# TRANSPORTATION AND ASSIGNMENT PROBLEMS

- The Transportation Model

- Solution of a Transportation Problem

- The Assignment Model

- Solution of the Assignment Model

# TRANSPORTATION AND ASSIGNMENT PROBLEMS

## OVERVIEW

- Part of a larger class of linear programming problems known as network flow models.

- Possess special mathematical features that enabled development of very efficient, unique solution methods.

- Methods are variations of traditional simplex procedure.

# THE TRANSPORTATION MODEL CHARACTERISTICS

• A product is transported from a number of sources to a number of destinations at the minimum possible cost.

• Each source is able to supply a fixed number of units of the product, and each destination has a fixed demand for the product.

• The linear programming model has constraints for supply at each source and demand at each destination.

• All constraints are equalities in a balanced transportation model where supply equals demand.

• Constraints contain inequalities in unbalanced models where supply does not equal demand.

# TRANSPORTATION MODEL EXAMPLE
# PROBLEM DEFINITION AND DATA

- Problem:How many tons of wheat to transport from each grain elevator to    each mill on a monthly basis in order to minimize the total cost of transportation ?

- Data:

| Grain Elevator | Supply | | Mill | Demand |
|---|---|---|---|---|
| 1. Kansas City | 150 | | A. Chicago | 200 |
| 2. Omaha | 175 | | B. St.Louis | 100 |
| 3. Des Moines | 275 | | C. Cincinnati | 300 |
| Total | 600 tons | | Total | 600 tons |

| | Transport cost from Grain Elevator to Mill ($/ton) | | |
|---|---|---|---|
| Grain Elevator | A. Chicago | B. St. Louis | C. Cincinnati |
| 1. Kansas City | $6 | 8 | 10 |
| 2. Omaha | 7 | 11 | 11 |
| 3. Des Moines | 4 | 5 | 12 |

