Optimization Techniques

MATHEMATICAL METHODS OF OPTIMIZATION

Nonlinear Programming

Optimization Techniques

INTRODUCTION TO NONLINEAR PROGRAMMING

- Many business problems can be modeled only with nonlinear functions.
- Problems that fit the general linear programming format but contain nonlinear functions are termed nonlinear programming (NLP) problems.
- There is no single method that can solve general nonlinear programmming
- Solution methods are more complex than linear programming methods.
- Often difficult, if not impossible, to determine optimal solution.
- Solution techniques generally involve searching a solution surface for high or low points requiring the use of advanced mathematics.

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Optimal Value of a Single Nonlinear Function Basic Model

Profit function, Z, with volume independent of price:

$$Z = vp - c_f - vc_v$$

where v = sales volume

p = price

c_f = fixed cost

c_v = variable cost

Add volume/price relationship:

$$v = 1{,}500 - 24.6p$$

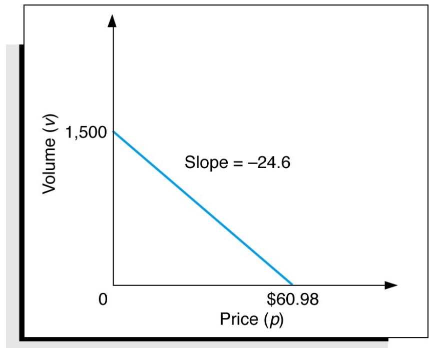

Figure 11.1
Linear Relationship of Volume to Price

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Optimal Value of a Single Nonlinear Function Expanding the Basic Model to a Nonlinear Model

With fixed cost (c_f = \$10,000) and variable cost (c_v = \$8):

$$Z = 1{,}696.8p - 24.6p^2 - 22{,}000$$

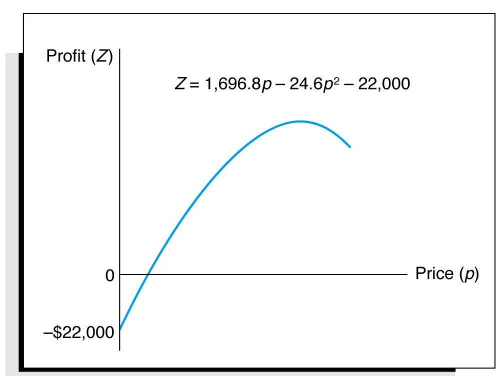

Figure 11.2
The Nonlinear Profit Function

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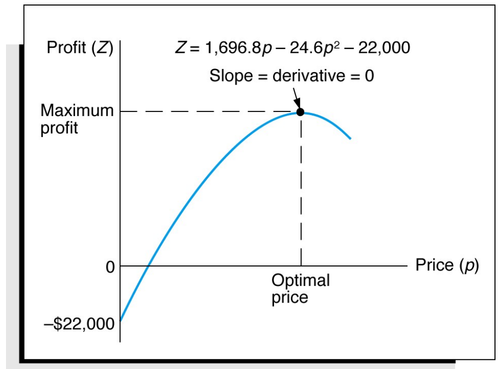
Optimal Value of a Single Nonlinear Function Maximum Point on a Curve

- The slope of a curve at any point is equal to the derivative of the curve's function.
- The slope of a curve at its highest point equals zero.

Figure 11.3
Maximum profit for the profit function



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Optimal Value of a Single Nonlinear Function Solution Using Calculus

nonlinearity!

$Z = 1{,}696.8 \cdot p - 24.6 \cdot p^2 - 22{,}000$

$dZ/dp = 1{,}696.8 - 49.2p$

$p = \$34.49$

$v = 1{,}500 - 24.6p$

$v = 651.6$ pairs of jeans

$Z = \$7{,}259.45$

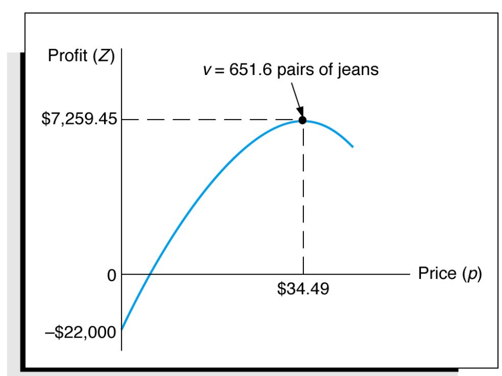

Figure 11.4
Maximum Profit, Optimal Price, and Optimal Volume

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Constrained Optimization in Nonlinear Problems Definition

