

# **TRANSPORTATION AND ASSIGNMENT PROBLEMS**

- The Transportation Model
- Solution of a Transportation Problem
- The Assignment Model
- Solution of the Assignment Model

# TRANSPORTATION AND ASSIGNMENT PROBLEMS

## OVERVIEW

- Part of a larger class of linear programming problems known as network flow models.
- Possess special mathematical features that enabled development of very efficient, unique solution methods.
- Methods are variations of traditional simplex procedure.

## THE TRANSPORTATION MODEL CHARACTERISTICS

- A product is transported from a number of sources to a number of destinations at the minimum possible cost.
- Each source is able to supply a fixed number of units of the product, and each destination has a fixed demand for the product.
- The linear programming model has constraints for supply at each source and demand at each destination.
- All constraints are equalities in a balanced transportation model where supply equals demand.
- Constraints contain inequalities in unbalanced models where supply does not equal demand.

# TRANSPORTATION MODEL EXAMPLE

## PROBLEM DEFINITION AND DATA

- Problem: How many tons of wheat to transport from each grain elevator to each mill on a monthly basis in order to minimize the total cost of transportation ?

- Data:

<u>Grain Elevator</u>	<u>Supply</u>	<u>Mill</u>	<u>Demand</u>
1. Kansas City	150	A. Chicago	200
2. Omaha	175	B. St. Louis	100
3. Des Moines	275	C. Cincinnati	300
Total	600 tons	Total	600 tons

Grain Elevator	Transport cost from Grain Elevator to Mill (\$/ton)		
	A. Chicago	B. St. Louis	C. Cincinnati
1. Kansas City	\$6	8	10
2. Omaha	7	11	11
3. Des Moines	4	5	12

