SIMPLEX METHOD (MATRIX FORM)
MATRIX ALGEBRA BASICS
REVISED SIMPLEX

Vectors

A special kind of matrix that plays an important role in matrix theory is the kind that has either a *single row* or a *single column*. Such matrices are often referred to as **vectors**. Thus

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]$$

is a **row vector**, and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(We use **boldface lowercase** letters to represent vectors**)**

is a **column vector**.

A **null vector 0** is either a row vector or a column vector whose elements are *all* 0s, that is,

$$\mathbf{0} = [0, 0, \ldots, 0] \qquad \text{or} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

One reason vectors play an important role in matrix theory is that any $m \times n$ matrix can be partitioned into either m row vectors or n column vectors, and important properties of the matrix can be analyzed in terms of these vectors.

Partitioning of matrices

Up to this point, matrices have been rectangular arrays of elements, each of which is a number. However, the notation and results are also valid if each element is itself a matrix.

For example, the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

may be written as

$$\mathbf{A} = [\mathbf{C}_1 \quad \mathbf{C}_2 \quad \mathbf{C}_3] \quad \text{where} \quad \mathbf{C}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \mathbf{C}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$\text{and} \quad \mathbf{C}_3 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

or as

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} \quad \text{where} \quad \mathbf{R}_1 = [a_{11} \quad a_{12} \quad a_{13}] \quad \text{and} \quad \mathbf{R}_2 = [a_{21} \quad a_{22} \quad a_{23}]$$

or as

$$\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2]$$

where

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

or where

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

The process of dividing a matrix into smaller matrices, or submatrices, is called partitioning and is usually denoted by a dotted line. The four partitions described would be denoted respectively as follows:

$$\left[\begin{array}{c:c:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]; \quad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hdashline a_{21} & a_{22} & a_{23} \end{array}\right]$$

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c:cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]$$

Matrix operations can then be performed with matrices whose elements are matrices, provided the rules of operation are valid for the given matrix and for the resulting submatrices.

Example : Calculate **AB**, given

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution. Partition the two matrices:

$$\mathbf{A} = \left[\begin{array}{c:cc} 6 & 0 & 1 \\ 2 & 1 & 0 \end{array}\right] = [\mathbf{A}_1 \quad \mathbf{A}_2] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc} 4 & 3 \\ \hdashline 0 & 1 \\ 1 & 0 \end{array}\right] = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = [4 \quad 3], \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$\mathbf{AB} = [\mathbf{A}_1 \quad \mathbf{A}_2] \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2$$

and

$$\mathbf{AB} = \left[\begin{bmatrix} 6 \\ 2 \end{bmatrix} [4 \quad 3] + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] = \left[\begin{bmatrix} 24 & 18 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$\mathbf{AB} = \begin{bmatrix} 25 & 18 \\ 8 & 7 \end{bmatrix}$$

Matrix Form of Linear Programming

Original Form of the Model

Maximize $Z = 3x_1 + 5x_2,$

subject to

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Augmented Form of the Model

Maximize $Z = 3x_1 + 5x_2,$

subject to

(1) $x_1 + x_3 = 4$

(2) $2x_2 + x_4 = 12$

(3) $3x_1 + 2x_2 + x_5 = 18$

and

$$x_j \ge 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

Maximize $Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \le \mathbf{b} \quad \text{and} \quad \mathbf{x} \ge \mathbf{0},$$

Maximize $Z = \mathbf{cx},$

subject to

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \ge \mathbf{0}$$

where $\mathbf{c}$ is the row vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$, $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors and $\mathbf{A}$ is the matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Maximize $Z = \mathbf{cx}$,

subject to

$\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$,

Maximize $Z = \mathbf{cx}$,

subject to

$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq \mathbf{0}$$

where $\mathbf{c}$ is the row vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$, $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors and $\mathbf{A}$ is the matrix

where $\mathbf{I}$ is the $m \times m$ identity matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$\begin{aligned} &(1) \quad x_1 + x_3 = 4 \\ &(2) \quad 2x_2 + x_4 = 12 \\ &(3) \quad 3x_1 + 2x_2 + x_5 = 18 \end{aligned}$$

and

$x_j \geq 0$, for $j = 1, 2, 3, 4, 5$.

