Presentation of Anasua Sarkar - 3

For Soft Computing (CSE/T/425E)

At Computer Science and Engineering Department, Jadavpur University, India

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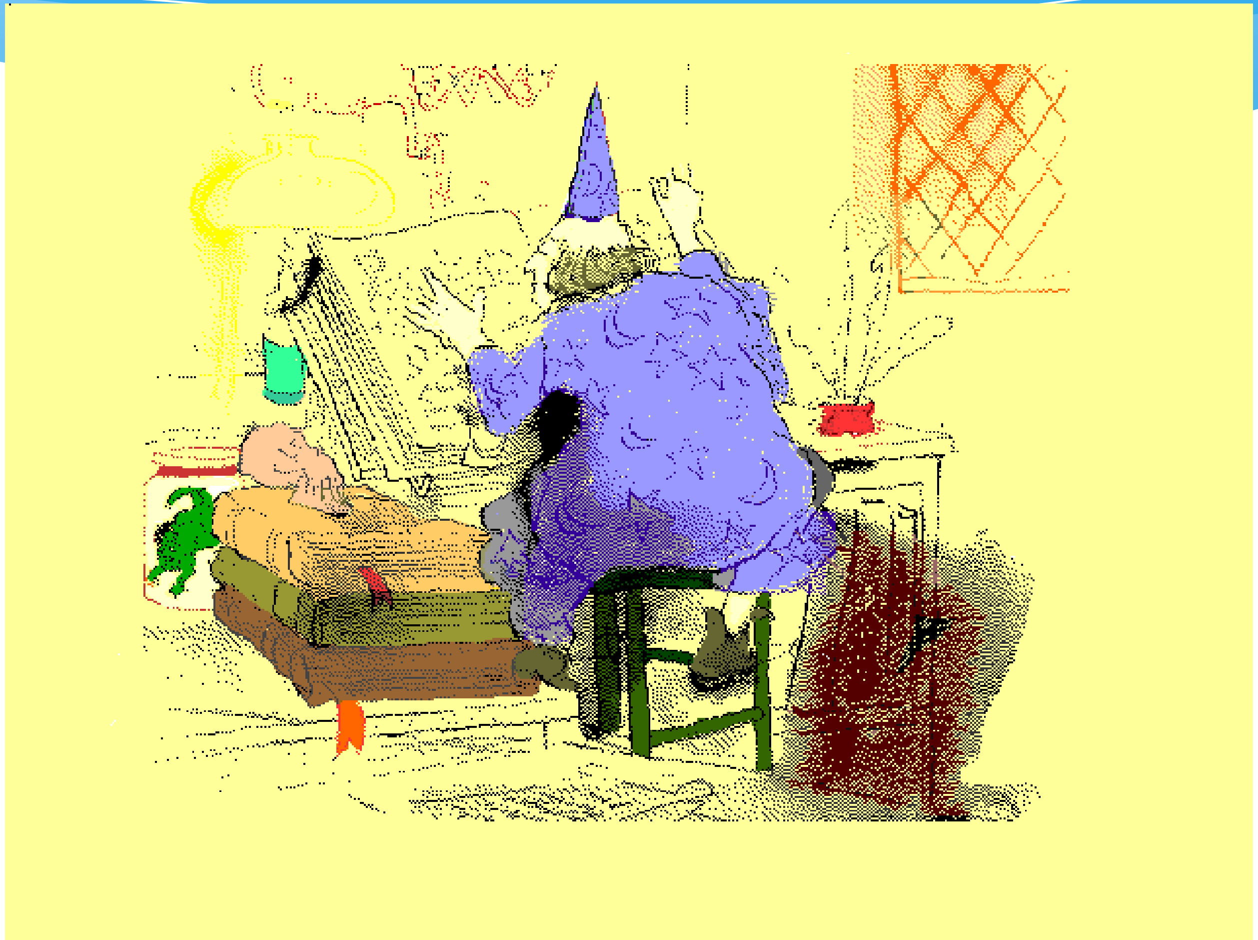
Topics of Interests

* Hard COMPUTING AND
* Soft Computing
* Reasoning under Uncertainty
* Fuzzy sets theory
* Type-2 Fuzzy sets
* Applications with Fuzzy Sets



Fuzzy Sets Theory

Boolean logic

Uses sharp distinctions. It forces us to draw a line between a members of class and non members.

Fuzzy logic

Reflects how people think. It attempt to model our senses of words, our decision making and our common sense -> more human and intelligent systems

Fuzzy Logic

- Fuzzy logic is a form of multi-valued logic derived from fuzzy set theory to deal with reasoning that is approximate rather than accurate. In contrast with "crisp logic", where binary sets have binary logic, fuzzy logic variables may have a truth value that ranges between 0 and 1 and is not constrained to the two truth values of classic propositional logic. Furthermore, when linguistic variables are used, these degrees may be managed by specific functions.

- Fuzzy logic emerged as a consequence of the 1965 proposal of fuzzy set theory by Lotfi Zadeh. Though fuzzy logic has been applied to many fields, from control theory to artificial intelligence, it still remains controversial among most statisticians, who prefer Bayesian logic, and some control engineers, who prefer traditional two-valued logic.

Fuzzy Logic

- Fuzzy logic and probabilistic logic are mathematically similar – both have truth values ranging between 0 and 1 – but conceptually distinct, due to different interpretations. Fuzzy logic corresponds to "degrees of truth", while probabilistic logic corresponds to "probability, likelihood"; as these differ, fuzzy logic and probabilistic logic yield different models of the same real-world situations.

- Both degrees of truth and probabilities range between [0, 1] and may seem similar at first. For example, let a 100 ml glass contain 30 ml of water. We may consider two concepts: Empty and Full. The meaning of each of them can be represented by fuzzy set. One might define the glass as being 0.7 empty and 0.3 full. Note that the concept of emptiness is subjective. We might equally well design a set membership function where the glass would be considered full for all values down to 50 ml.

- It is essential to realize that fuzzy logic uses truth degrees as a mathematical model of the vagueness phenomenon while probability is a mathematical model of ignorance. The same could be achieved using probabilistic methods, by defining a binary variable "full" that depends on a continuous variable that describes how full the glass is. There is no consensus on which method should be preferred in a specific situation.

Fuzzy Logic Operators

- Fuzzy Logic:
  - NOT (A) = 1 - A
  - A AND B = min(A, B)
  - A OR B = max(A, B)

Fuzzy Logic NOT

A	NOT A
0	1
0.25	0.75
0.5	0.5
0.75	0.25
1	0

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Fuzzy Logic AND

A AND B B A	0	0.25	0.5	0.75	1.0
0	0	0	0	0	0
0.25	0	0.25	0.25	0.25	0.25
0.5	0	0.25	0.5	0.5	0.5
0.75	0	0.25	0.5	0.75	0.75
1	0	0.25	0.5	0.75	1

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Fuzzy Logic OR

A OR B — B / A	0	0.25	0.5	0.75	1.0
0	0	0.25	0.5	0.75	1.0
0.25	0.25	0.25	0.5	0.75	1.0
0.5	0.5	0.5	0.5	0.75	1.0
0.75	0.75	0.75	0.75	0.75	1.0
1	1.0	1.0	1.0	1.0	1.0

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Misconceptions about Fuzzy concepts @FL2applications.ppt

Fuzziness is not *Vague*

we shall have a look at some propositions.

Dimitris is six feet tall

The first proposition (traditional) has a crisp truth value of either TRUE or FALSE.

He is tall

The second proposition is vague.

It does not provide sufficient information for us to make a decision, either fuzzy or crisp.

We do not know the value of the pronoun.

Is it Dimitris, John or someone else?

Fuzziness is not *Vague*

Andrei is tall

The third proposition is a fuzzy proposition.

It is true to some degree depending in the context, i.e., the universe of discourse.

It might be *SomeWhat True* if we are referring to basketball players or it might be *Very True* if we are referring to horse-jockeys.

Fuzziness is not *Multi-valued logic*

The limitations of two-valued logic were recognised very early.

A number of different logic theories based on multiple values of truth have been formulated through the years.

For example, in three-valued logic three truth values have been employed.

These are TRUTH, FALSE, and UNKNOWN represented by 1, 0 and 0.5 respectively.

In 1921 the first N-valued logic was introduced.

The set of truth values T_n were assumed to be evenly divided over the closed interval [0,1].

Fuzzy logic may be considered as an extension of multi-valued logic but they are somewhat different.

Multi-valued logic is still based on exact reasoning whereas fuzzy logic is approximate reasoning.

Fuzziness is not *Probability*

This is better explained using an example.
Let X be the set of all liquids (i.e., the universe of discourse) .
Let L be a subset of X which includes all suitable for drinking liquids.
Suppose now that you find two bottles, A and B.
The labels do not provide any clues about the contents.
Bottle A label is marked as membership of L is 0.9.
The label of bottle B is marked as probability of L is 0.9.
Given that you have to drink from the one you choose, the problem is of how to interpret the labels.

Fuzziness is not *Probability*

Well, membership of 0.9 means that the contents of A are *fairly similar* to perfectly potable liquids.

If, for example, a perfectly liquid is pure water then bottle A might contain, say, tonic water.

Probability of 0.9 means something completely different.

You have a 90% chance that the contents are potable and 10% chance that the contents will be unsavoury, some kind of acid maybe.

Hence, with bottle A you might drink something that is not pure but with bottle B you might drink something deadly. So choose bottle A.

Fuzziness is not *Probability*

Opening both bottles you observe beer (bottle A) and hydrochloric acid (bottle B).
The outcome of this observation is that the membership stays the same whereas the probability drops to zero.

All in all:
probability measures the likelihood that a future event will occur,
fuzzy logic measures the ambiguity of events that have already occurred.

In fact, fuzzy sets and probability exist as parts of a greater Generalized Information Theory.
This theory also includes:
Dempster-Shafer evidence theory,
possibility theory,
and so on.

Fuzzy Sets

- Extension of Classical Sets
- Fuzzy sets - 1965 Lotfi Zadeh as an extension of classical notation set.
- Not just a membership value of in the set and out the set, 1 and 0
  - but partial membership value, between 1 and 0

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Classical Set vs Fuzzy set

Let X be the universe of discourse and its elements be denoted as x.
In the classical set theory, crisp set A of X is defined as function $f_A(x)$ called the the characteristic function of A

$$f_A(x) : X \to \{0,1\},\ \text{where } f_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

In the fuzzy theory, fuzzy set A of universe of discourse X is defined by function $\mu_A(x)$ called the membership function of set A

$$\begin{aligned} \mu_A(x) : X \to [0,1], \mathit{where}\ \mu_A(x) &= 1 \ \text{ if } x \text{ is totally in } A; \\ \mu_A(x) &= 0 \ \text{ if } x \text{ is not in } A; \\ 0 < \mu_A(x) &< 1 \ \text{ if } x \text{ is partly in } A. \end{aligned}$$

Classical set theory

$A = \{x \mid x > 6\},$

- A Set is any well defined collection of objects.
- An object in a set is called an element or member of that set.
- Sets are defined by a simple statement,
- Describing whether a particular element having a certain property belongs to that particular set.
    - $A = \{a_1,a_2,a_3,\ldots\ldots,a_n\}$
- If the elements a_i ($i = 1,2,3,\ldots.,n$) of a set A are subset of universal set X, then set A can be represented for all elements $x \in X$ by its characteristics function
- $\mu_A(x) = 1$ if $x \in X$ otherwise 0

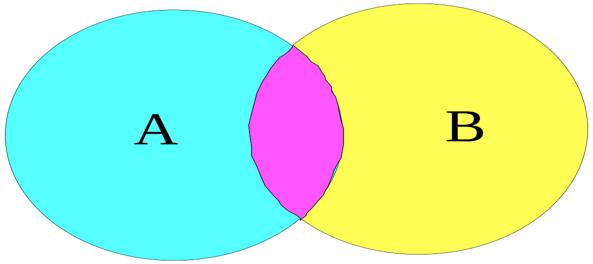
Operations on classical set theory

- ***Union:*** the union of two sets A and B is given as
  - A U B = { x | x ϵ A or x ϵ B }
  -
- ***Intersection:*** the intersection of two sets A and B is given as
  - A ∩ B = { x | x ϵ A and x ϵ B }
  -
- ***Complement:*** It is denoted by Ã and is defined as
- Ã = { x | x does not belongs A and x ϵ X }

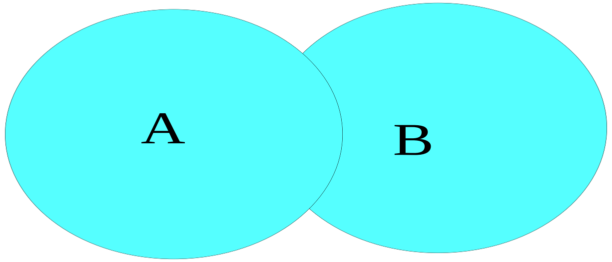
Law of contradiction	$A \cap \overline{A} = \emptyset$
Law of the excluded middle	$A \cup \overline{A} = X$
Idempotency	$A \cap A = A,\ A \cup A = A$
Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cap B = B \cap A,\ A \cup B = B \cup A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Absorption of complement	$A \cup (\overline{A} \cap B) = A \cup B$ $A \cap (\overline{A} \cup B) = A \cap B$
DeMorgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Set operations example using Venn diagrams

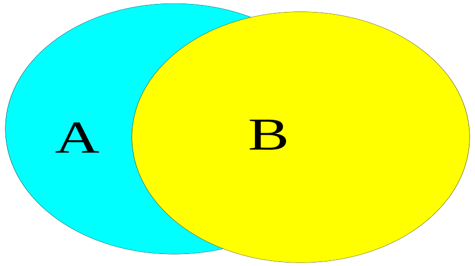
Intersection



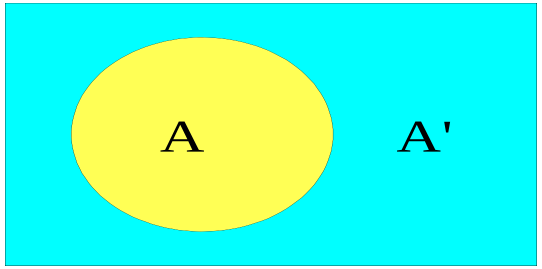
Union



Difference



Complement



Theory of Fuzzy Sets

Given any set 'S' and an element 'e', there is a very natural predicate, $\mu_s(e)$ called as the *belongingness predicate*.

The predicate is such that,

$$\mu_s(e) = 1, \qquad iff\ e \in S$$
$$= 0, \qquad otherwise$$

For example, $S = \{1, 2, 3, 4\}$, $\mu_s(1) = 1$ and $\mu_s(5) = 0$

A predicate $P(x)$ also defines a set naturally.

$$S = \{x \mid P(x) \text{ is } true\}$$

For example, $even(x)$ defines $S = \{x \mid x \text{ is even}\}$

Classical Set vs Fuzzy set

No	Name	Height (cm)	Degree of Membership of “tall men”	
			Crisp	Fuzzy
1	Boy	206	1	1
2	Martin	190	1	1
3	Dewanto	175	0	0.8
4	Joko	160	0	0.7
5	Kom	155	0	0.4

Fuzzy Set Theory (contd.)

In Fuzzy theory

$$\mu_s(e) = [0, 1]$$

Fuzzy set theory is a generalization of classical set theory *aka* called Crisp Set Theory.

In real life, *belongingness* is a fuzzy concept.

Example: Let, T = “tallness”

$$\mu_T(\text{height}=6.0\text{ft}) = 1.0$$
$$\mu_T(\text{height}=3.5\text{ft}) = 0.2$$

An individual with height 3.5ft is “tall” with a degree *0.2*

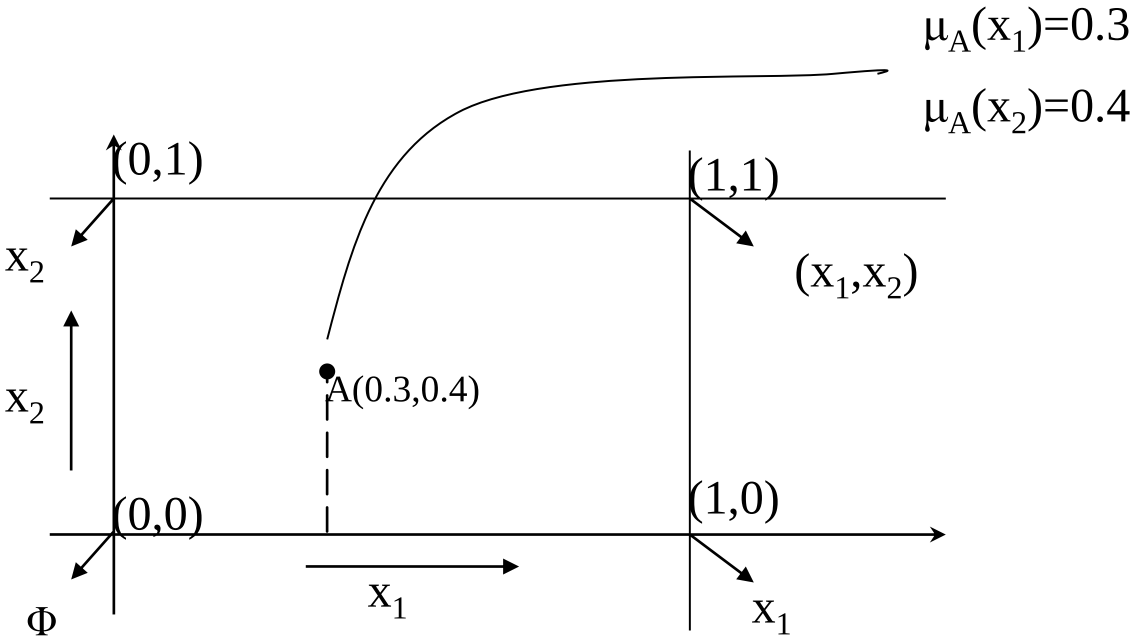
Representation of Fuzzy sets

Let $U = \{x_1, x_2, \ldots, x_n\}$

$|U| = n$

The various sets composed of elements from U are presented as points on and inside the n-dimensional hypercube. The crisp sets are the corners of the hypercube.