# TRANSPORTATION MODEL EXAMPLE

## MODEL FORMULATION

$$\text{minimize } Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$$

$$\text{subject to } x_{1A} + x_{1B} + x_{1C} = 150$$

$$x_{2A} + x_{2B} + x_{2C} = 175$$

$$x_{3A} + x_{3B} + x_{3C} = 275$$

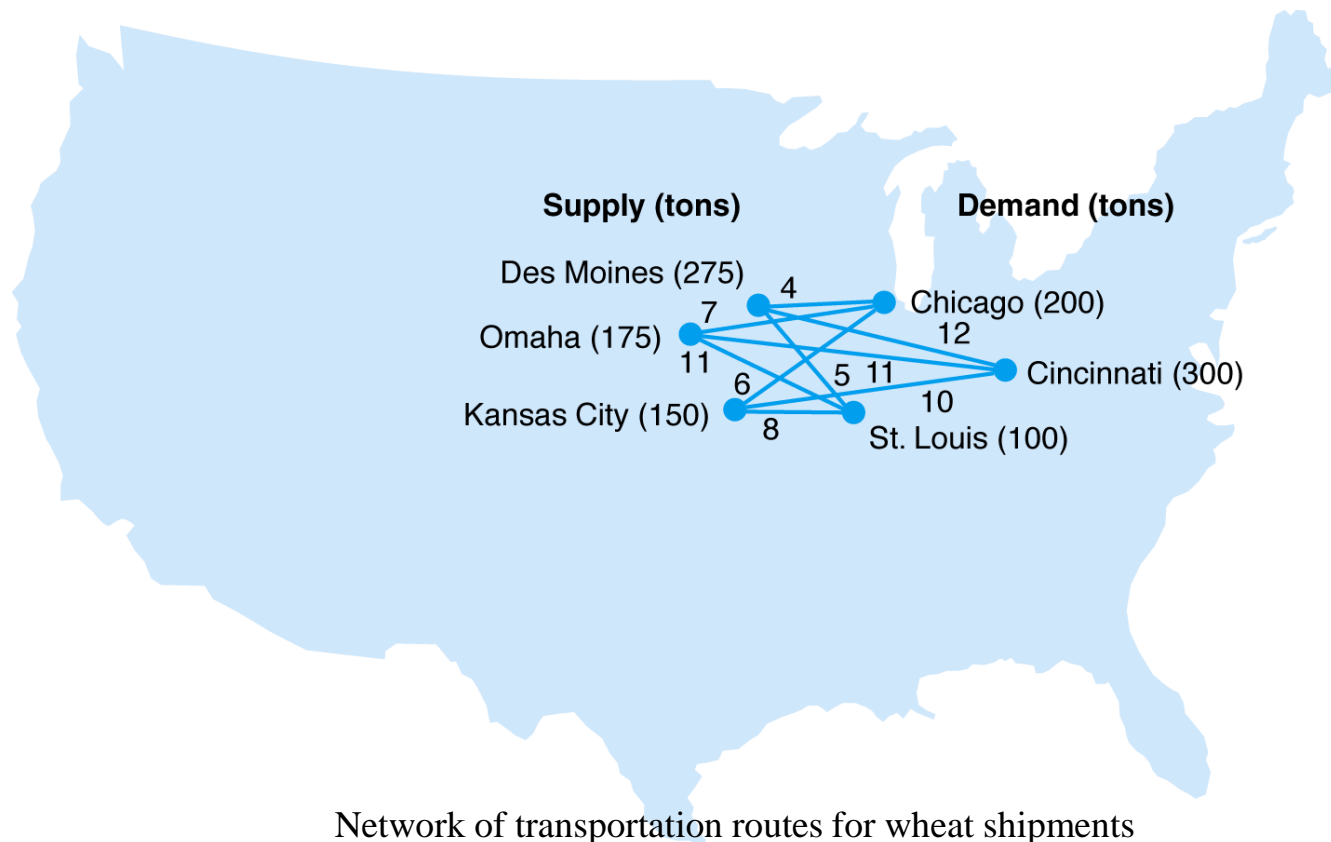
$$x_{1A} + x_{2A} + x_{3A} = 200$$

$$x_{1B} + x_{2B} + x_{3B} = 100$$

$$x_{1C} + x_{2C} + x_{3C} = 300$$

$$x_{ij} \geq 0$$

where  $x_{ij}$  = tons of wheat  
from each grain elevator,  $i$ ,  
 $i = 1, 2, 3$ , to each mill  $j$ ,  $j$   
= A,B,C



# SOLUTION OF THE TRANSPORTATION MODEL

## TABLEAU FORMAT

- Transportation problems are solved manually within a *tableau* format.
- Each cell in a transportation tableau is analogous to a decision variable that indicates the amount allocated from a source to a destination.
- The supply and demand values along the outside rim of a tableau are called *rim values*.

To From	A		B		C		Supply
1		6		8		10	150
2		7		11		11	175
3		4		5		12	275
<b>Demand</b>	200		100		300		600

The Transportation  
Tableau

# SOLUTION OF THE TRANSPORTATION MODEL

## SOLUTION METHODS

- Transportation models do not start at the origin where all decision values are zero; they must instead be given an *initial feasible solution*.
- Initial feasible solution determination methods include:
  - northwest corner method
  - minimum cell cost method
  - Vogel's Approximation Method
- Methods for solving the transportation problem itself include:
  - stepping-stone method and
  - modified distribution method.

# THE NORTHWEST CORNER METHOD

- In the northwest corner method the largest possible allocation is made to the cell in the upper left-hand corner of the tableau , followed by allocations to adjacent feasible cells.

The Initial NW Corner Solution

From \ To				Supply
	A	B	C	
1	6	8	10	150
	150			
2	7	11	11	175
	50	100	25	
3	4	5	12	275
			275	
<b>Demand</b>	200	100	300	600

- The initial solution is complete when all rim requirements are satisfied.
- Transportation cost is computed by evaluating the objective function:

$$\begin{aligned}
 Z &= \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C} \\
 &= 6(150) + 8(0) + 10(0) + 7(50) + 11(100) + 11(25) + 4(0) + 5(0) + 12(275) \\
 &= \$5,925
 \end{aligned}$$



# THE NORTHWEST CORNER METHOD

## SUMMARY OF STEPS

1. Allocate as much as possible to the cell in the upper left-hand corner, subject to the supply and demand conditions.
2. Allocate as much as possible to the next adjacent feasible cell.
3. Repeat step 2 until all rim requirements are met.

# THE MINIMUM CELL COST METHOD (1 OF 3)

- In the minimum cell cost method as much as possible is allocated to the cell with the minimum cost followed by allocation to the feasible cell with minimum cost.

<b>To</b> <b>From</b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>Supply</b>
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
<b>Demand</b>	200	100	300	600

The Initial Minimum Cell Cost Allocation

<b>To</b> <b>From</b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>Supply</b>
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
<b>Demand</b>	200	100	300	600

The Second Minimum Cell Cost Allocation

# THE MINIMUM CELL COST METHOD (2 OF 3)

- The complete initial minimum cell cost solution; total cost = \$4,550.
- The minimum cell cost method will provide a solution with a lower cost than the northwest corner solution because it considers cost in the allocation process.

From \ To				Supply
	A	B	C	
1	6	8	10	150
		25	125	
2	7	11	11	175
			175	
3	4	5	12	275
	200	75		
Demand	200	100	300	600

The Initial Solution

## THE MINIMUM CELL COST METHOD

### SUMMARY OF STEPS

(3 OF 3)

1. Allocate as much as possible to the feasible cell with the minimum transportation cost, and adjust the rim requirements.
2. Repeat step 1 until all rim requirements have been met.

# VOGEL'S APPROXIMATION METHOD (VAM)

(1 OF 5)

- Method is based on the concept of *penalty cost* or *regret*.
- A penalty cost is the difference between the largest and the next largest cell cost in a row (or column).
- In VAM the first step is to develop a penalty cost for each source and destination.
- Penalty cost is calculated by subtracting the minimum cell cost from the next higher cell cost in each row and column.

<div><div></div><div>To</div></div> <div>From</div>	A		B		C		Supply
1		6		8		10	150
2		7		11		11	175
3		4		5		12	275
Demand	200		100		300		600
	2		3		1		

The VAM Penalty Costs

# VOGEL'S APPROXIMATION METHOD (VAM)

## (2 OF 5)

- VAM allocates as much as possible to the minimum cost cell in the row or column with the largest penalty cost.