- If a nonlinear problem contains one or more constraints it becomes a constrained optimization model or a *nonlinear programming* model.
- A nonlinear programming model has the same general form as the linear programming model except that the objective function *and/or* the constraint(s) are nonlinear.
- Solution procedures are much more complex and no guaranteed procedure exists.

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Constrained Optimization in Nonlinear Problems Graphical Interpretation (1 of 3)

- Effect of adding constraints to nonlinear problem:

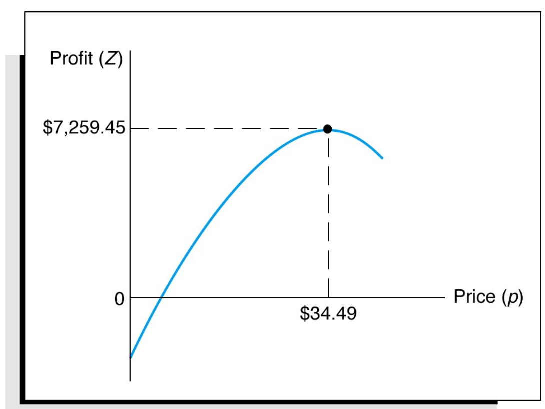

Figure 11.5
Nonlinear Profit Curve for the Profit Analysis Model

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Constrained Optimization in Nonlinear Problems Graphical Interpretation (2 of 3)

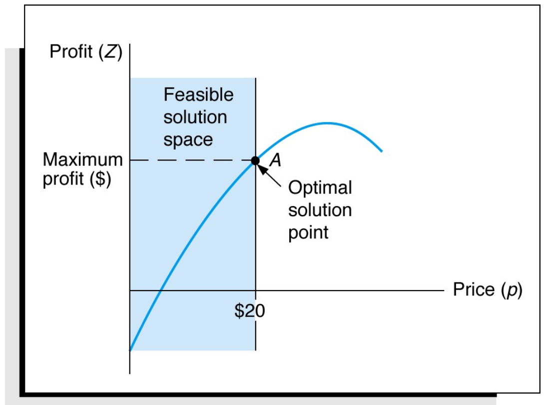

Figure 11.6
A Constrained Optimization Model

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Constrained Optimization in Nonlinear Problems Graphical Interpretation (3 of 3)

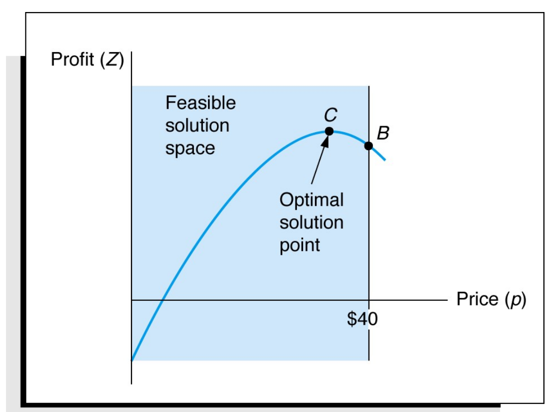

Figure 11.7
A Constrained Optimization Model with a Solution Point Not on the Constraint Boundary

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Constrained Optimization in Nonlinear Problems Characteristics

- Unlike linear programming, solution is often not on the boundary of the feasible solution space.
- Cannot simply look at points on the solution space boundary but must consider other points on the surface of the objective function.
- This greatly complicates solution approaches.
- Solution techniques can be very complex.

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INTRODUCTION TO NONLINEAR PROGRAMMING

- We optimized one-variable nonlinear functions using the 1st and 2nd derivatives.
- We will use the same concept here extended to functions with more than one variable.

