ASSIGNMENT-3

RAHUL R PAI
20110155

1. Singularities: The configurations for which the rank of Jacobian which maps dq to dx reduces are called singularities. The loss of rank represents loss of degree of ~~freedom~~ freedom.

Singular configurations are usually found at the boundary of the workspace or where the end effector cannot reach with small disturbances.

When the determinant of a manipulator jacobian is zero, then the configuration is a singular one. So, if the determinant is close to zero, the robot is close to a singular configuration. Also, the inverse kinematics problem will not have unique solution.

4.2. DH parameters for Stanford Manipulator

Transformation	d	θ (rad)	a	α (rad)
0 → 1	0	q_1	l_1	0
1 → 2	0	q_2	l_2	$\pi/2$
2 → 3	l_3	q_3	0	0

$$\therefore \text{Jacobian} = \begin{bmatrix} q_3 c_{12} - l_2 s_{12} - l_1 s_1 & q_3 c_{12} - l_2 s_{12} & s_{12} & 0 \\ q_3 s_{12} + l_2 c_{12} + l_1 c_1 & q_3 s_{12} + l_2 c_{12} & -c_{12} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{12} \\ 0 & 0 & 0 & -c_{12} \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Test case 1

$q_1 = \pi/4,\ q_2 = \pi/4,\ l_3 = 0.2,\ l_1 = 1,\ l_2 = 1,\ q_3 = 0$

$\therefore$ Calculated jacobian $= \begin{bmatrix} -1.7071 & -1 & 1 & 0 \\ 0.9071 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

From code:

```
[[-1.70710678e+00 -1.00000000e+00  1.00000000e+00 -0.00000000e+00]
 [ 9.07106781e-01  2.00000000e-01 -1.48965927e-11  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  4.89658886e-12  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  1.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -1.48965927e-11]
 [ 1.00000000e+00  1.00000000e+00  0.00000000e+00  4.89658886e-12]]
```

SCARA Manipulator: DH values: (3rd frame: End effector)

Transformations	d	θ(rad)	a	α(rad)
$0 \to 1$	0	q_1	l_1	0
$1 \to 2$	0	q_2	l_2	π
$2 \to 3$	l_3	q_3	0	0

Jacobian: $\begin{bmatrix} -l_2 s_{12} - l_1 s_1 & -l_2 s_{12} & 0 & 0 \\ l_2 c_{12} + l_1 c_1 & l_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$

Test case:

$$q_1 = \pi/6 \quad q_2 = \pi/6 \quad q_3 = 0, \; l_1 = 1, \; l_2 = 1, \; l_3 = 1$$

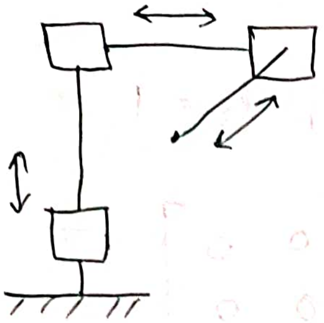
Calculated Jacobian:

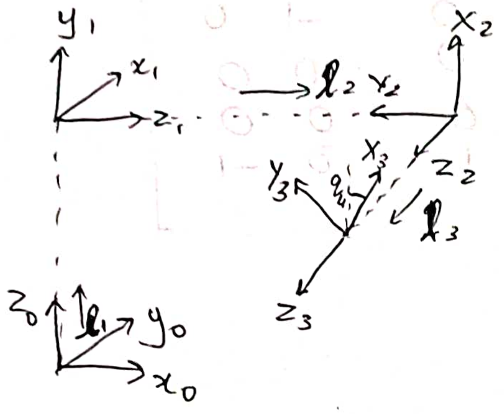
$$\begin{bmatrix} -1.366 & -0.866 & 0 & 0 \\ 1.366 & 0.5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

From code:

```
[[-1.36602540e+00 -8.66025404e-01 -1.79114065e-13  0.00000000e+00]
 [ 1.36602540e+00  5.00000000e-01  1.03411554e-13  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00 -1.00000000e+00 -0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -1.79114065e-13]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  1.03411554e-13]
 [ 1.00000000e+00  1.00000000e+00  0.00000000e+00 -1.00000000e+00]]
```