From \ To	A	B	C	Supply	
1	6	8	10	150	2
2	7	11	11	175	
3	4	5	12	275	1
<b>Demand</b>	200	100	300	600	
	2	3	2		

The Initial VAM  
Allocation

# VOGEL'S APPROXIMATION METHOD (VAM)

## (3 OF 5)

- After each VAM cell allocation, all row and column penalty costs are recomputed.

The Second  
M Allocation

<b>To</b> <b>From</b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>Supply</b>	
<b>1</b>	6	8	10	150	4
<b>2</b>	7	11	11	175	
<b>3</b>	4	5	12	275	8
<b>Demand</b>	200	100	300	600	
	2		2		

# VOGEL'S APPROXIMATION METHOD (VAM) (4 OF 5)

- Recomputed penalty costs after the third allocation.

From \ To	A		B		C		Supply
1		6		8		10	150
					150		
2		7		11		11	175
	175						
3		4		5		12	275
	25		100		150		
Demand	200		100		300		600

The Third VAM  
Allocation



# VOGEL'S APPROXIMATION METHOD (VAM) (5 OF 5)

- The initial VAM solution; total cost = \$5,125
- VAM and minimum cell cost methods both provide better initial solutions than does the northwest corner method.

From \ To				Supply
	A	B	C	
1	6 150	8 150	10 150	150
2	7 175	11 175	11 175	175
3	4 25	5 100	12 150	275
<b>Demand</b>	200	100	300	600

The Initial VAM  
Solution

## VOGEL'S APPROXIMATION METHOD (VAM)

### SUMMARY OF STEPS

1. Determine the penalty cost for each row and column.
2. Select the row or column with the highest penalty cost.
3. Allocate as much as possible to the feasible cell with the lowest transportation cost in the row or column with the highest penalty cost.
4. Repeat steps 1, 2, and 3 until all rim requirements have been met.

# THE STEPPING-STONE SOLUTION METHOD (1 OF 12)

- Once an initial solution is derived, the problem must be solved using either the stepping-stone method or the modified distribution method (MODI).
- The initial solution used as a starting point in this problem is the minimum cell cost method solution because it had the minimum total cost of the three methods used.

<div><div></div><div>To</div></div> <div>From</div>	A		B		C		Supply
1		6		8		10	150
			25		125		
2		7		11		11	175
					175		
3		4		5		12	275
	200		75				
Demand	200		100		300		600

The Minimum  
Cell Cost Solution

# THE STEPPING-STONE SOLUTION METHOD

## (2 OF 12)

- The stepping-stone method determines if there is a cell with no allocation that would reduce cost if used.

<div><div>To</div><div>From</div></div>	A		B		C		Supply
1	+1	6		8		10	150
2		7		11		11	175
3		4		5		12	275
Demand	200		100		300		600

The Allocation of One Ton to Cell 1A

# THE STEPPING-STONE SOLUTION METHOD (3 OF 12)

- Must subtract one ton from another allocation along that row.

From \ To	A		B		C		Supply
1	+1	6	-1	8		10	150
			25		125		
2		7		11		11	175
					175		
3		4		5		12	275
	200		75				
<b>Demand</b>	200		100		300		600

The Subtraction of  
One Ton from  
Cell 1B

# THE STEPPING-STONE SOLUTION METHOD (4 OF 12)

- A requirement of this solution method is that units can only be added to and subtracted from cells that already have allocations, thus one ton must be added to a cell as shown.

<div><div></div><div>To</div></div> <div>From</div>		A		B		C		Supply
1	↑	+1	6	→	-1		10	150
2			7				11	175
3	↓	-1	4	←	+1		12	275
Demand		200			100		300	600

The Addition of One Ton to Cell 3B and the Subtraction of One Ton from Cell 3A

# THE STEPPING-STONE SOLUTION METHOD (5 OF 12)

- An empty cell that will reduce cost is a potential entering variable.
- To evaluate the cost reduction potential of an empty cell, a closed path connecting used cells to the empty cells is identified.

To From		A	B	C	Supply
1		6	8	10	150
			25	125	
2		7	11	11	175
				175	
3		4	5	12	275
		200	75		
Demand		200	100	300	600

The Stepping-Stone Path for Cell 2A

$$2A \rightarrow 2C \rightarrow 1C \rightarrow 1B \rightarrow 3B \rightarrow 3A$$

$$+ \$7 - 11 + 10 - 8 + 5 - 4 = -\$1$$

# THE STEPPING-STONE SOLUTION METHOD (6 OF 12)

- The remaining stepping-stone paths and resulting computations for cells 2B and 3C.

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

2B → 2C → 1C → 1B  
 + \$11 - 11 + 10 - 8 = +\$2

The Stepping-Stone Path  
for Cell 2B

The Stepping-Stone Path for  
Cell 3C

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

3C → 1C → 1B → 3B  
 + \$12 - 10 + 8 - 5 = +\$5



# THE STEPPING-STONE SOLUTION METHOD (7 OF 12)

- After all empty cells are evaluated, the one with the greatest cost reduction potential is the entering variable.
- A tie can be broken arbitrarily.

