11.9.4.4

EE23BTECH11027 - K RAHUL*

Derivations and results:

$$log\left(1-x\right)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots \tag{1}$$

Let $x\left(n\right)$ be some expression in discrete time domain, whose Z transform, $X(z)$ shall be obtained for future reference.

For $x\left(n\right)=\frac{1}{n+c}u\left(n\right)$, where $c\geq 2,\ c\in\mathbb{N}$

$$X\left(z\right)=\sum_{n=-\infty}^{n=+\infty}x\left(n\right)z^{-n} \tag{2}$$

$$=\sum_{n=0}^{n=+\infty}\frac{1}{n+c}z^{-n} \tag{3}$$

$$=z^c\sum_{n=0}^{n=+\infty}\frac{1}{n+c}z^{-(n+c)} \tag{4}$$

$$X\left(z\right)=z^c\Big(-log\left(1-z^{-1}\right)-z^{-1}-\frac{z^{-2}}{2}-\frac{z^{-3}}{3}-\ldots-\frac{z^{-(c-1)}}{c-1}\Big)(\text{Using}(1)) \tag{5}$$

For $x\left(n\right)=\frac{1}{n+1}u\left(n\right)$,

$$X\left(z\right)=\sum_{n=-\infty}^{n=+\infty}x\left(n\right)z^{-n} \tag{6}$$

$$=\sum_{n=0}^{n=+\infty}\frac{1}{n+1}z^{-n} \tag{7}$$

$$=z\sum_{n=0}^{n=+\infty}\frac{1}{n+1}z^{-(n+1)} \tag{8}$$

$$=-zlog\left(1-z^{-1}\right)(\text{Using}(1)) \tag{9}$$

Let $D\left(z\right)$ be some expression in Z domain, whose inverse Z transform, $d(n)$ shall be obtained for future reference.

For $D\left(z\right)=z^2log\left(1-z^{-1}\right)$

$$d\left(n\right)=\frac{1}{2\pi j}\oint_C z^{n+1}\log\left(1-z^{-1}\right)dz \tag{10}$$

$$=\frac{-1}{2\pi j}\oint_C z^{n+1}\Big(z^{-1}+\frac{z^{-2}}{2}+\frac{z^{-3}}{3}+\ldots+\frac{z^{-(n+1)}}{n+1}+\frac{z^{-(n+2)}}{n+2}+\ldots\Big)dz \tag{11}$$

Making the substitution $z=e^{jt}\implies dz=je^{jt}$

$$=\frac{-1}{2\pi}\int_0^{2\pi}e^{(n+2)jt}\Big(e^{-jt}+\frac{e^{-2jt}}{2}+\frac{e^{-3jt}}{3}+..+\frac{z^{-(n+2)}jt}{n+2}+..\Big)dz \tag{12}$$

$$=\frac{-1}{n+2} \tag{13}$$

For $D\left(z\right)=\frac{z^k}{1-z^{-1}}$, where $k\in\mathbb{R}$

$$d\left(n\right)=\lim_{x\to 1}z^{n+k-1}(\text{Residue Theorem}) \tag{14}$$

$$=1 \tag{15}$$

Question:

Find sum to n terms of the following series: $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots$

Solution:

Symbol	Description	Value
$x(n)$	n^{th} term of series	

TABLE 0
PARAMETERS

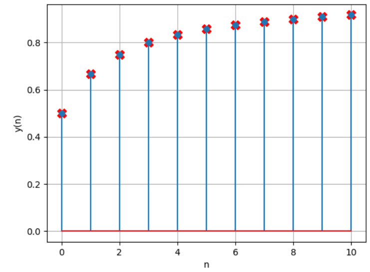

Fig. 0. Stem Plot of $y(n)$ v/s n

$$x(n) = \frac{1}{(n+1)(n+2)} u(n) \tag{16}$$
$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) u(n) \tag{17}$$
$$\tag{18}$$

Using (5) and (9), we get,

$$X(z) = -z log\left(1 - z^{-1}\right) + z^2 log\left(1 - z^{-1}\right) + z \tag{19}$$
$$= z(z-1) log\left(1 - z^{-1}\right) + z \tag{20}$$
$$Y(z) = X(z) U(z) \tag{21}$$
$$= z^2 log\left(1 - z^{-1}\right) + \frac{z}{1 - z^{-1}} \tag{22}$$
$$\tag{23}$$

Using (13) and (15),

$$y(n) = 1 - \frac{1}{n+2} \tag{24}$$