5.





$$H_0^3 = H_0^1 H_1^2 H_2^3 = [Z_1][X_1][Z_2][X_2][Z_3][$$

$$Z_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH parameters :

	d	θ	r	α
$0 \to 1$	l_1	$\pi/2$	0	$\pi/2$
$1 \to 2$	l_2	$\pi/2$	0	$\pi/2$
$2 \to 3$	l_3	q_4	0	0

$$Z_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H_1^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H_2^3 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} z_0 \\ 0 \end{bmatrix} \quad J_2 = \begin{bmatrix} z_1 \\ 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix} \quad J_4 = \begin{bmatrix} z_3 \times (o_3 - o_3) \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ z_3 \end{bmatrix}$$

$$z_0 = R_0^0 \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = R_0^1 \hat{k} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad z_2 = R_0^2 \hat{k} \quad z_3 = R_0^3 \hat{k}$$

$$H_0^2 = H_0^1 H_1^2 = \begin{bmatrix} 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_0^3 = H_0^2 H_2^3 = \begin{bmatrix} s_4 & c_4 & 0 & l_2 \\ 0 & 0 & 1 & l_3 \\ c_4 & -s_4 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad z_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

End-effector position $= o_3 - o_0 = \begin{bmatrix} l_2 \\ l_3 \\ l_1 \end{bmatrix}$

Test case : $l_1 = 1$, $l_2 = 0.5$, $l_3 = 0.75$, $q_4 = 0$

$\therefore$ Calculated Jacobian $= J$

From Code :

```
[[ 0.00000000e+00  1.00000000e+00  9.79317772e-12  9.79317772e-12]
 [ 0.00000000e+00 -4.89658886e-12  1.00000000e+00  1.00000000e+00]
 [ 1.00000000e+00  4.89658886e-12 -4.89658886e-12 -4.89658886e-12]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]]
```

Test case 2:

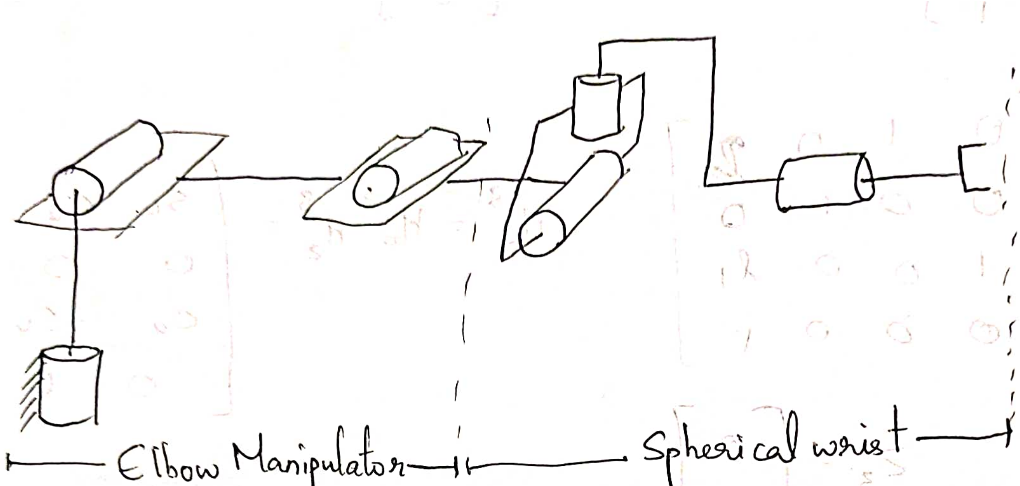
$l_1 = 0.25, \quad l_2 = 0.3, \quad l_3 = 1, \quad q_4 = 0$