From \ To				Supply
	A	B	C	
1	<div> <div>+</div> <div>←</div> <div>6</div> <div>→</div> <div>8</div> <div>→</div> <div>10</div> </div>	25	125	150
2	7	11	11	175
3	<div> <div>←</div> <div>4</div> <div>→</div> <div>5</div> <div>→</div> <div>12</div> </div>	75		275
Demand	200	100	300	600

The Stepping-Stone  
Path for Cell 1A

# THE STEPPING-STONE SOLUTION METHOD (8 OF 12)

- When reallocating units to the entering variable (cell), the amount is the minimum amount subtracted on the stepping-stone path.
- At each iteration one variable enters and one leaves (just as in the simplex method).

To From	A		B		C		Supply
1		6		8		10	150
	25				125		
2		7		11		11	175
					175		
3		4		5		12	275
	175		100				
Demand	200		100		300		600

The Second Iteration of  
the Stepping-Stone  
Method

# THE STEPPING-STONE SOLUTION METHOD

## (9 OF 12)

- Check to see if the solution is optimal.

From \ To	A	B	C	Supply
1	- ← 6 ↓ 25	8	+ 10 ↑ 125	150
2	+ 7	→ 11	- 11	175
3	4	5	12	275
Demand	200	100	300	600

$$2A \rightarrow 2C \rightarrow 1C \rightarrow 1A$$

$$+ \$7 - 11 + 10 - 6 = \$0$$

The Stepping-Stone Path for  
Cell 2A

The Stepping-  
Stone Path for Cell

1B

From \ To	A	B	C	Supply
1	- 6 ↑ 25	8	+ 10 ↓ 125	150
2	7	11	11	175
3	+ 4	→ 5	- 12	275
Demand	200	100	300	600

$$1B \rightarrow 3B \rightarrow 3A \rightarrow 1A$$

$$+ \$8 - 5 + 4 - 6 = +\$1$$

# THE STEPPING-STONE SOLUTION METHOD (10 OF 12)

- Continuing check for optimality.

From \ To	A	B	C	Supply
1	- 6 25	8	+ 10 125	150
2	7	+ 11	- 11 175	175
3	+ 4 175	- 5 100	12 275	275
<b>Demand</b>	200	100	300	600

$$2B \rightarrow 3B \rightarrow 3A \rightarrow 1A \rightarrow 1C \rightarrow 2C$$

$$+ \$11 - 5 + 4 - 6 + 10 - 11 = +\$3$$

The Stepping-Stone  
Path for Cell 2B

From \ To	A	B	C	Supply
1	+ 6 25	8	- 10 125	150
2	7	11	11 175	175
3	- 4 175	5 100	+ 12 275	275
<b>Demand</b>	200	100	300	600

The Stepping-Stone Path for Cell 3C

$$3C \rightarrow 3A \rightarrow 1A \rightarrow 1C$$

$$+ \$12 - 4 + 6 - 10 = +\$4$$

## THE STEPPING-STONE SOLUTION METHOD (11 OF 12)

- The stepping-stone process is repeated until none of the empty cells will reduce costs (i.e., an optimal solution).
- In example, evaluation of four paths indicates no cost reductions, therefore Table 19 solution is optimal.
- Solution and total minimum cost :

$$x_{1A} = 25 \text{ tons}, x_{2C} = 175 \text{ tons}, x_{3A} = 175 \text{ tons}, x_{1C} = 125 \text{ tons}, x_{3B} = 100 \text{ tons}$$

$$\begin{aligned} Z &= \$6(25) + 8(0) + 10(125) + 7(0) + 11(0) + 11(175) + 4(175) + 5(100) + 12(0) \\ &= \$4,525 \end{aligned}$$

# THE STEPPING-STONE SOLUTION METHOD (12 OF 12)

- A multiple optimal solution occurs when an empty cell has a cost change of zero and all other empty cells are positive.
- An alternate optimal solution is determined by allocating to the empty cell with a zero cost change.
- Alternate optimal total minimum cost also equals \$4,525.

The Alternative  
Optimal Solution

<b>To</b> <b>From</b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>Supply</b>
<b>1</b>	6	8	10	150
<b>2</b>	25	11	11	175
<b>3</b>	175	5	12	275
<b>Demand</b>	200	100	300	600

## THE STEPPING-STONE SOLUTION METHOD

### SUMMARY OF STEPS

1. Determine the stepping-stone paths and cost changes for each empty cell in the tableau.
2. Allocate as much as possible to the empty cell with the greatest net decrease in cost.
3. Repeat steps 1 and 2 until all empty cells have positive cost changes that indicate an optimal solution.

# THE MODIFIED DISTRIBUTION METHOD (MODI)

(1 OF 6)

- MODI is a modified version of the stepping-stone method in which math equations replace the stepping-stone paths.

- In the table, the extra left-hand column with the  $u_i$  symbols and the extra top row with the  $v_j$  symbols represent values that must be computed.

