1 p_U(t - \tau) d\tau \quad (4.12)$$

$$= \int_{t-1}^1 1 d\tau \quad (4.13)$$

$$= 2 - t \quad (4.14)$$

Fig. 4.5: The PDF of T Fig. 4.6: The CDF of T

Therefore, we have:

$$p_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 2 - x, & x \in (1, 2) \\ 0, & x \in (2, \infty) \end{cases} \quad (4.15)$$

To find the CDF, we use:

$$F_T(x) = \int_{-\infty}^x p_T(t) dt \quad (4.16)$$

We get:

$$F_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{x^2}{2}, & x \in (0, 1) \\ -\frac{x^2}{2} + 2x - 1, & x \in (1, 2) \\ 1, & x \in (2, \infty) \end{cases} \quad (4.17)$$

4.5 Verify your result for the PDF through a plot.

Solution: Execute the following code:

```
python3 pdf_plot.py
```

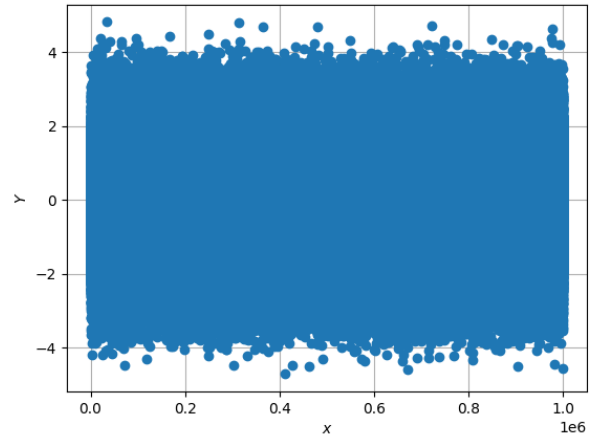
The theoretical PDF is plotted in Fig. 4.5

4.6 Verify your result for the CDF through a plot.

Solution: Execute the following code:

```
python3 cdf_plot.py
```

The theoretical CDF is plotted in Fig. 4.6

Fig. 5.2: The Plot of Y

where $A = 5$ dB, $X_1 \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

Solution: Use the functions 'bernoulli' and 'maxlike' in **exrand.c**:

```
gcc exrand.c
./a.out
```

5.2 Plot Y .

Y is plotted in Fig. 5.2

5.3 Guess how to estimate X from Y .

Solution: To estimate X from Y , we define the following function:

$$\text{sgn}(y) = \begin{cases} -1, & y \in (-\infty, 0] \\ 1, & y \in (0, \infty) \end{cases} \quad (5.2)$$

5 MAXIMAL LIKELIHOOD

5.1 Generate

$$Y = AX + N, \quad (5.1)$$

Using `sgny`, we can operate on Y to find corresponding values of X .

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.3)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.4)$$

Solution: Use the function "maxlike_proberr" in **exrand.c** to find the respective probabilities:

```
gcc exrand.c
./a.out
P_(e|0) = 0.312414
P_(e|1) = 0.310985
```

5.5 Find P_e .

Solution: Assume a general value of A . Our estimation function predicts that the data points above the x axis correspond to $X = 1$, and the data points below the x -axis correspond to $X = -1$. This isn't always the case, as $Y = AX + N$, and the N causes some spill-over. We have:

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.5)$$

$$= \Pr(AX + N < 0|X = 1) \quad (5.6)$$

$$= \Pr(N < -A) \quad (5.7)$$

$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (5.8)$$

$$= \int_A^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (5.9)$$

$$= Q_N(A) \quad (5.10)$$

where Q_N is the Q -function of the normal distribution.