$U=\{x_1,x_2\}$



A fuzzy set A is represented by a point in the n-dimensional space as the point $\{\mu_A(x_1), \mu_A(x_2), \ldots\ldots \mu_A(x_n)\}$

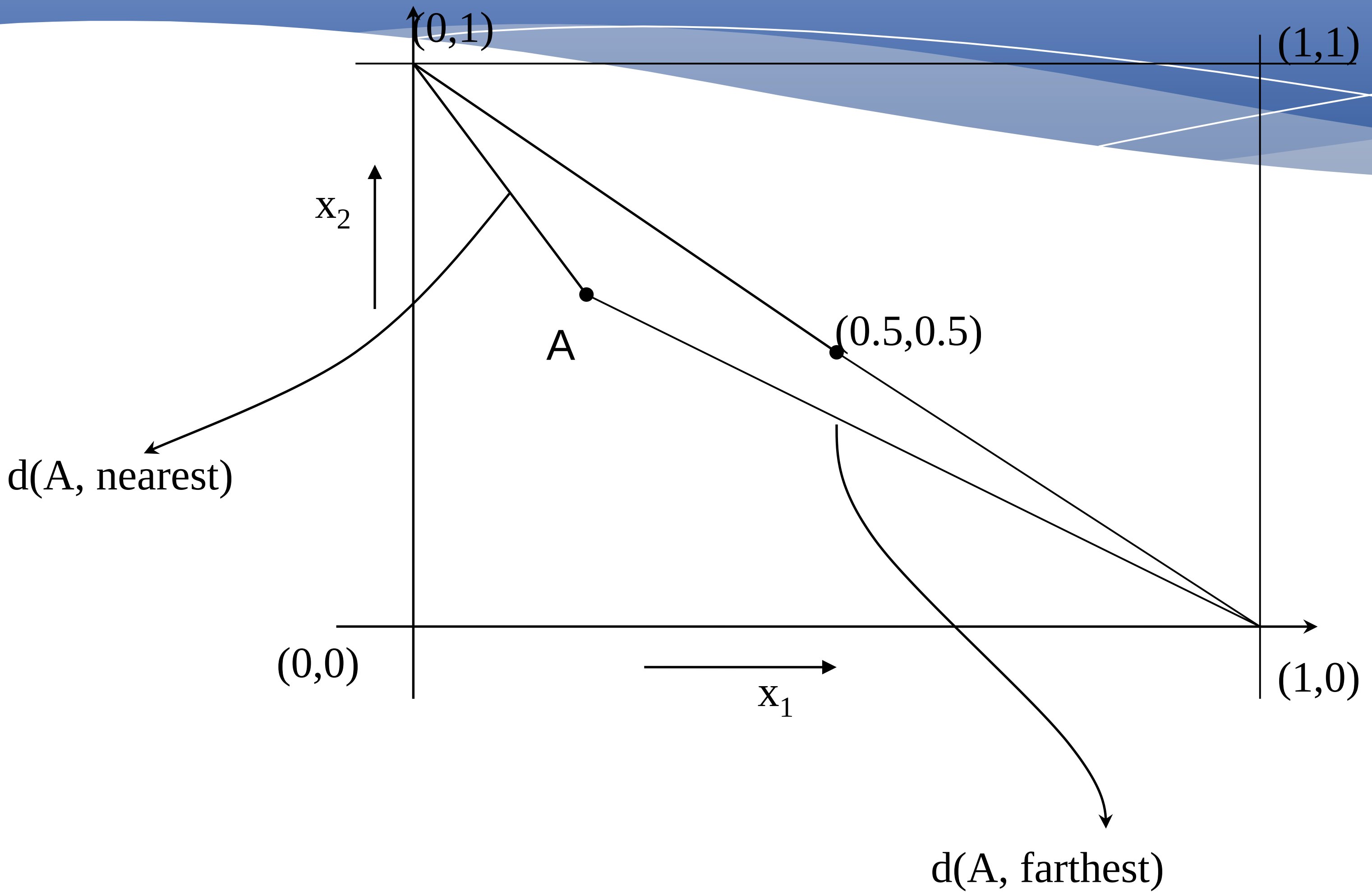
Degree of fuzziness

The centre of the hypercube is the *most fuzzy* set. Fuzziness decreases as one nears the corners

Measure of fuzziness

Called the entropy of a fuzzy set

$$\underbrace{E}_{\text{Entropy}}(\overbrace{S}^{\text{Fuzzy set}}) = d(S, \underbrace{nearest}_{\text{Nearest corner}}) / d(S, \overbrace{farthest}^{\text{Farthest corner}})$$


(0,1)
(1,1)
x_2
A
(0.5,0.5)
d(A, nearest)
(0,0)
x_1
(1,0)
d(A, farthest)

Definition

Distance between two fuzzy sets

$$d(S_1, S_2) = \sum_{i=1}^{n} \underbrace{| \mu_{s_1}(x_i) - \mu_{s_2}(x_i) |}_{L_1 \text{ - norm}}$$

Let C = fuzzy set represented by the centre point

d(c,nearest) = |0.5-1.0| + |0.5 – 0.0|

= 1

= d(C,farthest)

=> E(C) = 1

Definition

Cardinality of a fuzzy set

$$m(s) = \sum_{i=1}^{n} \mu_s(x_i)$$ (generalization of cardinality of classical sets)

Union, Intersection, complementation, subset hood

$$\mu_{s_1 \cup s_2}(x) = \max(\mu_{s_1}(x), \mu_{s_2}(x)), \forall x \in U$$

$$\mu_{s_1 \cap s_2}(x) = \min(\mu_{s_1}(x), \mu_{s_2}(x)), \forall x \in U$$

$$\mu_{s^c}(x) = 1 - \mu_s(x)$$

Example of Operations on Fuzzy Set

Let us define the following:

Universe $U=\{X_1, X_2, X_3\}$

Fuzzy sets

$A=\{0.2/X_1, 0.7/X_2, 0.6/X_3\}$ and

$B=\{0.7/X_1, 0.3/X_2, 0.5/X_3\}$

Then Cardinality of A and B are computed as follows:

Cardinality of $A=|A|=0.2+0.7+0.6=1.5$

Cardinality of $B=|B|=0.7+0.3+0.5=1.5$

While distance between A and B

$d(A,B)=|0.2-0.7)+|0.7-0.3|+|0.6-0.5|=1.0$

What does the cardinality of a fuzzy set mean? In crisp sets it means the number of elements in the set.

Example of Operations on Fuzzy Set (cntd.)

Universe $U=\{X_1, X_2, X_3\}$

Fuzzy sets $A=\{0.2/X_1, 0.7/X_2, 0.6/X_3\}$ and $B=\{0.7/X_1, 0.3/X_2, 0.5/X_3\}$

$A \cup B = \{0.7/X_1, 0.7/X_2, 0.6/X_3\}$

$A \cap B = \{0.2/X1, 0.3/X2, 0.5/X3\}$

$A^c = \{0.8/X_1, 0.3/X_2, 0.4/X_3\}$

$B^c = ?$

Fuzzy Set Operators

- Fuzzy Set:
  - Union
  - Intersection
  - Complement
  - Containment or subset
- Many possible definitions
  - we introduce one possibility

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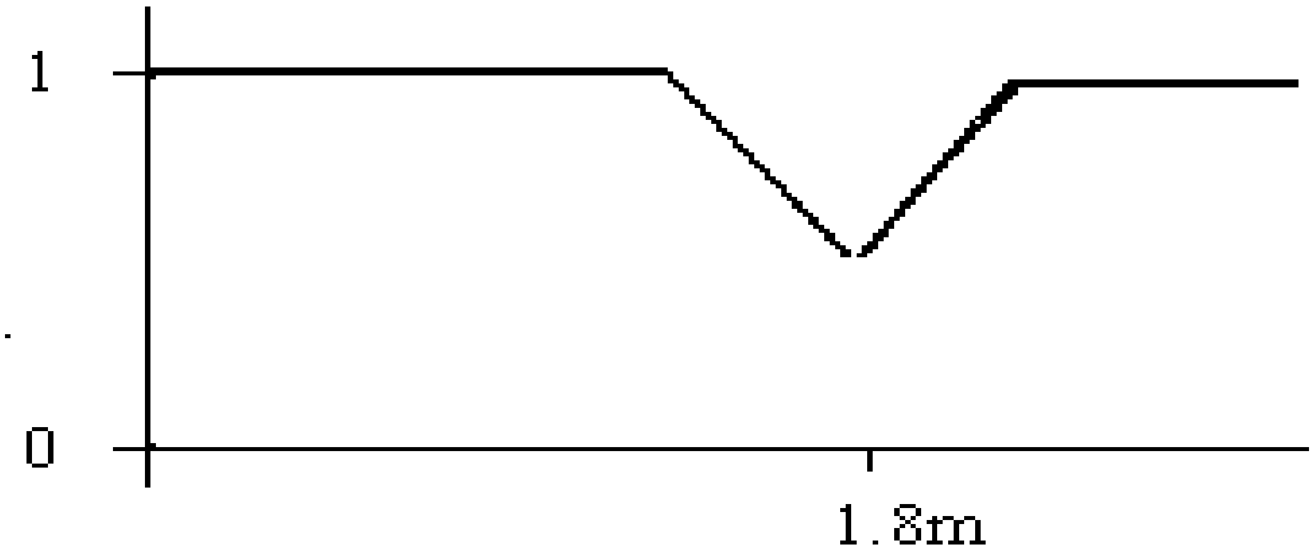
•Fuzzy Set Union

- Union ($f_A(x)$ and $f_B(x)$) = max ($f_A(x)$, $f_B(x)$)
- Union (Tall(x) and Short(x))

$C = A \cup B$ $\mu_C(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x).$



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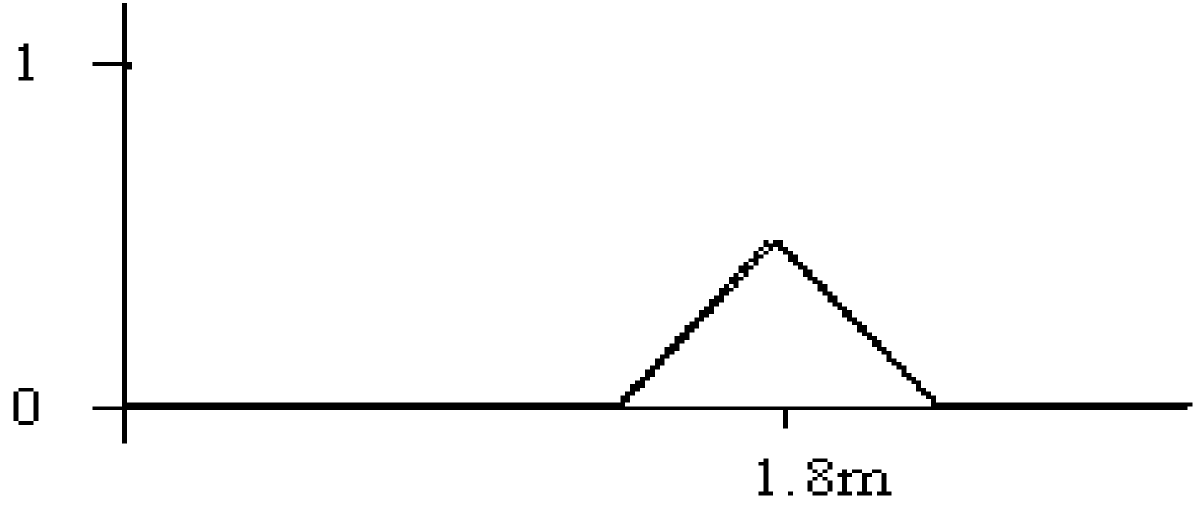
•Fuzzy Set Intersection

- Intersection ($f_A(x)$ and $f_B(x)$) = min ($f_A(x)$, $f_B(x)$)
- Intersection (Tall(x) and Short(x))

$C = A \cap B$ $\mu_C(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x).$



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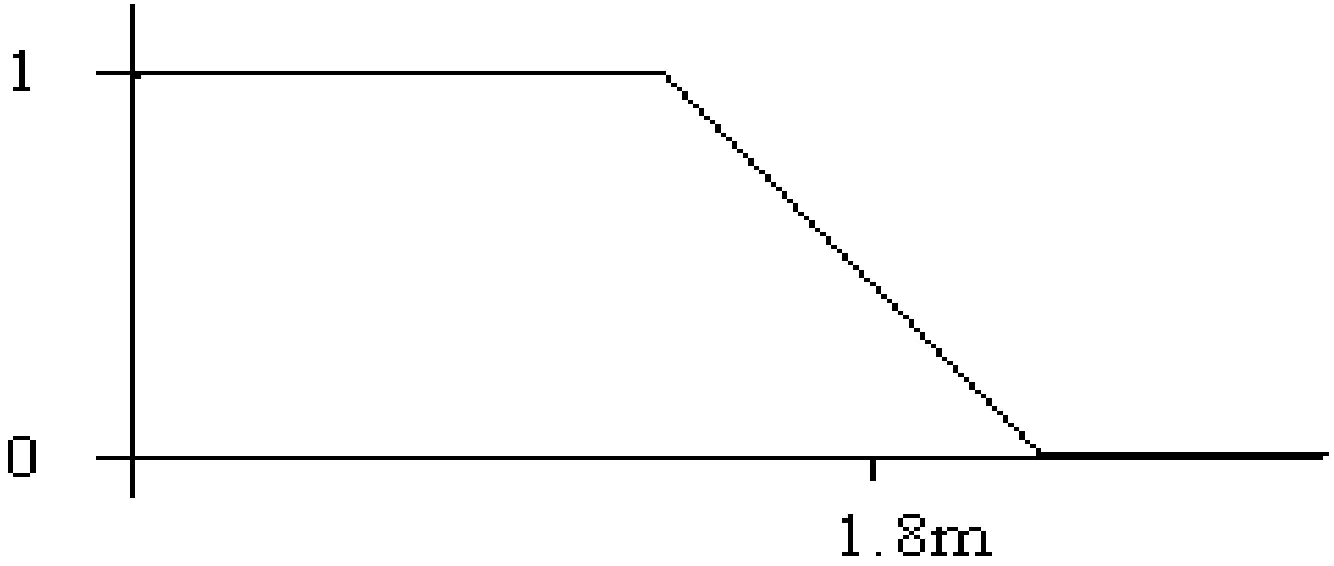
Fuzzy Set Complement

- Complement($f_A(x)$) = 1 - $f_A(x)$
- Not (Tall(x))

$\overline{A}$ ($\neg A$, NOT A)

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x).$$



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Fuzzy Set Subset

Fuzzy set A is **contained** in fuzzy set B (or, equivalently, A is a **subset** of B, or A is smaller than or equal to B) if and only if $\mu_A(x) \le \mu_B(x)$ for all x. In symbols,

$$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x). \tag{2.12}$$

□

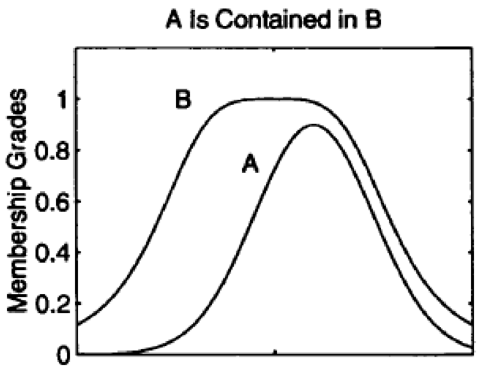


Figure 2.5. *The concept of $A \subseteq B$.* (MATLAB file: `subset.m`)

Definition 2.17 *Cartesian product and co-product*

Let A and B be fuzzy sets in X and Y, respectively. The **Cartesian product** of A and B, denoted by $A \times B$, is a fuzzy set in the product space $X \times Y$ with the membership function

$$\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y)). \tag{2.16}$$

Similarly, the **Cartesian co-product** $A + B$ is a fuzzy set with the membership function

$$\mu_{A+B}(x, y) = \max(\mu_A(x), \mu_B(y)). \tag{2.17}$$

Both $A \times B$ and $A + B$ are characterized by two-dimensional MFs, which are explored

Meaning of fuzzy subset

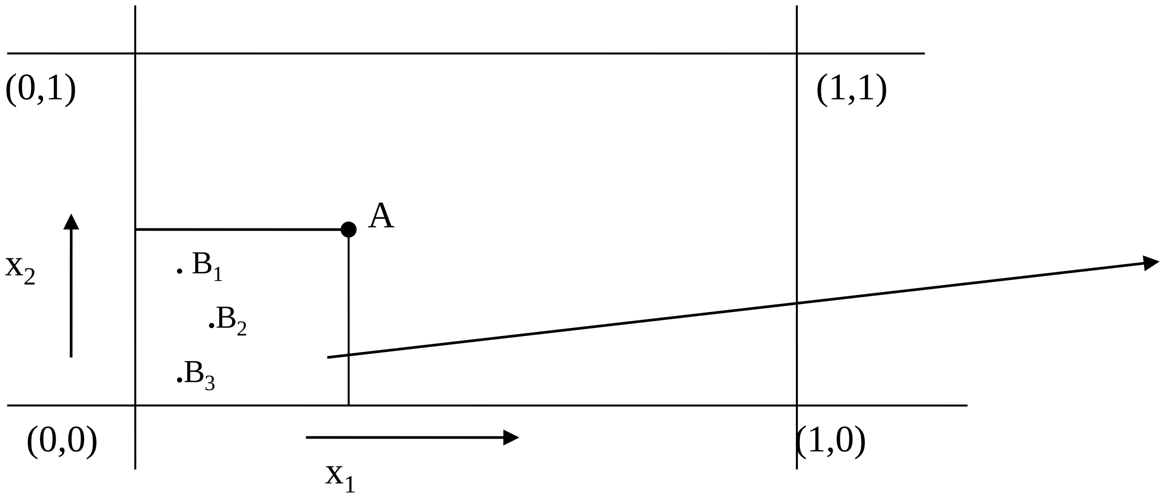
Suppose, following classical set theory we say

$$B \subset A$$

if

$$\mu_B(x) \leq \mu_A(x) \forall x$$

Consider the n-hyperspace representation of A and B



Region where $\mu_B(x) \leq \mu_A(x)$

Operations on Fuzzy Sets

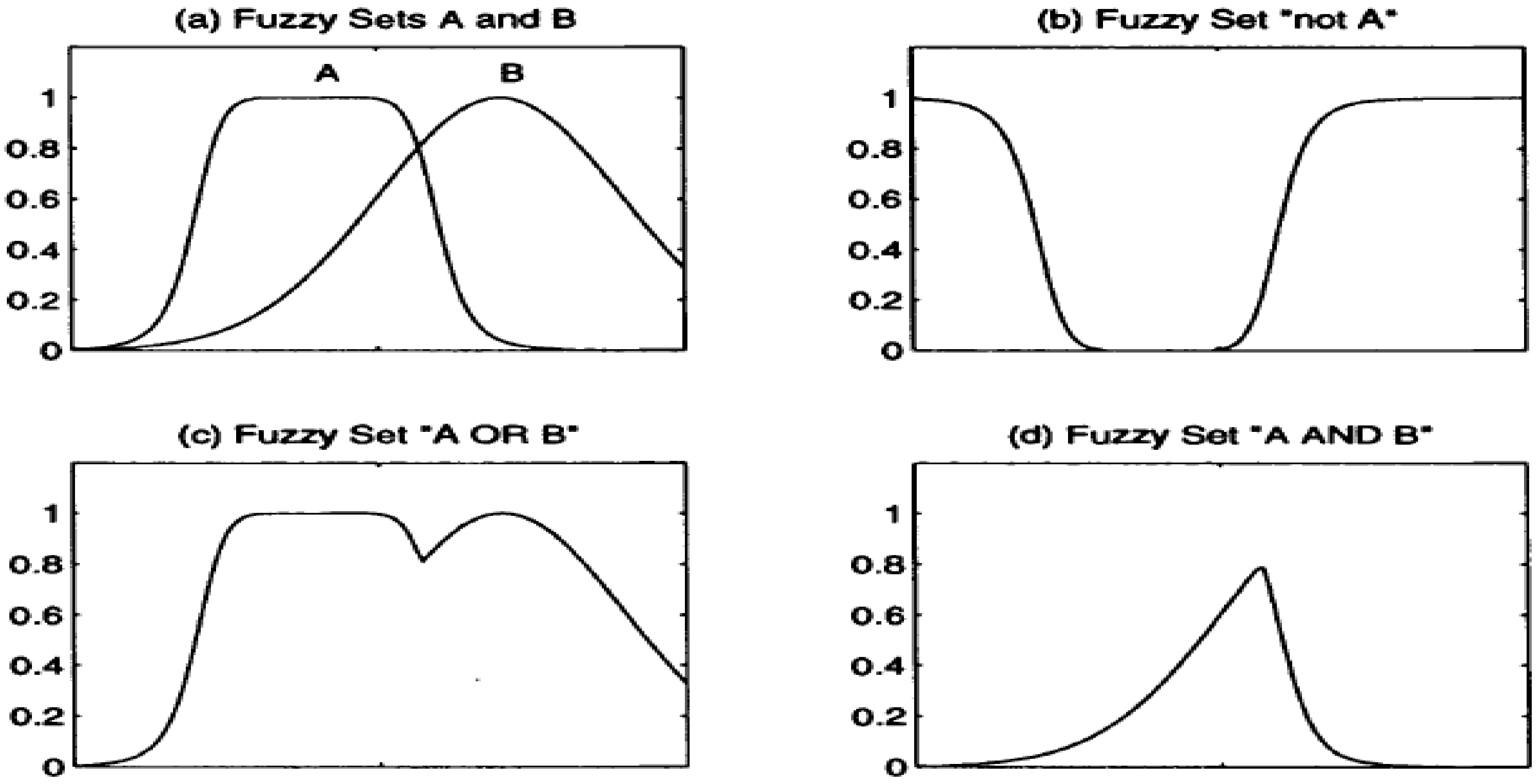


Figure 2.6. *Operations on fuzzy sets: (a) two fuzzy sets A and B; (b) $\overline{A}$; (c) $A \cup B$; (d) $A \cap B$.* (MATLAB file: `fuzsetop.m`)

Laws of Set Theory

- The laws of Crisp set theory also holds for fuzzy set theory (verify them)
- These laws are listed below:
  - Commutativity: $A \cup B = B \cup A$
  - Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$
  - Distributivity: $A \cup (B \cap C) = (A \cap C) \cup (B \cap C)$
    $A \cap (B \cup C) = (A \cup C) \cap (B \cup C)$
  - De Morgan's Law: $(A \cup B)^C = A^C \cap B^C$
    $(A \cap B)^C = A^C \cup B^C$

Distributivity Property Proof

Let Universe $U=\{x_1,x_2,\ldots x_n\}$

$$p_i = \mu_{AU(B\cap C)}(x_i)$$
$$= \max[\mu_A(x_i), \mu_{(B\cap C)}(x_i)]$$
$$= \max[\mu_A(x_i), \min(\mu_B(x_i), \mu_C(x_i))]$$

$$q_i = \mu_{(AUB)\cap(AUC)}(x_i)$$
$$= \min[\max(\mu_A(x_i), \mu_B(x_i)), \max(\mu_A(x_i), \mu_C(x_i))]$$

Distributivity Property Proof

Case I: $0<\mu_C<\mu_B<\mu_A<1$

$p_i = \max[\mu_A(x_i), \min(\mu_B(x_i),\mu_C(x_i))] = \max[\mu_A(x_i), \mu_C(x_i)] = \mu_A(x_i)$

$q_i = \min[\max(\mu_A(x_i), \mu_B(x_i)), \max(\mu_A(x_i), \mu_C(x_i))] = \min[\mu_A(x_i), \mu_A(x_i)] = \mu_A(x_i)$

Case II: $0<\mu_C<\mu_A<\mu_B<1$

$p_i = \max[\mu_A(x_i), \min(\mu_B(x_i),\mu_C(x_i))] = \max[\mu_A(x_i), \mu_C(x_i)] = \mu_A(x_i)$

$q_i = \min[\max(\mu_A(x_i), \mu_B(x_i)), \max(\mu_A(x_i), \mu_C(x_i))] = \min[\mu_B(x_i), \mu_A(x_i)] = \mu_A(x_i)$

Prove it for rest of the 4 cases.