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MULTIVARIABLE UNCONSTRAINED OPTIMIZATION

- For functions with one variable, we use the 1st and 2nd derivatives.
- For functions with multiple variables, we use identical information that is the gradient and the Hessian.
- The gradient is the first derivative with respect to all variables whereas the Hessian is the equivalent of the second derivative

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THE GRADIENT

- Review of the gradient (∇):

For a function "f", of variables $x_1, x_2, \ldots, x_n$:

$$\nabla f = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right]$$

Example: $f = 15x_1 + 2(x_2)^3 - 3x_1(x_3)^2$

$$\nabla f = \left[15 - 3(x_3)^2 \quad 6(x_2)^2 \quad -6x_1x_3 \right]$$

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THE HESSIAN

- The Hessian (∇^2) of $f(x_1, x_2, \ldots, x_n)$ is:

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^{\;2}} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^{\;2}} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^{\;2}} \end{bmatrix}$$

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HESSIAN EXAMPLE

- Example (from previously):

$$f = 15x_1 + 2(x_2)^3 - 3x_1(x_3)^2$$

$$\nabla f = \begin{bmatrix} 15 - 3(x_3)^2 & 6(x_2)^2 & -6x_1x_3 \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} 0 & 0 & -6x_3 \\ 0 & 12x_2 & 0 \\ -6x_3 & 0 & -6x_1 \end{bmatrix}$$

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UNCONSTRAINED OPTIMIZATION

The optimization procedure for multivariable functions is:

1. Solve for the gradient of the function equal to zero to obtain candidate points.
2. Obtain the Hessian of the function and evaluate it at each of the candidate points
   - If the result is "positive definite" (defined later) then the point is a local minimum.
   - If the result is "negative definite" (defined later) then the point is a local maximum.

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POSITIVE/NEGATIVE DEFINITE

- A matrix is "positive definite" if all of the eigenvalues of the matrix are positive (> 0)

- A matrix is "negative definite" if all of the eigenvalues of the matrix are negative (< 0)

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POSITIVE/NEGATIVE SEMI-DEFINITE

- A matrix is "positive semi-definite" if all of the eigenvalues are non-negative (≥ 0)
- A matrix is "negative semi-definite" if all of the eigenvalues are non-positive (≤ 0)

Optimization Techniques

EIGEN VALUE AND EIGEN VECTOR DEFINITIONS

Definition 1: A nonzero vector x is an *eigenvector* (or *characteristic vector*) of a square matrix A if there exists a scalar λ such that Ax = λx. Then λ is an *eigenvalue* (or *characteristic value*) of A.

Note: The zero vector can not be an eigenvector even though A0 = λ0. But λ = 0 can be an eigenvalue.

Example:

$$Show\ x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} is\ an\ eigenvector\ for\ A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

$$Solution: Ax = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$But\ \ for\ \ \lambda = 0,\ \ \lambda x = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Thus, x\ is\ an\ eigenvector\ of\ A, and\ \lambda = 0\ is\ an\ eigenvalue.$$

Optimization Techniques

GEOMETRIC INTERPRETATION OF EIGENVALUES AND EIGENVECTORS

An $n \times n$ matrix A multiplied by $n \times 1$ vector x results in another $n \times 1$ vector y=Ax. Thus A can be considered as a transformation matrix.

In general, a matrix acts on a vector by changing both its magnitude and its direction. However, a matrix may act on certain vectors by changing only their magnitude, and leaving their direction unchanged (or possibly reversing it). These vectors are the eigenvectors of the matrix.

A matrix acts on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is the eigenvalue associated with that eigenvector.

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EIGENVALUES

Let x be an eigenvector of the matrix A. Then there must exist an eigenvalue λ such that $Ax = \lambda x$ or, equivalently,

$$Ax - \lambda x = 0 \quad \text{or}$$

$$(A - \lambda I)x = 0$$

If we define a new matrix $B = A - \lambda I$, then

$$Bx = 0$$

If B has an inverse then $x = B^{-1}0 = 0$. But an eigenvector cannot be zero.

Thus, it follows that x will be an eigenvector of A if and only if B does not have an inverse, or equivalently $\det(B)=0$, or

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation of A. Its roots determine the eigenvalues of A.