# TRANSPORTATION MODEL EXAMPLE
## MODEL FORMULATION

minimize $Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$

subject to $x_{1A} + x_{1B} + x_{1C} = 150$

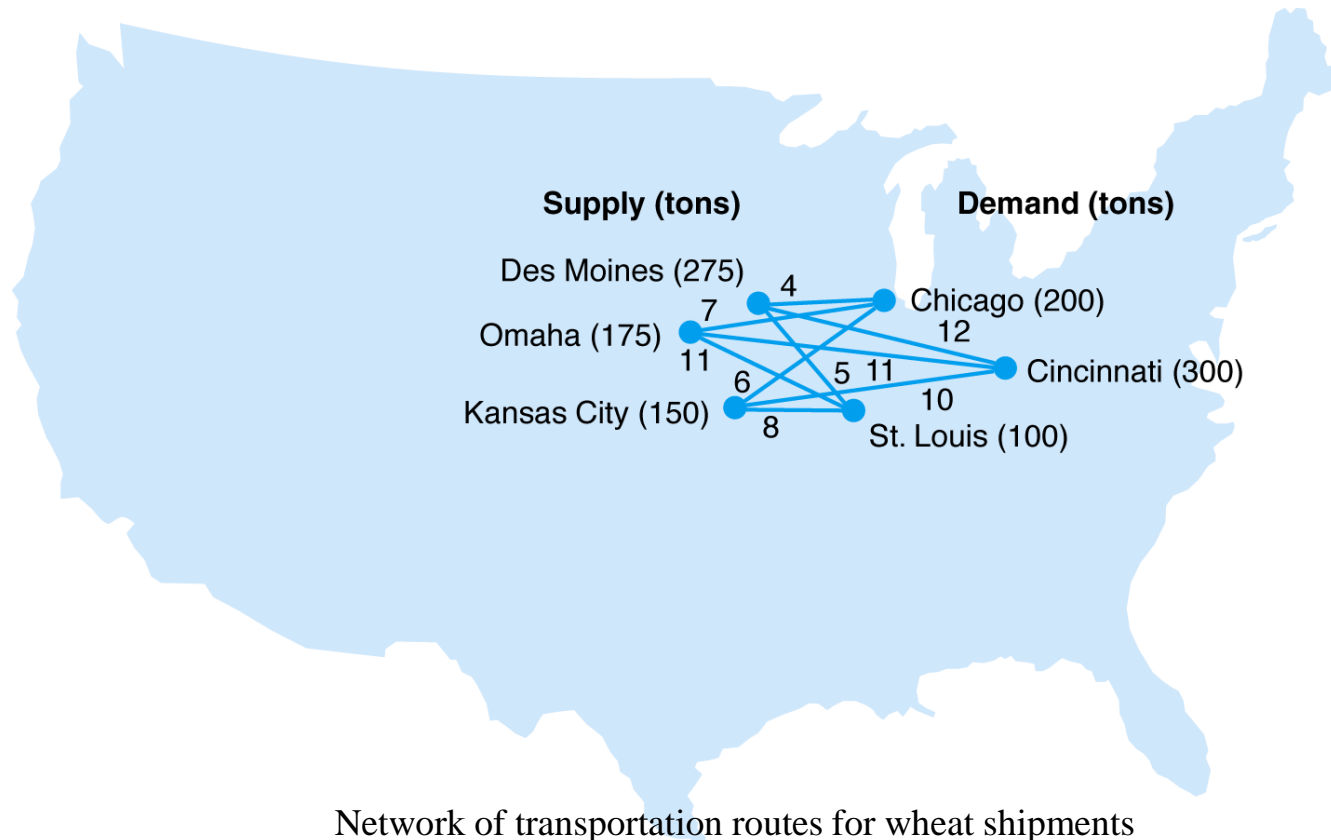$x_{2A} + x_{2B} + x_{2C} = 175$

$x_{3A} + x_{3B} + x_{3C} = 275$

$x_{1A} + x_{2A} + x_{3A} = 200$

$x_{1B} + x_{2B} + x_{3B} = 100$

$x_{1C} + x_{2C} + x_{3C} = 300$

$x_{ij} \geq 0$

where $x_{ij}$ = tons of wheat from each grain elevator, i, i = 1, 2, 3, to each mill j, j = A,B,C

**Supply (tons)**

Des Moines (275)

Omaha (175)

Kansas City (150)

**Demand (tons)**

Chicago (200)

Cincinnati (300)

St. Louis (100)

4  7  12  11  5  11  6  10  8

Network of transportation routes for wheat shipments

# SOLUTION OF THE TRANSPORTATION MODEL
## TABLEAU FORMAT

• Transportation problems are solved manually within a *tableau* format.

• Each cell in a transportation tableau is analogous to a decision variable that indicates the  amount allocated from a source to a destination.

• The supply and demand values along the outside rim of a tableau are called *rim values*.

The Transportation Tableau

| From \ To | A | B | C | Supply |
|-----------|-----|-----|-----|--------|
| 1 | 6 | 8 | 10 | 150 |
| 2 | 7 | 11 | 11 | 175 |
| 3 | 4 | 5 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

# SOLUTION OF THE TRANSPORTATION MODEL

## SOLUTION METHODS

• Transportation models do not start at the origin where all decision values are zero; they must instead be given an *initial feasible solution.*

• Initial feasible solution determination methods include:

    - northwest corner method

    - minimum cell cost method

    - Vogel's Approximation Method

• Methods for solving the transportation problem itself include:

    - stepping-stone method and

    - modified distribution method.

# THE NORTHWEST CORNER METHOD

- In the northwest corner method the largest possible allocation is made to the cell in the upper left-hand corner of the tableau , followed by allocations to adjacent feasible cells.

The Initial NW Corner Solution

| From \ To | A | | B | | C | | Supply |
|---|---|---|---|---|---|---|---|
| **1** | 150 | 6 | | 8 | | 10 | 150 |
| **2** | 50 | 7 | 100 | 11 | 25 | 11 | 175 |
| **3** | | 4 | | 5 | 275 | 12 | 275 |
| **Demand** | 200 | | 100 | | 300 | | 600 |

- The initial solution is complete when all rim requirements are satisfied.

- Transportation cost is computed by evaluating the objective function:

$$Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$$

$$= 6(150) + 8(0) + 10(0) + 7(50) + 11(100) + 11(25) + 4(0) + 5(0) + !2(275)$$

$$= \$5,925$$

# THE NORTHWEST CORNER METHOD

# SUMMARY OF STEPS

1. Allocate as much as possible to the cell in the upper left-hand corner, subject to the supply and demand conditions.

2. Allocate as much as possible to the next adjacent feasible cell.

3. Repeat step 2 until all rim requirements are met.

- In the minimum cell cost method as much as possible is allocated to the cell with the minimum cost followed by allocation to the feasible cell with minimum cost.

| From \ To | A | B | C | Supply |
|-----------|-----|-----|-----|--------|
| 1 | 6 | 8 | 10 | 150 |
| 2 | 7 | 11 | 11 | 175 |
| 3 | 4 200 | 5 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

The Initial Minimum Cell Cost Allocation

| From \ To | A | B | C | Supply |
|-----------|-----|-----|-----|--------|
| 1 | 6 | 8 | 10 | 150 |
| 2 | 7 | 11 | 11 | 175 |
| 3 | 4 200 | 5 75 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

The Second Minimum Cell Cost Allocation

- The complete initial minimum cell cost solution; total cost = $4,550.

- The minimum cell cost method will provide a solution with a lower cost than the northwest corner solution because it considers cost in the allocation process.

The Initial Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | | 6   25 | 8   125 | 10   150 |
| 2 | | 7 | 11 | 11   175   175 |
| 3 | 4   200 | 5   75 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

## THE MINIMUM CELL COST METHOD SUMMARY OF STEPS (3 OF 3)

1. Allocate as much as possible to the feasible cell with the minimum transportation cost, and adjust the rim requirements.

2. Repeat step 1 until all rim requirements have been met.

- Method is based on the concept of *penalty cost* or *regret*.

- A penalty cost is the difference between the largest and the next largest cell cost in a row (or column).

- In VAM the first step is to develop a penalty cost for each source and destination.

- Penalty cost is calculated by subtracting the minimum cell cost from the next higher cell cost in each row and column.

The VAM Penalty Costs

| From \ To | A | B | C | Supply | |
|-----------|-----|-----|-----|--------|---|
| 1 | 6 | 8 | 10 | 150 | 2 |
| 2 | 7 | 11 | 11 | 175 | 4 |
| 3 | 4 | 5 | 12 | 275 | 1 |
| Demand | 200 | 100 | 300 | 600 | |
| | 2 | 3 | 1 | | |

- VAM allocates as much as possible to the minimum cost cell in the row or column with the largest penalty cost.

