Random Numbers

Rahul Ramachandran

June 28, 2022

Outline

Problems

1. (1.3) Find the theoretical expression for $F_U(x) = \Pr(U \le x)$, where U is the uniform random variable between 0 and 1.
2. (1.5) Verify the results for the mean and variance of a uniform distribution theoretically given that $E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$
3. (2.2) Find the CDF of X, where X is a standard normal variable
4. (2.3) Find the CDF of X, where X is a standard normal variable
5. (2.5) Find the mean and variance of $p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$
6. (3.1) Find a theoretical expression for $F_V(x)$, where $V = -2\ln(1-U)$.

Solution (1.3)

U is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1}$$

Solution (1.3)

Therefore, we have:

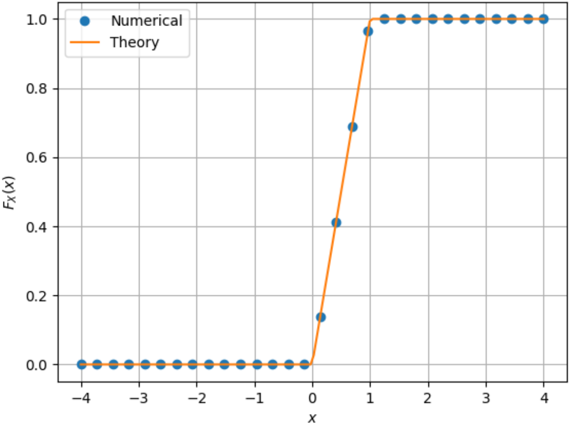
$$F_U(x) = \int_0^x U(x)dx \tag{2}$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{3}$$

Figure (1.3)

The empirical and theoretical CDFs are plotted below



Solution (1.5)

Since

$$dF_U(x) = p_U(x)dx \tag{4}$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \tag{5}$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{6}$$

Solution (1.5)

Therefore, from Equations **??** and **??**, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{7}$$

$$= \int_0^1 x^2 dx \tag{8}$$

$$= \frac{1}{3} \tag{9}$$

Similarly,

$$E[U^2] = \int_{-\infty}^{\infty} x p_U(x) dx \tag{10}$$

$$= \int_0^1 x dx \tag{11}$$

$$= \frac{1}{2} \tag{12}$$

Solution (1.5)

Therefore, the mean is $\frac{1}{2}$, and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{13}$$

$$= \frac{1}{12} \tag{14}$$

Solution (2.2/2.3)

We have:

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \tag{15}$$

Therefore,

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx \tag{16}$$

$$= \frac{\text{erf}\left(\frac{x}{\sqrt{2}}\right) + 1}{2} \tag{17}$$

where,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt \tag{18}$$

Figure (2.2)

The empirical and theoretical PDFs are plotted below

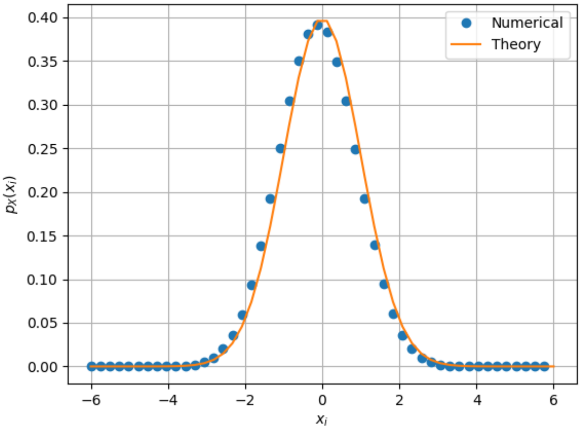
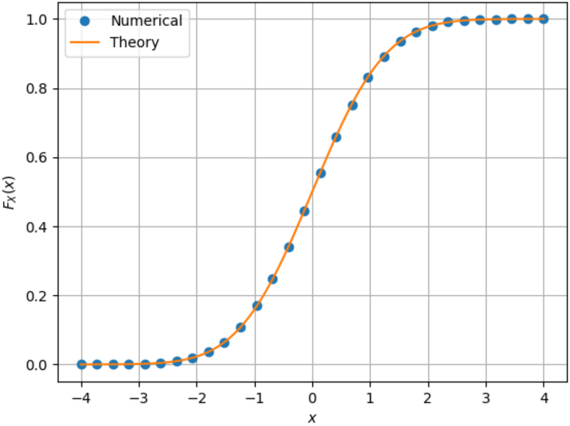


Figure (2.3)

The empirical and theoretical CDFs are plotted below



Solution (2.5)

We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{19}$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{20}$$

$$= 0 \tag{21}$$

Solution (2.5)

Also,

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) && (22)\\
&= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) && (23)\\
&= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} && (24)\\
&= 1 && (25)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{var}(X) &= E[X^2] - E[X]^2 && (26)\\
&= 1 && (27)
\end{aligned}
$$

Solution (2.5)

Therefore, the mean is 0 and the variance is 1. Running the empirical code in **./codes/exrancd.c**, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

Solution (3.2)

We have:

$$
\begin{aligned}
F_V(x) &= \Pr\left(V \le x\right) && (28)\\
&= \Pr\left(-2\ln(1-U) \le x\right) && (29)\\
&= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) && (30)\\
&= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) && (31)\\
&= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) && (32)
\end{aligned}
$$

Solution (3.2)

Therefore,

$$F_V(x) = \begin{cases} 0, & 1-\exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & 1-\exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1-\exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \tag{33}$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1-\exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \tag{34}$$

Figure (3.2)

The empirical and theoretical CDFs are plotted below