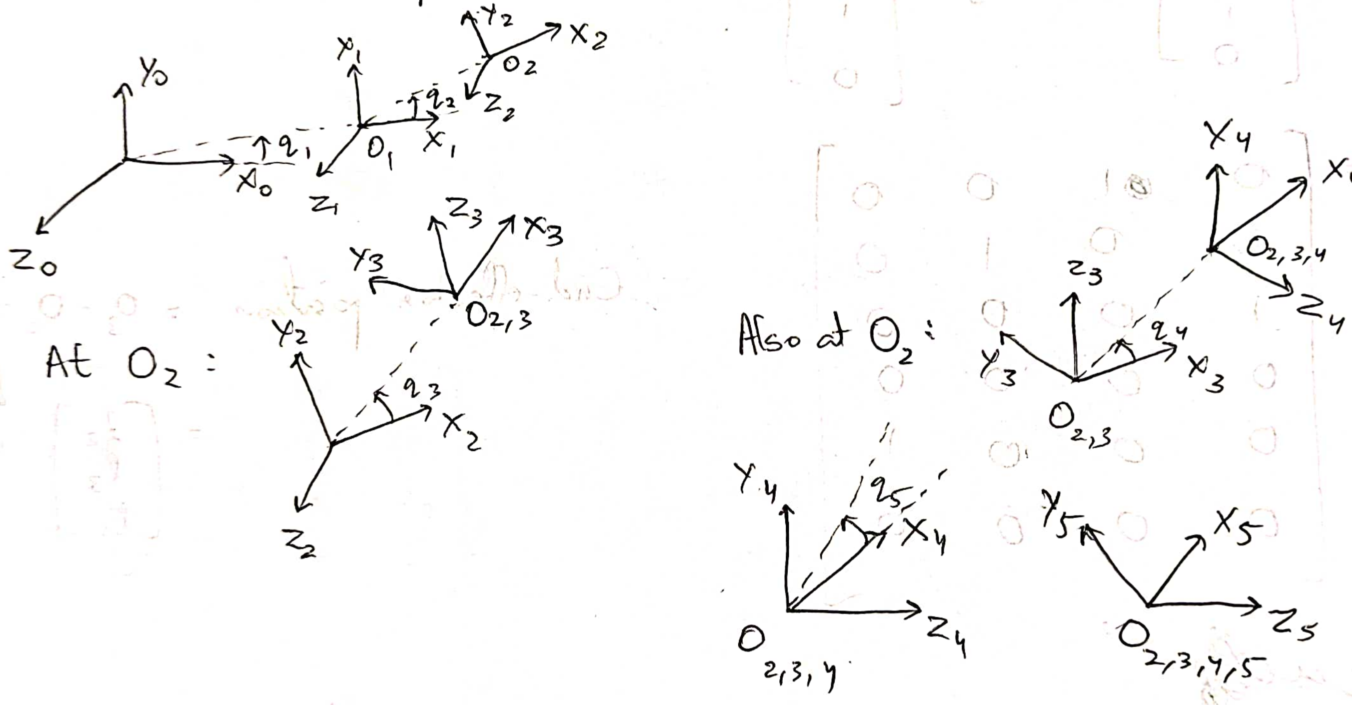
Calculated Jacobian $= J$

From code:


```
[[ 0.00000000e+00  1.00000000e+00  9.79317772e-12  9.79317772e-12]
 [ 0.00000000e+00 -4.89658886e-12  1.00000000e+00  1.00000000e+00]
 [ 1.00000000e+00  4.89658886e-12 -4.89658886e-12 -4.89658886e-12]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]]
```

6.





DH parameters:

	d	θ	a	α
$0 \to 1$	0	q_1	l_1	0
$1 \to 2$	0	q_2	l_2	0
$2 \to 3$	0	q_3	0	$-\pi/2$
$3 \to 4$	0	q_4	0	$\pi/2$
$4 \to 5$	0	q_5	0	0

$(d_5 = 0)$

$$J = [J_1 \quad J_2 \quad J_3 \quad J_4 \quad J_5 \quad J_6]$$

$$J_1 = \begin{bmatrix} z_0 \times (O_5 - O_0) \\ z_0 \end{bmatrix} \quad J_2 = \begin{bmatrix} z_1 \times (O_5 - O_1) \\ z_1 \end{bmatrix} \quad J_3 = \begin{bmatrix} z_2 \times (O_5 - O_2) \\ z_2 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} z_3 \times (O_5 - O_3) \\ z_3 \end{bmatrix} \quad J_5 = \begin{bmatrix} z_4 \times (O_5 - O_4) \\ z_4 \end{bmatrix} \quad J_6 = \begin{bmatrix} z_5 \times (O_5 - O_5) \\ z_5 \end{bmatrix}$$

