Evaluating Volume under Surfaces

Aim: To evaluate the volume under surface using double integral and to visualize the same using MatLab.

Statement of the problem: Evaluate and visualize the volume represented by the double integral

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx. \tag{1}$$

Above integral represents volume of the region below the surface $z = f(x, y)$ and above the plane $z = 0$. This integral can also be setup in the following way (by changing the order of integration of x and y):

$$\int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy. \tag{2}$$

Solution Approach: We evaluate the double integrals by repeated application of the symbolic toolbox command for integration that, on applying twice, will read as**:**

$$\text{volume} = \text{int}\big(\text{int}(\text{f(x, y)}, y, y_1(x), y_2(x)\big), x, x_1, x_2\big).$$

$$\text{volume} = \text{int}\big(\text{int}(\text{f(x, y)}, x, x_1(y), x_2(y)\big), y, y_1, y_2\big)$$

Further, to visualize the volume in MatLab we make use of two additional MatLab functions (provided by MathWorks) viz. "viewSolid" and "viewSolidone". These supporting function files can be downloaded from the link **ftp://10.30.2.53/MATLAB/**.

The first function "viewSolid" is used to visualize the integrals in which the order of integration is as given in (1) and "viewSolidone" is for the integrals of the form (2).

What follows is the syntax for using "viewSolid" and "viewSolidone" commands:

---

viewSolid(z,0,f(x,y),y,y1(x),y2(x),x,x1,x2)

viewSolidone(z,0,f(x,y),x,x1(y),x2(y),y,y1,y2)

It should be observed that the "viewSolid" command is used when y1 and y2 are functions of x whereas x1 and x2 are constants. The "viewSolidone" command is used in the reverse case. Now we consider few examples for illustration of the approach mentioned above.

Example 1:

Set up a double integral to find the volume of a sphere of unit radius.

Solution:

Let the sphere be $x^2 + y^2 + z^2 = 1$. We know that due to the symmetry the volume of the sphere is 8 times its volume in the first octant. Thus we setup a double integral to find the volume below the surface of the sphere in the first octant only and write the total volume as:

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy\, dx\,.$$

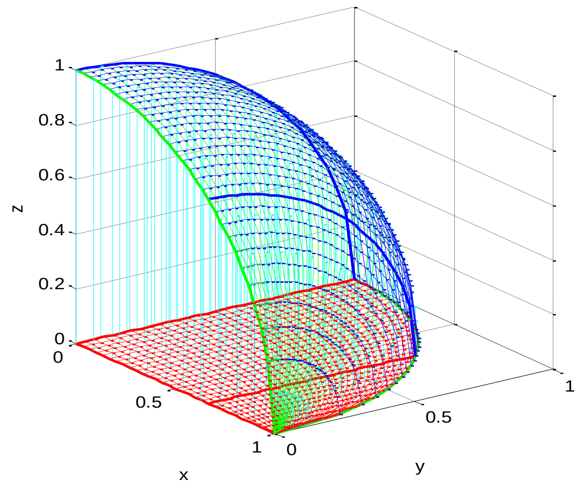
MATLAB Code:

```
clc
clear all
syms x y z
vol=8*int(int(sqrt(1-x^2-y^2),y,0,sqrt(1-x^2)),x,0,1)
viewSolid(z,0+0*x*y,sqrt(1-x^2-y^2),y,0+0*x,sqrt(1-x^2),x,0,1);
axis equal; grid on;
```

```
clc
clear all
syms x y z
vol=8*int(int(sqrt(1-x^2-y^2),y,0,sqrt(1-x^2)),x,0,1)
viewSolid(z,0+0*x*y,sqrt(1-x^2-y^2),y,0+0*x,sqrt(1-x^2),x,0,1); axis equal; grid on;
```

Output:

```
vol =

(4*pi)/3
```



Example 2:

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the lines $y = 2x$ and the parabola $y = x^2$

Solution:

The double integral for this problem can be setup as:

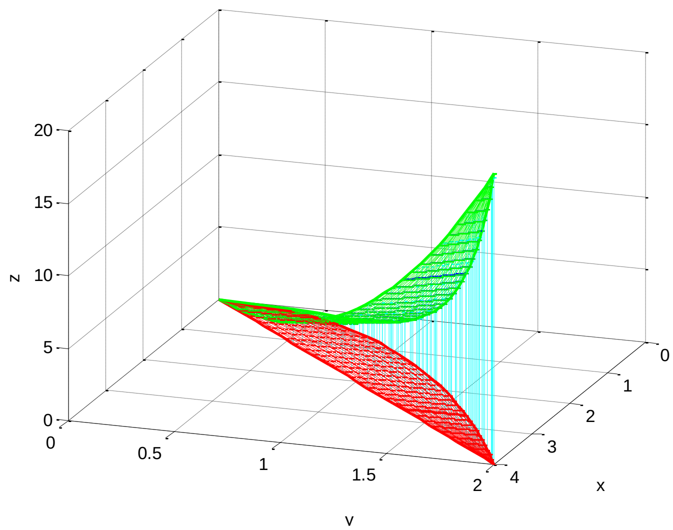
$$\int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

```
clc
clear all
syms x y z
vol = int(int(x^2+y^2, x,y/2,sqrt(y)), y, 0, 4)

viewSolidone(z,0+0*x*y,x^2+y^2,x,y/2,sqrt(y),y,0,4);

grid on;
```

Output:

vol =

216/35



Converting Cartesian to polar coordinates

Example 3:

Find the volume of the solid bounded by the plane z=0 and the paraboloid
$z = 1 - x^2 - y^2$

Sol:

By changing the coordinates from Cartesian to Polar we get

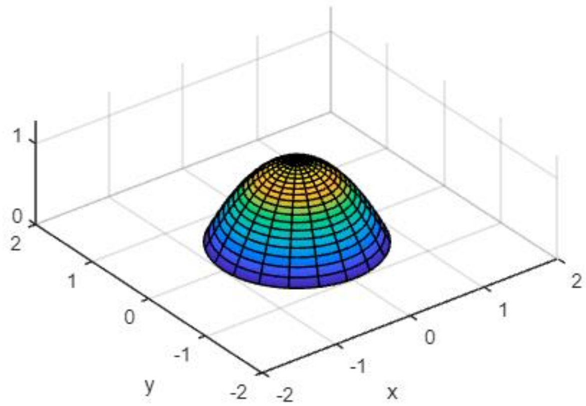
$$V = \iint_D (1 - x^2 - y^2)dA = \int_0^{2\pi} \int_0^1 (1 - r^2)rdr\theta$$

```
clc
clear all
syms r theta
V = int(int((1-r^2)*r, r, 0, 1), theta, 0, 2*pi)
fsurf(r*cos(theta),r*sin(theta), 1-r^2, [0 1 0 2*pi], 'MeshDensity', 20)
axis equal; axis([-2 2 -2 2 0 1.3])
xticks(-2:2); yticks(-2:2); zticks(0:1.3)
xlabel('x'); ylabel('y')
```

Output

V = pi/2

Figure window:



Example 4:

Evaluate $\iint_R y\sin(xy)\ dA$ where $R = [1,2]\times[0,\pi]$

MATLAB Code:

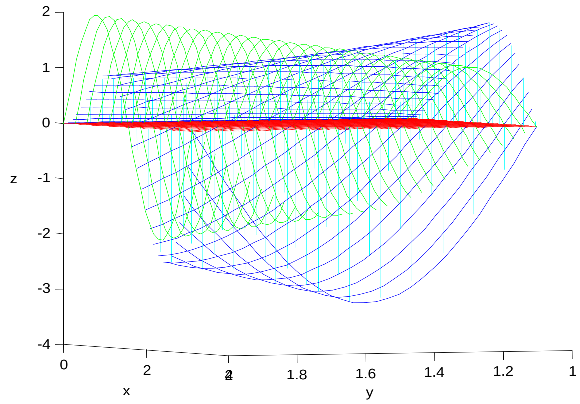
```
clc
clear all
syms x y z
viewSolidone(z,0+0*x+0*y,y*sin(x*y),x,1+0*y,2+0*y,y,0,pi)
int(int(y*sin(x*y),x,1,2),y,0,pi)
```

Output:

In the Command window:

```
>> ans
    0
```

In the Figure window:



Example 5:

Consider the following mathematical problem

$$\text{Evaluate } \iint_R (x-3y^2)\,dA \ \text{ where } \ R=\{(x,y)\,|\,0\le x\le 2,\ 1\le y\le 2\}$$

MATLAB Code:

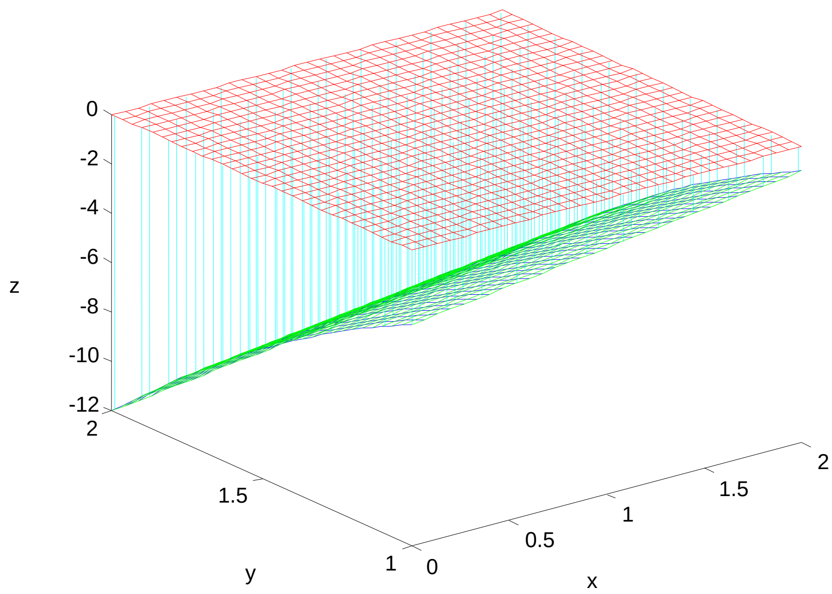
```
clc
clear all
syms x y z
viewSolid(z,0+0*x+0*y,x-3*y^2+0*y,y,1+0*x,2+0*y,x,0,2)
int(int(x-3*y^2+0*y,y,1,2),x,0,2)
```

Output:

In the Command window:

```
>> ans
      -12
```

In the Figure window:



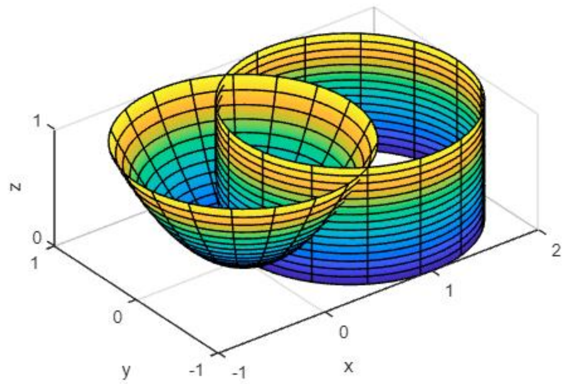
Example 6

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the xy- plane, and inside the cylinder $x^2 + y^2 = 2x$

```
clc
clear all
syms r theta z r1
V = int(int((r^2)*r, r, 0,2*cos(theta) ), theta, -pi/2, pi/2)
r = 2*cos(theta), x = r*cos(theta), y = r*sin(theta)
fsurf(x,y,z, [0 2*pi 0 1], 'MeshDensity', 16)
axis equal; xlabel('x'); ylabel('y'); zlabel('z')
zticks(0:1.5)
hold on
fsurf(r1*cos(theta),r1*sin(theta),r1^2, [0 1 0 2*pi], 'MeshDensity', 20)
```

Output:

V = (3*pi)/2



Evaluating triple integrals

Aim: Evaluating triple integrals (Cartesian, Cylindrical and Spherical coordinates) and visualizing regions using Matlab.

MATLAB Syntax used

int(f,v)	uses the symbolic object v as the variable of integration, rather than the variable determined by symvar
fill(X,Y,C)	fill(X,Y,C) creates filled polygons from the data in X and Y with vertex color specified by C.
fliplr(A)	If A is a row vector, then fliplr(A) returns a vector of the same length with the order of its elements reversed. If A is a column vector, then fliplr(A) simply returns A.
fsurf(f)	fsurf(f) creates a surface plot of the function z = f(x,y) over the default interval [-5 5] for x and y.
fsurf(f,xyinterval)	fsurf(f,xyinterval) plots over the specified interval. To use the same interval for both x and y, specify xyinterval as a two-element vector of the form [min max]. To use different intervals, specify a four-element vector of the form [xmin xmax ymin ymax].