T-norm

A *fuzzy intersection/t-norm* i is a binary operation on the unit interval that satisfies at least the following axioms for all $a, b, d \in [0, 1]$:

Axiom i1. $i(a, 1) = a$ (*boundary condition*).

Axiom i2. $b \leq d$ implies $i(a, b) \leq i(a, d)$ (*monotonicity*).

Axiom i3. $i(a, b) = i(b, a)$ (*commutativity*).

Axiom i4. $i(a, i(b, d)) = i(i(a, b), d)$ (*associativity*).

Let us call this set of axioms the *axiomatic skeleton for fuzzy intersections/t-norms*.

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T-Conorm

A *fuzzy union/t-conorm* u is a binary operation on the unit interval that satisfies at least the following axioms for all $a, b, d \in [0, 1]$:

Axiom u1. $u(a, 0) = a$ (*boundary condition*).

Axiom u2. $b \leq d$ implies $u(a, b) \leq u(a, d)$ (*monotonicity*).

Axiom u3. $u(a, b) = u(b, a)$ (*commutativity*).

Axiom u4. $u(a, u(b, d)) = u(u(a, b), d)$ (*associativity*).

Since this set of axioms is essential for fuzzy unions, we call it the *axiomatic skeleton for fuzzy unions/t-conorms*.

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Cartesian Product

- The intersection and union operations can also be used to assign memberships on the Cartesian product of two sets.
- Consider, as an example, the fuzzy membership of a set, *G*, of liquids that taste *good* and the set, *LA*, of cities close to Los Angeles

$$\mu_G = 0.0/\text{Swamp Water} + 0.5/\text{Radish Juice} + 0.9/\text{Grape Juice}$$

$$\mu_{LA} = 0.0/\text{LA} + 0.5/\text{Chicago} + 0.8/\text{New York} + 0.9/\text{London}$$

Cartesian Product

- We form the set...

  = liquids that taste *good AND* cities that are *close* to LA

- The following table results...

	LosAngeles(0.0)	Chicago (0.5)	New York (0.8)	London(0.9)	
Swamp Water (0.0)	0.00	0.00	0.00	0.00	
Radish Juice (0.5)	0.00	0.25	0.40	0.45	
Grape Juice (0.9)	0.00	0.45	0.72	0.81	

Single argument Fuzzy Operations - Hedge

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Concentration set operator

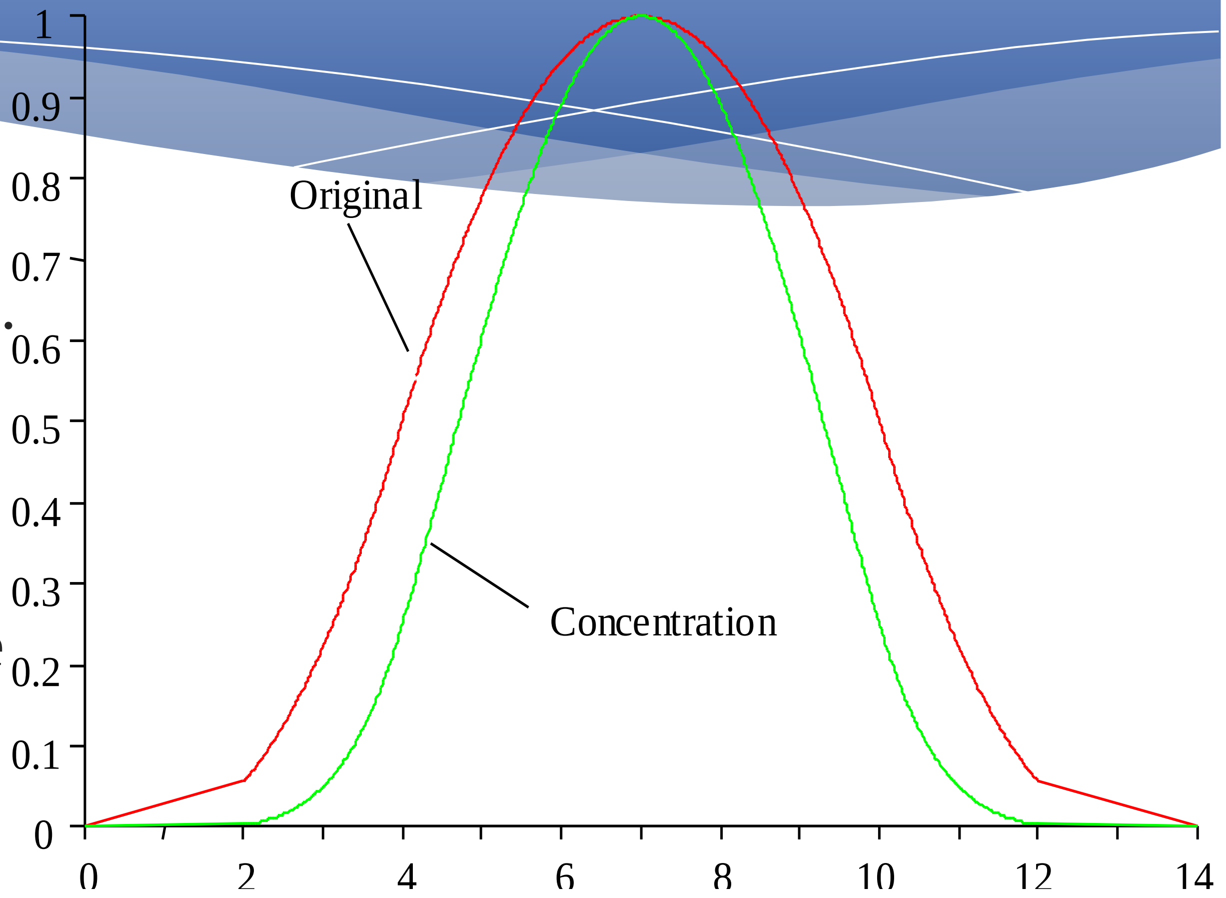
CON(A)

$\mu_{CON(A)}=(\mu_A(x))^2$

This operation reduces the membership grade of elements that have small membership grades.

If TALL=-.125/5+0.5/6+0.875/6.5+1/7+1/7.5+1/8 then

VERY TALL = 0.0165/5+0.25/6+0.76/6.5+1/7+1/7.5+1/8 since VERY TALL=TALL^2.



Dilation set operator

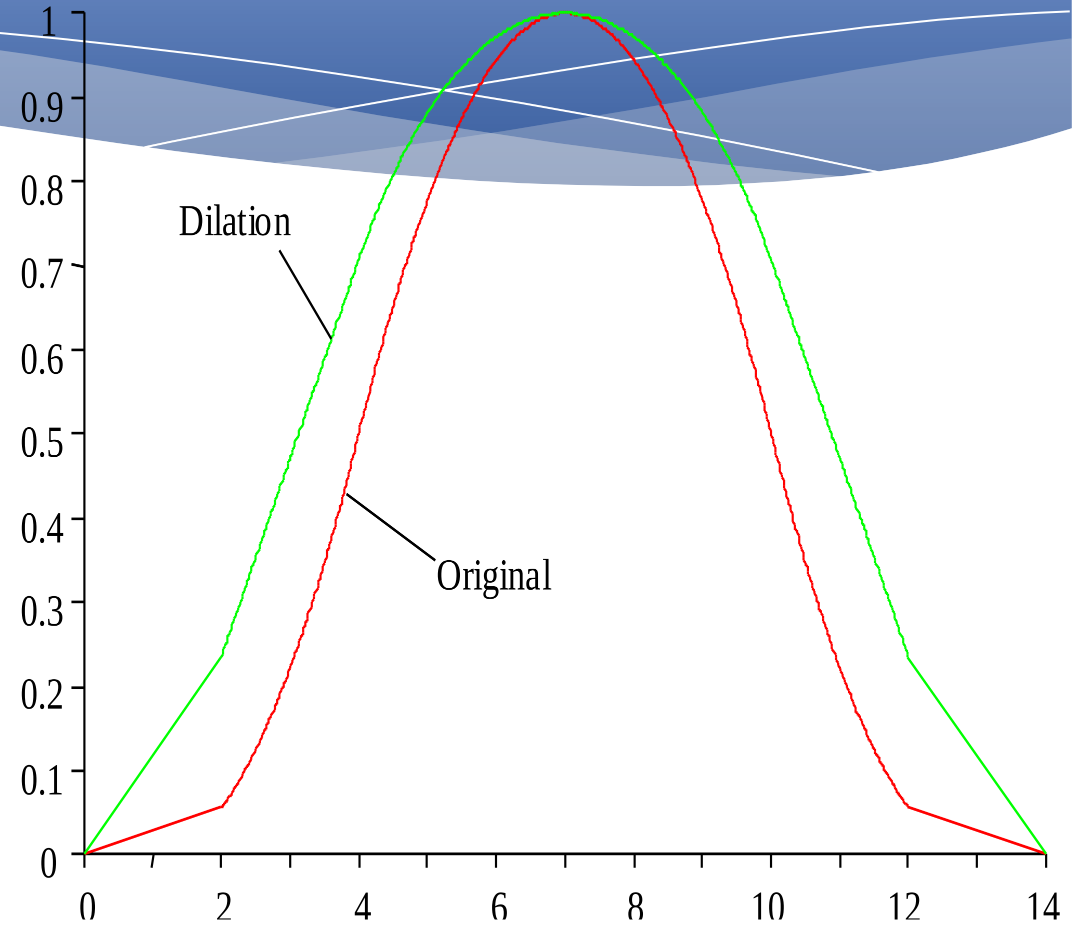
DIL(A) $\mu_{DIL(A)}=(\mu_A(x))^{0.5}$

This operation increases the membership grade of elements that have small membership grades.

It is the inverse of the concentration operation.

If TALL=-.125/5+0.5/6+0.875/6.5+1/7+1/7.5+1/8 then

MORE or LESS TALL = 0.354/5+0.707/6+0.935/6.5+1/7+1/7.5+1/8 since MORE or LESS TALL=TALL$^{0.5}$.

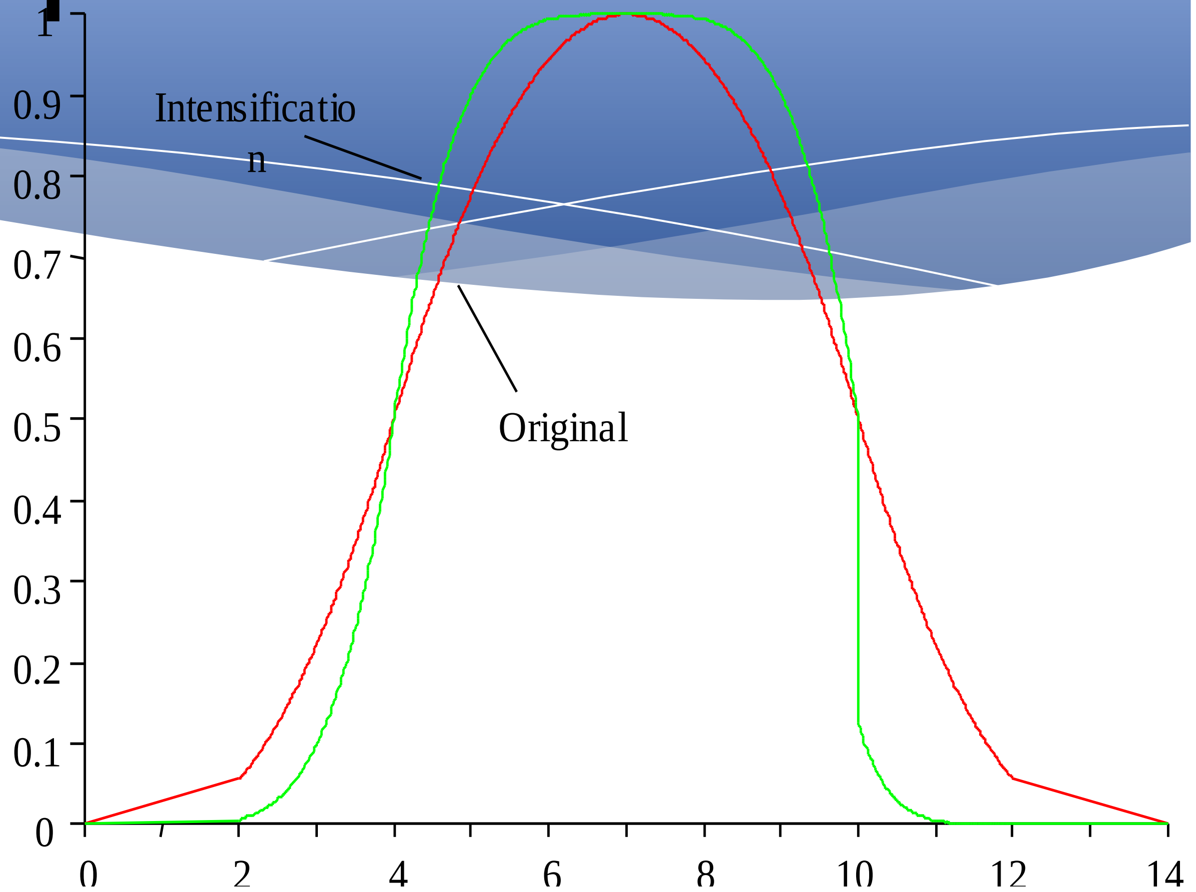


Intensification set operator

This operation raises the membership grade of those elements within the 0.5 points and This operation reduces the membership grade of those elements outside the crossover (0.5) point.

Hence, intensification amplifies the signal within the bandwidth while reducing the `noise'.

If TALL = -.125/5+0.5/6+0.875/6.5+1/7+1/7.5+1/8 then INT(TALL) = 0.031/5+0.5/6+0.969/6.5+1/7+1/7.5+1/8.



$$\mu_{INT(A)}(x) = \begin{cases} 2(\mu_A(x))^2 \text{ for } 0 \le \mu_A(x) \le 0.5 \\ 1-2(1-\mu_A(x))^2 \text{ for } 0.5 < \mu_A(x) \le 1 \end{cases}$$

Normalization set operator

$\mu_{NORM(A)}(x)=\mu_A(x)/\max\{\mu_A(x)\}$ where the max function returns the maximum membership grade for all elements of x.

If the maximum grade is <1, then all membership grades will be increased.

If the maximum is 1, then the membership grades remain unchanged.

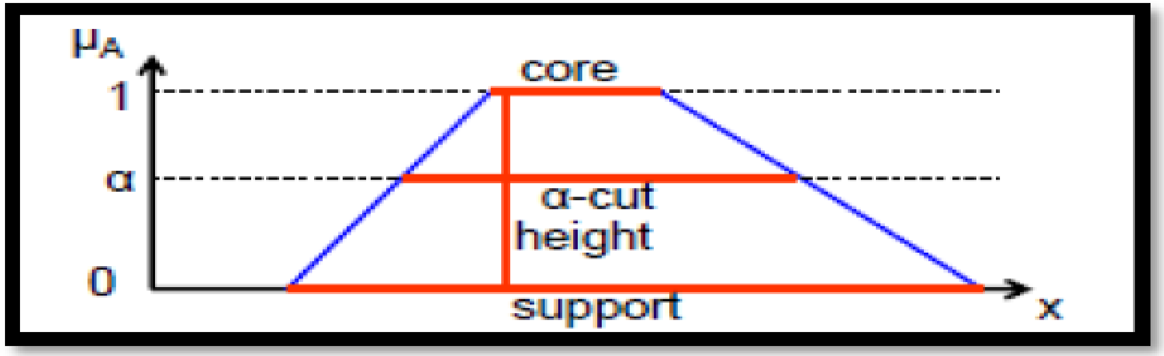
NORM(TALL) = TALL since the maximum is 1

MF: basic concepts

- **Fuzzy singleton**- A fuzzy set with a membership function that is unity at a one particular point and zero everywhere else.
- **Singleton output function**- An output function that is given by a spike at a single number rather than a continuous curve. In the Fuzzy Logic Toolbox it is only supported as part of a zero-order Sugeno model.

Fuzzy Membership Function: Basic Concepts

- **Support**: elements having non-zero degree of membership.
- **Core**: set with elements having degree of 1.
- **α-Cut**: set of elements with degree >= α.
- **Height**: maximum degree of membership.



Membership Functions in the Fuzzy Logic Toolbox

Support, Crossover, Singleton

<u>Support of a fuzzy set:</u>

The <u>support of a fuzzy set</u> is the <u>set of all elements</u> of the universe of discourse that their grade of membership is greater than zero.

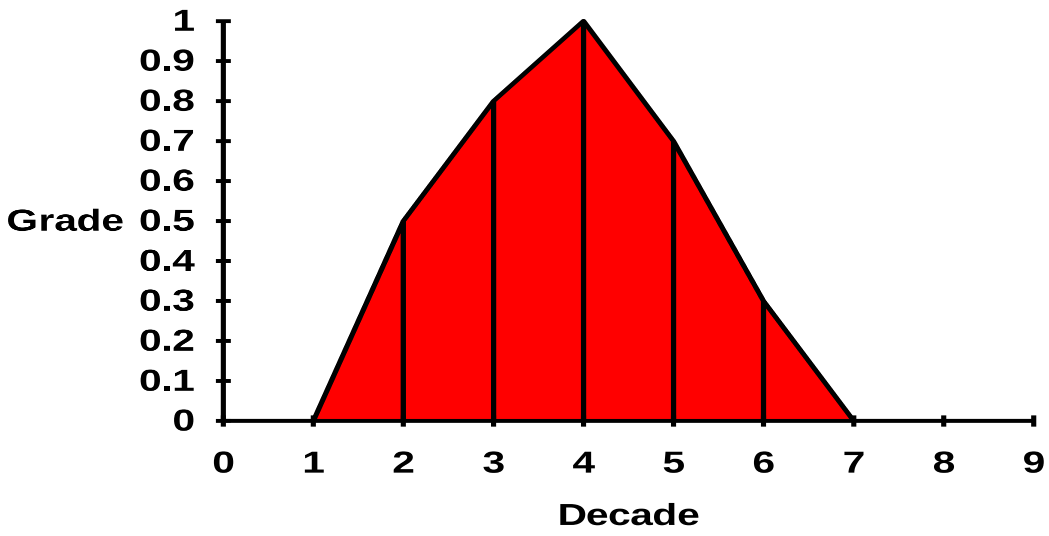
For X_2 the support is {2,3,4,5,6}.