The Initial VAM Allocation

| From \ To | A | B | C | Supply | |
|---|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 150 | 2 |
| 2 | 7 / 175 | 11 | 11 | 175 | |
| 3 | 4 | 5 | 12 | 275 | 1 |
| Demand | 200 | 100 | 300 | 600 | |
| | 2 | 3 | 2 | | |

- After each VAM cell allocation, all row and column penalty costs are recomputed.

The Second
VAM Allocation

| To<br>From | A | B | C | Supply | |
|---|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 150 | 4 |
| 2 | 7<br>175 | 11 | 11 | 175 | |
| 3 | 4 | 5<br>100 | 12 | 275 | 8 |
| Demand | 200 | 100 | 300 | 600 | |
| | 2 | | 2 | | |

- Recomputed penalty costs after the third allocation.

**The Third VAM Allocation**

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 | 10 / 150 | 150 |
| 2 | 7 / 175 | 11 | 11 | 175 |
| 3 | 4 / 25 | 5 / 100 | 12 / 150 | 275 |
| Demand | 200 | 100 | 300 | 600 |

- The initial VAM solution; total cost = $5,125

- VAM and minimum cell cost methods both provide better initial solutions than does the northwest corner method.

The Initial VAM Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 | 10 150 | 150 |
| 2 | 7 175 | 11 | 11 | 175 |
| 3 | 4 25 | 5 100 | 12 150 | 275 |
| Demand | 200 | 100 | 300 | 600 |

# VOGEL'S APPROXIMATION METHOD (VAM) SUMMARY OF STEPS

1. Determine the penalty cost for each row and column.

2. Select the row or column with the highest penalty cost.

3. Allocate as much as possible to the feasible cell with the lowest transportation cost in the row or column with the highest penalty cost.

4. Repeat steps 1, 2, and 3 until all rim requirements have been met.

- Once an initial solution is derived, the problem must be solved using either the stepping-stone method or the modified distribution method (MODI).

- The initial solution used as a starting point in this problem is the minimum cell cost method solution because it had the minimum total cost of the three methods used.

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 150 |
| | | 25 | 125 | |
| 2 | 7 | 11 | 11 | 175 |
| | | | 175 | |
| 3 | 4 | 5 | 12 | 275 |
| | 200 | 75 | | |
| Demand | 200 | 100 | 300 | 600 |

The Minimum Cell Cost Solution

- The stepping-stone method determines if there is a cell with no allocation that would reduce cost if used.

| From \ To | A | B | C | Supply | |
|---|---|---|---|---|---|
| 1 | +1 ⌐6 | 8 25 | 10 125 | 150 | 151 |
| 2 | 7 | 11 | 11 175 | 175 | |
| 3 | 4 200 | 5 75 | 12 | 275 | |
| Demand | 200 | 100 | 300 | 600 | |

The Allocation of One Ton to Cell 1A

- Must subtract one ton from another allocation along that row.

| From \ To | A | B | C | Supply |
|-----------|---|---|---|--------|
| 1 | +1 ... 6 | −1 ... 8 / 25 | 10 / 125 | 150 |
| 2 | 7 | 11 | 11 / 175 | 175 |
| 3 | 4 / 200 | 5 / 75 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

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The Subtraction of One Ton from Cell 1B

- A requirement of this solution method is that units can only be added to and subtracted from cells that already have allocations, thus one ton must be added to a cell as shown.

The Addition of One Ton to Cell 3B and the Subtraction of One Ton from Cell 3A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | +1 → 6 → −1 → 8 | | 10 | |
| 1 | | 25 | 125 | 150 |
| 2 | 7 | 11 | 11 | |
| 2 | | | 175 | 175 |
| 3 | −1 ← 4 ← +1 → 5 | | 12 | |
| 3 | 200 | 75 | | 275 |
| Demand | 200 | 100 | 300 | 600 |

- An empty cell that will reduce cost is a potential entering variable.

- To evaluate the cost reduction potential of an empty cell, a closed path connecting used cells to the empty cells is identified.

The Stepping-Stone Path for Cell 2A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8  25 | 10  125 | 150 |
| 2 | 7 | 11 | 11  175 | 175 |
| 3 | 4  200 | 5  75 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

2A → 2C → 1C → 1B → 3B → 3A
+ \$7 − 11 + 10 − 8 + 5 − 4 = −\$1

- The remaining stepping-stone paths and resulting computations for cells 2B and 3C.

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | | 6 + ← 8 − / 25 | 10 / 125 | 150 |
| 2 | | 7 − → 11 + / | 11 / 175 | 175 |
| 3 | 4 / 200 | 5 / 75 | 12 / | 275 |
| Demand | 200 | 100 | 300 | 600 |

2B → 2C → 1C → 1B
+ $11 − 11 + 10 − 8 = +$2

The Stepping-Stone Path
for Cell 2B

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | | 6 + ← 8 − / 25 | 10 / 125 | 150 |
| 2 | | 7 / 11 | 11 / 175 | 175 |
| 3 | 4 − / 200 | 5 → + / 75 | 12 / | 275 |
| Demand | 200 | 100 | 300 | 600 |

The Stepping-Stone Path for Cell 3C

3C → 1C → 1B → 3B
+ $12 − 10 + 8 − 5 = +$5

- After all empty cells are evaluated, the one with the greatest cost reduction potential is the entering variable.

