Assignment 14

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Outline

Problem Statement

Papoulis 15.4

Show that the probability of extinction of a population given that the zeroth generation has size m is given by π_0^m, where π_0 is the extinction probability. Show that the probability that the population grows indefinitely in that case is $1 - \pi_0^m$.

Definitions

- $P_Y(z) = E(z^Y)$ is called the common moment generating function, and is defined as:

$$P_Y(z) = \sum_{k=0}^{\infty} p_k z^k \tag{1}$$

  where $p_k = \Pr(Y = k)$
- A **branching process** is a process consisting of independent reproducing individuals. Each individual lives a single unit of time, and has Y_i offspring
- X_n maps to the total size of the n^{th} generation, and equals $Y_1 + Y_2 + \ldots Y_{X_{n-1}}$
- The **extinction probability** π_0 is the probability that the k^{th} generation has no offspring, for some finite k. It equals $\lim_{n\to\infty} P_{X_n}(0)$

Common MGF

We have

$$X_n = \sum_{i=1}^{X_{n-1}} Y_i \tag{2}$$

where Y_i are i.i.d. Also,

$$P_{X_n}(z) = \sum_{k=0}^{\infty} \Pr\left(X_n = k\right) z^k \tag{3}$$

This can be rewritten as:

$$P_{X_n}(z) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \Pr\left(X_n = k | X_{n-1} = i\right) \Pr\left(X_{n-1} = i\right) z^k \tag{4}$$

using the Law of Total Probability.

Common MGF

From 4, we get:

$$P_{X_n}(z) = \sum_{i=0}^{\infty} \Pr\left(X_{n-1} = i\right) \sum_{k=0}^{\infty} \Pr\left(X_n = k | X_{n-1} = i\right) z^k \tag{5}$$

$$\tag{6}$$

When $X_n = Y_1 + Y_2 + \ldots Y_i$, observe that:

$$E(z^{X_n}) = E(z^{Y_1 + Y_2 + \ldots Y_i}) \tag{7}$$

$$= \left[E(z^{Y_1})\right]^i \tag{8}$$

since Y_i are i.i.d . Therefore,

$$P_{X_n}(z) = \sum_{i=0}^{\infty} \Pr\left(X_{n-1} = i\right) \left[P_{Y_1}(z)\right]^i \tag{9}$$

$$= P_{X_{n-1}}(P_{Y_1}(z)) \tag{10}$$

Common MGF

Assuming that $X_0 = Y_1$, (assuming the zeroth generation has size 1) we have:

$$P_{X_n}(z) = P_{X_{n-1}}(P_{X_0}(z)) \tag{11}$$

Therefore:

$$\begin{aligned} P_{X_n}(z) &= P_{X_{n-1}}(P_{X_0}(z)) && (12)\\ &= P_{X_{n-2}}(P_{X_0}(P_{X_0}(z))) && (13)\\ &= P_{X_0}(P_{X_0}(\dots P_{X_0}(z)) && (14)\\ &= P_{X_0}(P_{X_{n-1}}(z)) && (15) \end{aligned}$$

Extinction Probability

From the definition of extinction probability, we have:

$$\pi_0 = \lim_{n\to\infty} P_{X_n}(0) \tag{16}$$
$$= \lim_{n\to\infty} P_{X_{n-1}}(P_{X_0}(0)) \tag{17}$$

Using Equation 15:

$$\pi_0 = \lim_{n\to\infty} P_{X_0}(P_{X_{n-1}}(0)) \tag{18}$$
$$= P_{X_0}(\pi_0) \tag{19}$$

Hence, the extinction probability can be obtained by solving the equation $\pi_0 = P_{X_0}(\pi_0)$

Solution

If the zeroth generation has size m, we can split it into m independent and identical generations of size 1. Therefore, since π_0 is the extinction probability of a single generation, we have:

$$
\begin{aligned}
p_e &= \lim_{n\to\infty} P_{X_0^{(1)}}(P_{X_{n-1}^{(1)}}(0)) \times \ldots \times \lim_{n\to\infty} P_{X_0^{(m)}}(P_{X_{n-1}^{(m)}}(0)) && (20)\\
&= \pi_0 \times \pi_0 \times \ldots \pi_0 && (21)\\
&= \pi_0^m && (22)
\end{aligned}
$$

where p_e is the extinction probability for the specified generation. We then have:

$$
\begin{aligned}
p_s &= 1 - p_e && (23)\\
&= 1 - \pi_0^m && (24)
\end{aligned}
$$

where p_s is the probability that the population will survive and grow indefinitely.