Additionally, a fuzzy set has *compact support* if its support is finite.

<u>Crossover point:</u>

The element of a fuzzy set that has a grade of membership equal to 0.5 is known as the crossover point.

For X_2 the crossover point is 2.



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Support, Crossover, Singleton

Fuzzy singleton:

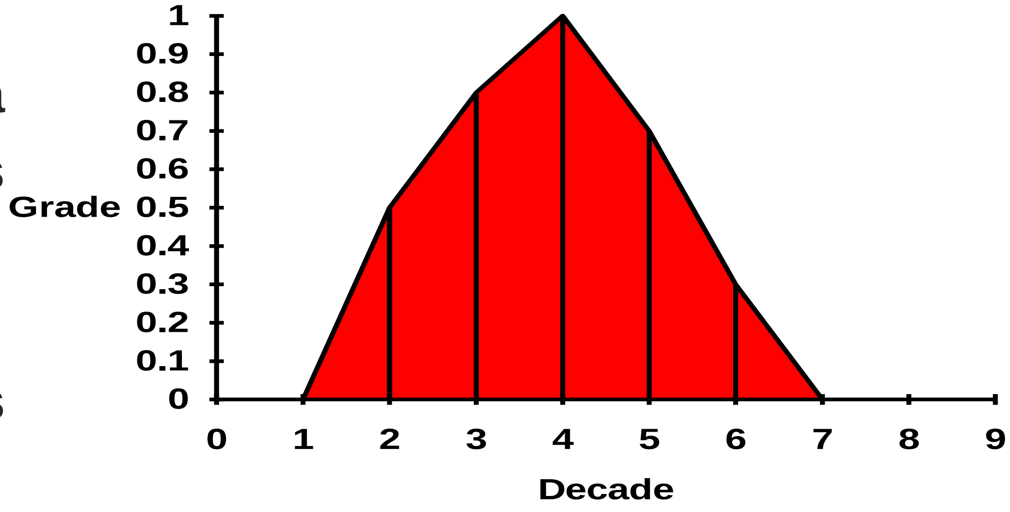
The fuzzy set whose support is a single point in the universe of discourse with grade of membership equal to one is known as the fuzzy singleton.

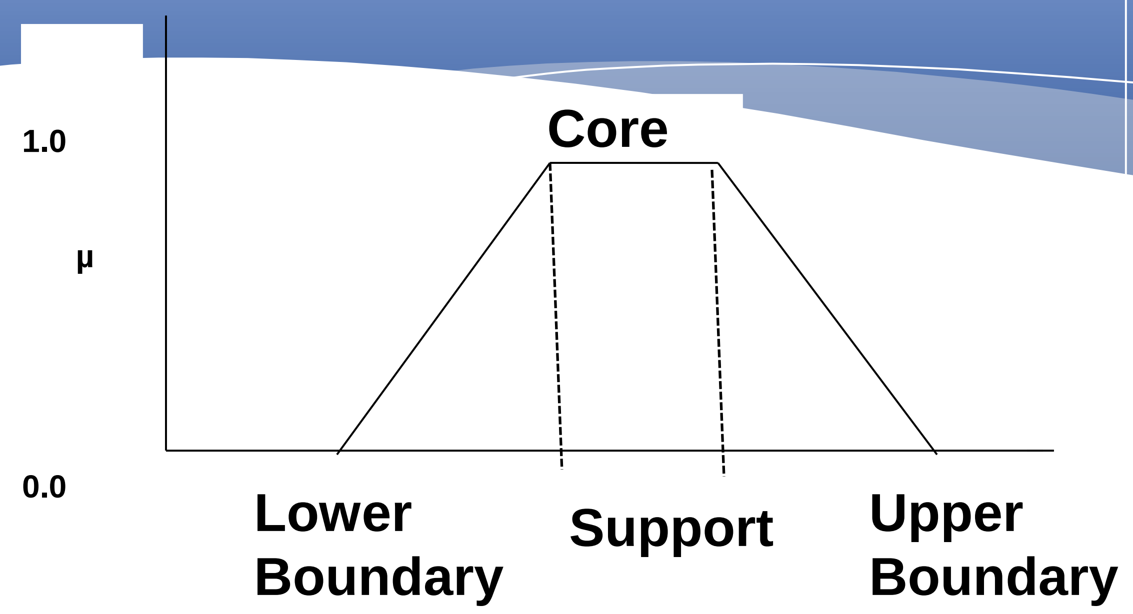
α-Level sets:

The fuzzy set that contains the elements which have a grade of membership greater than the α-level set is known as the α-Level set.

For X_2 the α-Level set when α=0.6 is {3,4,5}.

Whereas for X_2 the α-Level set when α=0.4 is {2,3,4,5}.





Core
1.0
μ
0.0
Lower Boundary
Support
Upper Boundary

Fuzzy Membership Functions

Fuzzy Set Membership Function

Membership Function

- The membership function fully defines the fuzzy set
- A membership function provides a measure of *the degree of similarity* of an element to a fuzzy set

Membership functions can

- either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
- Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)

Membership Functions

- The Fuzzy Logic Toolbox includes 11 built-in membership function types. These 11 functions are, in turn, built from several basic functions:
  - o Piecewise linear functions.
  - o Gaussian distribution function.
  - o Sigmoid curve.
  - o Quadratic polynomial curves.
  - o Cubic polynomial curves.

- The simplest membership functions are formed using straight lines. These straight line membership functions have the advantage of simplicity.
  - o Triangular membership function: trimf.
  - o Trapezoidal membership function: trapmf.

- Two membership functions are built on the Gaussian distribution curve: a simple Gaussian curve and a two-sided composite of two different Gaussian curves. The two functions are gaussmf and gauss2mf.

- The generalized bell membership function is specified by three parameters and has the function name gbellmf.

- Sigmoidal membership function: sigmf.

- Polynomial based curves: Three related membership functions are the Z, S, and Pi curves, all named because of their shape (The functions zmf, smf and pimf).

- Fuzzy Logic Toolbox also allows you to create your own membership functions.

Example:

```
x = (0:0.1:10)';
y1 = trapmf (x, [2 3 7 9]);
y2 = trapmf (x, [3 4 6 8]);
y3 = trapmf (x, [4 5 5 7]);
y4 = trapmf (x, [5 6 4 6]);
plot (x, [y1 y2 y3 y4]);
```

Fuzzy Set Membership Functions (Continue)

Triangular membership function

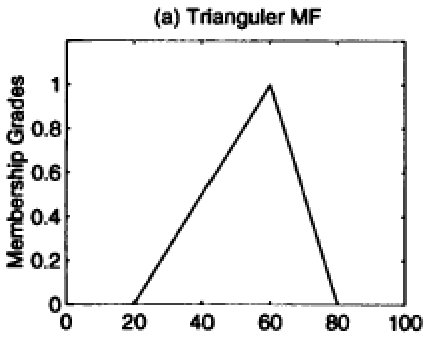
A *triangular* membership function is specified by three parameters {a, b, c}

a, b and c represent the x coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)

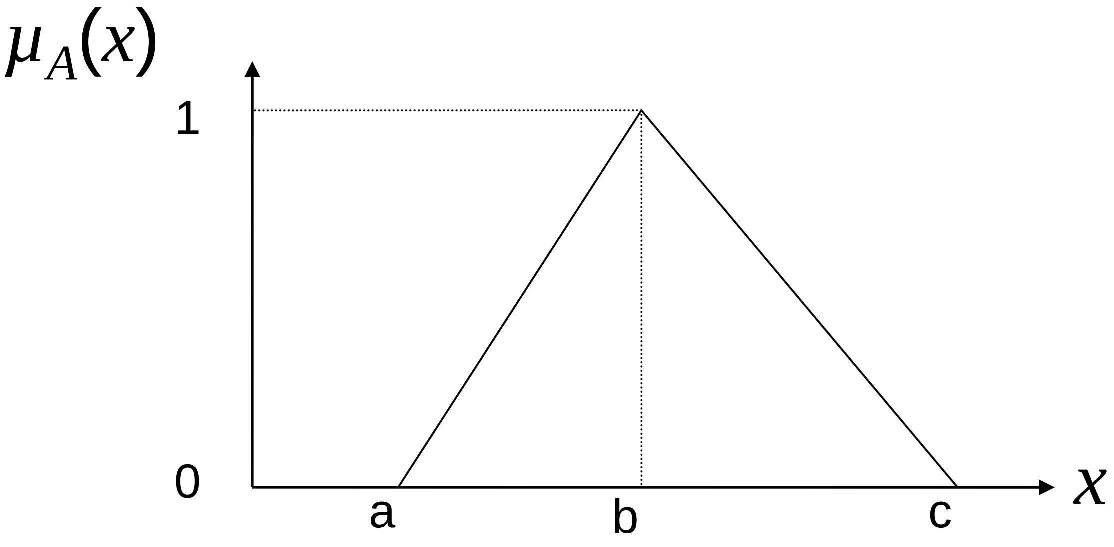
$$\text{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a. \\ \dfrac{x-a}{b-a}, & a \le x \le b. \\ \dfrac{c-x}{c-b}, & b \le x \le c. \\ 0, & c \le x. \end{cases}$$

$$\text{triangle}(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

$$a < b < c$$



$\text{triangle}(x;20,60,80)$



@Fuzzy Set Theory.pptx

Fuzzy Set Membership Functions (Continue)

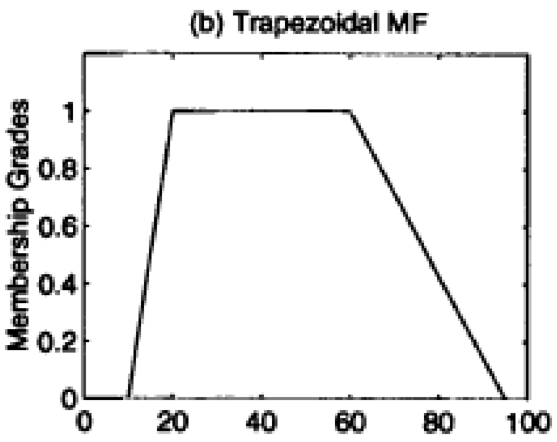
Trapezoid membership function

A *trapezoidal* membership function is specified by four parameters {a, b, c, d} as follows:

$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0, & x \leq a. \\ \dfrac{x-a}{b-a}, & a \leq x \leq b. \\ 1, & b \leq x \leq c. \\ \dfrac{d-x}{d-c}, & c \leq x \leq d. \\ 0, & d \leq x. \end{cases}$$

$$\text{trapezoid}(x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right).$$

$a < b \leq c < d$



$$\text{trapezoid}(x; 10, 20, 60, 95).$$

Gaussian membership function

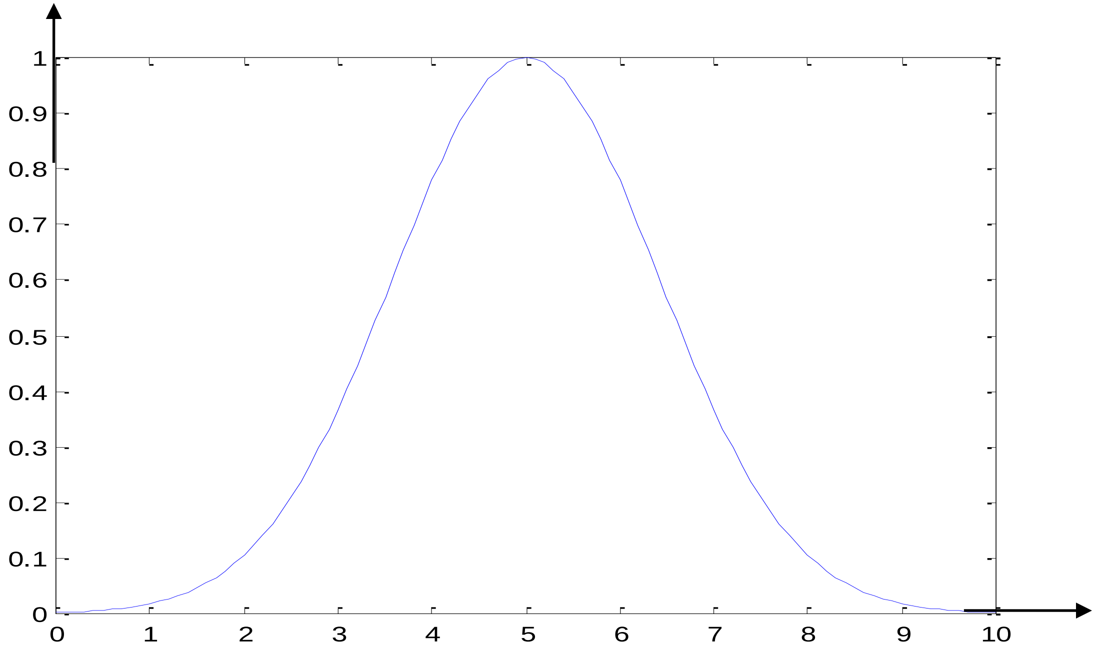
$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

c: centre
s: width
m: fuzzification factor (e.g., *m*=2)

$$\text{gaussian}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$

$\text{gaussian}(x;50,20)$.