- A tie can be broken arbitrarily.

The Stepping-Stone Path for Cell 1A

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | + ← 6 — | 8 25 | 10 125 | 150 |
| 2 | 7 | 11 | 11 175 | 175 |
| 3 | 4 → + 200 | 5 75 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

- When reallocating units to the entering variable (cell), the amount is the minimum amount subtracted on the stepping-stone path.

- At each iteration one variable enters and one leaves (just as in the simplex method).

The Second Iteration of the Stepping-Stone Method

| From \ To | A | | B | | C | | Supply |
|---|---|---|---|---|---|---|---|
| 1 | 25 | 6 | | 8 | 125 | 10 | 150 |
| 2 | | 7 | | 11 | 175 | 11 | 175 |
| 3 | 175 | 4 | 100 | 5 | | 12 | 275 |
| Demand | 200 | | 100 | | 300 | | 600 |

- Check to see if the solution is optimal.

| To From | A | B | C | Supply |
|---|---|---|---|---|
| 1 | − ← 6   8 + 10 ↑   25 | | 125 | 150 |
| 2 | + 7 → 11 → − 11   175 | | | 175 |
| 3 | 4   175 | 5   100 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

2A → 2C → 1C → 1A
+ \$7 − 11 + 10 − 6 = \$0

The Stepping-Stone Path for Cell 2A

| To From | A | B | C | Supply |
|---|---|---|---|---|
| 1 | − → 6 → + 8   25 | | 10   125 | 150 |
| 2 | 7 | 11 | 11   175 | 175 |
| 3 | + ← 4 − 5   175 | 100 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

The Stepping-Stone Path for Cell 1B

1B → 3B → 3A → 1A
+ \$8 − 5 + 4 − 6 = +\$1

- Continuing check for optimality.

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | − 6 25 | 8 → + | 10 125 | 150 |
| 2 | 7 | + ← 11 − | 11 175 | 175 |
| 3 | + ← 4 − 175 | 5 100 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

2B → 3B → 3A → 1A → 1C → 2C
+ $11 − 5 + 4 − 6 + 10 − 11 = +$3

The Stepping-Stone
Path for Cell 2B

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | + ← 6 25 | 8 − | 10 125 | 150 |
| 2 | 7 | 11 | 11 175 | 175 |
| 3 | − ← 4 175 | 5 + 100 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

The Stepping-Stone
Path for Cell 3C

3C → 3A → 1A → 1C
+ $12 − 4 + 6 − 10 = +$4

- The stepping-stone process is repeated until none of the empty cells will reduce costs (i.e., an optimal solution).

- In example, evaluation of four paths indicates no cost reductions, therefore Table 19 solution is optimal.

- Solution and total minimum cost :

$x_{1A}$ = 25 tons, $x_{2C}$ = 175 tons, $x_{3A}$ = 175 tons, $x_{1C}$ = 125 tons, $x_{3B}$ = 100 tons

$Z = \$6(25) + 8(0) + 10(125) + 7(0) + 11(0) + 11(175) + 4(175) + 5(100) + 12(0)$

$= \$4,525$

- A multiple optimal solution occurs when an empty cell has a cost change of zero and all other empty cells are positive.

- An alternate optimal solution is determined by allocating to the empty cell with a zero cost change.

- Alternate optimal total minimum cost also equals $4,525.

The Alternative Optimal Solution

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 | 10<br>150 | 150 |
| 2 | 7<br>25 | 11 | 11<br>150 | 175 |
| 3 | 4<br>175 | 5<br>100 | 12 | 275 |
| Demand | 200 | 100 | 300 | 600 |

# THE STEPPING-STONE SOLUTION METHOD
# SUMMARY OF STEPS

1. Determine the stepping-stone paths and cost changes for each empty cell in the tableau.

2. Allocate as much as possible to the empty cell with the greatest net decrease in cost.

3. Repeat steps 1 and 2 until all empty cells have positive cost changes that indicate an optimal solution.

- MODI is a modified version of the stepping-stone method in which math equations replace the stepping-stone paths.

- In the table, the extra left-hand column with the $u_i$ symbols and the extra top row with the $v_j$ symbols represent values that must be computed.