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EIGENVALUES: EXAMPLES

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: −1, − 2

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

λ = 2 is an eigenvector of multiplicity 3.

Optimization Techniques

EIGENVECTORS

To each distinct eigenvalue of a matrix A there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If λ_i is an eigenvalue then the corresponding eigenvector x_i is the solution of $(A - \lambda_i I)x_i = 0$

Example 1 (cont.):

$$\lambda = -1: (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2: (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

Optimization Techniques

EIGENVECTORS

Example 2 (cont.): Find the eigenvectors of

Recall that λ = 2 is an eigenvector of multiplicity 3.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solve the homogeneous linear system represented by

$$(2I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_1 = s, x_3 = t$. The eigenvectors of λ = 2 are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

s and *t* not both zero.

Optimization Techniques

PROPERTIES OF EIGENVALUES AND EIGENVECTORS

Definition: The trace of a matrix A, designated by tr(A), is the sum of the elements on the main diagonal.

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: A matrix is singular if and only if it has a zero eigenvalue (A square matrix that does not have a matrix inverse. A matrix is singular iff its determinant is 0).

Property 3: The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

Property 4: If λ is an eigenvalue of A and A is invertible, then 1/λ is an eigenvalue of matrix A^{-1}.

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PROPERTIES OF EIGENVALUES AND EIGENVECTORS

Property 5: If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA where k is any arbitrary scalar.

Property 6: If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for any positive integer k.

Property 8: If λ is an eigenvalue of A then λ is an eigenvalue of A^T.

Property 9: The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.

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LINEARLY INDEPENDENT EIGENVECTORS

Theorem: Eigenvectors corresponding to distinct (that is, different) eigenvalues are linearly independent.

Theorem: If λ is an eigenvalue of multiplicity k of an $n \times n$ matrix A then the number of linearly independent eigenvectors of A associated with λ is given by $m = n - r(A- \lambda I)$. Furthermore, $1 \le m \le k$.

Example 2 (cont.): The eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$ *s* and *t* not both zero.

$\lambda = 2$ has *two linearly independent eigenvectors*

Optimization Techniques

EXAMPLE MATRIX

Given the matrix A:

$$A = \begin{bmatrix} 2 & 4 & 5 \\ -5 & -7 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

The eigenvalues of A are:

$\lambda_1 = -3.702$ $\lambda_2 = -2$ $\lambda_3 = 2.702$

This matrix is negative definite

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UNCONSTRAINED NLP EXAMPLE

Consider the problem:

Minimize $f(x_1, x_2, x_3) = (x_1)^2 + x_1(1 - x_2) + (x_2)^2 - x_2x_3 + (x_3)^2 + x_3$

First, we find the gradient with respect to x_i:

$$\nabla f = \begin{bmatrix} 2x_1 + 1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 + 1 \end{bmatrix}$$

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UNCONSTRAINED NLP EXAMPLE

Next, we set the gradient equal to zero:

$$\nabla f = 0 \qquad \Rightarrow \qquad \begin{bmatrix} 2x_1 + 1 - x_2 \\ -x_1 + 2_2 - x_3 \\ -x_2 + 2x_3 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, we have a system of 3 equations and 3 unknowns. When we solve, we get:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

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UNCONSTRAINED NLP EXAMPLE

So we have only one candidate point to check.

Find the Hessian:

$$\nabla^2 f = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Optimization Techniques

UNCONSTRAINED NLP EXAMPLE

The eigenvalues of this matrix are:

$$\lambda_1 = 3.414 \qquad \lambda_2 = 0.586 \qquad \lambda_3 = 2$$

All of the eigenvalues are > 0, so the Hessian is positive definite.

So, the point $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ is a minimum

Optimization Techniques

UNCONSTRAINED NLP EXAMPLE

Unlike in Linear Programming, unless we know the shape of the function being minimized or can determine whether it is convex, <u>we cannot tell whether this point is the global minimum</u> or if there are function values smaller than it.

Optimization Techniques

9.2: Taylor Series

Brook Taylor
1685 - 1731

Brook Taylor was an accomplished musician and painter. He did research in a variety of areas, but is most famous for his development of ideas regarding infinite series.