$O_5 = O_4 = O_3 = O_2$ are the same. $z_0 = R_0^0 \hat{k}$, $z_1 = R_0^1 \hat{k}$, $z_2 = R_0^2 \hat{k}$, $z_3 = R_0^3 \hat{k}$, $z_4 = R_0^4 \hat{k}$, $z_5 = R_0^5 \hat{k}$

$$J_1 = \begin{bmatrix} z_0 \times (O_5 - O_0) \\ z_0 \end{bmatrix} \quad J_2 = \begin{bmatrix} z_1 \times (O_5 - O_1) \\ z_1 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 \\ z_2 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 0 \\ z_3 \end{bmatrix} \quad J_5 = \begin{bmatrix} 0 \\ z_4 \end{bmatrix} \quad J_6 = \begin{bmatrix} 0 \\ z_5 \end{bmatrix} \quad z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H_0^5 = H_0^1 H_1^2 H_2^3 H_3^4 H_4^5$$

$$Z_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H_0^1 = Z_1 X_1 = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H_1^2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H_2^3 = \begin{bmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_4=\begin{bmatrix} C_4 & -S_4 & 0 & 0\\ S_4 & C_4 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad X_4=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \Rightarrow H_3^4=\begin{bmatrix} C_4 & 0 & S_4 & 0\\ S_4 & 0 & -C_4 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Z_5=\begin{bmatrix} C_5 & -S_5 & 0 & 0\\ S_5 & C_5 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad X_5=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \Rightarrow H_4^5=\begin{bmatrix} C_5 & -S_5 & 0 & 0\\ S_5 & C_5 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_0^2=H_0^1H_1^2=\begin{bmatrix} C_{12} & -S_{12} & 0 & l_2C_{12}+l_1C_1\\ S_{12} & C_{12} & 0 & l_2S_{12}+l_1S_1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_0^3=H_0^2H_2^3=\begin{bmatrix} C_{123} & 0 & -S_{123} & l_2C_{12}+l_1C_1\\ S_{123} & 0 & C_{123} & l_2S_{12}+l_1S_1\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_0^4=H_0^3H_3^4=\begin{bmatrix} C_4C_{123} & -S_{123} & S_4C_{123} & l_2C_{12}+l_1C_1\\ C_4S_{123} & C_{123} & S_4S_{123} & l_2S_{12}+l_1S_1\\ -S_4 & 0 & C_4 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_0^5=H_0^4H_4^5=\begin{bmatrix} C_4C_{123}C_5-S_{123}S_5 & -S_5C_4C_{123}-S_{123}C_5 & S_4C_{123} & l_2C_{12}+l_1C_1\\ C_4C_5S_{123}+C_{123}S_5 & -S_5C_4C_{123}+C_5C_{123} & S_4S_{123} & l_2S_{12}+l_1S_1\\ -S_4C_5 & S_4C_5 & C_4 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$Z_1=\begin{bmatrix}0\\0\\1\end{bmatrix} \quad Z_2=\begin{bmatrix}0\\0\\1\end{bmatrix} \quad Z_3=\begin{bmatrix}-S_{123}\\C_{123}\\0\end{bmatrix} \quad Z_4=\begin{bmatrix}S_4C_{123}\\S_4S_{123}\\C_4\end{bmatrix} \quad Z_5=\begin{bmatrix}S_4C_{123}\\S_4S_{123}\\C_4\end{bmatrix}$$

$$O_5 - O_0 = \begin{bmatrix} l_2 C_{12} + l_1 C_1 \\ l_2 S_{12} + l_1 S_1 \\ 0 \end{bmatrix} \quad O_1 - O_0 = \begin{bmatrix} l_1 C_1 \\ l_1 S_1 \\ 0 \end{bmatrix} \Rightarrow O_5 - O_1 = \begin{bmatrix} l_2 C_{12} \\ l_2 S_{12} \\ 0 \end{bmatrix}$$

