

Random Numbers

Rahul Ramachandran

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Abstract—This manual provides a simple introduction to the generation of random numbers

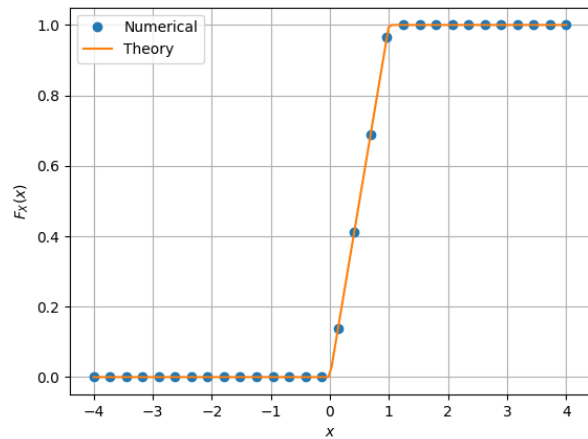


Fig. 1.2: The CDF of U

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
raw/master/manual/codes/coeffs.h
gcc exrand.c
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution:

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

The following code plots Fig. 1.2

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.2)$$

Therefore, we have:

$$F_U(x) = \int_0^x U(x) dx \quad (1.3)$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.4)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and

variance of U .

Solution: Add the following function to coeffs.h

```
double variance(char *str)
{
    int i=0,c;
    FILE *fp;
    double x, temp=0.0;

    fp = fopen(str,"r");
    //get numbers from file
    while(fscanf(fp,"%lf",&x)!=EOF)
    {
        //Count numbers in file
        i=i+1;
        //Add all numbers in file
        temp = temp+x*x;
    }
    double mn = mean(str);
    fclose(fp);
    temp = temp/(i-1);
    return temp - mn*mn ;

}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution: Since

$$dF_U(x) = p_U(x)dx \quad (1.8)$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \quad (1.9)$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.10)$$

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x)dx \quad (1.11)$$

$$= \int_0^1 x^2 dx \quad (1.12)$$

$$= \frac{1}{3} \quad (1.13)$$

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \quad (1.14)$$

$$= \int_0^1 x dx \quad (1.15)$$

$$= \frac{1}{2} \quad (1.16)$$

Therefore, the mean is $\frac{1}{2}$, and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.17)$$

$$= \frac{1}{12} \quad (1.18)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Add the following line to **exrand.c** and execute the code:

```
gaussian("gau.dat", 1000000);
gcc exrand.c
./a.out
```

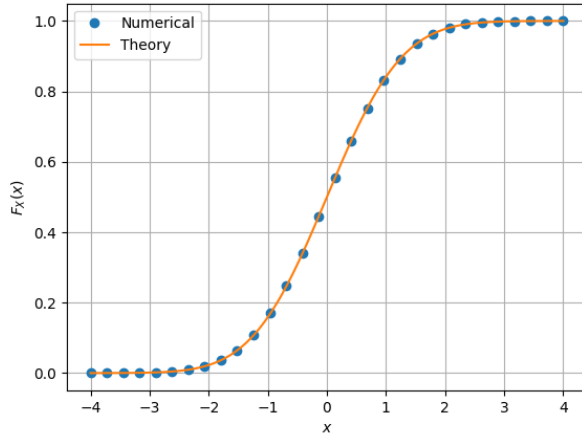
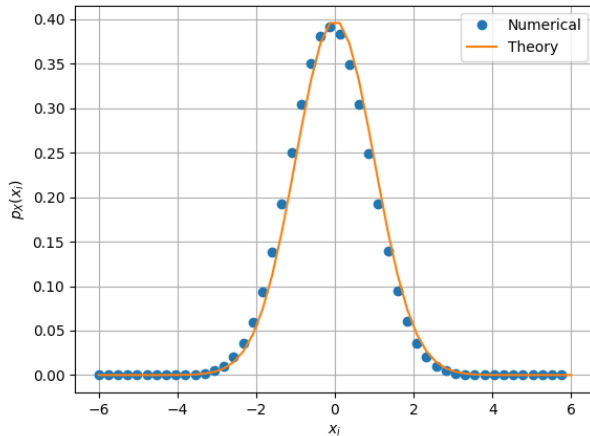
2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

Fig. 2.2: The CDF of X Fig. 2.3: The PDF of X

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/pdf_plot.py
python3 pdf_plot.py
```