Similarly,

$$P_{e|1} = Q_N(A) \quad (5.11)$$

Therefore,

$$P_e = P_{e|0} \times \Pr(X = 1) + P_{e|1} \times \Pr(X = -1) \quad (5.12)$$

$$= \frac{1}{2}P_{e|0} + \frac{1}{2}P_{e|1} \quad (5.13)$$

$$= \frac{1}{2}Q_N(A) + \frac{1}{2}Q_N(A) \quad (5.14)$$

$$= Q_N(A) \quad (5.15)$$

5.6 Verify by plotting the theoretical P_e .

The graph of P_e is plotted in Fig. 5.6

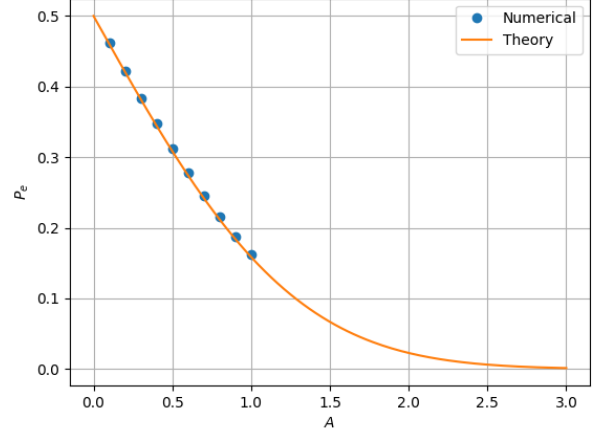


Fig. 5.6: The Plot of P_e

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: Use the function "chi" in **exrand.c** and execute:

```
gcc exrand.c
./a.out
```

Define the functions "chi_pdf" and "chi_cdf" in **functions.py** and execute:

```
python3 cdf_plot.py
python3 pdf_plot.py
```

The graphs are plotted in Fig. 6.1

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

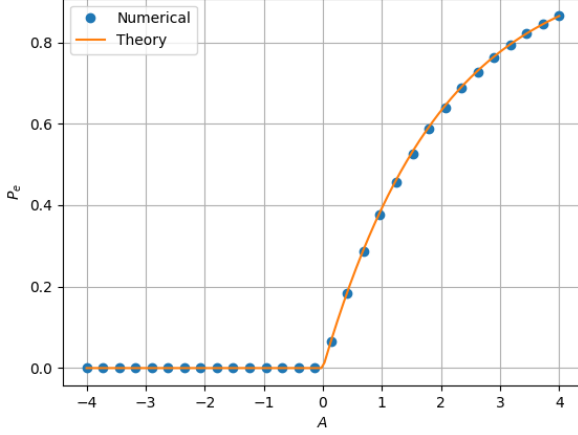
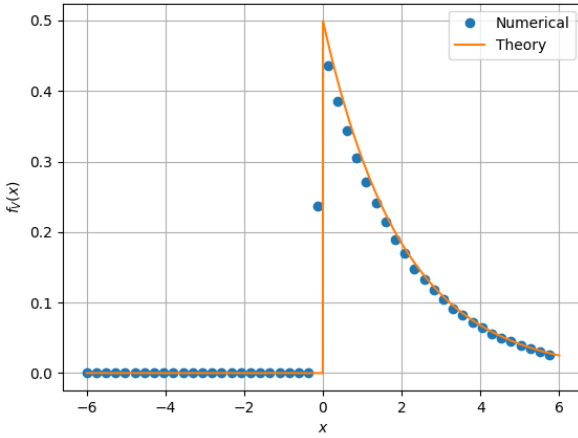
find α .

Solution: We will assume that X_1 and X_2 are i.i.d. Let

$$X_1 = r \cos \theta \quad (6.3)$$

$$X_2 = r \sin \theta \quad (6.4)$$

$$(6.5)$$

Fig. 6.1: The CDF of V Fig. 6.1: The PDF of V

The Jacobian Matrix is then defined as:

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \quad (6.6)$$

$$J = \begin{pmatrix} \frac{\delta r \cos \theta}{\delta r} & \frac{\delta r \cos \theta}{\delta \theta} \\ \frac{\delta r \sin \theta}{\delta r} & \frac{\delta r \sin \theta}{\delta \theta} \end{pmatrix} \quad (6.7)$$

$$J = \begin{pmatrix} \cos \theta & -R \sin \theta \\ \sin \theta & R \cos \theta \end{pmatrix} \quad (6.8)$$

$$\Rightarrow |J| = R \quad (6.9)$$

Then,

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \quad (6.10)$$

$$= \frac{1}{2\pi} e^{-\frac{(x_1^2 + x_2^2)}{2}} \quad (6.11)$$

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} \quad (6.12)$$

Now, since

$$f_{r, \theta}(r, \theta) = |J|f_{X_1, X_2}(x_1, x_2) \quad (6.13)$$

we have:

$$f_{r, \theta}(r, \theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \quad (6.14)$$

Therefore,

$$f_R(r) = \int_0^{2\pi} f_{r, \theta}(r, \theta) d\theta \quad (6.15)$$

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta \quad (6.16)$$

$$= r e^{-\frac{r^2}{2}} \quad (6.17)$$

$$(6.18)$$

We then have:

$$F_R(r) = \Pr(R \leq r) \quad (6.19)$$

$$= \int_0^r f_R(r) dr = 1 - e^{-\frac{r^2}{2}} \quad (6.20)$$

$F_V(x)$ is given by:

$$F_V(x) = F_{X_1^2 + X_2^2}(x) \quad (6.21)$$

$$= F_{R^2} x \quad (6.22)$$

$$= \Pr(R^2 \leq x) \quad (6.23)$$

$$= \Pr(R \leq \sqrt{x}) \quad (6.24)$$

Therefore,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \geq 0 \end{cases} \quad (6.25)$$

Comparing with Eq. 6.1 we get:

$$\alpha = \frac{1}{2} \quad (6.26)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.27)$$

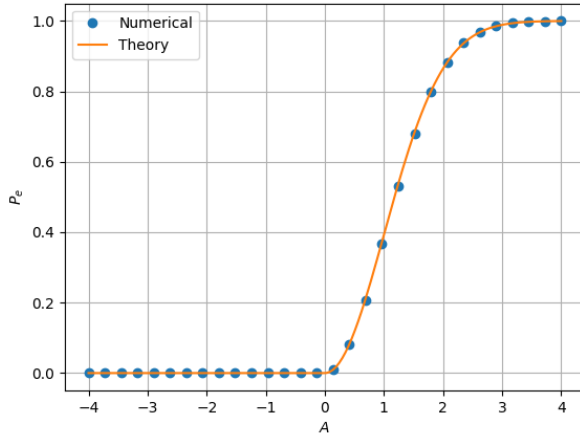


Fig. 6.3: The CDF of A

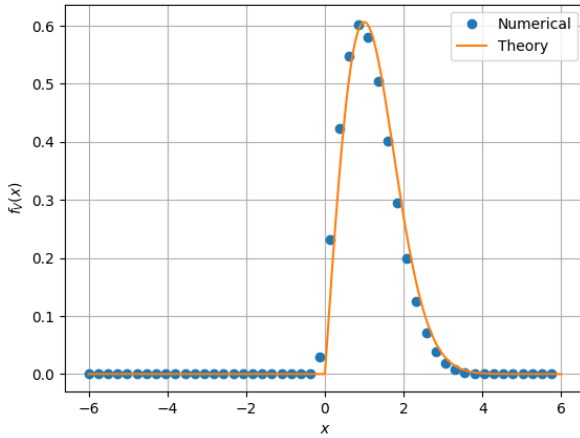


Fig. 6.3: The PDF of A

Solution: Use the function "ray" in **exrand.c** and execute:

```
gcc exrand.c
./a.out
```

Add the functions "ray_pdf" and "ray_cdf" to **functions.py** and execute the below files:

```
python3 pdf_plot.py
python3 cdf_plot.py
```

The graphs are plotted in Fig. 6.3

7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Raleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x) dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.