- Computed for all cells with allocations :

$$u_i + v_j = c_{ij} = \text{unit transportation cost for cell } ij.$$

The Minimum Cell Cost  
Initial Solution

	$v_j$	$v_A =$	$v_B =$	$v_C =$	
$u_i$	<div> <div>To</div> <div>From</div> </div>	A	B	C	Supply
$u_1 =$	1	6	8	10	150
$u_2 =$	2	7	11	11	175
$u_3 =$	3	4	5	12	275
	Demand	200	100	300	600



# THE MODIFIED DISTRIBUTION METHOD (MODI)

## (2 OF 6)

- Formulas for cells containing allocations:

$$x_{1B}: u_1 + v_B = 8$$

$$x_{1C}: u_1 + v_C = 10$$

$$x_{2C}: u_2 + v_C = 11$$

$$x_{3A}: u_3 + v_A = 4$$

$$x_{3B}: u_3 + v_B = 5$$

	$v_j$	$v_A = 7$	$v_B = 8$	$v_C = 10$	
$u_i$	<div> <div>To</div> <div>From</div> </div>	A	B	C	Supply
$u_1 = 0$	1	<div>6</div>	<div>8</div>	<div>10</div>	150
$u_2 = 1$	2	<div>7</div>	<div>11</div>	<div>11</div>	175
$u_3 = -3$	3	<div>4</div>	<div>5</div>	<div>12</div>	275
	<b>Demand</b>	200	100	300	600

The Initial Solution with All  $u_i$  and  $v_j$  Values

- Five equations with 6 unknowns, therefore let  $u_1 = 0$  and solve to obtain:

$$v_B = 8, v_C = 10, u_2 = 1, u_3 = -3, v_A = 7$$

## THE MODIFIED DISTRIBUTION METHOD (MODI)

(3 OF 6)

- Each MODI allocation replicates the stepping-stone allocation.
- Use following to evaluate all empty cells:

$$c_{ij} - u_i - v_j = k_{ij}$$

where  $k_{ij}$  equals the cost increase or decrease that would occur by allocating to a cell.

- For the empty cells in Table 26:

$$x_{1A}: k_{1A} = c_{1A} - u_1 - v_A = 6 - 0 - 7 = -1$$

$$x_{2A}: k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 7 = -1$$

$$x_{2B}: k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 8 = +2$$

$$x_{3C}: k_{3C} = c_{3C} - u_3 - v_C = 12 - (-3) - 10 = +5$$

# THE MODIFIED DISTRIBUTION METHOD (MODI)

(4 OF 6)

- After each allocation to an empty cell, the  $u_i$  and  $v_j$  values must be recomputed.

	$v_j$	$v_A =$	$v_B =$	$v_C =$	
$u_i$	<div> <div>To</div> <div>From</div> </div>	A	B	C	Supply
$u_1 =$	1	<div> <div>6</div> <div>25</div> </div>	<div> <div>8</div> </div>	<div> <div>10</div> <div>125</div> </div>	150
$u_2 =$	2	<div> <div>7</div> </div>	<div> <div>11</div> </div>	<div> <div>11</div> <div>175</div> </div>	175
$u_3 =$	3	<div> <div>4</div> <div>175</div> </div>	<div> <div>5</div> <div>100</div> </div>	<div> <div>12</div> </div>	275
	Demand	200	100	300	600

The Second Iteration of the MODI Solution Method

# THE MODIFIED DISTRIBUTION METHOD (MODI)

(5 OF 6)

- Recomputing  $u_i$  and  $v_j$  values:

$$x_{1A}: u_1 + v_A = 6, v_A = 6 \quad x_{1C}: u_1 + v_C = 10, v_C = 10 \quad x_{2C}: u_2 + v_C = 11, u_2 = 1$$

$$x_{3A}: u_3 + v_A = 4, u_3 = -2 \quad x_{3B}: u_3 + v_B = 5, v_B = 7$$

	$v_j$	$v_A = 6$	$v_B = 7$	$v_C = 10$	
$u_i$	<div>To From</div>	A	B	C	Supply
$u_1 = 0$	1	<div>6 25</div>	<div>8</div>	<div>10 125</div>	150
$u_2 = 1$	2	<div>7</div>	<div>11</div>	<div>11 175</div>	175
$u_3 = -2$	3	<div>4 175</div>	<div>5 100</div>	<div>12</div>	275
	Demand	200	100	300	600

The New  $u_i$  and  $v_j$  Values for the Second Iteration

## THE MODIFIED DISTRIBUTION METHOD (MODI) (6 OF 6)

- Cost changes for the empty cells,  $c_{ij} - u_i - v_j = k_{ij}$ ;

$$x_{1B}: k_{1B} = c_{1B} - u_1 - v_B = 8 - 0 - 7 = +1$$

$$x_{2A}: k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 6 = 0$$

$$x_{2B}: k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 7 = +3$$

$$x_{3C}: k_{2B} = c_{2B} - u_3 - v_C = 12 - (-2) - 10 = +4$$

- Since none of the values are negative, solution obtained is optimal.
- Cell 2A with a zero cost change indicates a multiple optimal solution.

## THE MODIFIED DISTRIBUTION METHOD (MODI) SUMMARY OF STEPS

1. Develop an initial solution.
2. Compute the  $u_i$  and  $v_j$  values for each row and column.
3. Compute the cost change,  $k_{ij}$ , for each empty cell.
4. Allocate as much as possible to the empty cell that will result in the greatest net decrease in cost (most negative  $k_{ij}$ )
5. Repeat steps 2 through 4 until all  $k_{ij}$  values are positive or zero.