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc:ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Solving for a Basic Feasible Solution

For initialization,

$$\text{Maximize} \quad Z = \mathbf{c}\mathbf{x},$$

subject to

$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix}\mathbf{x} \\ \mathbf{x}_s\end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix}\mathbf{x} \\ \mathbf{x}_s\end{bmatrix} \geq \mathbf{0}$$

For any iteration,

$$\text{Maximize} \quad Z = c_B \mathbf{x}_B + c_N \mathbf{x}_N$$

subject to

$$[\mathbf{B}, \mathbf{N}]\begin{bmatrix}\mathbf{x}_\mathbf{B} \\ \mathbf{x}_\mathbf{N}\end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix}\mathbf{x}_\mathbf{B} \\ \mathbf{x}_\mathbf{N}\end{bmatrix} \geq \mathbf{0}$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & \cdots & N_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ N_{m1} & N_{m2} & \cdots & N_{mn} \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix} \quad \mathbf{x}_\mathrm{N} = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}$$

Solving for a Basic Feasible Solution

For initialization,

Maximize $Z = \mathbf{cx}$,

subject to

$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\end{bmatrix} = \mathbf{b} \text{ and } \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\end{bmatrix} \geq \mathbf{0}$$

$$\mathbf{x}_B = \mathbf{x} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{I}^{-1}\mathbf{b} = \mathbf{c}_B \mathbf{b}$$

For any iteration,

Maximize $Z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N$

subject to

$$[\mathbf{B}, \mathbf{N}]\begin{bmatrix}\mathbf{x}_B\\ \mathbf{x}_N\end{bmatrix} = \mathbf{b} \text{ and } \begin{bmatrix}\mathbf{x}_B\\ \mathbf{x}_N\end{bmatrix} \geq \mathbf{0}$$

$$Z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N = \mathbf{c}_B \mathbf{x}_B = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$
$$\mathbf{B}\mathbf{x}_B = \mathbf{b} - \mathbf{N}\mathbf{x}_N$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & \cdots & N_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ N_{m1} & N_{m2} & \cdots & N_{mn} \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}$$

Example

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\mathbf{x}_B = \mathbf{x} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{I}^{-1}\mathbf{b} = \mathbf{c}_B \mathbf{b}$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

Iteration 0

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \ \text{so} \ \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

$$\mathbf{c}_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

Iteration 1

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$\text{so} \ \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}, \ \mathbf{c}_B = [0, 5, 0], \ \text{so} \ Z = [0, 5, 0]\begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.$$

Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \ \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \ \mathbf{c}_B = [0, 5, 3], \ Z = [0, 5, 3]\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

Matrix Form of the Set of Equations in the Simplex Tableau

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

For the original set of equations, the matrix form is

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.$$

For any iteration,

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix}$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	**0** $\mathbf{I}$	0 $\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

Example

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\mathbf{c} = [3, 5], \ [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right], \ \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

For Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \ \boxed{\mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}},$$

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}},$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \boxed{[0, \tfrac{3}{2}, 1]},$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = \boxed{[0, 0]}$$

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}}, \quad \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \boxed{36}.$$

$$\begin{bmatrix} 1 & \boxed{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}} & \boxed{\mathbf{c}_B\mathbf{B}^{-1}} \\ 0 & \boxed{\mathbf{B}^{-1}\mathbf{A}} & \boxed{\mathbf{B}^{-1}} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}} \\ \boxed{\mathbf{B}^{-1}\mathbf{b}} \end{bmatrix}.$$

$$\left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ \hline 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix}$$

Summary of the Revised Simplex Method

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
			Coefficient of:						
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Z	Original Variables	Slack Variables	Right Side
			Coefficient of:			
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 0	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 0	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

1. Initialization (Iteration 0)

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc:ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}$$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

$$\mathbf{c}_B = [0, 0, 0], \qquad \text{so} \qquad Z = [0, 0, 0]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

Optimality test:

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 0, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - [3, 5] = [-3, -5]$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

2. Iteration 1

Step 1: Determine the entering basic variable

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}=[0,0,0]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\\3&2\end{bmatrix}-[3,5]=[-3,-5]$$

$$-c_2=-5<-3=-c_1$$

So x_2 is chosen to be the entering variable.

Step 2: Determine the leaving basic variable

$$\mathbf{B}^{-1}\mathbf{A}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}—&0\\—&2\\—&2\end{bmatrix}=\begin{bmatrix}—&0\\—&2\\—&2\end{bmatrix}$$

$$a_{12}=0 \qquad a_{22}=2 \qquad a_{32}=2$$

$$b_2/a_{22}=\frac{12}{2} \qquad b_3/a_{32}=\frac{18}{2}$$

So the number of the pivot row r =2

Thus, x_4 is chosen to be the leaving variable.

Step 3: Determine the new BF solution

The new set of basic variables is $\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}$

To obtain the new $\mathbf{B}^{-1}$, $\boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a_{12}}{a_{22}} \\ \dfrac{1}{a_{22}} \\ -\dfrac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{2} \\ -1 \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1},$$

So the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	**0** $\mathbf{I}$	0 $\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

Optimality test:

The nonbasic variables are x_1 and x_4.

