11.9.4.4

EE23BTECH11027 - K RAHUL*

Derivations and results:

$$log\,(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \tag{1}$$

Let $x(n)$ be some expression in discrete time domain, whose Z transform, $X(z)$ shall be obtained for future reference.
For $x(n) = \frac{1}{n+c}u(n)$, where $c \geq 2,\ c \in \mathbb{N}$

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x(n)\, z^{-n} \tag{2}$$
$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-n} \tag{3}$$
$$= z^c \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-(n+c)} \tag{4}$$
$$\tag{5}$$

Using, (1)

$$X(z) = z^c\Big(-log\left(1-z^{-1}\right) - z^{-1} - \frac{z^{-2}}{2} - \frac{z^{-3}}{3} - \dots - \frac{z^{-(c-1)}}{c-1}\Big) \tag{6}$$

For $x(n) = \frac{1}{n+1}u(n)$,

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x(n)\, z^{-n} \tag{7}$$
$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-n} \tag{8}$$
$$= z \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-(n+1)} \tag{9}$$
$$\tag{10}$$

Using (1),

$$X(z) = -zlog\left(1-z^{-1}\right) \tag{11}$$

Let $D(z)$ be some expression in Z domain, whose inverse Z transform, $d(n)$ shall be obtained for future reference.
For $D(z) = z^2 log\left(1-z^{-1}\right)$

$$d(n) = \frac{1}{2\pi j}\oint_C z^{n+1}\log\left(1-z^{-1}\right)dz \tag{12}$$
$$= \frac{-1}{2\pi j}\oint_C z^{n+1}\Big(z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \dots + \frac{z^{-(n+1)}}{n+1} + \frac{z^{-(n+2)}}{n+2} + \dots\Big)dz \tag{13}$$

Making the substitution $z = e^{jt} \implies dz = je^{jt}$

$$d(n) = \frac{-1}{2\pi}\int_0^{2\pi} e^{(n+2)jt}\Big(e^{-jt} + \frac{e^{-2jt}}{2} + \frac{e^{-3jt}}{3} + .. + \frac{z^{-(n+2)}jt}{n+2} + ..\Big)dz \tag{14}$$
$$= \frac{-1}{n+2} \tag{15}$$

For $D(z) = \dfrac{z^k}{1-z^{-1}}$, where $k \in \mathbb{R}$

$$d(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n+k-1}}{1-z^{-1}}dz \tag{16}$$

Using Residue Theorem,

$$d(n) = \lim_{z\to 1}\left(\frac{z^{n+k-1}}{1-z^{-1}}\right)\left(1-z^{-1}\right) \tag{17}$$
$$= 1 \tag{18}$$

Question:
Find sum to n terms of the following series: $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\dots$

Solution:

Symbol	Description	Value
$x(n)$	n^{th} term of series	

TABLE 0
PARAMETERS

$$x(n) = \frac{1}{(n+1)(n+2)}u(n) \tag{19}$$
$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right)u(n) \tag{20}$$
$$\tag{21}$$

Using (6) and (11), we get,

$$X(z) = -zlog\left(1-z^{-1}\right) + z^2log\left(1-z^{-1}\right) + z \tag{22}$$
$$= z(z-1)\,log\left(1-z^{-1}\right) + z \tag{23}$$
$$Y(z) = X(z)\,U(z) \tag{24}$$
$$= z^2log\left(1-z^{-1}\right) + \frac{z}{1-z^{-1}} \tag{25}$$
$$\tag{26}$$

Using (15) and (18),

$$y(n) = 1 - \frac{1}{n+2} \tag{27}$$

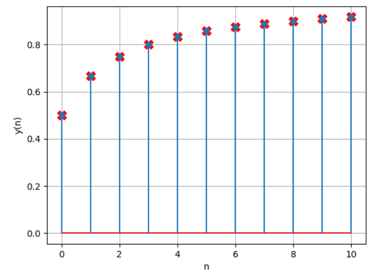

Fig. 0. Stem Plot of $y(n)$ v/s n