Greg Kelly, Hanford High School, Richland, Washington

Optimization Techniques

Suppose we wanted to find a fourth degree polynomial of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

that approximates the behavior of $f(x) = \ln(x+1)$ at $x = 0$

If we make $P(0) = f(0)$, and the first, second, third and fourth derivatives the same, then we would have a pretty good approximation.

→

Optimization

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \qquad f(x) = \ln(x+1)$$

$$f(x) = \ln(x+1) \qquad P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(0) = \ln(1) = 0 \qquad P(0) = a_0 \rightarrow \boxed{a_0 = 0}$$

$$f'(x) = \frac{1}{1+x} \qquad P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$f'(0) = \frac{1}{1} = 1 \qquad P'(0) = a_1 \rightarrow \boxed{a_1 = 1}$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$f''(0) = -\frac{1}{1} = -1 \qquad P''(0) = 2a_2 \rightarrow \boxed{a_2 = -\frac{1}{2}}$$

→

Optimization

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \qquad f(x) = \ln(x+1)$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$f''(0) = -\frac{1}{1} = -1 \qquad P''(0) = 2a_2 \rightarrow \boxed{a_2 = -\frac{1}{2}}$$

$$f'''(x) = 2 \cdot \frac{1}{(1+x)^3} \qquad P'''(x) = 6a_3 + 24a_4 x$$

$$f'''(0) = 2 \qquad P'''(0) = 6a_3 \rightarrow \boxed{a_3 = \frac{2}{6}}$$

$$f^{(4)}(x) = -6\frac{1}{(1+x)^4} \qquad P^{(4)}(x) = 24a_4$$

$$f^{(4)}(0) = -6 \qquad P^{(4)}(0) = 24a_4 \rightarrow \boxed{a_4 = -\frac{6}{24}}$$

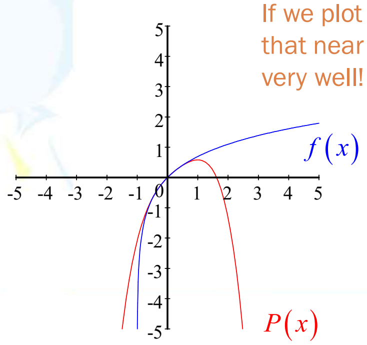
→

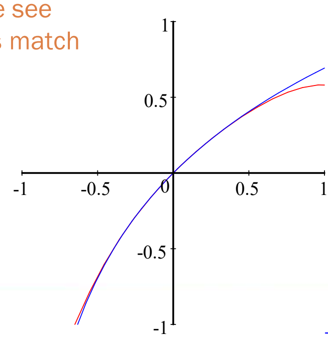
$$P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4 \qquad f(x)=\ln(x+1)$$

$$P(x)=0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4$$

$$P(x)=0+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4} \qquad f(x)=\ln(x+1)$$

If we plot both functions, we see that near zero the functions match very well!





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Our polynomial: $$0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4$$

has the form: $$f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3+\frac{f^{(4)}(0)}{24}x^4$$

or: $$\frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

This pattern occurs no matter what the original function was!

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Optimization

Maclaurin Series:

(generated by f at $x = 0$)

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

If we want to center the series (and it's graph) at some point other than zero, we get the Taylor Series:

Taylor Series:

(generated by f at $x = a$)

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

→

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example: $y = \cos x$

$f(x) = \cos x \quad f(0) = 1$

$f'(x) = -\sin x \quad f'(0) = 0$

$f''(x) = -\cos x \quad f''(0) = -1$

$f'''(x) = \sin x \quad f'''(0) = 0$

$f^{(4)}(x) = \cos x \quad f^{(4)}(0) = 1$

$$P(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \cdots$$

$$\boxed{P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots}$$

→

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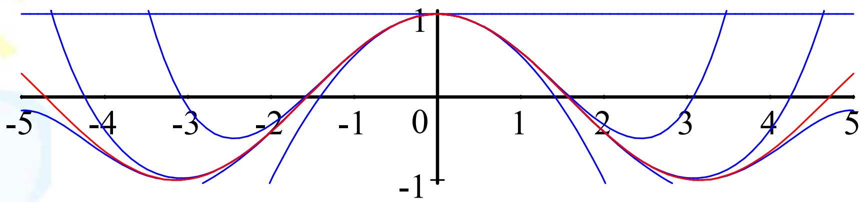
$y = \cos x$

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots$$



The more terms we add, the better our approximation.