$\mu_A(x)$

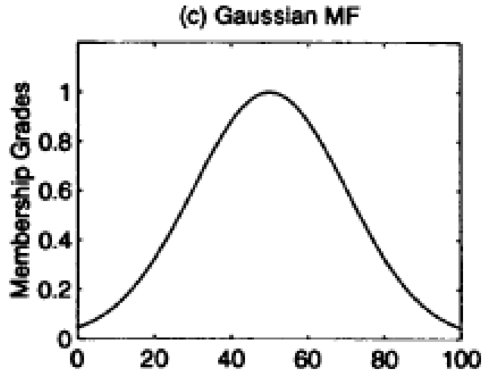


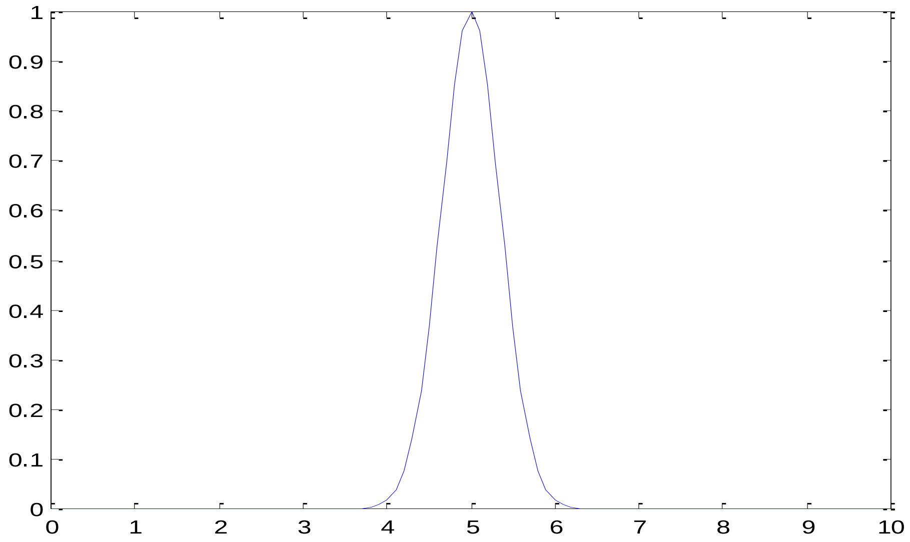
x

$c=5$

$s=2$

$m=2$

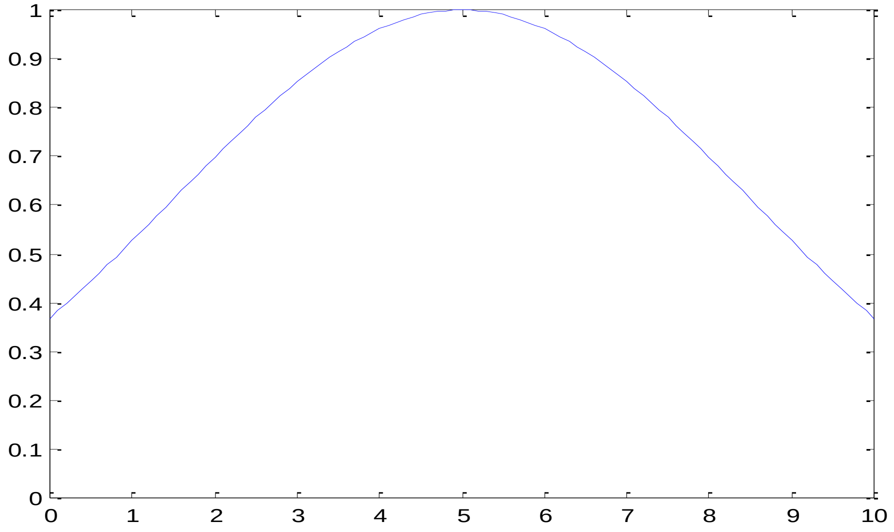




$c=5$

$s=0.5$

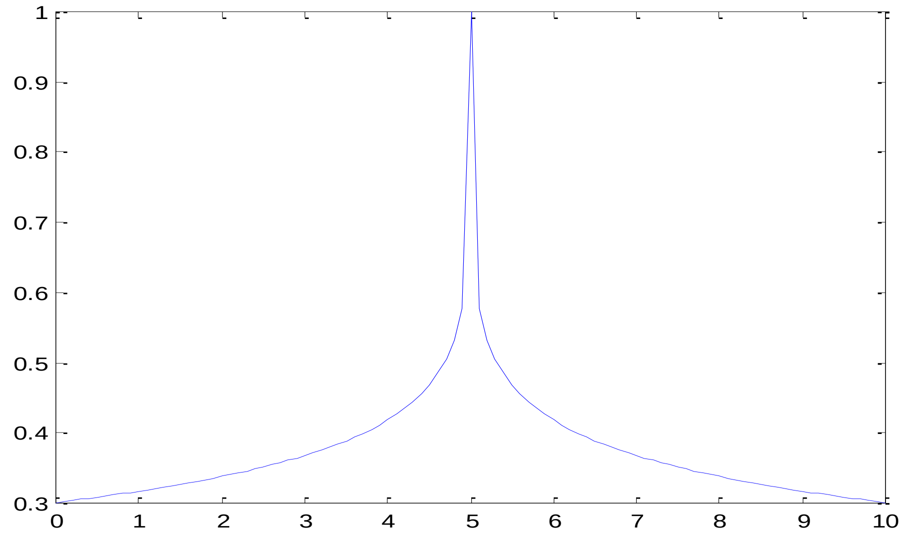
$m=2$



$c=5$

$s=5$

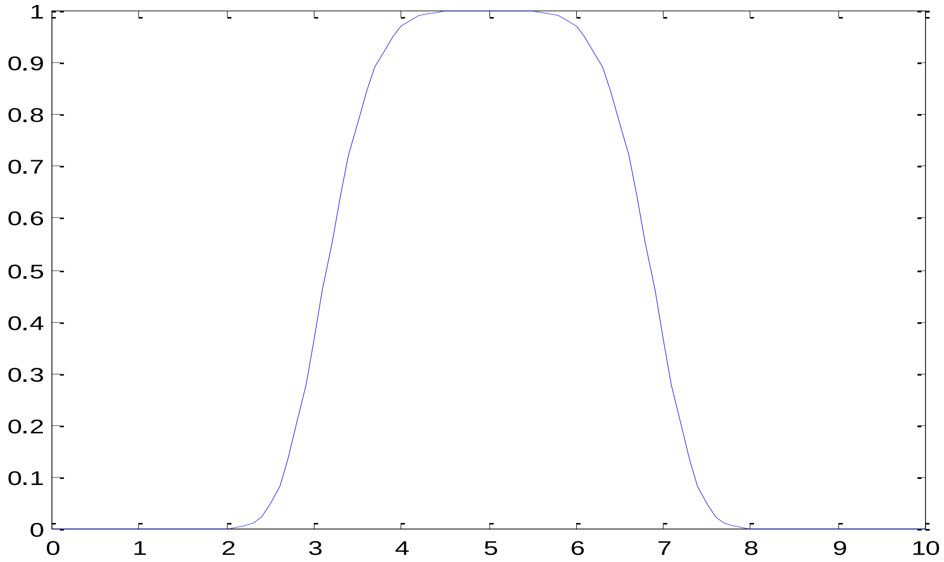
$m=2$



$c=5$

$s=2$

$m=0.2$



$c=5$

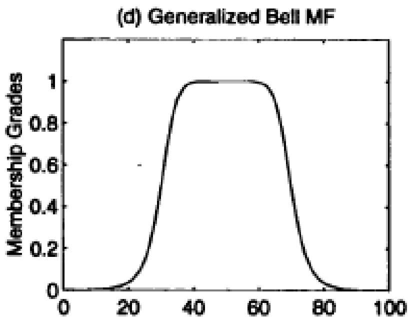
$s=5$

$m=5$

Generalized Bell Membership Function

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}},$$

$$bell(x; 20, 4, 50).$$

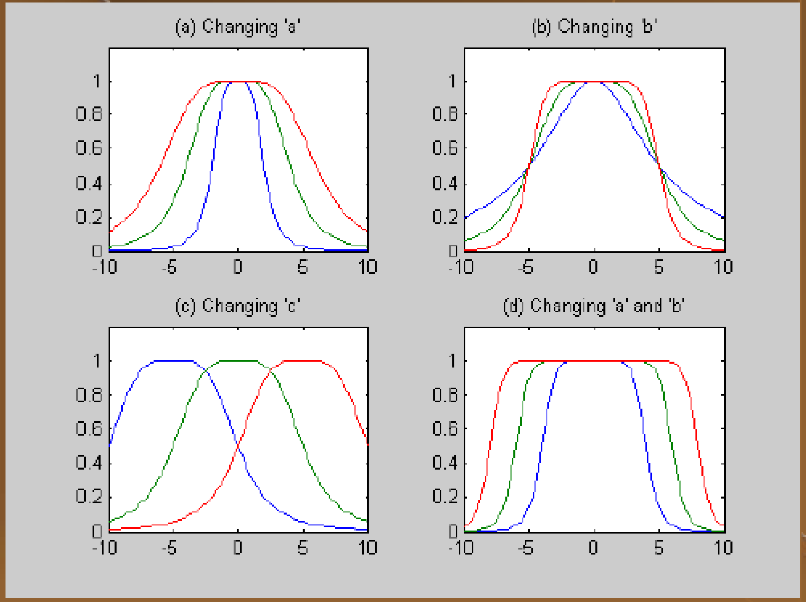


If b is –ve, its shape becomes upside down bell.
This MF is direct generalization of Cauchy distribution **= Cauchy MF.**
Adjust c and a to vary center and width f MF
and adjaust b to control slopes at crossover points.
With one more parameter, it has more degree of freedom than Gaussian MF.

- Gaussian MFs and bell MFs achieve smoothness, they are unable to specify asymmetric Mfs which are important in many applications

- Asymmetric & close MFs can be synthesized using either the absolute difference or the product of two sigmoidal functions

- Change of parameters in the generalized bell MF

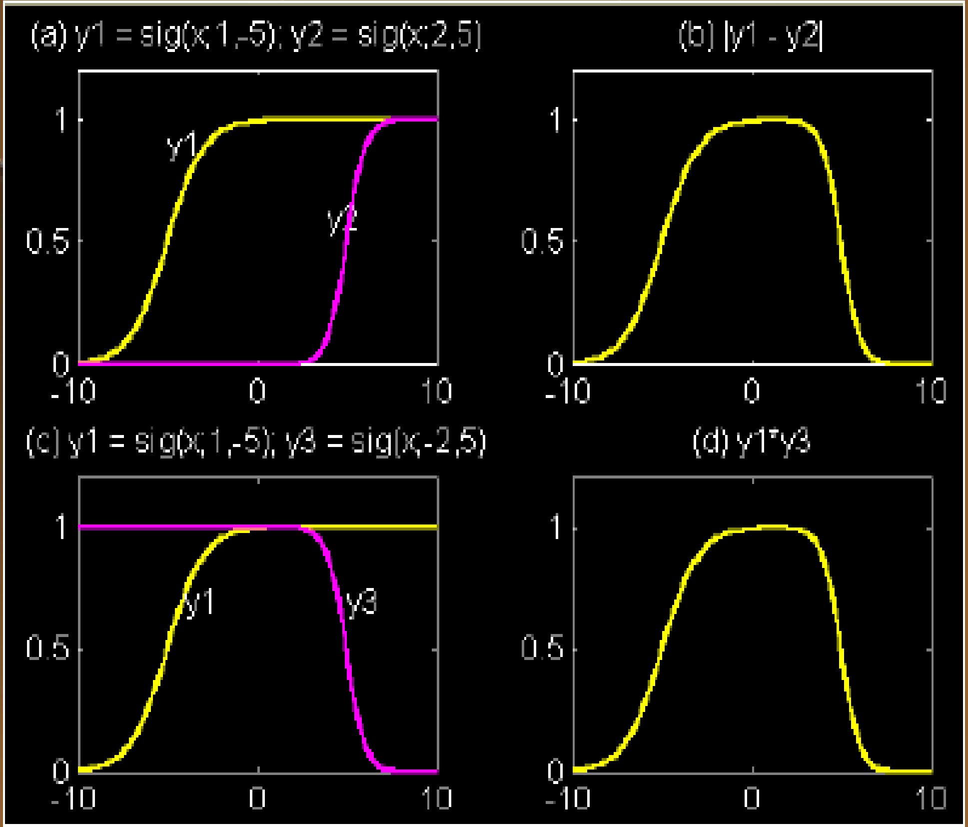


- Sigmoidal MF: $$\mathrm{sigmf}(x;a,c) = \frac{1}{1+e^{-a(x-c)}}$$

Extensions:

Abs. difference of two sig. MF

Product of two sig. MF



A controls slope at crossover point x==c.

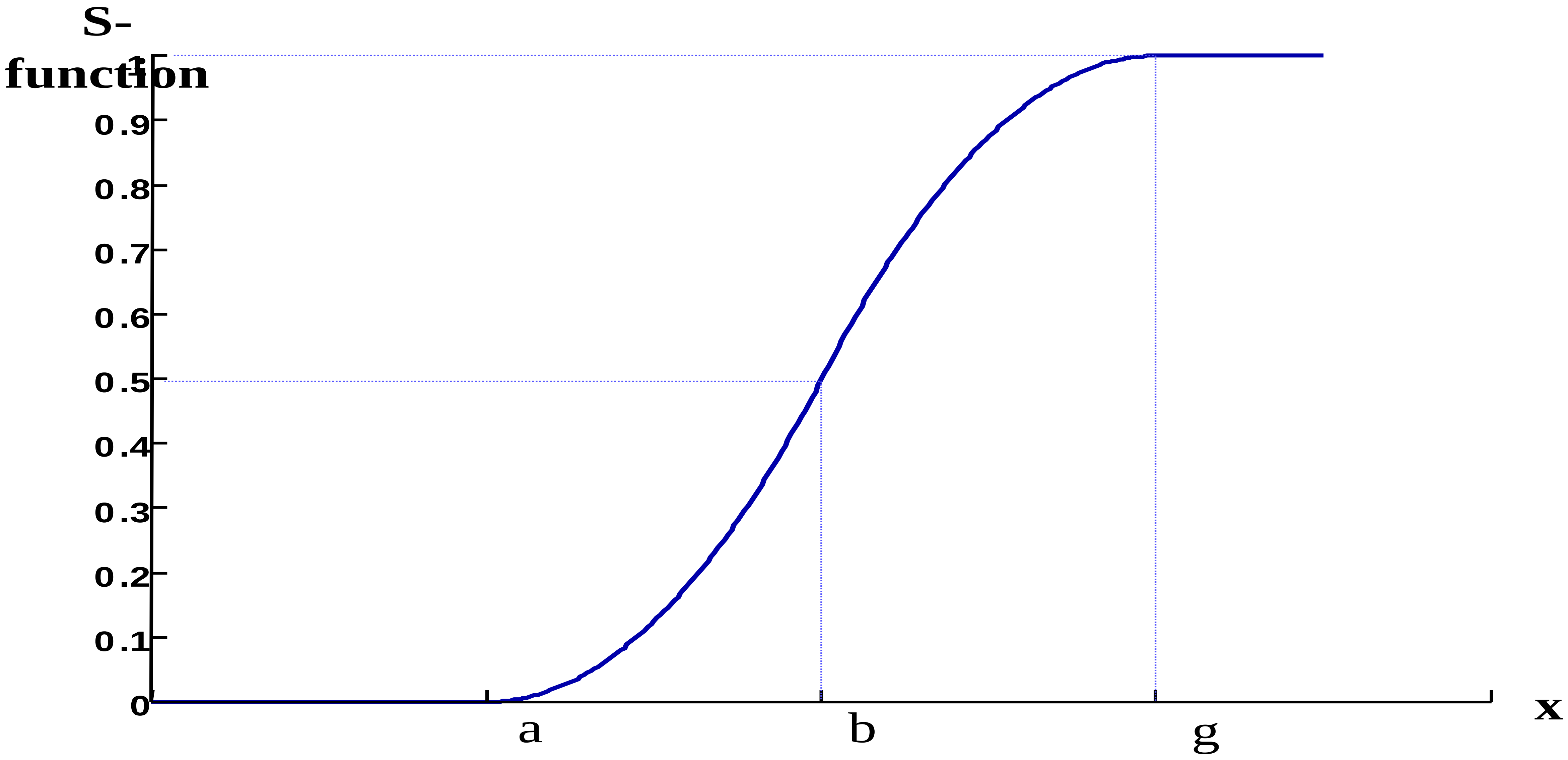
It is inherently open right or open left, so appropriate for “very large” or “very negative” concepts.

2 simple ways to get close MF using sigmoidal MF:

The S-Function

$$S(x;\alpha,\beta,\gamma)=\begin{cases} 0 \; for\, x \le \alpha \\ 2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 \; for\, \alpha \langle x \le \beta \\ 1-2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 \; for\, \beta \langle x \le \gamma \\ 1 \; for\, x \rangle \gamma \end{cases}$$

The S-Function



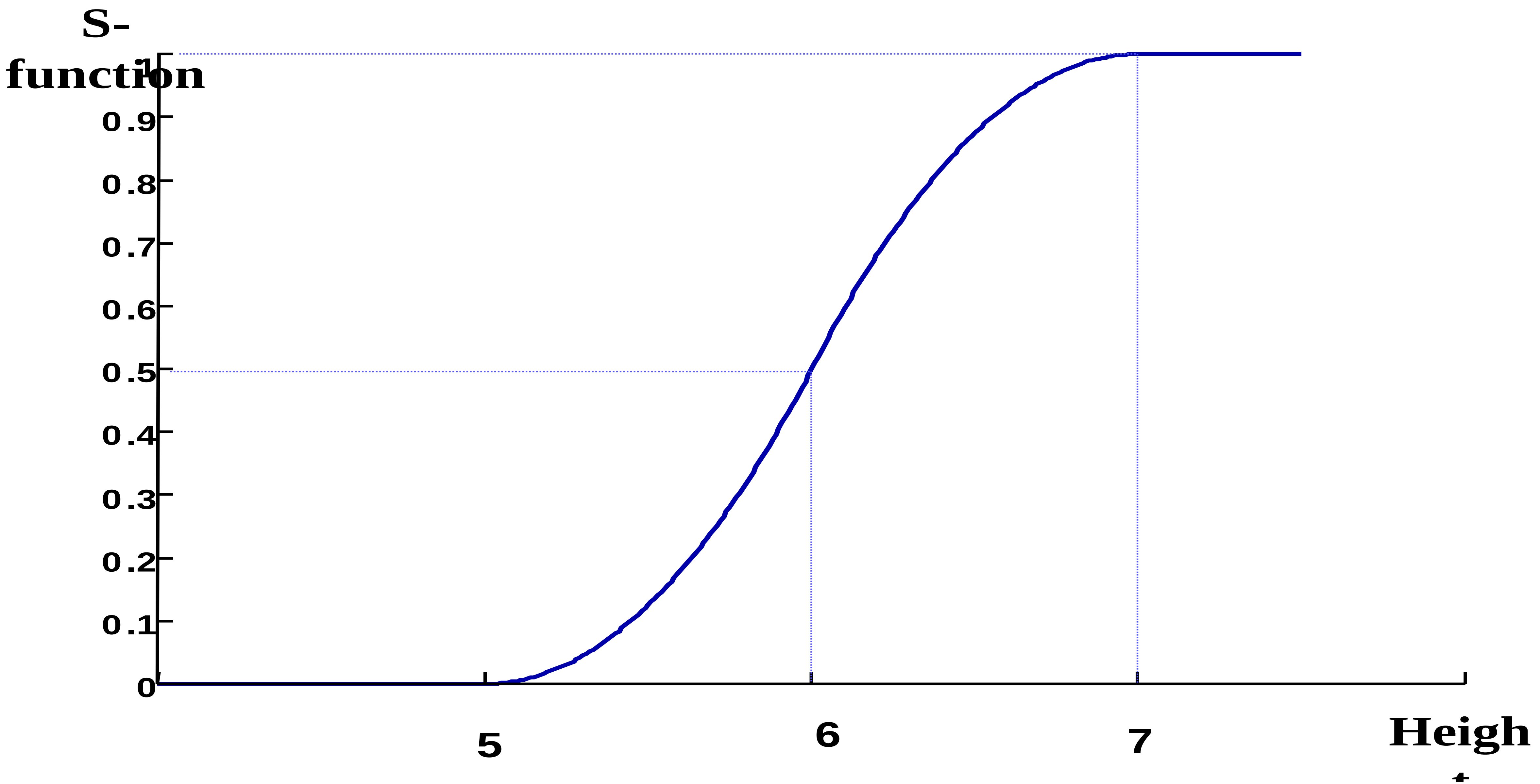
The S-Function

- As one can see the S-function is flat at a value of 0 for x≤a and at 1 for x≥g. In between a and g the S-function is a quadratic function of x.
- To illustrate the S-function we shall use the fuzzy proposition *Dimitris is tall*.
- We assume that:
- Dimitris is an adult
- The universe of discource are normal people (i.e., excluding the extremes of basketball players etc.)
- then we may assume that anyone less than 5 feet is not tall (i.e., a=5) and anyone more than 7 feet is tall (i.e., g=7).
- Hence, b=6.
- Anyone between 5 and 7 feet has a membership function which increases monotonically with his height.

S-Function

$$S(x;5,6,7) = \begin{cases} 0 \; for \, x \leq 5 \\ \left(\dfrac{x-5}{2}\right)^2 \; for \, 5 \langle x \leq 6 \\ 1-\left(\dfrac{x-7}{2}\right)^2 \; for \, 6 \langle x \leq 7 \\ 1 \; for \, x \rangle 7 \end{cases}$$

Hence the membership of 6 feet tall people is 0.5, whereas for 6.5 feet tall people increases to 0.9.



Pi-Function

The π-MF with two parameters a and c is a π-shaped MF defined via the S and Z-MFs introduced earlier, as follows:

$$\pi(x;a,c) = \begin{cases} S(x;c-a,c), & \text{for } x \leq c \\ Z(x;c,c+a), & \text{for } x > c, \end{cases} \tag{2.67}$$

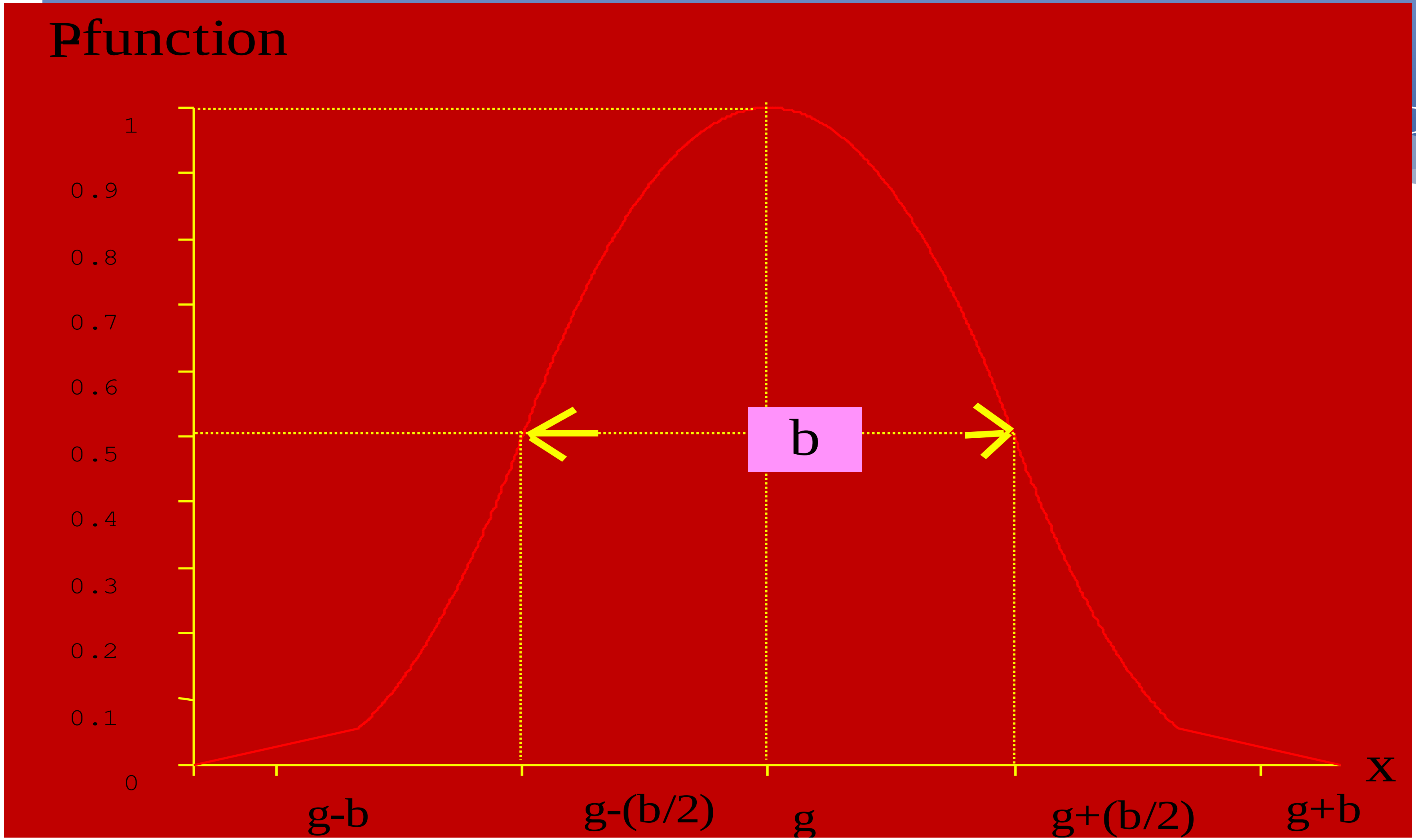
where c is the center and a (> 0) is the spread on each side of the MF. (a)

The **two-sided π-MF** is an extension of the π-MF introduced previously; it is defined with four parameters a, b, c, and d:

$$\text{ts\_}\pi(x;a,b,c,d) = \begin{cases} 0, & \text{for } x \leq a. \\ S(x,a,b), & \text{for } a < x < b. \\ 1, & \text{for } b \leq x \leq c. \\ Z(x,c,d), & \text{for } c < x < d. \\ 0, & \text{for } d \leq x. \end{cases} \tag{2.68}$$

$$\Pi(x;\beta,\gamma) = \begin{cases} S(x;\gamma-\beta,\gamma-\frac{\beta}{2},\gamma) \text{ for x} \leq \gamma \\ 1-S(x;\gamma,\gamma+\frac{\beta}{2},\gamma+\beta) \text{ for x} \geq \gamma \end{cases}$$

Pi-Function



The Pi-function goes to zero at $\gamma < \beta$, and the 0.5 point is at $\gamma = (\beta/2)$.