- Computed for all cells with allocations :

$$u_i + v_j = c_{ij} = \text{unit transportation cost for cell }_{ij}.$$

The Minimum Cell Cost Initial Solution

| $v_j$ | | $v_A =$ | $v_B =$ | $v_C =$ | |
|---|---|---|---|---|---|
| $u_i$ | To<br>From | A | B | C | Supply |
| $u_1 =$ | 1 | 6 | 8<br>25 | 10<br>125 | 150 |
| $u_2 =$ | 2 | 7 | 11 | 11<br>175 | 175 |
| $u_3 =$ | 3 | 4<br>200 | 5<br>75 | 12 | 275 |
| | Demand | 200 | 100 | 300 | 600 |

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- Formulas for cells containing allocations:

$x_{1B}$:  $u_1 + v_B = 8$

$x_{1C}$:  $u_1 + v_C = 10$

$x_{2C}$:  $u_2 + v_C = 11$

$x_{3A}$:  $u_3 + v_A = 4$

$x_{3B}$:  $u_3 + v_B = 5$

| $u_i$ | $v_j$ / From / To | $v_A = 7$ / A | $v_B = 8$ / B | $v_C = 10$ / C | Supply |
|---|---|---|---|---|---|
| $u_1 = 0$ | 1 | 6 | 8 / 25 | 10 / 125 | 150 |
| $u_2 = 1$ | 2 | 7 | 11 | 11 / 175 | 175 |
| $u_3 = -3$ | 3 | 4 / 200 | 5 / 75 | 12 | 275 |
| | Demand | 200 | 100 | 300 | 600 |

The Initial Solution with All ui and vj Values

- Five equations with 6 unknowns, therefore let $u_1 = 0$ and solve to obtain:

$v_B = 8$,  $v_C = 10$,  $u_2 = 1$,  $u_3 = -3$,  $v_A = 7$

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- Each MODI allocation replicates the stepping-stone allocation.

- Use following to evaluate all empty cells:

$$c_{ij} - u_i - v_j = k_{ij}$$

where $k_{ij}$ equals the cost increase or decrease that would occur by allocating to a cell.

- For the empty cells in Table 26:

$$x_{1A}: \ k_{1A} = c_{1A} - u_1 - v_A = 6 - 0 - 7 = -1$$

$$x_{2A}: \ k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 7 = -1$$

$$x_{2B}: \ k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 8 = +2$$

$$x_{3C}: \ k_{3C} = c_{3C} - u_3 - v_C = 12 - (-3) - 10 = +5$$

- After each allocation to an empty cell, the $u_i$ and $v_j$ values must be recomputed.

| $u_i$ | $v_j$ / From (To) | $v_A =$ / A | $v_B =$ / B | $v_C =$ / C | Supply |
|---|---|---|---|---|---|
| $u_1 =$ | 1 | 6 / 25 | 8 | 10 / 125 | 150 |
| $u_2 =$ | 2 | 7 | 11 | 11 / 175 | 175 |
| $u_3 =$ | 3 | 4 / 175 | 5 / 100 | 12 | 275 |
| | Demand | 200 | 100 | 300 | 600 |

The Second Iteration of the MODI Solution Method

- Recomputing $u_i$ and $v_j$ values:

$x_{1A}$: $u_1 + v_A = 6$, $v_A = 6$     $x_{1C}$: $u_1 + v_C = 10$, $v_C = 10$     $x_{2C}$: $u_2 + v_C = 11$, $u_2 = 1$

$x_{3A}$: $u_3 + v_A = 4$, $u_3 = -2$     $x_{3B}$: $u_3 + v_B = 5$, $v_B = 7$

| $u_i$ | $v_j$ From \ To | $v_A = 6$ A | $v_B = 7$ B | $v_C = 10$ C | Supply |
|---|---|---|---|---|---|
| $u_1 = 0$ | 1 | 6 <br> 25 | 8 | 10 <br> 125 | 150 |
| $u_2 = 1$ | 2 | 7 | 11 | 11 <br> 175 | 175 |
| $u_3 = -2$ | 3 | 4 <br> 175 | 5 <br> 100 | 12 | 275 |
| | Demand | 200 | 100 | 300 | 600 |

The New ui and vj Values for the Second Iteration

- Cost changes for the empty cells, $c_{ij} - u_i - v_j = k_{ij}$;

$$x_{1B}: k_{1B} = c_{1B} - u_1 - v_B = 8 - 0 - 7 = +1$$

$$x_{2A}: k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 6 = 0$$

$$x_{2B}: k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 7 = +3$$

$$x_{3C}: k_{2B} = c_{2B} - u_3 - v_C = 12 - (-2) - 10 = +4$$

- Since none of the values are negative, solution obtained is optimal.

- Cell 2A with a zero cost change indicates a multiple optimal solution.

# THE MODIFIED DISTRIBUTION METHOD (MODI)
# SUMMARY OF STEPS

1. Develop an initial solution.

2. Compute the $u_i$ and $v_j$ values for each row and column.

3. Compute the cost change, $k_{ij}$, for each empty cell.

4. Allocate as much as possible to the empty cell that will result in the greatest net decrease in cost (most negative $k_{ij}$)

5. Repeat steps 2 through 4 until all $k_{ij}$ values are positive or zero.

- When demand exceeds supply a dummy row is added to the tableau.

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 150 |
| 2 | 7 | 11 | 11 | 175 |
| 3 | 4 | 5 | 12 | 275 |
| Dummy | 0 | 0 | 0 | 50 |
| Demand | 200 | 100 | 350 | 650 |

An Unbalanced Model
(Demand . Supply)

- When supply exceeds demand, a dummy column is added to the tableau.