Hint: On the TI-89, the factorial symbol is:

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example: $y = \cos(2x)$

Rather than start from scratch, we can use the function that we already know:

$$P(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \frac{(2x)^{10}}{10!} \cdots$$

→

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example: $y=\cos(x)$ at $x=\frac{\pi}{2}$

$$f(x)=\cos x \quad f\left(\frac{\pi}{2}\right)=0$$

$$f'(x)=-\sin x \quad f'\left(\frac{\pi}{2}\right)=-1$$

$$f''(x)=-\cos x \quad f''\left(\frac{\pi}{2}\right)=0$$

$$f'''(x)=\sin x \quad f'''\left(\frac{\pi}{2}\right)=1$$

$$f^{(4)}(x)=\cos x \quad f^{(4)}\left(\frac{\pi}{2}\right)=0$$

$$P(x)=0-1\left(x-\frac{\pi}{2}\right)+\frac{0}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+\cdots$$

$$\boxed{P(x)=-\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}+\cdots}$$

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There are some Maclaurin series that occur often enough that they should be memorized. They are on your formula sheet.

→

Optimization

When referring to Taylor polynomials, we can talk about **number of terms**, **order** or **degree**.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is a polynomial in **3 terms**.

It is a **4th order** Taylor polynomial, because it was found using the 4th derivative.

It is also a **4th degree** polynomial, because x is raised to the 4th power.

The **3rd order** polynomial for $\cos x$ is $1 - \frac{x^2}{2!}$, but it is **degree 2**.

A recent AP exam required the student to know the difference between *order* and *degree*.

→

Optimization

Tech

The TI-89 finds Taylor Polynomials:

taylor (expression, variable, order, [point])

$\underbrace{\text{F3} \quad 9}$

F3 9

taylor $\left(\cos(x), x, 6\right)$ $\quad \frac{-x^6}{720} + \frac{x^4}{24} - \frac{x^2}{2} + 1$

taylor $\left(\cos(2x), x, 6\right)$ $\quad \frac{-4x^6}{45} + \frac{2x^4}{3} - 2x^2 + 1$

taylor $\left(\cos(x), x, 5, \pi/2\right)$ $\quad \frac{-(2x-\pi)^5}{3840} + \frac{(2x-\pi)^3}{48} - \frac{2x-\pi}{2}$

π

Optimization Techniques

Summary

- A first order approximation of $f(x)$ **around point** $\bar{x}$ is given by
- $f_1(x) = f(x') + f'(x') * (x - x')$
- *Taylor Approximation around a vector x*
- $f_1(\boldsymbol{x}) = f(\boldsymbol{x}^k) + \nabla f(\boldsymbol{x}^k)^T(\boldsymbol{x} - \boldsymbol{x}^k)$
- $f1(\boldsymbol{x}) = f(\boldsymbol{x}^k) + \sum_{j=1}^{n} \frac{\partial f(x^k)}{\partial x_j}(x_j - x_j^k)$
- Determine first order Taylor approximation of the function
  - $f(x_1, x_2) = x_1^4 + x_1^2 + 2x_2^2 - 2x_1x_2$ around a point (x_1, x_2)=(1,1)

Optimization Techniques

Summary

- A second order approximation of $f(x)$ **around point** $\bar{x}$ for single variable is given by
- $f_2(x) = f(x') + f'(x')(x - x') + \frac{1}{2} f''(x')(x - x')^2$
- *A second order Taylor Approximation around a vector x*
- $f_2(\boldsymbol{x}) = f(\boldsymbol{x}^k) + \nabla f(\boldsymbol{x}^k)^T(\boldsymbol{x} - \boldsymbol{x}^k) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^k)^T H(\boldsymbol{x}^k)(\boldsymbol{x} - \boldsymbol{x}^k)$
- $f1(\boldsymbol{x}) = f(\boldsymbol{x}^k) + \sum_{j=1}^{n} \frac{\partial f(x^k)}{\partial x_j} * (x_j - x_j^k) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(x^k)}{\partial x_i \partial x_j}(x_i - x_i^k)(x_j - x_j^k)$

