

# Random Numbers

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*Abstract*—This manual provides a simple introduction to the generation of random numbers

## 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
raw/master/manual/codes/coeffs.h
gcc exrand.c
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:**

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/cdf_plot.py
python3 cdf_plot.py
```

The following code plots Fig. 1.2

- 1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:**  $U$  is given by

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.2)$$

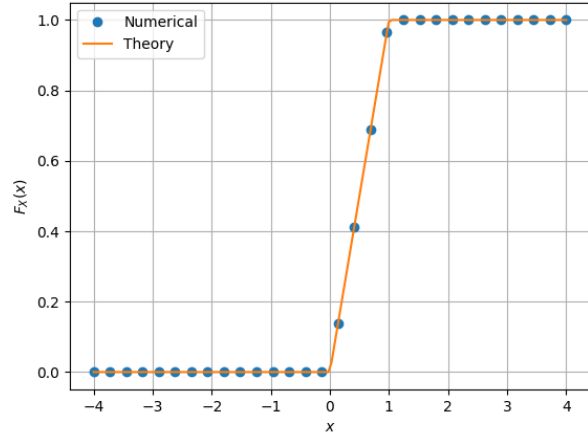


Fig. 1.2: The CDF of  $U$

Therefore, we have:

$$F_U(x) = \int_0^x U(x)dx \quad (1.3)$$

Computing the integral, we get:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.4)$$

- 1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:** Add the following function to coeffs.h

```
double variance(char *str)
{
    int i=0,c;
    FILE *fp;
```

```
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x*x;
}
double mn = mean(str);
fclose(fp);
temp = temp/(i-1);
return temp - mn*mn ;

}
```

Following the steps mentioned below gives the required result:

```
gcc exrand.c
./a.out
mean = 0.500031
variance = 0.083247
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

**Solution:** Since

$$dF_U(x) = p_U(x)dx \quad (1.8)$$

we have:

$$E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \quad (1.9)$$

Also,

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.10)$$

Therefore, from Equations 1.9 and 1.10, we have:

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x)dx \quad (1.11)$$

$$= \int_0^1 x^2 dx \quad (1.12)$$

$$= \frac{1}{3} \quad (1.13)$$

Similarly,

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \quad (1.14)$$

$$= \int_0^1 x dx \quad (1.15)$$

$$= \frac{1}{2} \quad (1.16)$$

Therefore, the mean is  $\frac{1}{2}$ , and the variance equals:

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.17)$$

$$= \frac{1}{12} \quad (1.18)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:** Add the following line to **exrand.c** and execute the code:

```
gaussian("gau.dat", 1000000);
gcc exrand.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of  $X$  is plotted in Fig. 2.2

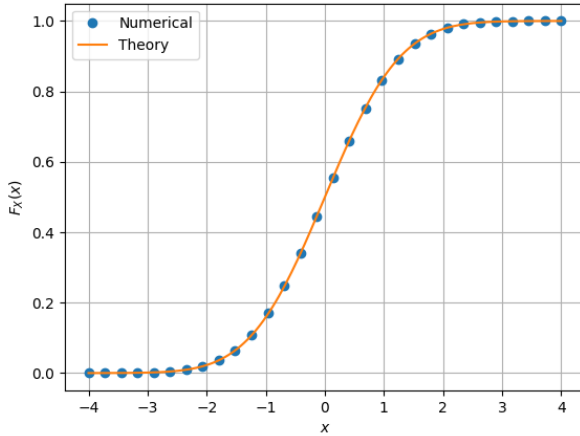
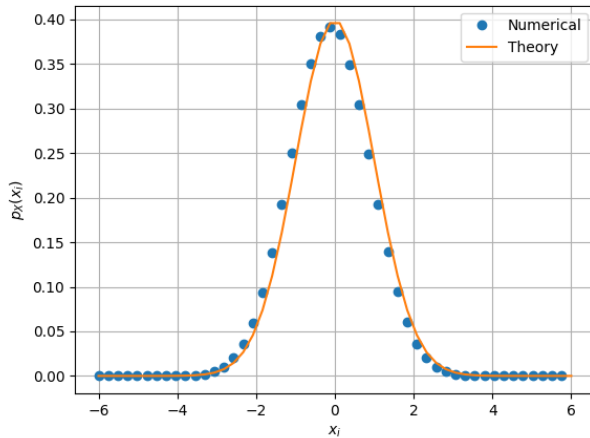
2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

**Solution:** The PDF of  $X$  is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
raw/master/manual/codes/pdf_plot.py
python3 pdf_plot.py
```

Fig. 2.2: The CDF of  $X$ Fig. 2.3: The PDF of  $X$ 

2.4 Find the mean and variance of  $X$  by writing a C program.

**Solution:** Use the main and variance functions in **coeffs.h**, and execute the code below

```
gcc exrand.c
./a.out
```

We get

```
mean = 0.000685
variance = 1.000025
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

**Solution:** We have:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.4)$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (2.5)$$

$$= 0 \quad (2.6)$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.8)$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (2.9)$$

$$= 1 \quad (2.10)$$

Hence,

$$\text{var}(X) = E[X^2] - E[X]^2 \quad (2.11)$$

$$= 1 \quad (2.12)$$

Therefore, the mean is 0 and the variance is 1. Running the empirical code in **./codes/exrand.c**, we get mean = 0.000685 and variance = 1.000025, which closely matches the theoretical values.

### 3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF. **Solution:**

Add the following function to **coeffs.h**:

```
void logarithmic(char *str){
    int i=0,c;
    FILE *fp, *fp2;
    double x, temp=0.0;

    fp = fopen("uni.dat","r");
    fp2 = fopen(str,"w");
    //get numbers from file
    while(fscanf(fp,"%lf",&x)!=EOF)
    {
        temp = -2*log(1-x);
        fprintf(fp2,"%lf\n",temp);
    }
}
```

```

}

fclose(fp);
fclose(fp2);

return ;

}

```

Using this function in **exrand.c** prints the numbers in **log.dat**

3.2 Find a theoretical expression for  $F_V(x)$ .

**Solution:** We have:

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \quad (3.7)$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \quad (3.8)$$

The CDF of  $V$  is plotted in Fig. 3.2

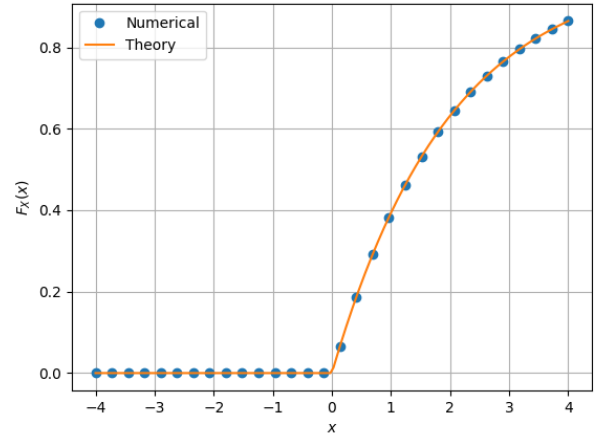


Fig. 3.2: The CDF of  $V$