$$z_0 \times (O_5 - O_0) = \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ O_x & O_y & O_z \end{bmatrix} = -O_y \hat{i} + O_x \hat{j}$$

$$= -(l_2 S_{12} + l_1 S_1)\hat{i} + (l_2 C_{12} + l_1 C_1)\hat{j}$$

$$z_1 \times (O_5 - O_1) = -O_y \hat{i} + O_x \hat{j} = -l_2 S_{12} \hat{i} + l_2 C_{12} \hat{j}$$

$$J = \begin{bmatrix} -l_2 S_{12} - l_1 S_1 & -l_2 S_{12} & 0 & 0 & 0 & 0 \\ l_2 C_{12} + l_1 C_1 & l_2 C_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S_{123} & S_4 C_{123} & S_4 C_{123} \\ 0 & 0 & 0 & C_{123} & S_4 S_{123} & S_4 S_{123} \\ 1 & 1 & 1 & 0 & C_4 & C_4 \end{bmatrix}$$

End effector position : $O_5 - O_0$

Test case 1: $q_1 = \pi/2,\ q_2 = \pi/3,\ q_3 = \pi/4,\ q_4 = \pi/6,\ q_5 = \pi/8$

$l_1 = 1 \quad l_2 = 1$

Calculated Jacobian :

$$\begin{bmatrix} -1.5 & -0.5 & 0 & 0 & 0 & 0 \\ -0.866 & -0.866 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2588 & -0.4829 & -0.4829 \\ 0 & 0 & 0 & -0.9659 & -0.1294 & -0.1294 \\ 1 & 1 & 1 & 0 & 0.866 & 0.866 \end{bmatrix}$$

From code :

```
[[-1.50000000e+00 -5.00000000e-01  0.00000000e+00 -0.00000000e+00
  -0.00000000e+00 -0.00000000e+00]
 [-8.66025404e-01 -8.66025404e-01  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  2.58819045e-01
  -4.82962913e-01 -4.82962913e-01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -9.65925826e-01
  -1.29409523e-01 -1.29409523e-01]
 [ 1.00000000e+00  1.00000000e+00  1.00000000e+00  4.89658886e-12
   8.66025404e-01  8.66025404e-01]]
```

Test Case 2:

$q_1 = \pi/8 \quad q_2 = \pi/4 \quad q_3 = \pi, \quad q_4 = \pi/2, \quad q_5 = \pi/6, \quad l_1 = l_2 = 1$

$\therefore$ Calculated Jacobian:

$$\begin{bmatrix} -1.30656 & -0.92387 & 0 & 0 & 0 & 0 \\ 1.30656 & 0.38268 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9238 & -0.382 & -0.382 \\ 0 & 0 & 0 & -0.382 & 0.9238 & 0.9238 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

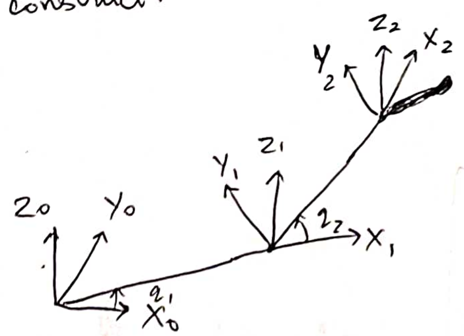
From code:

```
[[-1.30656296e+00 -9.23879533e-01  0.00000000e+00 -0.00000000e+00
  -0.00000000e+00 -0.00000000e+00]
 [ 1.30656296e+00  3.82683432e-01  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  9.23879533e-01
  -3.82683432e-01 -3.82683432e-01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -3.82683432e-01
  -9.23879533e-01 -9.23879533e-01]
 [ 1.00000000e+00  1.00000000e+00  1.00000000e+00  4.89658886e-12
   4.89658886e-12  4.89658886e-12]]
```

7. 2R Manipulators :

- Direct drive : They have motors attached directly to joints of the 2R Manipulator. Does not involve transmission elements between actuators & joints. The behaviour of system is predictable.
- Remotely Driven: Have motors attached to base, rotation of links controlled from these using belts or other means. More compact & Low weight robot can be made.
- 5-bar parallelogram arrangement: Made from 5 links connected together in a closed chain. It is cheaper and easier to construct.