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}=[0,5,0]\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & \text{—}\\ 0 & \text{—}\\ 3 & \text{—}\end{bmatrix}-[3,\text{—}]=[-3,\text{—}],$$

$$\mathbf{c}_B\mathbf{B}^{-1}=[0,5,0]\begin{bmatrix}\text{—} & 0 & \text{—}\\ \text{—} & \frac{1}{2} & \text{—}\\ \text{—} & -1 & \text{—}\end{bmatrix}=[\text{—},\tfrac{5}{2},\text{—}],$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of:			Right Side
			Z	Original Variables	Slack Variables	
0	Z $\mathbf{x}_B$	(0) $(1, 2, \ldots, m)$	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	**0** $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) $(1, 2, \ldots, m)$	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

3. Iteration 2

Step 1: Determine the entering basic variable

x_1 is chosen to be the entering variable.

Step 2: Determine the leaving basic variable

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & - \\ 0 & - \\ 3 & - \end{bmatrix} = \begin{bmatrix} 1 & - \\ 0 & - \\ 3 & - \end{bmatrix}. \qquad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Coefficient of: Original Variables	Coefficient of: Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) $(1, 2, \ldots, m)$	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	**0** $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) $(1, 2, \ldots, m)$	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

The ratio 4/1 > 6/3 indicate that x5 is the leaving basic variable

Step 3: Determine the new BF solution

The new set of basic variables is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a'_{11}}{a'_{31}} \\ -\dfrac{a'_{21}}{a'_{31}} \\ \dfrac{1}{a'_{31}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{3} \\ 0 \\ \dfrac{1}{3} \end{bmatrix}. \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore, the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}.$$

Optimality test:

The nonbasic variables are x_4 and x_5.

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} - & \frac{1}{3} & -\frac{1}{3} \\ - & \frac{1}{2} & 0 \\ - & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [-, \tfrac{3}{2}, 1].$$

Relationship between the initial and final simplex tableaux

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Z	Coefficient of: Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Initial Tableau

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $$\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$$

Combined: $$\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$$

Final Tableau

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*].$

Other rows: $$\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*].$$

Combined: $$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right].$$

$$\mathbf{z}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} \qquad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1} \qquad Z^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{A}^* = \mathbf{B}^{-1}\mathbf{A} \qquad \mathbf{S}^* = \mathbf{B}^{-1} \qquad \mathbf{b}^* = \mathbf{B}^{-1}\mathbf{b}$$

$$(1)\quad \mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \mid \mathbf{y}^* \mid \mathbf{y}^*\mathbf{b}] = [\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \mid \mathbf{c}_B\mathbf{B}^{-1} \mid \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}]$$

$$(2)\quad \mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \mid \mathbf{S}^* \mid \mathbf{S}^*\mathbf{b}] = [\mathbf{B}^{-1}\mathbf{A} \mid \mathbf{B}^{-1} \mid \mathbf{B}^{-1}\mathbf{b}]$$

■ **TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Initial Tableau

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$

For iteration 1:

$$\mathbf{S}^* = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{y}^* = [0, \ \tfrac{5}{2}, \ 0]$$

$$\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [-3, \ -5 \mid 0, \ 0, \ 0 \mid 0] + [0, \ \tfrac{5}{2}, \ 0]\left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [-3, \ 0, \ \boxed{0, \ \tfrac{5}{2}, \ 0}, \ 30]$$

$$\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 3 & 0 & 0 & -1 & 1 & 6 \end{array}\right]$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Initial Tableau

Row 0: $\mathbf{t} = [-3, \ -5 \ \vdots \ 0, \ 0, \ 0 \ \vdots \ 0] = [-\mathbf{c} \ \vdots \ \mathbf{0} \ \vdots \ 0].$

Other rows: $\mathbf{T} = \left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \ \vdots \ \mathbf{I} \ \vdots \ \mathbf{b}].$

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c:c:c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$

For iteration 2:

$$\mathbf{S}^* = \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad \mathbf{y}^* = [0, \ \tfrac{3}{2}, \ 1]$$

$$\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [-3, \ -5 \ \vdots \ 0, \ 0, \ 0 \ \vdots \ 0] + [0, \ \tfrac{3}{2}, \ 1]\left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [0, \ 0, \ \boxed{0, \ \tfrac{3}{2}, \ 1}, \ 36]$$

$$\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\left[\begin{array}{cc:ccc:c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = \begin{bmatrix} 0 & 0 & \boxed{\begin{matrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{matrix}} & 2 \\ 0 & 1 & & 6 \\ 1 & 0 & & 2 \end{bmatrix}$$

- Computationally effective modification of the standard Simplex Method.
- Unfortunately our examples are too small to make the computational savings significant. You may even get an impression that it is more time consuming.
- However for large problems, particularly where n is much greater than m, the saving could be significant.

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- Hughes-McMakee-notes\chapter-05.pdf