# THE UNBALANCED TRANSPORTATION MODEL

## (1 OF 2)

- When demand exceeds supply a dummy row is added to the tableau.

From \ To				Supply
	A	B	C	
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Dummy	0	0	0	50
Demand	200	100	350	650

An Unbalanced Model  
(Demand . Supply)

# THE UNBALANCED TRANSPORTATION MODEL

## (2 OF 2)

- When supply exceeds demand, a dummy column is added to the tableau.
- The dummy column (or dummy row) has no effect on the initial solution methods or the optimal solution methods.

From \ To	A	B	C	Dummy	Supply
1	6	8	10	0	150
2	7	11	11	0	175
3	4	5	12	0	375
<b>Demand</b>	200	100	300	100	700

An Unbalanced Model (Supply . Demand)



# DEGENERACY

(1 OF 3)

- In a transportation tableau with  $m$  rows and  $n$  columns, there must be  $m + n - 1$  cells with allocations; if not, it is *degenerate*.
- The tableau in the figure does not meet the condition since  $3 + 3 - 1 = 5$  cells and there are only 4 cells with allocations.

From \ To				Supply
	A	B	C	
1	6	8	10	150
		100	50	
2	7	11	11	250
			250	
3	4	5	12	200
	200			
Demand	200	100	300	600

The Minimum Cell Cost Initial Solution

# DEGENERACY (2 OF 3)

- In a degenerate tableau, all the stepping-stone paths or MODI equations cannot be developed.
- To rectify a degenerate tableau, an empty cell must artificially be treated as an occupied cell.

The Initial Solution

<b>To</b> <b>From</b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>Supply</b>
<b>1</b>	0 <span>6</span>	100 <span>8</span>	50 <span>10</span>	150
<b>2</b>	<span>7</span>	<span>11</span>	250 <span>11</span>	250
<b>3</b>	200 <span>4</span>	<span>5</span>	<span>12</span>	200
<b>Demand</b>	200	100	300	600

# DEGENERACY (3 OF 3)

- The stepping-stone path s and cost changes for this tableau:

$$2A \ 2C \ 1C \ 1A$$

$$x_{2A}: \ 7 - 11 + 10 - 6 = 0$$

$$2B \ 2C \ 1C \ 1B$$

$$x_{2B}: \ 11 - 11 + 10 - 8 = +2$$

$$3B \ 1B \ 1A \ 3A$$

$$x_{3B}: \ 5 - 8 + 6 - 4 = -1$$

$$3C \ 1C \ 1A \ 3A$$

$$x_{3C}: \ 12 - 10 + 6 - 4 = +4$$

<b>To</b>				
<b>From</b>	<b>A</b>	<b>B</b>	<b>C</b>	<b>Supply</b>
		6	8	10
<b>1</b>	100		50	150
		7	11	11
<b>2</b>			250	250
		4	5	12
<b>3</b>	100	100		200
<b>Demand</b>	200	100	300	600

The Second Stepping-Stone Iteration

## PROHIBITED ROUTES

- A prohibited route is assigned a large cost such as  $M$ .
- When the prohibited cell is evaluated, it will always contain the cost  $M$ , which will keep it from being selected as an entering variable.

# THE ASSIGNMENT MODEL

## CHARACTERISTICS

- Special form of linear programming model similar to the transportation model.
- Supply at each source and demand at each destination limited to one unit.
- In a balanced model supply equals demand.
- In an unbalanced model supply does not equal demand.

# THE ASSIGNMENT MODEL

## EXAMPLE PROBLEM DEFINITION AND DATA

Problem: Assign four teams of officials to four games in a way that will minimize total distance traveled by the officials. Supply is always one team of officials, demand is for only one team of officials at each game.

Officials	Game Sites			
	RALEIGH	ATLANTA	DURHAM	CLEMSON
A	210	90	180	160
B	100	70	130	200
C	175	105	140	170
D	80	65	105	120

# THE ASSIGNMENT MODEL

## EXAMPLE PROBLEM MODEL FORMULATION

Minimize  $Z = 210x_{AR} + 90x_{AA} + 180x_{AD} + 160x_{AC} + 100x_{BR} + 70x_{BA} + 130x_{BD} + 200x_{BC} + 175x_{CR} + 105x_{CA} + 140x_{CD} + 170x_{CC} + 80x_{DR} + 65x_{DA} + 105x_{DD} + 120x_{DC}$

subject to

$$x_{AR} + x_{AA} + x_{AD} + x_{AC} = 1$$

$$x_{BR} + x_{BA} + x_{BD} + x_{BC} = 1$$

$$x_{CR} + x_{CA} + x_{CD} + x_{CC} = 1$$

$$x_{DR} + x_{DA} + x_{DD} + x_{DC} = 1$$

$$x_{AR} + x_{BR} + x_{CR} + x_{DR} = 1$$

$$x_{AA} + x_{BA} + x_{CA} + x_{DA} = 1$$

$$x_{AD} + x_{BD} + x_{CD} + x_{DD} = 1$$

$$x_{AC} + x_{BC} + x_{CC} + x_{DC} = 1$$

$$x_{ij} \geq 0$$

# SOLUTION OF THE ASSIGNMENT MODEL

## (1 OF 7)

- An *assignment problem* is a special form of the transportation problem where all supply and demand values equal one.
- Example: assigning four teams of officials to four games in a way that will minimize distance traveled by the officials.