    - The dummy column (or dummy row) has no effect on the initial solution methods or the optimal solution methods.

| From \ To | A | B | C | Dummy | Supply |
|---|---|---|---|---|---|
| 1 | 6 | 8 | 10 | 0 | 150 |
| 2 | 7 | 11 | 11 | 0 | 175 |
| 3 | 4 | 5 | 12 | 0 | 375 |
| Demand | 200 | 100 | 300 | 100 | 700 |

An Unbalanced Model (Supply . Demand)

- In a transportation tableau with *m* rows and *n* columns, there must be $m + n - 1$ cells with allocations; if not, it is *degenerate*.

- The tableau in the figure does not meet the condition since $3 + 3 - 1 = 5$ cells and there are only 4 cells with allocations.

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 | 8 <br> 100 | 10 <br> 50 | 150 |
| 2 | 7 | 11 | 11 <br> 250 | 250 |
| 3 | 4 <br> 200 | 5 | 12 | 200 |
| Demand | 200 | 100 | 300 | 600 |

The Minimum Cell Cost Initial Solution

- In a degenerate tableau, all the stepping-stone paths or MODI equations cannot be developed.

-To rectify a degenerate tableau, an empty cell must artificially be treated as an occupied cell.

The Initial Solution

| From \ To | A | | B | | C | | Supply |
|-----------|-----|---|-----|----|-----|----|--------|
| 1 | 0 | 6 | 100 | 8 | 50 | 10 | 150 |
| 2 | | 7 | | 11 | 250 | 11 | 250 |
| 3 | 200 | 4 | | 5 | | 12 | 200 |
| Demand | 200 | | 100 | | 300 | | 600 |

- The stepping-stone path s and cost changes for this tableau:

2A  2C  1C  1A

$x_{2A}$:   $7 - 11 + 10 - 6 = 0$

2B  2C  1C  1B

$x_{2B}$:   $11 - 11 + 10 - 8 = +2$

3B  1B  1A  3A

$x_{3B}$:   $5 - 8 + 6 - 4 = -1$

3C  1C  1A  3A

$x_{3C}$:   $12 - 10 + 6 - 4 = +4$

| From \ To | A | B | C | Supply |
|---|---|---|---|---|
| 1 | 6 <br> 100 | 8 | 10 <br> 50 | 150 |
| 2 | 7 | 11 | 11 <br> 250 | 250 |
| 3 | 4 <br> 100 | 5 <br> 100 | 12 | 200 |
| Demand | 200 | 100 | 300 | 600 |

The Second Stepping-Stone Iteration

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# PROHIBITED ROUTES

- A prohibited route is assigned a large cost such as *M*.

- When the prohibited cell is evaluated, it will always contain the cost M, which will keep it from being selected as an entering variable.

# THE ASSIGNMENT MODEL

## CHARACTERISTICS

• Special form of linear programming model similar to the transportation model.

• Supply at each source and demand at each destination limited to one unit.

• In a balanced model supply equals demand.

• In an unbalanced model supply does not equal demand.

# THE ASSIGNMENT MODEL
# EXAMPLE PROBLEM DEFINITION AND DATA

Problem: Assign four teams of officials to four games in a way that will minimize total distance traveled by the officials. Supply is always one team of officials, demand is for only one team of officials at each game.

| Officials | Game Sites | | | |
| --- | --- | --- | --- | --- |
| | RALEIGH | ATLANTA | DURHAM | CLEMSON |
| A | 210 | 90 | 180 | 160 |
| B | 100 | 70 | 130 | 200 |
| C | 175 | 105 | 140 | 170 |
| D | 80 | 65 | 105 | 120 |

## THE ASSIGNMENT MODEL
## EXAMPLE PROBLEM MODEL FORMULATION

Minimize $Z = 210x_{AR} + 90x_{AA} + 180x_{AD} + 160x_{AC} + 100x_{BR} + 70x_{BA} + 130x_{BD} + 200x_{BC} + 175x_{CR} + 105x_{CA} + 140x_{CD} + 170x_{CC} + 80x_{DR} + 65x_{DA} + 105x_{DD} + 120x_{DC}$

subject to

$$x_{AR} + x_{AA} + x_{AD} + x_{AC} = 1$$

$$x_{BR} + x_{BA} + x_{BD} + x_{BC} = 1$$

$$x_{CR} + x_{CA} + x_{CD} + x_{CC} = 1$$

$$x_{DR} + x_{DA} + x_{DD} + x_{DC} = 1$$

$$x_{AR} + x_{BR} + x_{CR} + x_{DR} = 1$$

$$x_{AA} + x_{BA} + x_{CA} + x_{DA} = 1$$

$$x_{AD} + x_{BD} + x_{CD} + x_{DD} = 1$$

$$x_{AC} + x_{BC} + x_{CC} + x_{DC} = 1$$

$$x_{ij} \geq 0$$

- An *assignment problem* is a special form of the transportation problem where all supply and demand values equal one.

- Example: assigning four teams of officials to four games in a way that will minimize distance traveled by the officials.

| Officials | Game Sites | | | |
|---|---|---|---|---|
| | Raleigh | Atlanta | Durham | Clemson |
| A | 210 | 90 | 180 | 160 |
| B | 100 | 70 | 130 | 200 |
| C | 175 | 105 | 140 | 170 |
| D | 80 | 65 | 105 | 120 |

The Travel Distances to Each Game for Each Team of Officials

- An *opportunity cost table* is developed by first subtracting the minimum value in each row from all other row values (*row reductions*) and then repeating this process for each column.

| Officials | Game Sites | | | |
|---|---|---|---|---|
| | Raleigh | Atlanta | Durham | Clemson |
| A | 120 | 0 | 90 | 70 |
| B | 30 | 0 | 60 | 130 |
| C | 70 | 0 | 35 | 65 |
| D | 15 | 0 | 40 | 55 |

The Assignment Tableau with Row Reductions

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- The minimum value in each column is subtracted from all column values (*column reductions*).