To find the CDF theoretically, consider

2.4 Find the mean and variance of X by writing a C program.

Solution: Use the main and variance functions in **coeffs.h**, and execute the code below

```
gcc exrand.c
./a.out
```

We get

mean = 0.000685
variance = 1.000025

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.4)$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (2.5)$$

$$= 0 \quad (2.6)$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.8)$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (2.9)$$

$$= 1 \quad (2.10)$$

Hence,

$$\text{var}(X) = E[X^2] - E[X]^2 \quad (2.11)$$

$$= 1 \quad (2.12)$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in **./codes/exrand.c**, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

2.6 Find the theoretical CDF of X

Solution: To find the theoretical CDF, consider:

$$Q_X(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \quad (2.13)$$

$$= \frac{\text{erfc}\left(\frac{x}{\sqrt{2}}\right)}{2} \quad (2.14)$$

The CDF is then:

$$F_X(x) = 1 - Q_X(x) \quad (2.15)$$

$$= 1 - \frac{\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)}{2} \quad (2.16)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF. **Solution:**

Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
    int i=0,c;
    FILE *fp, *fp2;
    double x, temp=0.0;

    fp = fopen("uni.dat","r");
    fp2 = fopen(str, "w");
    //get numbers from file
    while(fscanf(fp,"%lf",&x)!=EOF)
    {
        temp = -2*log(1-x);
        fprintf(fp2,"%lf\n",temp);
    }

    fclose(fp);
    fclose(fp2);

    return ;
}
```

Using this function in **exrand.c** prints the numbers in **log.dat**

3.2 Find a theoretical expression for $F_V(x)$.

Solution: We have:

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

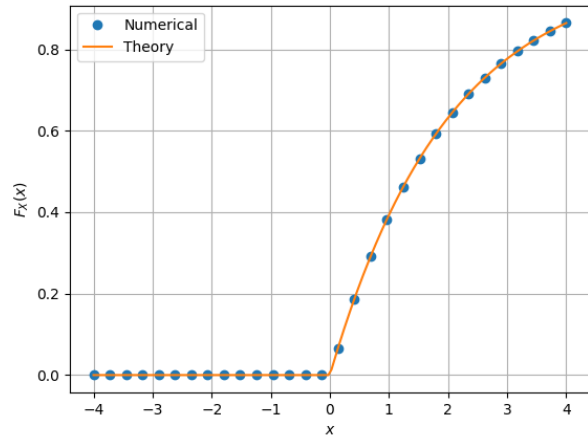


Fig. 3.2: The CDF of V

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \quad (3.7)$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \quad (3.8)$$

The CDF of V is plotted in Fig. 3.2

4 TRIANGULAR DISTRIBUTIONS

4.1 Generate

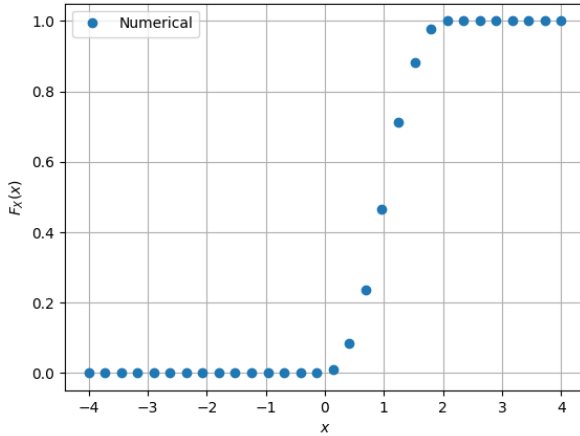
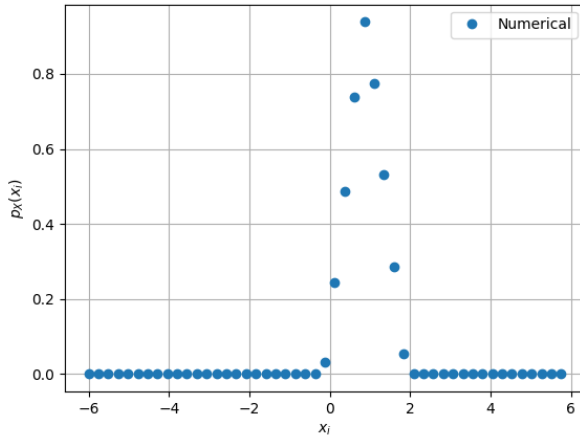
$$T = U_1 + U_2 \quad (4.1)$$

Solution: Use the function 'triangular' in **exrand.c** and execute the following code:

```
wget https://github.com/Rahuboy/AI1110/blob
/main/RandomNumbers/codes/coeffs.h
wget https://github.com/Rahuboy/AI1110/blob
/main/RandomNumbers/codes/exrand.c
gcc exrand.c
./a.out
```

4.2 Find the CDF of T .

Solution:

Fig. 4.2: The CDF of T Fig. 4.3: The PDF of T

```
wget https://github.com/Rahuboy/AI1110/
blob/main/RandomNumbers/codes/
cdf_plot.py
python3 cdf_plot.py
```

The above code plots Fig. 4.2

4.3 Find the PDF of T .

