Assignment 14

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Problem Statement

Papoulis 15.4

Show that the probability of extinction of a population given that the zeroth generation has size m is given by π_0^m , where π_0 is the extinction probability. Show that the probability that the population grows indefinitely in that case is $1 - \pi_0^m$.



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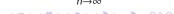
Definitions

• $P_Y(z) = E(z^Y)$ is called the common moment generating function, and is defined as:

$$P_Y(z) = \sum_{k=0}^{\infty} p_k z^k \tag{1}$$

where $p_k = \Pr(Y = k)$

- A branching process is a process consisting of independent reproducing individuals. Each individual lives a single unit of time, and has Y_i offspring
- X_n maps to the total size of the n^{th} generation, and equals $Y_1 + Y_2 + ... Y_{X_{n-1}}$
- The **extinction probability** π_0 is the probability that the k^{th} generation has no offspring, for some finite k. It equals $\lim_{n\to\infty} P_{X_n}(0)$



Common MGF

We have

$$X_n = \sum_{i=1}^{X_{n-1}} Y_i$$
 (2)

where Y_i are i.i.d. Also,

$$P_{X_n}(z) = \sum_{k=0}^{\infty} \Pr(X_n = k) z^k$$
(3)

This can be rewritten as:

$$P_{X_n}(z) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \Pr(X_n = k | X_{n-1} = i) \Pr(X_{n-1} = i) z^k$$
 (4)

using the Law of Total Probability.



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Common MGF

From 4, we get:

$$P_{X_n}(z) = \sum_{i=0}^{\infty} \Pr(X_{n-1} = i) \sum_{k=0}^{\infty} \Pr(X_n = k | X_{n-1} = i) z^k$$
 (5)

When $X_n = Y_1 + Y_2 + \dots Y_i$, observe that:

$$E(z^{X_n}) = E(z^{Y_1 + Y_2 + \dots Y_i})$$
 (7)

$$= \left[E(z^{Y_1}) \right]^i \tag{8}$$

since Y_i are i.i.d . Therefore,

$$P_{X_n}(z) = \sum_{i=0}^{\infty} \Pr(X_{n-1} = i) [P_{Y_1}(z)]^i$$
 (9)

$$= P_{X_{n-1}}(P_{Y_1}(z)) (10)$$

(6)

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Common MGF

Assuming that $X_0 = Y_1$, (assuming the zeroth generation has size 1) we have:

$$P_{X_n}(z) = P_{X_{n-1}}(P_{X_0}(z))$$
 (11)

Therefore:

$$P_{X_n}(z) = P_{X_{n-1}}(P_{X_0}(z))$$
 (12)

$$= P_{X_{n-2}}(P_{X_0}(P_{X_0}(z)))$$
 (13)

$$= P_{X_0}(P_{X_0}(\dots P_{X_0}(z)) \tag{14}$$

$$= P_{X_0}(P_{X_{n-1}}(z)) (15)$$



Extinction Probability

From the definition of extinction probability, we have:

$$\pi_0 = \lim_{n \to \infty} P_{X_n}(0) \tag{16}$$

$$= \lim_{n \to \infty} P_{X_{n-1}}(P_{X_0}(0)) \tag{17}$$

Using Equation 15:

$$\pi_0 = \lim_{n \to \infty} P_{X_0}(P_{X_{n-1}}(0)) \tag{18}$$

$$=P_{X_0}(\pi_0) \tag{19}$$

Hence, the extinction probability can be obtained by solving the equation $\pi_0 = P_{X_0}(\pi_0)$



Solution

If the zeroth generation has size m, we can split it into m independent and identical generations of size 1. Therefore, since π_0 is the extinction probability of a single generation, we have:

$$p_{e} = \lim_{n \to \infty} P_{X_{0}^{(1)}}(P_{X_{n-1}^{(1)}}(0)) \times \dots \times \lim_{n \to \infty} P_{X_{0}^{(m)}}(P_{X_{n-1}^{(m)}}(0))$$
(20)

$$=\pi_0\times\pi_0\times\ldots\pi_0\tag{21}$$

$$=\pi_0^m \tag{22}$$

where p_e is the extinction probability for the specified generation. We then have:

$$p_{\rm s}=1-p_{\rm e} \tag{23}$$

$$=1-\pi_0^m \tag{24}$$

where p_s is the probability that the population will survive and grow indefinitely.

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