- Assignments can be made in the table wherever a zero is present.

- An *optimal solution* results when each of the four teams can be assigned to a different game.

- Table 36 does not contain an optimal solution

| Officials | Game Sites | | | |
|---|---|---|---|---|
| | Raleigh | Atlanta | Durham | Clemson |
| A | 105 | 0 | 55 | 15 |
| B | 15 | 0 | 25 | 75 |
| C | 55 | 0 | 0 | 10 |
| D | 0 | 0 | 5 | 0 |

The Tableau with Column Reductions

- An optimal solution occurs when the number of independent unique assignments equals the number of rows and columns.

- If the number of unique assignments is less than the number of rows (or columns) a line test must be used.

| Officials | Raleigh | Atlanta | Durham | Clemson |
|-----------|---------|---------|--------|---------|
| A | 105 | 0 | 55 | 15 |
| B | 15 | 0 | 25 | 75 |
| C | 35 | 0 | 0 | 10 |
| D | 0 | 0 | 5 | 0 |

Game Sites

The Opportunity Cost Table with the Line Test

- In a line test all zeros are crossed out by horizontal and vertical lines; the minimum uncrossed value is subtracted from all other uncrossed values and added to values where two lines cross.

| Officials | Raleigh | Atlanta | Durham | Clemson |
|-----------|---------|---------|--------|---------|
| A | 90 | 0 | 40 | 0 |
| B | 0 | 0 | 10 | 60 |
| C | 55 | 15 | 0 | 10 |
| D | 0 | 15 | 5 | 0 |

Game Sites

The Second Iteration

- At least four lines are required to cross out all zeros in table 38.

- This indicates an optimal solution has been reached.

- Assignments and distances:

| Assignment | Distance | Assignment | Distance |
|---|---|---|---|
| Team A → Atlanta | 90 | Team A → Clemson | 160 |
| Team B → Raleigh | 100 | Team B → Atlanta | 70 |
| Team C → Durham | 140 | Team C → Durham | 140 |
| Team D → Clemson | 120 | Team D → Raleigh | 80 |
| Total | 450 miles | Total | 450 miles |

- If in initial assignment team A went to Clemson, result is the same; resulting assignments represent multiple optimal solutions.

- When supply exceeds demand, a dummy column is added to the tableau.

- When demand exceeds supply, a dummy row is added to the tableau.

- The addition of a dummy row or column does not affect the solution method.

- A prohibited assignment is given a large relative cost of $M$ so that it will never be selected.

|  | Game Sites | | | | |
| Officials | Raleigh | Atlanta | Durham | Clemson | Dummy |
| --- | --- | --- | --- | --- | --- |
| A | 210 | 90 | 180 | 160 | 0 |
| B | 100 | 70 | 130 | 200 | 0 |
| C | 175 | 105 | 140 | 170 | 0 |
| D | 80 | 65 | 105 | 120 | 0 |
| E | 95 | 115 | 120 | 100 | 0 |

An Unbalanced Assignment Tableau with a Dummy Column

## SOLUTION OF THE ASSIGNMENT MODEL
## SUMMARY OF SOLUTION STEPS

1. Perform row reductions.

2. Perform column reductions.

3  In the completed opportunity cost table, cross out all zeros using the minimum number of horizontal and/or vertical lines.

4. If fewer than $m$ lines are required, subtract the minimum uncrossed value from all other uncrossed values, and add the same value to all cells where two lines intersect.

5. Leave all other values unchanged and repeat step 3.

6. If $m$ lines are required, the tableau contains the optimal solution. If fewer than $m$ lines are required, repeat step 4.

# THE ASSIGNMENT PROBLEM WITH MORE EXAMPLES

# THE ASSIGNMENT PROBLEM

- In many business situations, management needs to assign - personnel to jobs, - jobs to machines, - machines to job locations, or - salespersons to territories.

- Consider the situation of assigning $n$ jobs to $n$ machines.

- When a job i (=1,2,....,n) is assigned to machine j (=1,2, .....n) that incurs a cost Cij.

- The objective is to assign the jobs to machines at the least possible total cost.

# THE ASSIGNMENT PROBLEM

- This situation is a special case of the Transportation Model And it is known as the *assignment problem*.

- Here, jobs represent "sources" and machines represent "destinations."

- The supply available at each source is 1 unit And demand at each destination is 1 unit.