```
wget https://github.com/Rahuboy/AI1110/
blob/main/RandomNumbers/codes/
pdf_plot.py
python3 pdf_plot.py
```

The above code plots Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: When

$$Z = X + Y \quad (4.2)$$

where X , Y and Z are random variables, we have:

$$p_Z(t) = (p_X * p_Y)(t) \quad (4.3)$$

$$= \int_{-\infty}^{\infty} p_X(\tau) p_Y(t - \tau) d\tau \quad (4.4)$$

Here, $p_X(t) = p_Y(t) = p_U(t)$. Therefore:

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t - \tau) d\tau \quad (4.5)$$

$$= \int_0^1 p_U(t - \tau) d\tau \quad (4.6)$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau \quad (4.7)$$

$$= \int_0^t p_U(t - \tau) d\tau \quad (4.8)$$

$$= \int_0^t 1 d\tau \quad (4.9)$$

$$= t \quad (4.10)$$

when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau \quad (4.11)$$

$$= \int_{t-1}^1 p_U(t - \tau) d\tau \quad (4.12)$$

$$= \int_{t-1}^1 1 d\tau \quad (4.13)$$

$$= 2 - t \quad (4.14)$$

Therefore, we have:

$$p_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 2 - x, & x \in (1, 2) \\ 0, & x \in (2, \infty) \end{cases} \quad (4.15)$$

To find the CDF, we use:

$$F_T(x) = \int_{-\infty}^x p_T(t) dt \quad (4.16)$$

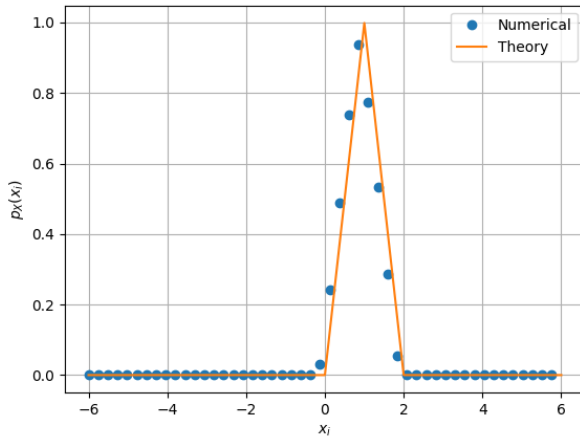


Fig. 4.5: The PDF of T

We get:

$$F_T(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{x^2}{2}, & x \in (0, 1) \\ -\frac{x^2}{2} + 2x - 1, & x \in (1, 2) \\ 1, & x \in (2, \infty) \end{cases} \quad (4.17)$$

4.5 Verify your result for the PDF through a plot.

Solution: Execute the following code:

```
python3 pdf_plot.py
```

The theoretical PDF is plotted in Fig. 4.5

4.6 Verify your result for the CDF through a plot.

Solution: Execute the following code:

```
python3 cdf_plot.py
```

The theoretical CDF is plotted in Fig. 4.6

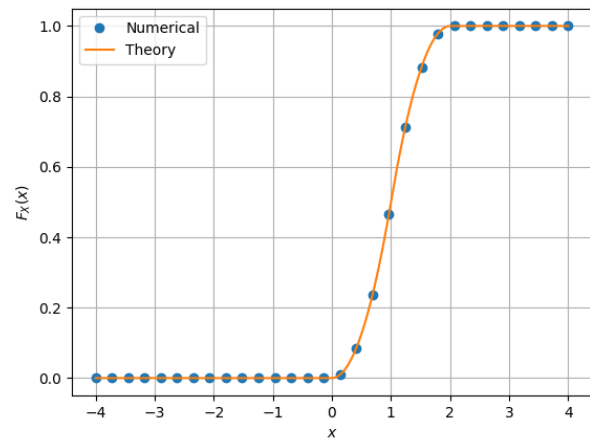


Fig. 4.6: The CDF of T