Officials	Game Sites			
	Raleigh	Atlanta	Durham	Clemson
A	210	90	180	160
B	100	70	130	200
C	175	105	140	170
D	80	65	105	120

The Travel Distances to Each Game for Each Team of Officials



## SOLUTION OF THE ASSIGNMENT MODEL (2 OF 7)

- An *opportunity cost table* is developed by first subtracting the minimum value in each row from all other row values (*row reductions*) and then repeating this process for each column.

Officials	Game Sites			
	Raleigh	Atlanta	Durham	Clemson
A	120	0	90	70
B	30	0	60	130
C	70	0	35	65
D	15	0	40	55

The Assignment Tableau with Row Reductions

# SOLUTION OF THE ASSIGNMENT MODEL

## (3 OF 7)

- The minimum value in each column is subtracted from all column values (*column reductions*).
- Assignments can be made in the table wherever a zero is present.
- An *optimal solution* results when each of the four teams can be assigned to a different game.
- Table 36 does not contain an optimal solution

Officials	Game Sites			
	Raleigh	Atlanta	Dur ham	Clemson
A	105	0	55	15
B	15	0	25	75
C	55	0	0	10
D	0	0	5	0

The Tableau with Column Reductions

# SOLUTION OF THE ASSIGNMENT MODEL (4 OF 7)

- An optimal solution occurs when the number of independent unique assignments equals the number of rows and columns.
- If the number of unique assignments is less than the number of rows (or columns) a line test must be used.

Officials	Game Sites			
	Raleigh	Atlanta	Durham	Clemson
A	105	0	55	15
B	15	0	25	75
C	35	0	0	10
D	0	0	5	0

The Opportunity Cost Table with the Line Test

## SOLUTION OF THE ASSIGNMENT MODEL (5 OF 7)

- In a line test all zeros are crossed out by horizontal and vertical lines; the minimum uncrossed value is subtracted from all other uncrossed values and added to values where two lines cross.

Officials	Game Sites			
	Raleigh	Atlanta	Durham	Clemson
A	90	0	40	0
B	0	0	10	60
C	55	15	0	10
D	0	15	5	0

The Second Iteration

## SOLUTION OF THE ASSIGNMENT MODEL (6 OF 7)

- At least four lines are required to cross out all zeros in table 38.
- This indicates an optimal solution has been reached.
- Assignments and distances:

<u>Assignment</u>	<u>Distance</u>	<u>Assignment</u>	<u>Distance</u>
Team A → Atlanta	90	Team A → Clemson	160
Team B → Raleigh	100	Team B → Atlanta	70
Team C → Durham	140	Team C → Durham	140
Team D → Clemson	120	Team D → Raleigh	80
Total	450 miles	Total	450 miles

- If in initial assignment team A went to Clemson, result is the same; resulting assignments represent multiple optimal solutions.

## SOLUTION OF THE ASSIGNMENT MODEL (7 OF 7)

- When supply exceeds demand, a dummy column is added to the tableau.
- When demand exceeds supply, a dummy row is added to the tableau.
- The addition of a dummy row or column does not affect the solution method.
- A prohibited assignment is given a large relative cost of  $M$  so that it will never be selected.

Officials	Game Sites				
	Raleigh	Atlanta	Dur ham	Clemson	Dummy
A	210	90	180	160	0
B	100	70	130	200	0
C	175	105	140	170	0
D	80	65	105	120	0
E	95	115	120	100	0

An Unbalanced Assignment Tableau with a Dummy Column

# SOLUTION OF THE ASSIGNMENT MODEL

## SUMMARY OF SOLUTION STEPS

1. Perform row reductions.

2. Perform column reductions.

3 In the completed opportunity cost table, cross out all zeros using the minimum number of horizontal and/or vertical lines.

4. If fewer than  $m$  lines are required, subtract the minimum uncrossed value from all other uncrossed values, and add the same value to all cells where two lines intersect.

5. Leave all other values unchanged and repeat step 3.

6. If  $m$  lines are required, the tableau contains the optimal solution. If fewer than  $m$  lines are required, repeat step 4.

# THE ASSIGNMENT PROBLEM WITH MORE EXAMPLES





# THE ASSIGNMENT PROBLEM

- In many business situations, management needs to assign - personnel to jobs, - jobs to machines, - machines to job locations, or - salespersons to territories.
- Consider the situation of assigning  $n$  jobs to  $n$  machines.
- When a job  $i$  ( $=1,2,\dots,n$ ) is assigned to machine  $j$  ( $=1,2,\dots,n$ ) that incurs a cost  $C_{ij}$ .
- The objective is to assign the jobs to machines at the least possible total cost.



