11.9.4.4

EE23BTECH11027 - K RAHUL*

Derivations and results:

Symbol	Description
x(n)	$\frac{1}{n+c}$, $c \geq 2$
$w(n)$	$\frac{1}{n+1}$
$D(z)$	$z^2 log\left(1 - z^{-1}\right)$

TABLE 0
Notations

$$log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \tag{1}$$

Let $x(n)$ and $w(n)$ be some expression as described in the table in discrete time domain, whose Z transform, $X(z)$ and $W(z)$ respectively, shall be obtained for future reference.

For $x(n) = \frac{1}{n+c} u(n), \forall c \geq 2, c \in \mathbb{N}$

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x(n) z^{-n} \tag{2}$$

$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-n} \tag{3}$$

$$= z^c \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-(n+c)} \tag{4}$$

Using, (1)

$$X(z) = z^c\left(-log\left(1 - z^{-1}\right) - z^{-1} - \frac{z^{-2}}{2} - \frac{z^{-3}}{3} - \dots - \frac{z^{-(c-1)}}{c-1}\right) \forall c \geq 2, c \in \mathbb{N} \tag{5}$$

For $w(n) = \frac{1}{n+1} u(n)$,

$$W(z) = \sum_{n=-\infty}^{n=+\infty} w(n) z^{-n} \tag{6}$$

$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-n} \tag{7}$$

$$= z \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-(n+1)} \tag{8}$$

Using (1),

$$W(z) = -zlog\left(1 - z^{-1}\right) \tag{9}$$

Let $D(z)$ be some expression as described in the table in Z domain, whose inverse Z transform, $d(n)$ shall be obtained for future reference.

For $D(z) = z^2 log\left(1 - z^{-1}\right)$

$$d(n) = \frac{1}{2\pi j} \oint_C z^{n+1} \log\left(1 - z^{-1}\right) dz \tag{10}$$

$$= \frac{-1}{2\pi j} \oint_C z^{n+1}\left(z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \dots + \frac{z^{-(n+1)}}{n+1} + \frac{z^{-(n+2)}}{n+2} + \dots\right) dz \tag{11}$$

Making the substitution $z = e^{jt} \implies dz = je^{jt} dt$

$$d(n) = \frac{-1}{2\pi} \int_0^{2\pi} e^{(n+2)jt}\left(e^{-jt} + \frac{e^{-2jt}}{2} + \frac{e^{-3jt}}{3} + .. + \frac{z^{-(n+2)} jt}{n+2} + ..\right) dt \tag{12}$$

$$= \frac{-1}{n+2} u(n) \tag{13}$$

$$u(n) \overset{Z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{14}$$

$$u(n+1) \overset{Z}{\longleftrightarrow} \frac{z}{1 - z^{-1}} \quad |z| > 1 \tag{15}$$

Question:

Find sum to n terms of the following series: $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\dots$

Solution:

Symbol	Description	Value
$x(n)$	n^{th} term of series	$\frac{1}{(n+1)(n+2)}u(n)$
$y(n)$	Sum of n terms of series	?

TABLE 0
PARAMETERS

$$x(n) = \frac{1}{(n+1)(n+2)}u(n) \tag{16}$$
$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right)u(n) \tag{17}$$

Using (5) and (9), we get,

$$X(z) = -zlog\left(1-z^{-1}\right) + z^2log\left(1-z^{-1}\right) + z \tag{18}$$
$$= z(z-1)\,log\left(1-z^{-1}\right) + z \tag{19}$$
$$Y(z) = X(z)\,U(z) \tag{20}$$
$$= z^2log\left(1-z^{-1}\right) + \frac{z}{1-z^{-1}} \tag{21}$$

Using (13) and (15),

$$y(n) = u(n+1) - \frac{1}{n+2}u(n),\ n \geq 0 \tag{22}$$

Since $y(n)$ is only defined for $n \geq 0$, the above expression can be equivalently written as

$$y(n) = \left(1 - \frac{1}{n+2}\right)u(n) \tag{23}$$

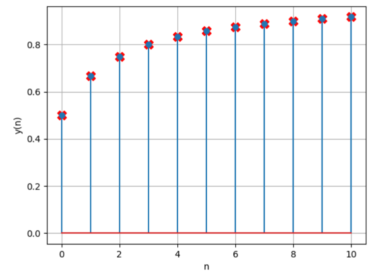

Fig. 0. Stem Plot of $y(n)$ v/s n