$f(x_1, x_2) = x_1^4 + x_1^2 + 2x_2^2 - 2x_1x_2$ around a point (x_1, x_2)=(1,1)

Optimization Techniques

METHOD OF SOLUTION

- In the previous example, when we set the gradient equal to zero, we had a system of 3 linear equations & 3 unknowns.
- For other problems, these equations could be nonlinear.
- Thus, the problem can become trying to solve a system of nonlinear equations, which can be very difficult.

Optimization Techniques

METHOD OF SOLUTION

- To avoid this difficulty, NLP problems are usually solved numerically.
- We will now look at examples of numerical methods used to find the optimum point for single-variable NLP problems. These and other methods may be found in any numerical methods reference.

Optimization Techniques

NEWTON'S METHOD

When solving the equation $f'(x) = 0$ to find a minimum or maximum, one can use the iteration step:

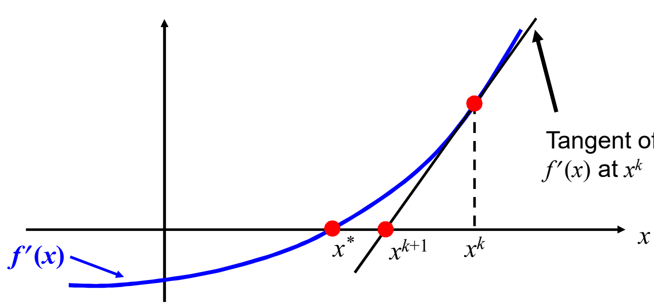
$$x^{k+1} = x^k - \frac{f'(x^k)}{f''(x^k)}$$

where k is the current iteration.

Iteration is continued until $|x^{k+1} - x^k| < \varepsilon$ where ε is some specified tolerance.

Optimization Techniques

NEWTON'S METHOD DIAGRAM



Newton's Method approximates $f'(x)$ as a straight line at x^k and obtains a new point (x^{k+1}), which is used to approximate the function at the next iteration. This is carried on until the new point is sufficiently close to x^*.

Optimization Techniques

NEWTON'S METHOD COMMENTS

- One must ensure that $f(x^{k+1}) < f(x^k)$ for finding a minimum and $f(x^{k+1}) > f(x^k)$ for finding a maximum.
- Disadvantages:
  - The initial guess is very important – if it is not close enough to the solution, the method may not converge
  - Both the first and second derivatives must be calculated

Optimization Techniques

REGULA-FALSI METHOD

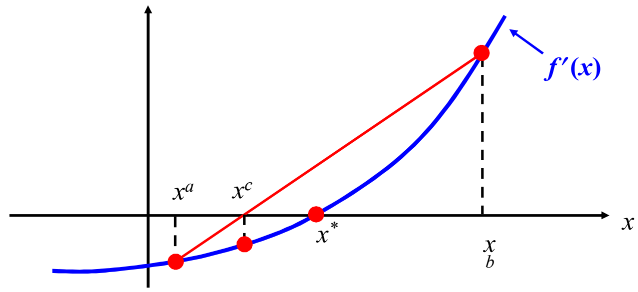
This method requires two points, x^a & x^b that bracket the solution to the equation $f'(x) = 0$.

$$x^c = x^b - \frac{f'(x^b) \cdot (x^b - x^a)}{f'(x^b) - f'(x^a)}$$

where x^c will be between x^a & x^b. The next interval will be x^c and either x^a or x^b, whichever has the sign opposite of x^c.

Optimization Techniques

REGULA-FALSI DIAGRAM



The Regula-Falsi method approximates the function $f'(x)$ as a straight line and interpolates to find the root.

Optimization Techniques

REGULA-FALSI COMMENTS

- This method requires initial knowledge of two points bounding the solution
- However, it does not require the calculation of the second derivative
- The Regula-Falsi Method requires slightly more iterations to converge than the Newton's Method