# THE ASSIGNMENT PROBLEM

| | | Machine | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | ……. | n | Source |
| Job | 1 | C11 | C12 | ……. | C1n | 1 |
| | 2 | C21 | C22 | ……. | C2n | 1 |
| | . | . | . | | . | . |
| | . | . | . | | . | . |
| | . | . | . | | . | . |
| | n | Cn1 | Cn2 | ……. | Cnn | 1 |
| Destination | | 1 | 1 | …… | 1 | |

The assignment model can be expressed mathematically as follows:

$X_{ij}$ =  0,  if the job j is not assigned to machine i

1,  if the job j is assigned to machine i

# THE ASSIGNMENT PROBLEM

$$\text{Min} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

(Sum of assignments from a source should be exactly equal to 1):

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad \text{For } i=1,2,\ldots,n$$

(Sum of assignments to a destination should be equal to the demanded quantity by that destination):

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad \text{For } j=1,2,\ldots,n$$

(Quantities to be assigned can be either 0 or 1):

$$X_{ij} = 0 \quad \text{or} \quad 1 \quad \text{For all } i \text{ and } j.$$

# THE ASSIGNMENT PROBLEM EXAMPLE

- Ballston Electronics manufactures small electrical devices.

- Products are manufactured on five different assembly lines (1,2,3,4,5).

- When manufacturing is finished, products are transported from the assembly lines to one of the five different inspection areas (A,B,C,D,E).

- Transporting products from five assembly lines to five inspection areas requires different times (in minutes)

# THE ASSIGNMENT PROBLEM EXAMPLE

| Assembly Line | Inspection Area | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| 1 | 10 | 4 | 6 | 10 | 12 |
| 2 | 11 | 7 | 7 | 9 | 14 |
| 3 | 13 | 8 | 12 | 14 | 15 |
| 4 | 14 | 16 | 13 | 17 | 17 |
| 5 | 19 | 11 | 17 | 20 | 19 |

Under current arrangement, assignment of inspection areas to the assembly lines are 1 to A, 2 to B, 3 to C, 4 to D, and 5 to E.

This arrangement requires 10+7+12+17+19 = 65 man minutes.

# HUNGARIAN METHOD EXAMPLE

| | | Machine | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | |
| | 1 | 5 | 7 | 9 | 1 |
| Job | 2 | 14 | 10 | 12 | 1 |
| | 3 | 15 | 13 | 16 | 1 |
| | | 1 | 1 | 1 | |

**Step 1:**  Select the smallest value in each row. Subtract this value from each value in that row

**Step 2:** Do the same for the columns that do not have any zero value.

# HUNGARIAN METHOD EXAMPLE

|  |  | Machine | | |
|---|---|---|---|---|
|  |  | 1 | 2 | 3 |
| Job | 1 | **5** | 7 | 9 |
|  | 2 | 14 | **10** | 12 |
|  | 3 | 15 | **13** | 16 |

|  |  | Machine | | |
|---|---|---|---|---|
|  |  | 1 | 2 | 3 |
| Job | 1 | 0 | 2 | 4 |
|  | 2 | 4 | 0 | **2** |
|  | 3 | 2 | 0 | 3 |

|  | Machine | | |
|---|---|---|---|
|  | 1 | 2 | 3 |
| 1 | **0** | 2 | 2 |
| 2 | 4 | 0 | **0** |
| 3 | 2 | **0** | 1 |

If not finished, continue with other columns.

# HUNGARIAN METHOD EXAMPLE

**Step 3:** Assignments are made at zero values.

- Therefore, we assign job 1 to machine 1; job 2 to machine 3, and job 3 to machine 2.

- Total cost is 5+12+13 = 30.

- It is not always possible to obtain a feasible assignment as in here.

# HUNGARIAN METHOD EXAMPLE 2

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | _1_ | 4 | 6 | 3 |
| 2 | 9 | _7_ | 10 | 9 |
| 3 | _4_ | 5 | 11 | 7 |
| 4 | 8 | 7 | 8 | _5_ |

|   | 1 | 2 | _3_ | 4 |
|---|---|---|---|---|
| 1 | 0 | 3 | _5_ | 2 |
| 2 | 2 | 0 | _3_ | 2 |
| 3 | 0 | 1 | _7_ | 3 |
| 4 | 3 | 2 | _3_ | 0 |

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | _0_ | 3 | 2 | 2 |
| 2 | 2 | _0_ | 0 | 2 |
| 3 | 0 | 1 | 4 | 3 |
| 4 | 3 | 2 | _0_ | 0 |

# HUNGARIAN METHOD EXAMPLE 2

- A feasible assignment is not possible at this moment.

- In such a case, The procedure is to draw a *minimum* number of *lines* through some of the rows and columns, *Such that all zero values are crossed out*.

# HUNGARIAN METHOD EXAMPLE 2

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 3 | 2 | 2 |
| 2 | 2 | 0 | 0 | 2 |
| 3 | 0 | 1 | 4 | 3 |
| 4 | 3 | 2 | 0 | 0 |

The next step is to select the smallest uncrossed out element.  This element is **subtracted from every uncrossed out element** and **added to every element at the intersection** of two lines.

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | **0** | 2 | 1 | 1 |
| 2 | 3 | 0 | **0** | 2 |
| 3 | 0 | **0** | 3 | 2 |
| 4 | 4 | 2 | 0 | **0** |

# HUNGARIAN METHOD EXAMPLE 2

- We can now easily assign to the zero values. Solution is to assign (1 to 1), (2 to 3), (3 to 2) and (4 to 4).

- If drawing lines do not provide an easy solution, then we should perform the task of drwaing lines one more time.

- Actually, we should continue drawing lines until a feasible assignment is possible.