Notice that the β parameter represents the bandwidth of the 0.5 points.

β=b

γ=g here.

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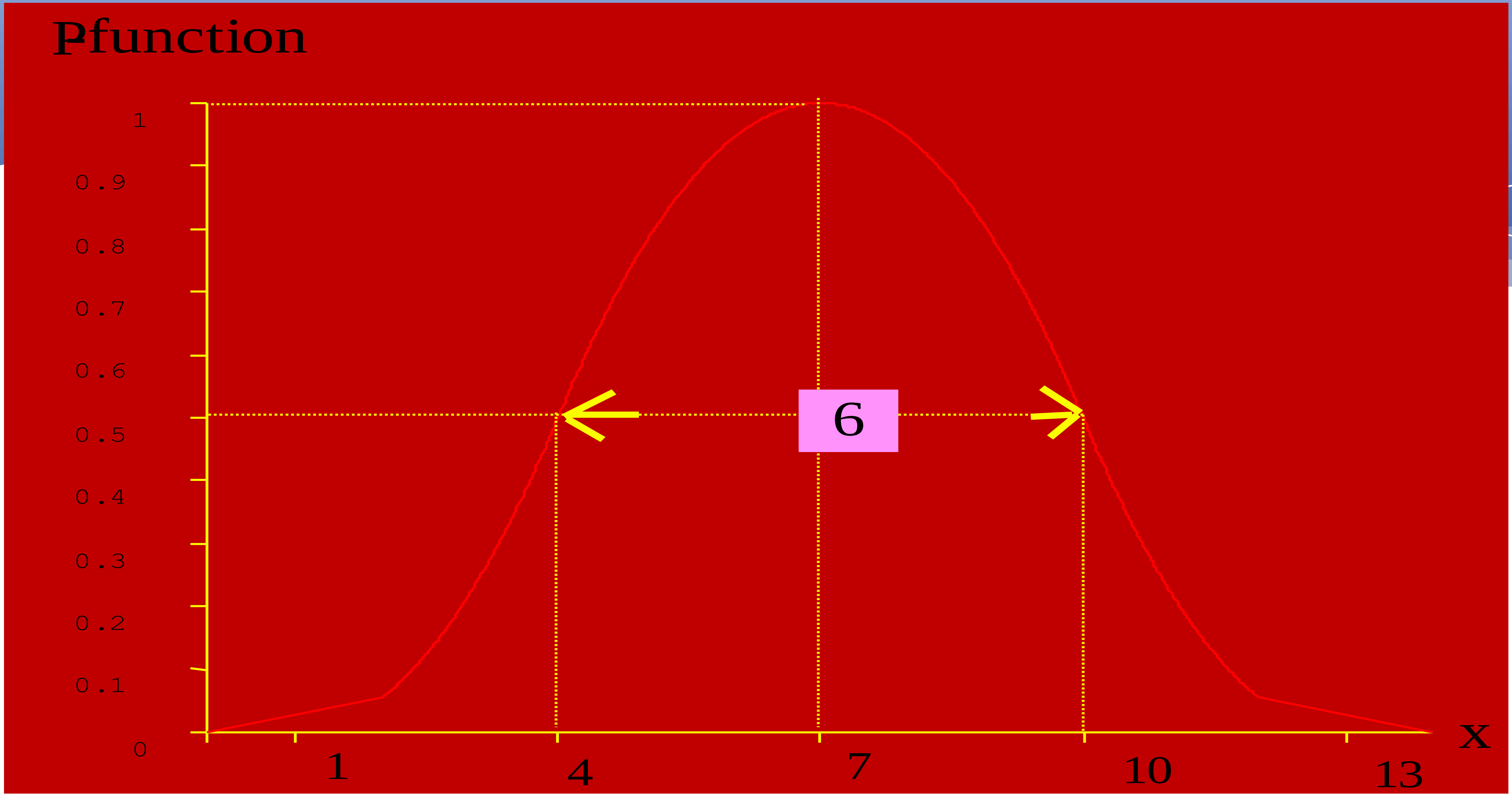
Pi-Function Example



$$\Pi(x;6,7)=\begin{cases} S(x;1,4,7) \text{ for x} \leq 7 \\ 1-S(x;7,10,13) \text{ for x} \geq 7 \end{cases}$$

The **S-MF** with two parameters l and r ($l < r$) is an S-shaped open-right MF defined by

$$\mathrm{S}(x;l,r) = \begin{cases} 0, & \text{for } x \le l. \\ 2(\frac{x-l}{r-l})^2, & \text{for } l < x \le \frac{l+r}{2}. \\ 1-2(\frac{r-x}{r-l})^2, & \text{for } \frac{l+r}{2} < x \le r. \\ 1, & \text{for } r < x. \end{cases} \tag{2.65}$$

The **Z-MF** with two parameters l and r ($l < r$) is a Z-shaped open-left MF defined by

$$\mathrm{Z}(x;l,r) = 1 - \mathrm{S}(x;l,r), \tag{2.66}$$

where $\mathrm{S}(x;l,r)$ is the S-MF in the previous exercise. Repeat (a) through (d) of Exercise 6 with the Z-MF.

The **two-sided Gaussian MF** is defined by

$$\text{ts\_gaussian}(x;c_1,\sigma_1,c_2,\sigma_2) = \begin{cases} \exp\left[-\frac{1}{2}\left(\frac{x-c_1}{\sigma_1}\right)^2\right], & \text{for } x \le c_1. \\ 1, & \text{for } c_1 < x < c_1. \\ \exp\left[-\frac{1}{2}\left(\frac{x-c_2}{\sigma_2}\right)^2\right], & \text{for } c_2 \le x. \end{cases} \tag{2.69}$$

Fuzzy Rules and Fuzzy reasoning

Linguistic variable, linguistic term

Principle of incompatibility

Linguistic variable: A *linguistic variable* is a variable whose values are sentences in a natural or artificial language.

A linguistic variable is characterized by a quintuple $(x, T(x), X, G, M)$ in which x is the name of the variable; $T(x)$ is the **term set** of x—that is, the set of its **linguistic values** or **linguistic terms**; X is the universe of discourse; G is a **syntactic rule** which generates the terms in $T(x)$; and M is a **semantic rule** which associates with each linguistic value A its meaning $M(A)$, where $M(A)$ denotes a fuzzy set in X.

For example, the values of the fuzzy variable *height* could be *tall, very tall, very very tall, somewhat tall, not very tall, tall but not very tall, quite tall, more or less tall.*
Tall is a ***linguistic*** **value or primary term**.

If ***age*** is a linguistic variable then its term set is
T(age) = { young, not young, very young, not very young,…… middle aged, not middle aged,… old, not old, very old, more or less old, not very old,…not very young and not very old,…}.

X=[0,100], primary terms, negation, hedges(very, more or less), connectives.

Contsructing MFs for composite linguistic terms

X=[0,100]

$$\mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + (\frac{x}{20})^4},$$

$$\mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + (\frac{x-100}{30})^6},$$

- *more or less old* $= \text{DIL}(old) = old^{0.5}$
$$= \int_X \sqrt{\frac{1}{1 + (\frac{x-100}{30})^6}} \Big/ x.$$
- *not young and not old* $= \neg young \cap \neg old$
$$= \int_X \left[1 - \frac{1}{1 + (\frac{x}{20})^4}\right] \wedge \left[1 - \frac{1}{1 + (\frac{x-100}{30})^6}\right] \Big/ x.$$
- *young but not too young* $= young \cap \neg young^2$
$$= \int_X \left[\frac{1}{1 + (\frac{x}{20})^4}\right] \wedge \left[1 - \left(\frac{1}{1 + (\frac{x}{20})^4}\right)^2\right] \Big/ x.$$
- *extremely old*
$$= \text{CON}(\text{CON}(\text{CON}(old))) = ((old^2)^2)^2 = \int_X \left[\frac{1}{1 + (\frac{x-100}{30})^6}\right]^8 \Big/ x.$$

Fuzzy Rules/Fuzzy implications/Fuzzy conditional statements

Fuzzy rules are useful for modeling human thinking, perception (Opinion,view) and judgment.

A fuzzy if-then rule is of the form "If x is A then y is B" where A and B are linguistic values defined by fuzzy sets on universes of discourse *X* and *Y*, respectively.

"x is A" is called ***antecedent or premise*** and "y is B" is called ***consequent or conclusion***.

Examples, for such a rule are –

If <u>pressure is high</u>, <u>then volume is small</u>.

If the road is slippery, then driving is dangerous.

If the fruit is ripe, then it is soft.

Binary fuzzy relation

- A binary fuzzy relation is a fuzzy set in $X \times Y$ which maps each element in $X \times Y$ to a membership value between *0* and *1*.
- Binary fuzzy relations are fuzzy sets in X × Y which map each element in X × Y to a membership grade between 0 and 1.
- Unary fuzzy relations are fuzzy sets with one dimensional MFs and binary fuzzy relations are fuzzy sets with two dimensional MFs.

- If X and Y are two universes of discourse, then

  $R = \{((x,y), \mu_R(x, y)) \mid (x,y) \in X \times Y\}$ is a binary fuzzy relation in $X \times Y$.

  $X \times Y$ indicates cartesian product of X and Y.
- The fuzzy rule "*If x is A then y is B*" may be abbreviated as $A \rightarrow B$ and is interpreted as $A \times B$.

  $\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$, $a = \mu_A(x)$, $b = \mu_B(y)$, f = Fuzzy implication function, $\mu_R(x,y)$= 2-D MF

- A fuzzy if then rule may be defined (Mamdani) as a binary fuzzy relation R on the product space $X \times Y$.

  $$R = A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) \textit{ T-norm } \mu_B(y) / (x,y).$$

@Fuzzy Set Theory.pptx

The Extension Principle of Zadeh

- General procedure to extend crisp domains of mathematical expressions to fuzzy domains.
- Given a formula *f(x)* and a fuzzy set *A* defined by, $\mu_A(x)$
- how do we compute the membership function of *f(A)* ?
- How this is done is what is called the *extension principle* (of professor Zadeh).
- What the extension principle says is that

$$f_\alpha(A) = f(A(\alpha)).$$

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n.$$

$$B = f(A) = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n,$$

$$y_i = f(x_i),\ i = 1, \ldots, n.$$

- The formal definition is: $[f(A)](y) = sup_{x|y=f(x)}\{\mu_A(x)\}$

Extension Principle - Example

Let $f(x) = ax+b$,

$a \in A = 1/2/3$, $b \in B = 2/3/5$, and $x = 6$. Then
$f(x) = 6A + B = 8/15/23$

Example 3.1 *Application of the extension principle to fuzzy sets with discrete universes*

Let

$$A = 0.1/-2 + 0.4/-1 + 0.8/0 + 0.9/1 + 0.3/2$$

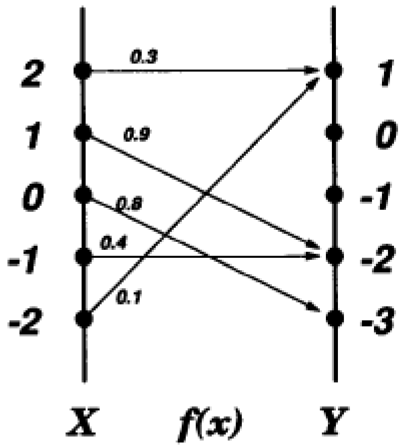
and

$$f(x) = x^2 - 3.$$

Upon applying the extension principle, we have

$$\begin{aligned} B &= 0.1/1 + 0.4/-2 + 0.8/-3 + 0.9/-2 + 0.3/1 \\ &= 0.8/-3 + (0.4 \vee 0.9)/-2 + (0.1 \vee 0.3)/1 \\ &= 0.8/-3 + 0.9/-2 + 0.3/1, \end{aligned}$$

where $\vee$ represents max. Figure 3.1 illustrates this example.



Extension principle on fuzzy sets with discrete universes.

Max-min composition

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively. The **max-min composition** of $\mathcal{R}_1$ and $\mathcal{R}_2$ is a fuzzy set defined by

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{[(x,z), \max_y \min(\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(y,z))] | x \in X, y \in Y, z \in Z\}, \quad (3.5)$$

or, equivalently,

$$\begin{aligned} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x,z) &= \max_y \min[\mu_{\mathcal{R}_1}(x,y), \mu_{\mathcal{R}_2}(y,z)] \\ &= \vee_y [\mu_{\mathcal{R}_1}(x,y) \wedge \mu_{\mathcal{R}_2}(y,z)], \end{aligned} \quad (3.6)$$

with the understanding that $\vee$ and $\wedge$ represent max and min, respectively.

When $\mathcal{R}_1$ and $\mathcal{R}_2$ are expressed as relation matrices, the calculation of $\mathcal{R}_1 \circ \mathcal{R}_2$ is almost the same as matrix multiplication, except that $\times$ and $+$ are replaced by $\wedge$ and $\vee$, respectively. For this reason, the max-min composition is also called the **max-min product**.

Definition 3.4 *Max-product composition*

Assuming the same notation as used in the definition of max-min composition, we can define **max-product composition** as follows:

$$\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x,z) = \max_y [\mu_{\mathcal{R}_1}(x,y) \mu_{\mathcal{R}_2}(y,z)]. \quad (3.8)$$

□

The following example demonstrates how to apply max-min and max-product composition and how to interpret the resulting fuzzy relations $\mathcal{R}_1 \circ \mathcal{R}_2$.

Let

$\mathcal{R}_1$ = "x is relevant to y"

$\mathcal{R}_2$ = "y is relevant to z"

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1,2,3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$, and $Z = \{a, b\}$. Assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ can be expressed as the following relation matrices:

$$\mathcal{R}_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix}$$

$$\mathcal{R}_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}.$$

Now we want to find $\mathcal{R}_1 \circ \mathcal{R}_2$, which can be interpreted as a derived fuzzy relation "x is relevant to z" based on $\mathcal{R}_1$ and $\mathcal{R}_2$. For simplicity, suppose that we are only interested in the degree of relevance between 2 ($\in X$) and a ($\in Z$). If we adopt max-min composition, then

$$\begin{aligned} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(2,a) &= \max(0.4 \wedge 0.9,\ 0.2 \wedge 0.2,\ 0.8 \wedge 0.5,\ 0.9 \wedge 0.7) \\ &= \max(0.4,\ 0.2,\ 0.5,\ 0.7) \\ &= 0.7 \text{ (by max-min composition).} \end{aligned}$$

On the other hand, if we choose max-product composition instead, we have

$$\begin{aligned} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(2,a) &= \max(0.4 \times 0.9,\ 0.2 \times 0.2,\ 0.8 \times 0.5,\ 0.9 \times 0.7) \\ &= \max(0.36,\ 0.04,\ 0.40,\ 0.63) \\ &= 0.63 \text{ (by max-product composition).} \end{aligned}$$

Inference Engine

- Fuzzy rules
  - based on fuzzy premises and fuzzy consequences

- e.g.
  - If height is Short and weight is Light then feet are Small
  - Short(height) AND Light(weight) => Small(feet)

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Fuzzy Reasoning

- **Approximate reasoning –** An inference procedure that derives conclusions from a set of if-then rules and known facts.
- Modus ponens (in traditional true value logic)- $A \rightarrow B.$ infers truth of a proposition B from truth of A.
- A=tomato is red
- B=tomato is ripe
- $A \rightarrow B.$ If it is true that "Tomato is red" then it is also true that "Tomato is ripe".

$$R = A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) \tilde{*} \mu_B(y)/(x,y),$$

A coupled with B

premise 1 (fact):	x is A,
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B.

premise 1 (fact):	x is A',
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B',

- Approximate manner = "if tomato is more or less red" then we may infer that "tomato is more or less ripe".
- This inference procedure is called – **Approximate reasoning** or **Fuzzy reasoning** or **Generalized Modus Ponens (GMP).**

$$B' = A' \circ R = A' \circ (A \rightarrow B).$$

premise 1 (fact):	x is A' and y is B',
premise 2 (rule):	if x is A and y is B then z is C,
consequence (conclusion):	z is C'.

"$A \times B \rightarrow C.$"

- R can be viewed as a fuzzy set with 2-D MFs.