8.



DH parameters :

	d	θ	a	α
$0 \to 1$	0	q_1	l_1	0
$1 \to 2$	0	q_2	l_2	0

$$J = [J_1 \ J_2] \qquad J_1 = \begin{bmatrix} z_0 \times (O_2 - O_0) \\ z_0 \end{bmatrix} \qquad J_2 = \begin{bmatrix} z_1 \times (O_2 - O_1) \\ z_1 \end{bmatrix}$$

$$z_0 = R_0^0 \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad H_0^2 = H_0^1 H_1^2$$

$$Z_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_1 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H_1^2 = \begin{bmatrix} C_2 & -S_2 & 0 & l_2C_2 \\ S_2 & C_2 & 0 & l_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^2 = H_0^1 H_1^2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_2C_{12} + l_1C_1 \\ S_{12} & C_{12} & 0 & l_2S_{12} + l_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad O_2 - O_0 = \begin{bmatrix} l_2C_{12} + l_1C_1 \\ l_2S_{12} + l_1S_1 \\ 0 \end{bmatrix}$$

$$O_2 - O_1 = \begin{bmatrix} l_2C_{12} \\ l_2S_{12} \\ 0 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad Z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_2S_{12} - l_1S_1 & -l_2S_{12} \\ l_2C_{12} + l_1C_1 & l_2C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

End effector: $O_2 - O_0$

$$x = l_2C_{12} + l_1C_1$$

$$y = l_2S_{12} + l_1S_1$$

Velocity $\dot{x} = J\dot{q}$

where $\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$

$$\dot{X} = J \times \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} (-l_2 s_{12} - l_1 s_1)\dot{q}_1 - l_2 s_{12}\dot{q}_2 \\ (l_2 c_{12} + l_1 c_1)\dot{q}_1 + l_1 c_{12}\dot{q}_2 \\ 0 \\ 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix}$$

$$\dot{x} = (-l_2 s_{12} - l_1 s_1)\dot{q}_1 - l_2 s_{12}\dot{q}_2 \qquad \omega_x = 0$$

$$\dot{y} = (l_2 c_{12} + l_1 c_1)\dot{q}_1 + l_2 c_{12}\dot{q}_2 \qquad \omega_y = 0$$

$$\dot{z} = 0 \qquad \omega_z = \dot{q}_1 + \dot{q}_2$$

Now, Torque $\tau = J^T F$

$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}$$

$$\therefore \quad \tau = \begin{bmatrix} (-l_2 s_{12} - l_1 s_1) F_x + (l_2 c_{12} + l_1 c_1) F_y + M_z \\ - l_2 s_{12} F_x + l_2 c_{12} F_y + M_z \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

10- Equations of motions provided $D(q)$ and $V(q)$

$$L = K - V$$

$$K = \frac{1}{2} q^T D(q) \dot{q}$$

$$V = V(q)$$

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \sum_j d_{kj}(q)\ddot{q}_j + \sum_{ij} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j$$

$$\frac{\partial L}{\partial q} = \frac{1}{2} \sum_{ij} \frac{\partial_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial V}{\partial q_k}$$

$$\tau = \sum_j d_{kj}(q)\ddot{q}_j + \sum_{ij}\left[\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i\dot{q}_j - \frac{\partial V}{\partial q_k}$$

Christoffel symbol, $c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial d_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$

$$\phi_R = \frac{\partial V}{\partial q_k}$$

$$\Rightarrow \tau = \sum_j d_{kj}(q)\ddot{q}_j + \sum_{ijk} c_{ijk}\dot{q}_i\dot{q}_j + \phi_k(q)$$

$$\Rightarrow \tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(\dot{q})$$