Note: We invite your suggestions for the improvement of the topic triple integrals(Matlab codes and contents). mail id : kaliyappan.m@vit.ac.in

Example 1

Evaluate the iterated integral

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz \, dy \, dx \, dz$$

Matlab code

```
syms x y z
sol = int(int(int(6*x*z,y,0,x+z),x,0,z),z,0,1)
```

Example 2

Evaluate the triple integral $\iiint_E 6xy\, dV$, where E lies under the plane z =1+x+y and above the region in the xy-pane bounded by the curves $y = \sqrt{x}$, y=0 and x=1.

Sol

Here $E=\{(x,y,z) | 0\le x\le 1, 0\le y\le \sqrt{x}, 0\le z\le 1+x+y\}$

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

Matlab code

```
syms x y z
sol = int(int(int(6*x*y,z,0,1+x+y),y,0,sqrt(x)),x,0,1)
viewSolid(z,0+0*x*y,1+x+y,y,0+0*x,sqrt(x),x,0,1);
axis equal; grid on;
```

Example 3

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane z =4, and above the paraboloid $z = 1 - (x^2 + y^2)$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Sol

In cylindrical coordinates the cylinder is r =1 and the paraboloid is z = 1- r^2, so we can write

$$E = \{(r, \theta, z) \,|\, 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is $f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$ where K is the proportionality constant.

The mass of E is

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

Matlab code

```
syms r z theta K
Ma= int(int(int((K*r)*r, z, 1-r^2,4), r ,0, 1),theta,0,2*pi) % integration

x = r*cos(theta), y = r*sin(theta), s = sym(4)
fsurf(x,y,1-r^2, [0 1 0 2*pi], 'g', 'EdgeColor', 'none');  % plotting paraboloid
hold on
fsurf(1*cos(theta), 1*sin(theta), r, 'y', [0  4  0 2*pi], 'EdgeColor', 'none')  % plotting cylinder of radius 1 with height z = 4
fsurf(x,y,s, [0 1 0 2*pi], 'b', 'EdgeColor', 'none'); % plotting circular  plane z=4.
hold on
axis equal; xlabel('x'); ylabel('y'); zlabel('z');
alpha 0.5
```

Example 4

Use Matlab to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$

Matlab code

```
syms r z theta
x = r*cos(theta); y = r*sin(theta);
fsurf(x,y,5-r^2,[0 sqrt(5) 0 2*pi], 'g', 'EdgeColor', 'none');
hold on
fsurf(x,y,r^2, [0 sqrt(5) 0 2*pi], 'y', 'EdgeColor', 'none');
axis equal; xlabel('x'); ylabel('y'); zlabel('z');
alpha 0.5
```

Example 5

Draw a sphere of radius 5 with centre at (0,0,0)

Matlab code

```
syms r z phi rho theta
rho=5
x= rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z= rho*cos(phi) ;
fsurf(x,y,z, [0 pi 0 2*pi], 'g', 'MeshDensity', 20);
```

Example 6

Draw a hemisphere of radius 3 with centre at (0,0,0)

Matlab code

```
syms r z phi rho theta
```

```
rho=3
x= rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z= rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2  0 2*pi], 'g', 'MeshDensity', 20);
```

Example 7

Evaluate $\iiint_E z\,dV$, where E is enclosed by the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.

Sol: By Changing Cartesian to Spherical coordinates we can get

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\rho^2\sin\phi\; d\rho\, d\theta\, d\phi$$

Matlab code

```
clc
clear all
syms r  phi rho theta
Sol=int(int(int((rho*cos(phi))*(rho)^2*sin(phi), rho,1,2), phi ,0, pi/2),theta,0,pi/2)
rho=1;
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'g', 'MeshDensity', 20);
hold on
rho=2;
x = rho*sin(phi)*cos(theta), y = rho*sin(phi)*sin(theta), z = rho*cos(phi) ;
fsurf(x,y,z, [0 pi/2 0 pi/2], 'b', 'MeshDensity', 20);
```

Exercise

1. **Evaluate the triple integral $\iiint_E y\,dV$, where E is bounded by the planes x =0, y = 0, z=0, and 2x+2y+z = 4.**
2. **Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$, where E is enclosed by the sphere $x^2+y^2+z^2=9$ in the first octant.**
3. **Evaluate $\iiint_E e^z\,dV$, where E is enclosed by the paraboloid $z=1+x^2+y^2$, the cylinder $x^2+y^2=5$, and the xy-plane.**