Assignment 11

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Outline

Problem Statement

Papoulis 5.50

A biased coin is tossed and the first outcome is noted. The tossing is continued until the outcome is the complement of the first outcome, thus completing the first run. Let X denote the length of the first run. Find the p.m.f of X, and show that

$$E\{X\} = \frac{p}{q} + \frac{q}{p}$$

Definitions

- As specified in the question, we let the random variable X map to the set $\{1, 2, \dots\}$ based on the length of the run. We take the length to equal the number of consecutive heads or tails.
- We let H represent the event of "heads" and T represent the event of "tails". A string of Hs and Ts represents consecutive heads and tails (for example, HT represents heads and then tails).
- Let $\Pr(H) = p = 1 - q$

Solution

Consider the case when X maps to 1. This represents two distinct events, HT and TH. Since these events are mutually exclusive, we have:

$$\begin{aligned}
\Pr(X = 1) &= \Pr(HT + TH) && (1)\\
&= \Pr(HT) + \Pr(TH) && (2)\\
&= \Pr(H)\Pr(T) + \Pr(H)\Pr(T) && (3)\\
&= 2pq && (4)
\end{aligned}$$

Solution

We can thus similarly find the probability mass function as under

$$
\begin{aligned}
p_X(k) &= \Pr(X = k) && (5)\\
&= \Pr(TT \dots TH + HH \dots HT) && (6)\\
&= \Pr(TT \dots TH) + \Pr(HH \dots HT) && (7)\\
&= q^k p + p^k q && (8)
\end{aligned}
$$

Solution

From this, the expectation value $E(X)$ is given by:

$$E(X) = \sum_{k=1}^{\infty} k \times p_X(k) \tag{9}$$

$$= \sum_{k=1}^{\infty} k \times (q^k p + p^k q) \tag{10}$$

$$= pq \times \left(\sum_{k=1}^{\infty} kp^{k-1} + \sum_{k=1}^{\infty} kq^{k-1} \right) \tag{11}$$

Solution

Equation 11 can be manipulated as under:

$$
\begin{aligned}
pq \times \left(\sum_{k=1}^{\infty} kp^{k-1} + \sum_{k=1}^{\infty} kq^{k-1} \right) &= pq \times \left(\frac{d}{dp} \sum_{k=1}^{\infty} p^k + \frac{d}{dq} \sum_{k=1}^{\infty} q^k \right) && (12)\\
&= pq \times \left(\frac{d}{dp} \left(\frac{p}{1-p} \right) + \frac{d}{dq} \left(\frac{q}{1-q} \right) \right) && (13)\\
&= pq \times \left(\frac{1}{(1-q)^2} + \frac{1}{(1-p)^2} \right) && (14)\\
&= pq \times \left(\frac{1}{p^2} + \frac{1}{q^2} \right) && (15)\\
&= \frac{p}{q} + \frac{q}{p} && (16)
\end{aligned}
$$