# THE ASSIGNMENT PROBLEM

- This situation is a special case of the Transportation Model And it is known as the *assignment problem*.
- Here, jobs represent “sources” and machines represent “destinations.”
- The supply available at each source is 1 unit And demand at each destination is 1 unit.



# THE ASSIGNMENT PROBLEM

	Machine					Source
		1	2	.....	n	
Job	1	C11	C12	.....	C1n	1
	2	C21	C22	.....	C2n	1
	.	.	.		.	.
	.	.	.		.	.
	.	.	.		.	.
	n	Cn1	Cn2	.....	Cnn	1
Destination		1	1	.....	1	

The assignment model can be expressed mathematically as follows:

$X_{ij} =$  0, if the job  $j$  is not assigned to machine  $i$   
           1, if the job  $j$  is assigned to machine  $i$



# THE ASSIGNMENT PROBLEM

$$\text{Min} \quad \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$$

(Sum of assignments from a source should be exactly equal to 1):

$$\sum_{j=1}^n X_{ij} = 1 \quad \text{For } i=1,2,\dots,n$$

(Sum of assignments to a destination should be equal to the demanded quantity by that destination):

$$\sum_{i=1}^n X_{ij} = 1 \quad \text{For } j=1,2,\dots,n$$

(Quantities to be assigned can be either 0 or 1):

$$X_{ij} = 0 \quad \text{or} \quad 1 \quad \text{For all } i \text{ and } j.$$



# THE ASSIGNMENT PROBLEM EXAMPLE

- Ballston Electronics manufactures small electrical devices.
- Products are manufactured on five different assembly lines (1,2,3,4,5).
- When manufacturing is finished, products are transported from the assembly lines to one of the five different inspection areas (A,B,C,D,E).
- Transporting products from five assembly lines to five inspection areas requires different times (in minutes)



# THE ASSIGNMENT PROBLEM EXAMPLE

Assembly Line	Inspection Area				
	A	B	C	D	E
1	10	4	6	10	12
2	11	7	7	9	14
3	13	8	12	14	15
4	14	16	13	17	17
5	19	11	17	20	19

Under current arrangement, assignment of inspection areas to the assembly lines are 1 to A, 2 to B, 3 to C, 4 to D, and 5 to E.

This arrangement requires  $10+7+12+17+19 = 65$  man minutes.



# HUNGARIAN METHOD EXAMPLE

		Machine			
		1	2	3	
Job	1	5	7	9	1
	2	14	10	12	1
	3	15	13	16	1
		1	1	1	

**Step 1:** Select the smallest value in each row.  
Subtract this value from each value in that row

**Step 2:** Do the same for the columns that do not have any zero value.



# HUNGARIAN METHOD EXAMPLE

		Machine		
		1	2	3
Job	1	<b>5</b>	7	9
	2	14	<b>10</b>	12
	3	15	<b>13</b>	16

		Machine		
		1	2	3
Job	1	0	2	4
	2	4	0	<b>2</b>
	3	2	0	3

	Machine		
	1	2	3
1	<b>0</b>	2	2
2	4	0	<b>0</b>
3	2	<b>0</b>	1

If not finished, continue with other columns.





# HUNGARIAN METHOD EXAMPLE

**Step 3:** Assignments are made at zero values.

- Therefore, we assign job 1 to machine 1; job 2 to machine 3, and job 3 to machine 2.
- Total cost is  $5+12+13 = 30$ .
- It is not always possible to obtain a feasible assignment as in here.



# HUNGARIAN METHOD EXAMPLE 2

	1	2	3	4
1	<u>1</u>	4	6	3
2	9	<u>7</u>	10	9
3	<u>4</u>	5	11	7
4	8	7	8	<u>5</u>

	1	2	<u>3</u>	4
1	0	3	<u>5</u>	2
2	2	0	<u>3</u>	2
3	0	1	<u>7</u>	3
4	3	2	<u>3</u>	0

	1	2	3	4
1	<u>0</u>	3	2	2
2	2	<u>0</u>	0	2
3	0	1	4	3
4	3	2	<u>0</u>	0



# HUNGARIAN METHOD EXAMPLE 2

- A feasible assignment is not possible at this moment.
- In such a case, The procedure is to draw a ***minimum*** number of ***lines*** through some of the rows and columns, ***Such that all zero values are crossed out.***



# HUNGARIAN METHOD EXAMPLE 2

	1	2	3	4
1	0	3	2	2
2	2	0	0	2
3	0	1	4	3
4	3	2	0	0

The next step is to select the smallest uncrossed out element. This element is *subtracted from every uncrossed out element* and *added to every element at the intersection* of two lines.

	1	2	3	4
1	<u>0</u>	2	1	1
2	3	0	<u>0</u>	2
3	0	<u>0</u>	3	2
4	4	2	0	<u>0</u>



# HUNGARIAN METHOD EXAMPLE 2

- We can now easily assign to the zero values. Solution is to assign (1 to 1), (2 to 3), (3 to 2) and (4 to 4).
- If drawing lines do not provide an easy solution, then we should perform the task of drawing lines one more time.
- Actually, we should continue drawing lines until a feasible assignment is possible.