Fuzzy Reasoning

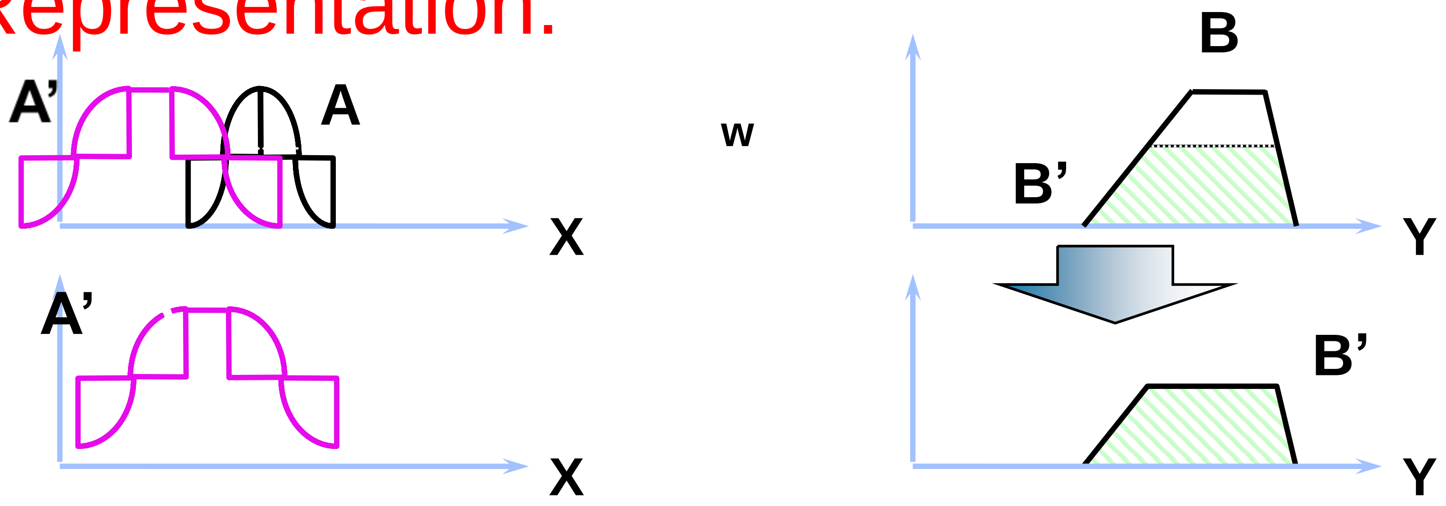
- Single rule with single antecedent

  Rule: if x is A then y is B

  Fact: x is A’

  Conclusion: y is B’

- Graphic Representation:



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Fuzzy Reasoning

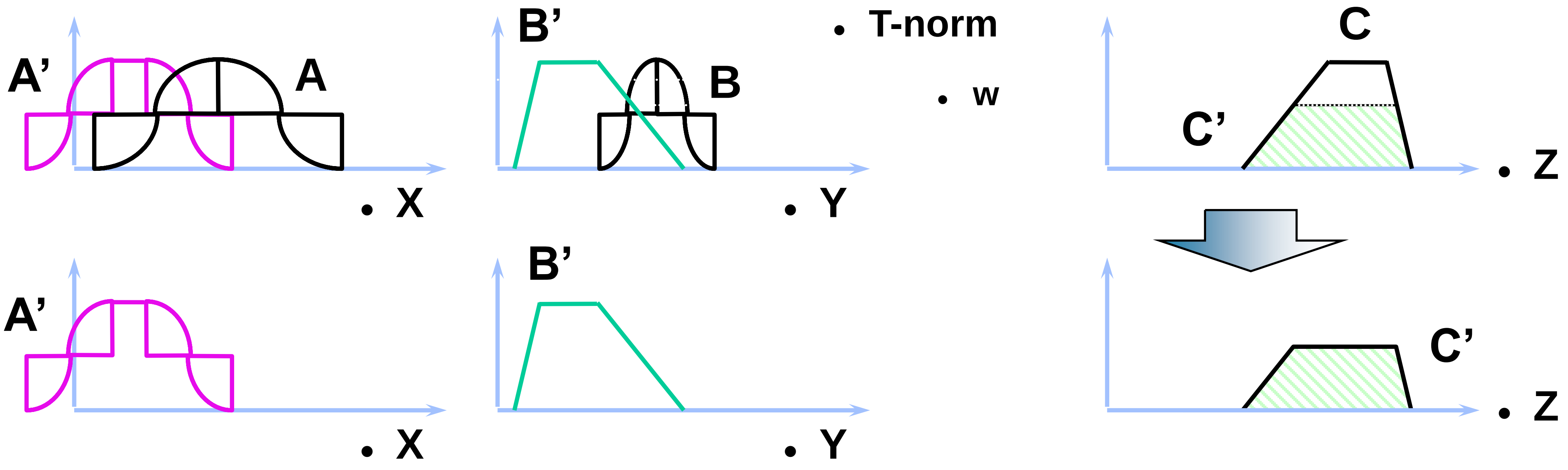
- Single rule with multiple antecedent

  Rule: if x is A and y is B then z is C

  Fact: x is A' and y is B'

  Conclusion: z is C'

- Graphic Representation:



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Fuzzy Reasoning

- Multiple rules with multiple antecedent

  Rule 1: if x is A_1 and y is B_1 then z is C_1

  Rule 2: if x is A_2 and y is B_2 then z is C_2

  Fact: x is A' and y is B'

  Conclusion: z is C'

- Graphic Representation: (next slide)

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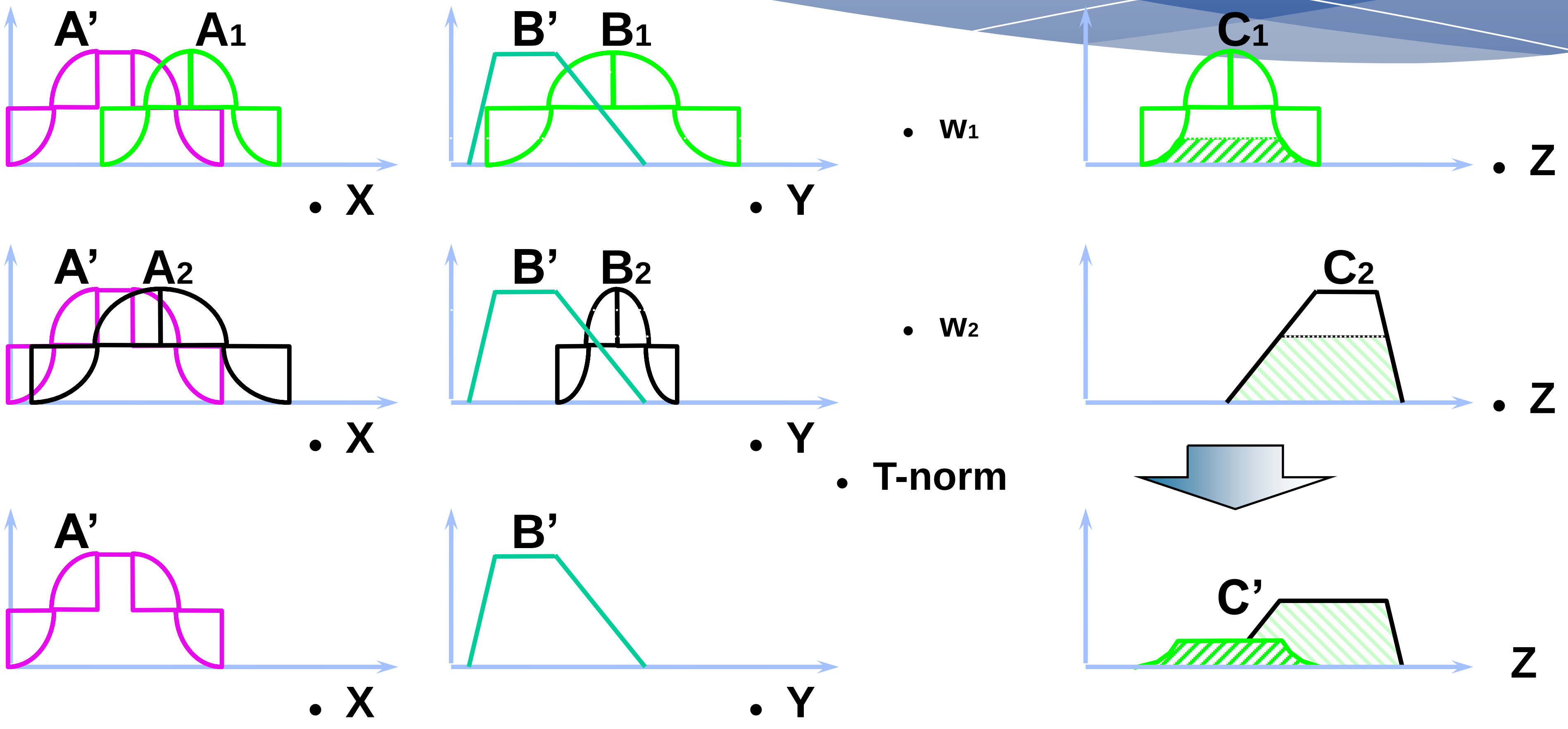
Fuzzy Reasoning

- Graphics representation:



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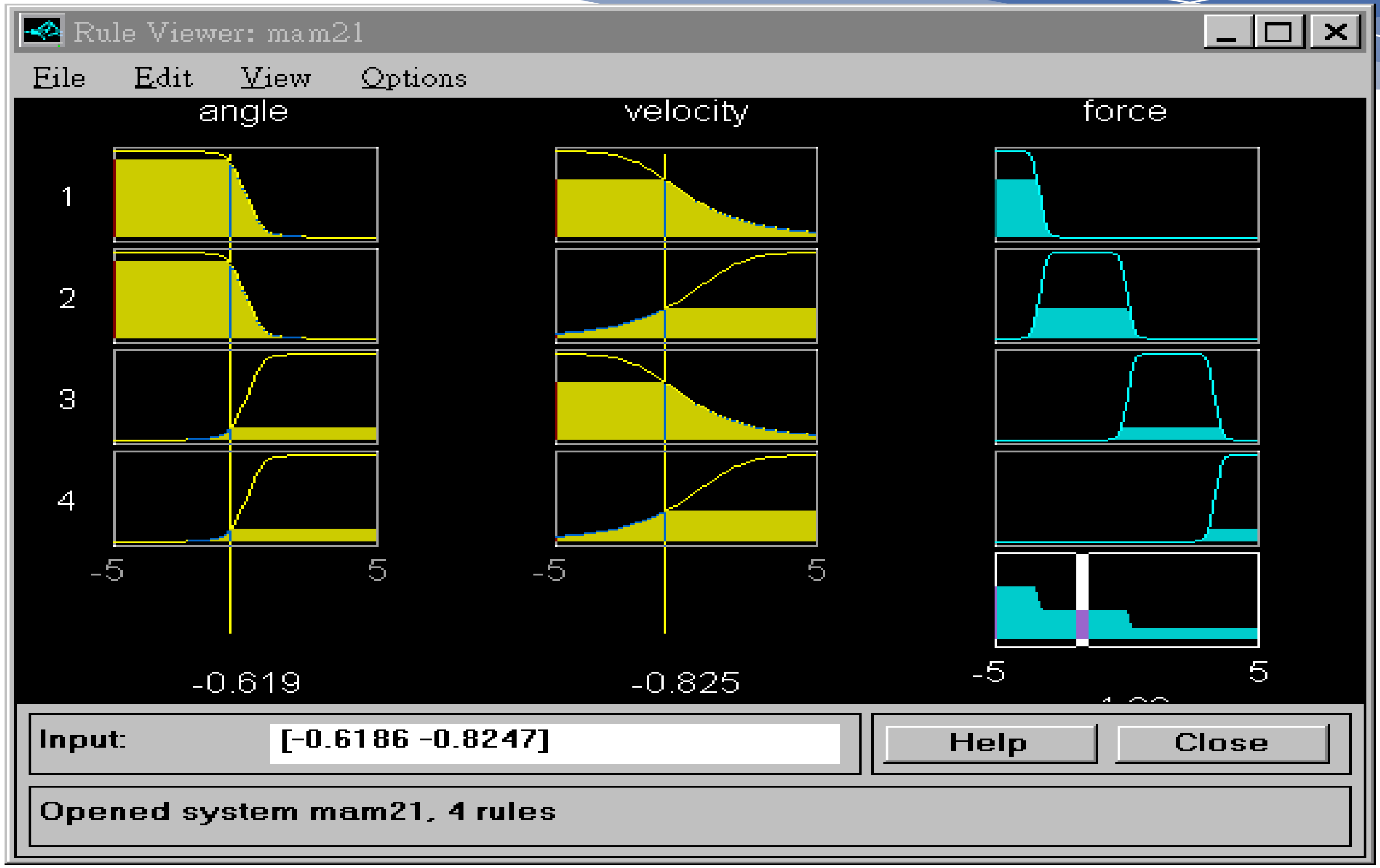
Fuzzy Reasoning: MATLAB Demo

- >> rule view mam21



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Process of Fuzzy reasoning

In summary, the process of fuzzy reasoning or approximate reasoning can be divided into four steps:

Degrees of compatibility Compare the known facts with the antecedents of fuzzy rules to find the degrees of compatibility with respect to each antecedent MF.

Firing strength Combine degrees of compatibility with respect to antecedent MFs in a rule using fuzzy AND or OR operators to form a firing strength that indicates the degree to which the antecedent part of the rule is satisfied.

Qualified (induced) consequent MFs Apply the firing strength to the consequent MF of a rule to generate a qualified consequent MF. (The qualified consequent MFs represent how the firing strength gets propagated and used in a fuzzy implication statement.)

Overall output MF Aggregate all the qualified consequent MFs to obtain an overall output MF.

These four steps are also employed in a fuzzy inference system, which is intro-

Other Variants

- Some terminology:
  - Degrees of compatibility (match)
  - Firing strength
  - Qualified (induced) MFs
  - Overall output MF

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Expert Systems: Fuzzy Inference

Contents

- FIS
- Mamdani Fuzzy Inference
  - Fuzzification of the input variables
  - Rule evaluation
  - Aggregation of the rule outputs
  - Defuzzification
- Sugeno Fuzzy Inference
- Mamdani or Sugeno?

Fuzzy Inference System

- FIS is a popular computing framework based on concepts of Fuzzy Set theory, Fuzzy if-then rules and Fuzzy reasoning.
- 3 conceptual components
- **Rule base** – selection of fuzzy rules
- **Database** – defines membership functions used in fuzzy rules
- **Reasoning Mechanism** – performs inference procedure.

- **Inputs –** Fuzzy inputs or Crisp inputs
- **Outputs** – Fuzzy sets (For controllers – Crisp outputs using defuzzification)

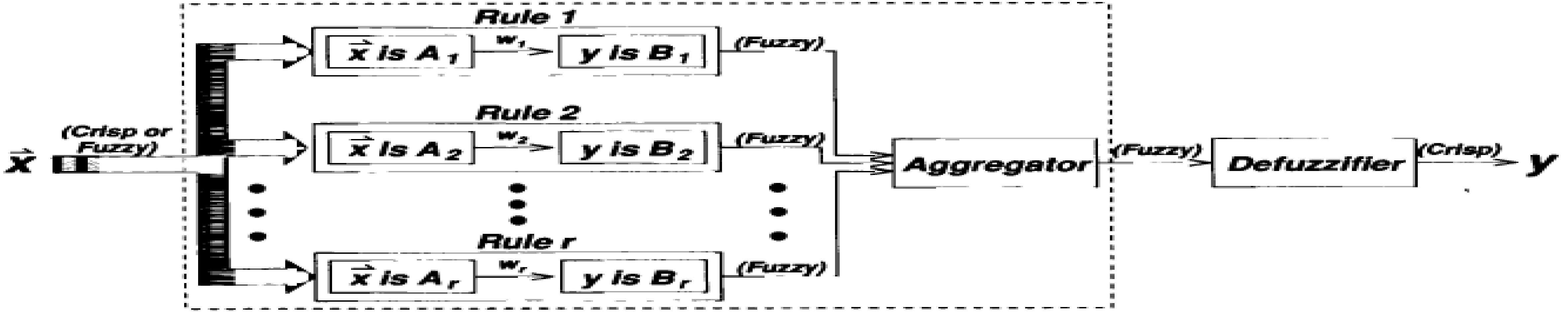

Figure 4.1. *Block diagram for a fuzzy inference system.*

Mamdani Fuzzy Inference

- The most commonly used fuzzy inference technique is the so-called **Mamdani** method.

- In 1975, Professor Ebrahim Mamdani of London University built one of the first fuzzy systems to control a steam engine and boiler combination. He applied a set of fuzzy rules supplied by experienced human operators.

- The Mamdani-style fuzzy inference process is performed in four steps:

  1. Fuzzification of the input variables
  2. Rule evaluation (inference)
  3. Aggregation of the rule outputs (composition)
  4. Defuzzification.

Mamdani FIS

- 2 FIs as two controllers to generate heat input to the boiler and throttle opening for engine cylinder, respectively to regulate the stream pressure in the boiler and the speed of the engine.
- Plant takes crisp values as inputs, so Defuzzifier is required.

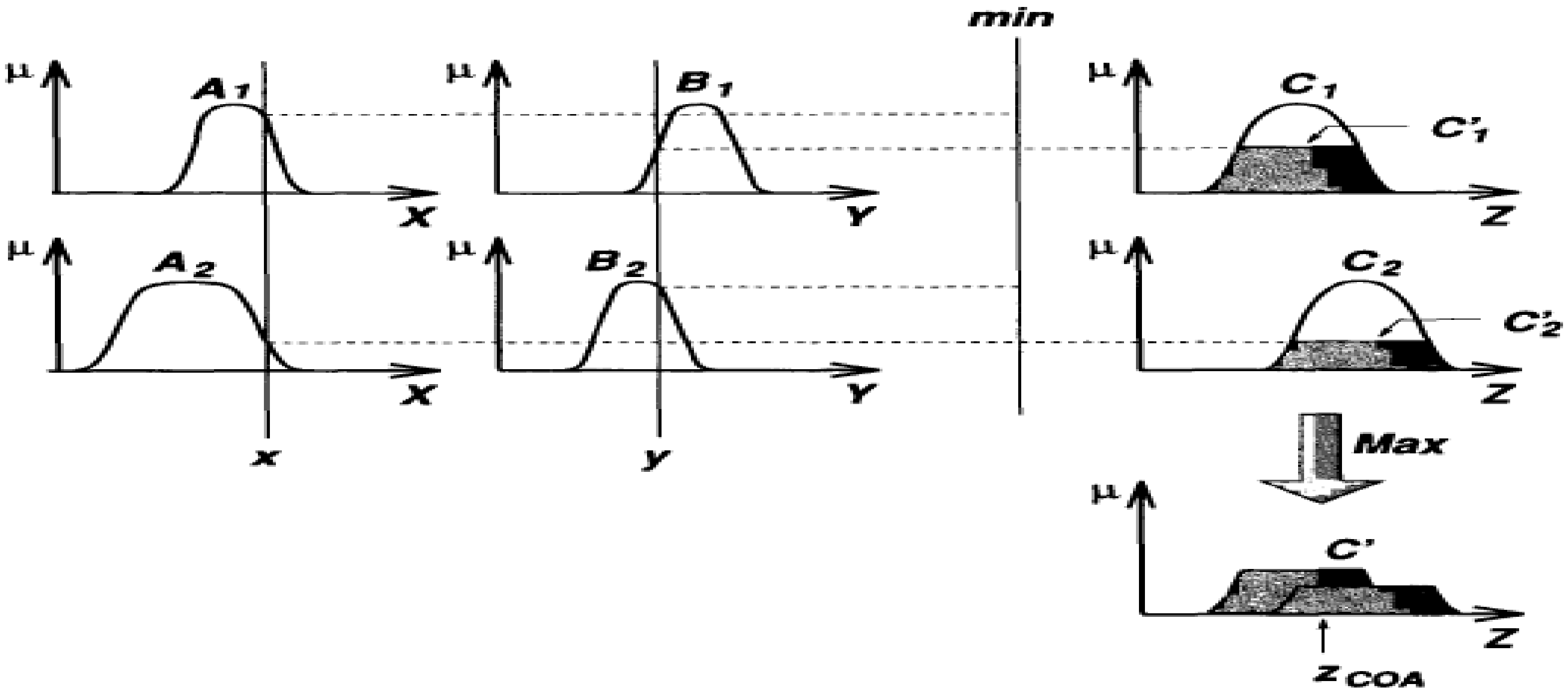

Figure 4.2. *The Mamdani fuzzy inference system using* min *and* max *for T-norm and T-conorm operators, respectively.*

Mamdani models

Single input single output Mamdani Fuzzy model

$$\begin{cases} \text{If } X \text{ is small then } Y \text{ is small.} \\ \text{If } X \text{ is medium then } Y \text{ is medium.} \\ \text{If } X \text{ is large then } Y \text{ is large.} \end{cases}$$

input and output universe are $[-10, 10]$ and $[0, 10]$,

2 input single output Mamdani Fuzzy model

$$\begin{cases} \text{If } X \text{ is small and } Y \text{ is small then } Z \text{ is negative large.} \\ \text{If } X \text{ is small and } Y \text{ is large then } Z \text{ is negative small.} \\ \text{If } X \text{ is large and } Y \text{ is small then } Z \text{ is positive small.} \\ \text{If } X \text{ is large and } Y \text{ is large then } Z \text{ is positive large.} \end{cases}$$

