11.9.4.4

EE23BTECH11027 - K RAHUL*

Derivations and results:

Symbol	Description
x(n)	$\frac{1}{n+c}u(n)\,,\ c \geq 2$
$w(n)$	$\frac{1}{n+1}u(n)$
$D(z)$	$z^2 log\left(1 - z^{-1}\right)$

TABLE 0
Notations

$$\delta(n) \overset{\mathcal{Z}}{\longleftrightarrow} 1 \tag{1}$$

$$\delta(n+k) \overset{\mathcal{Z}}{\longleftrightarrow} z^k, \forall k \in \mathbb{R} \tag{2}$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \tag{3}$$

Let $x(n)$ and $w(n)$ be some expression as described in the table in discrete time domain, whose Z transform, $X(z)$ and $W(z)$ respectively, shall be obtained for future reference.

For $x(n) = \dfrac{1}{n+c}u(n), \forall c \geq 2, c \in \mathbb{N}$

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x(n) z^{-n} \tag{4}$$

$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-n} \tag{5}$$

$$= z^c \sum_{n=0}^{n=+\infty} \frac{1}{n+c} z^{-(n+c)} \tag{6}$$

Using, (3)

$$X(z) = z^c\Big(-\log\left(1-z^{-1}\right) - z^{-1} - \frac{z^{-2}}{2} - \frac{z^{-3}}{3} - \ldots - \frac{z^{-(c-1)}}{c-1}\Big)\ \forall c \geq 2, c \in \mathbb{N} \tag{7}$$

For $w(n) = \dfrac{1}{n+1}u(n)$,

$$W(z) = \sum_{n=-\infty}^{n=+\infty} w(n) z^{-n} \tag{8}$$

$$= \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-n} \tag{9}$$

$$= z \sum_{n=0}^{n=+\infty} \frac{1}{n+1} z^{-(n+1)} \tag{10}$$

Using (3),

$$W(z) = -z \log\left(1 - z^{-1}\right) \tag{11}$$

Let $D(z)$ be some expression as described in the table in Z domain, whose inverse Z transform, $d(n)$ shall be obtained for future reference.

For $D(z) = z^k \log\left(1 - z^{-1}\right) \forall k \geq 1, k \in \mathbb{Z}$

$$D(z) = \left(-z^{k-1} - \frac{1}{2}z^{k-2} - \frac{1}{3}z^{k-3} - \frac{1}{4}z^{k-4} - \ldots\right) \tag{12}$$

Using (2),

$$d(n) = \Big(-\delta(n+k-1) - \frac{1}{2}\delta(n+k-2) - \frac{1}{3}\delta(n+k-3) - \frac{1}{4}\delta(n+k-4) - \ldots - \frac{1}{n+k}\delta(0) - \ldots\Big) \tag{13}$$

$$= \frac{-1}{n+k}u(n) \tag{14}$$

Question:

Find sum to n terms of the following series: $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots$

Solution:

Symbol	Description	Value
$x(n)$	n^{th} term of series	$\frac{1}{(n+1)(n+2)}u(n)$
$y(n)$	Sum of n terms of series	?

TABLE 0
Parameters

$$x(n) = \frac{1}{(n+1)(n+2)}u(n) \tag{15}$$
$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right)u(n) \tag{16}$$

Using (7) and (11), we get,

$$X(z) = -zlog\left(1 - z^{-1}\right) + z^2 \log\left(1 - z^{-1}\right) + z \tag{17}$$
$$= z(z-1)\log\left(1 - z^{-1}\right) + z \tag{18}$$
$$Y(z) = X(z)U(z) \tag{19}$$
$$= z^2 \log\left(1 - z^{-1}\right) + \frac{z}{1 - z^{-1}} \tag{20}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{21}$$
$$u(n+k) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^k}{1 - z^{-1}}, \forall k \in \mathbb{R}, \quad |z| > 1 \tag{22}$$

Using (22) and (14) ,

$$y(n) = u(n+1) - \frac{1}{n+2}u(n), \; n \geq 0 \tag{23}$$

Since $y(n)$ is only defined for $n \geq 0$, the above expression can be equivalently written as

$$y(n) = \left(1 - \frac{1}{n+2}\right)u(n) \tag{24}$$

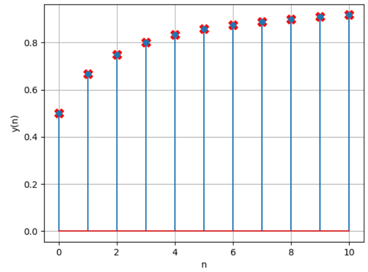

Fig. 0. Stem Plot of $y(n)$ v/s n