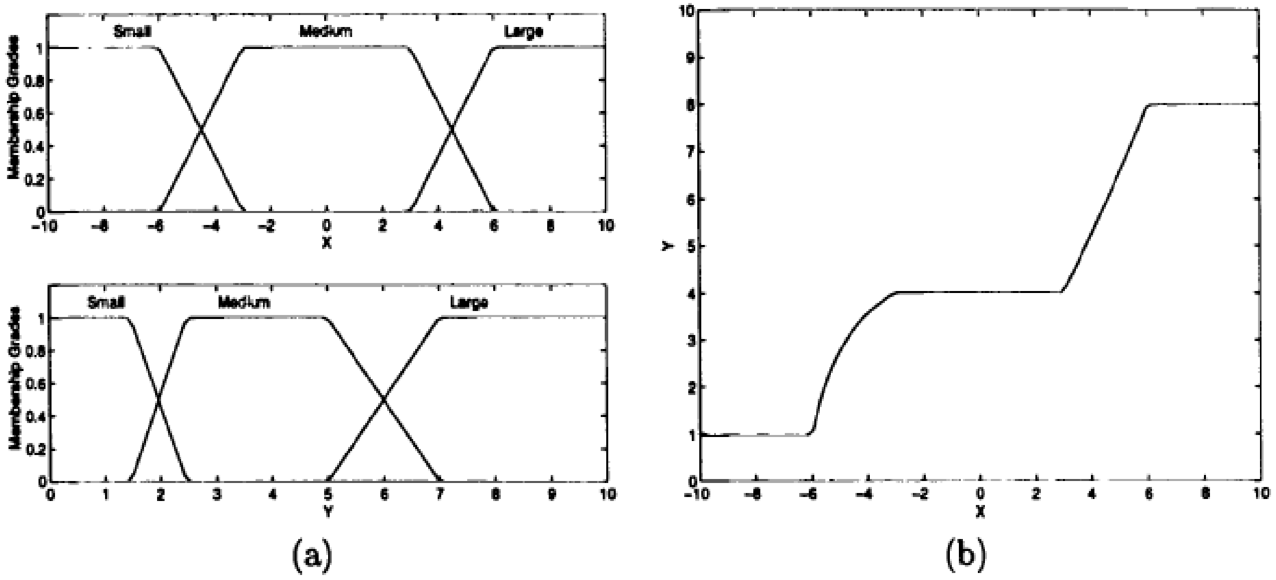

Figure 4.5. *Single-input single-output Mamdani fuzzy model in Example 4.1: (a) antecedent and consequent MFs; (b) overall input-output curve.* (MATLAB file: mam1.m)

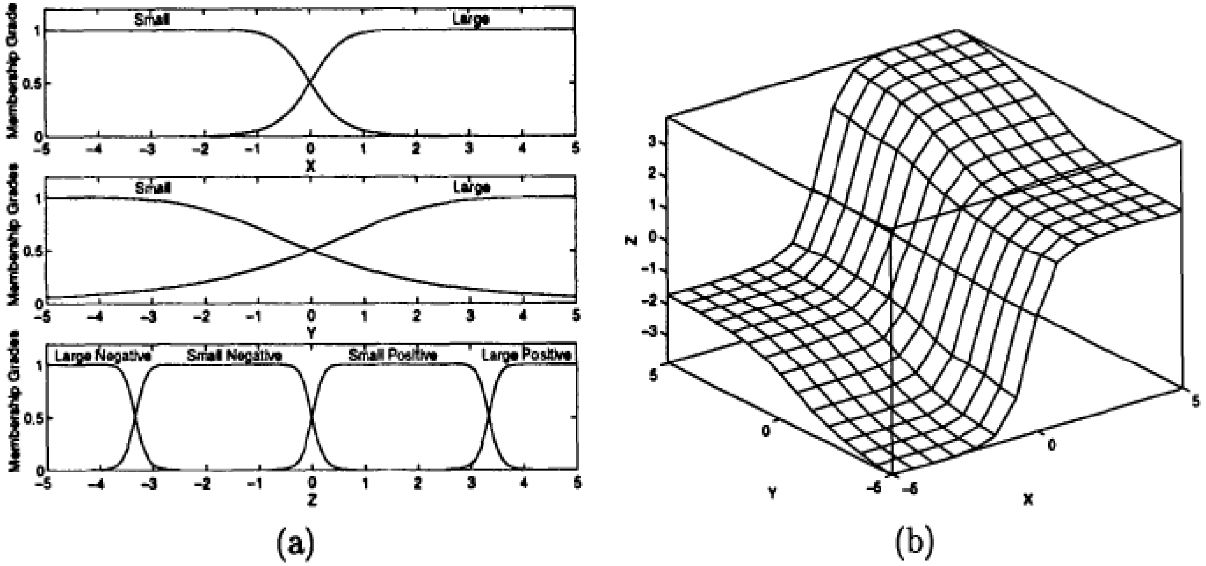

Figure 4.6. *Two-input single-output Mamdani fuzzy model in Example 4.2: (a) antecedent and consequent MFs; (b) overall input-output surface.* (MATLAB file: mam2.m)

Mamdani Fuzzy Inference

We examine a simple two-input one-output problem that includes three rules:

Rule: 1	IF x is A3	OR	y is B1	THEN	z is C1
Rule: 2	IF x is A2	AND	y is B2	THEN	z is C2
Rule: 3	IF x is A1			THEN	z is C3

Real-life example for these kinds of rules:

Rule: 1	IF project_funding is adequate	OR	project_staffing is small	THEN	risk is low
Rule: 2	IF project_funding is marginal	AND	project_staffing is large	THEN	risk is normal
Rule: 3	IF project_funding is inadequate			THEN	risk is high

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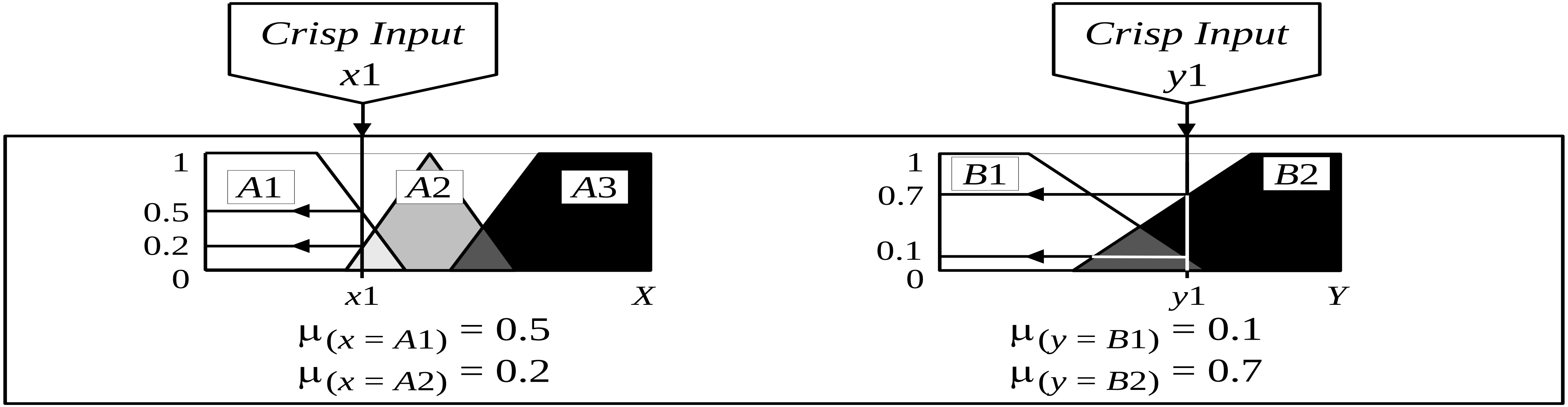
Step 1: Fuzzification

- The first step is to take the crisp inputs, x1 and y1 (*project funding* and *project staffing*), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.

- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.

RECALL: To evaluate the disjunction of the rule antecedents, we use the **OR** fuzzy operation. Typically, fuzzy expert systems make use of the classical fuzzy operation union:

$$\mu_{A \cup B}(x) = \max\,[\mu_A(x),\ \mu_B(x)]$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the **AND** fuzzy operation intersection:

$$\mu_{A \cap B}(x) = \min\,[\mu_A(x),\ \mu_B(x)]$$

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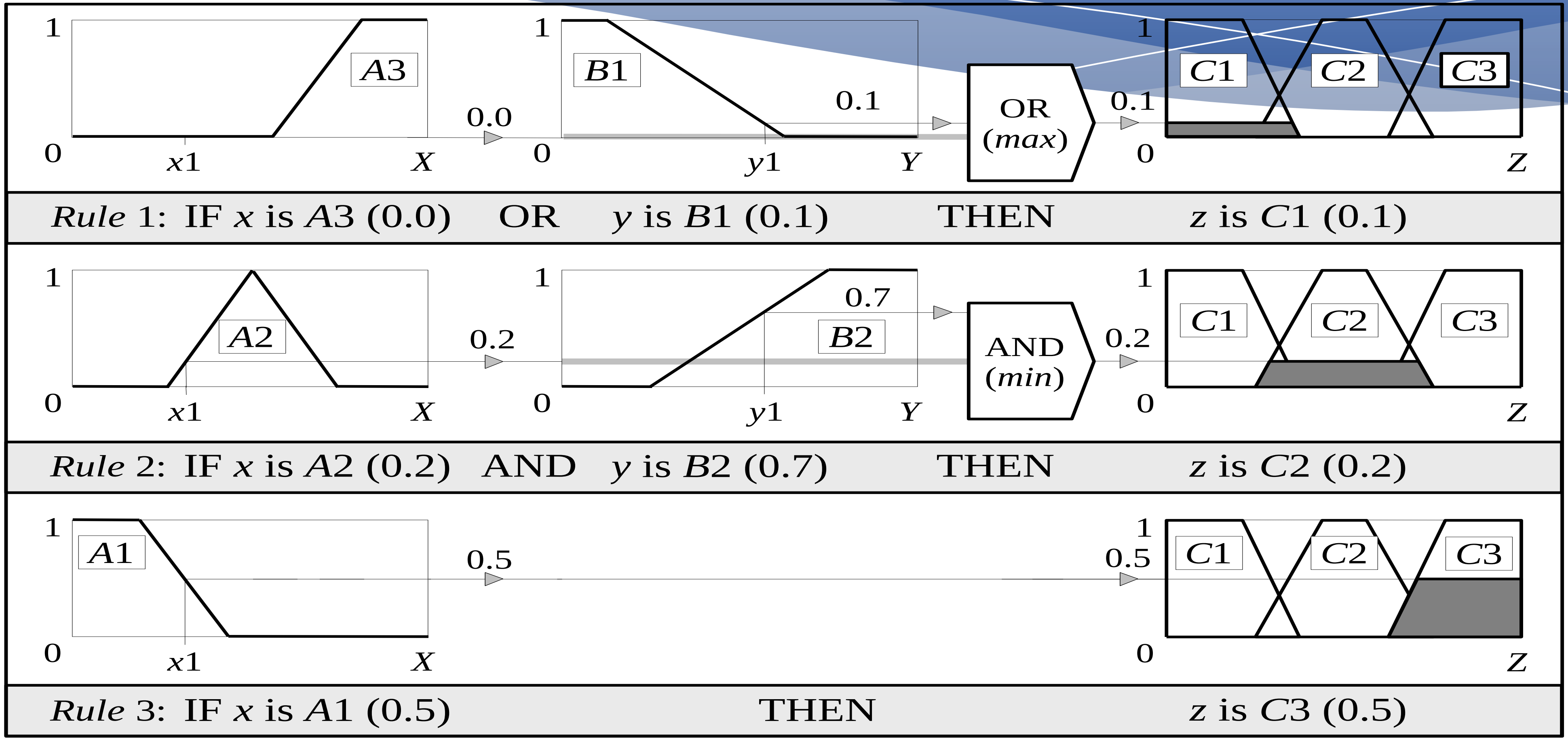
Step 2: Rule Evaluation



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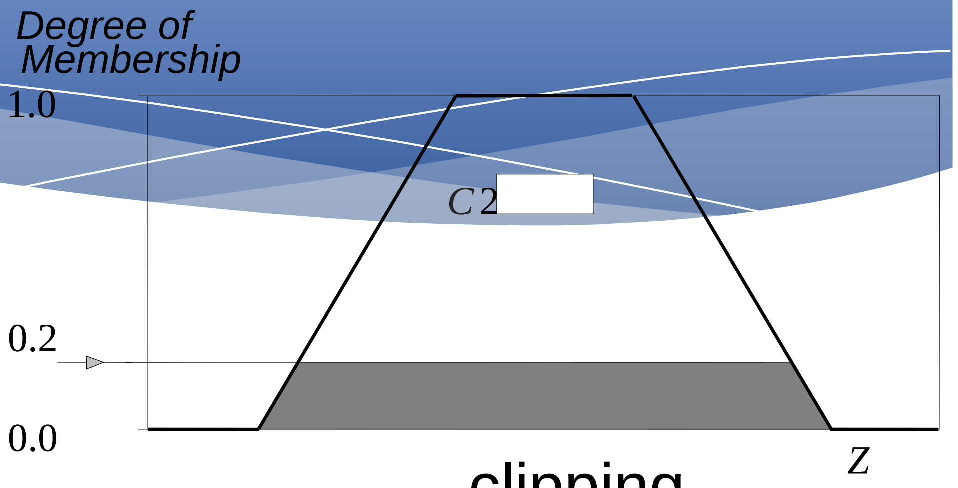
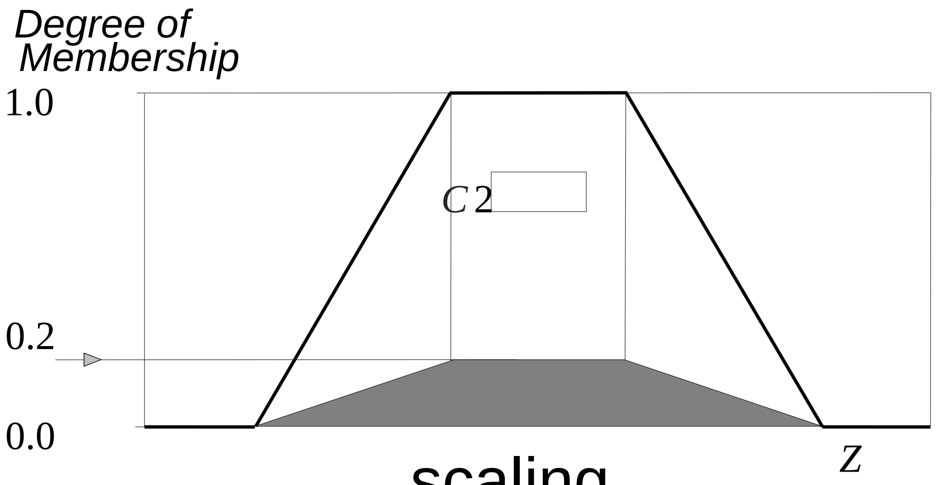
Step 2: Rule Evaluation

- Now the result of the antecedent evaluation can be applied to the membership function of the consequent.
- The most common method is to cut the consequent membership function at the level of the antecedent truth. This method is called **clipping** (alpha-cut).
  - Since the top of the membership function is sliced, the clipped fuzzy set loses some information.
  - However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.
- While clipping is a frequently used method, **scaling** offers a better approach for preserving the original shape of the fuzzy set.
  - The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent.
  - This method, which generally loses less information, can be very useful in fuzzy expert systems.



clipping



scaling

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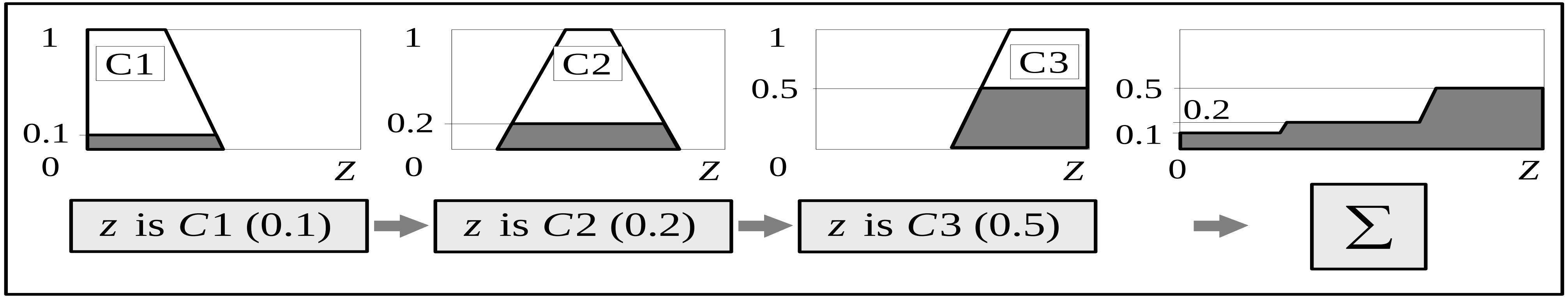
Step 3: Aggregation of the Rule Outputs

- Aggregation is the process of unification of the outputs of all rules.
- We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.
- The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.



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Step 4: Defuzzification

- The last step in the fuzzy inference process is defuzzification.
- Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.
- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.
- There are several defuzzification methods, but probably the most popular one is the **centroid technique**. It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this **centre of gravity** (**COG**) can be expressed as:

$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

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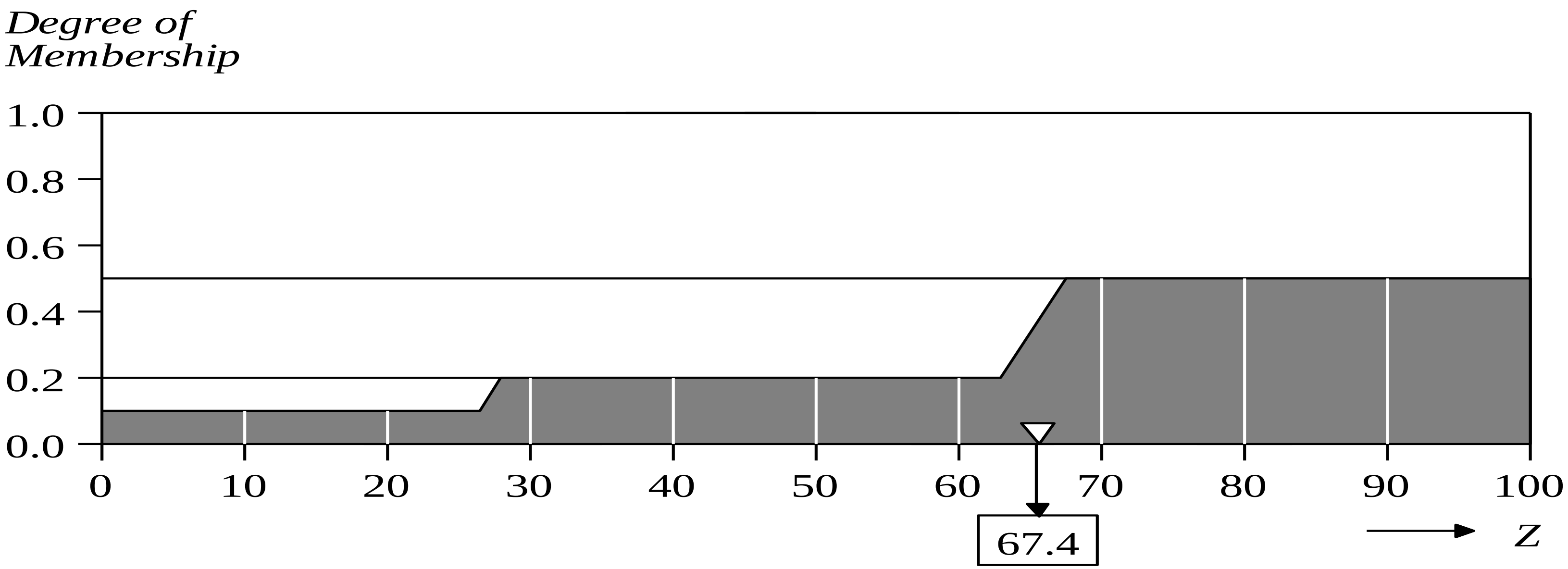
Step 4: Defuzzification

- Centroid defuzzification method finds a point representing the centre of gravity of the aggregated fuzzy set *A*, on the interval [*a*, *b*].
- A reasonable estimate can be obtained by calculating it over a sample of points.



$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$

Sugeno Fuzzy Model

- TSK fuzzy model by Tagasaki, Sugeno and Kang

$$\text{if } x \text{ is } A \text{ and } y \text{ is } B \text{ then } z = f(x, y),$$

- f=constant, **Zero-order Sugeno Fuzzy model**
- Special case of Mamdani FIS where rule's conseq specified by Fuzzy singleton
- f(x,y)= first-order polynomial -> First-order Sugenc Fuzzy model

- Single input Sugeno Fuzzy model

$$\begin{cases} \text{If } X \text{ is small then } Y = 0.1X + 6.4. \\ \text{If } X \text{ is medium then } Y = -0.5X + 4. \\ \text{If } X \text{ is large then } Y = X - 2. \end{cases}$$

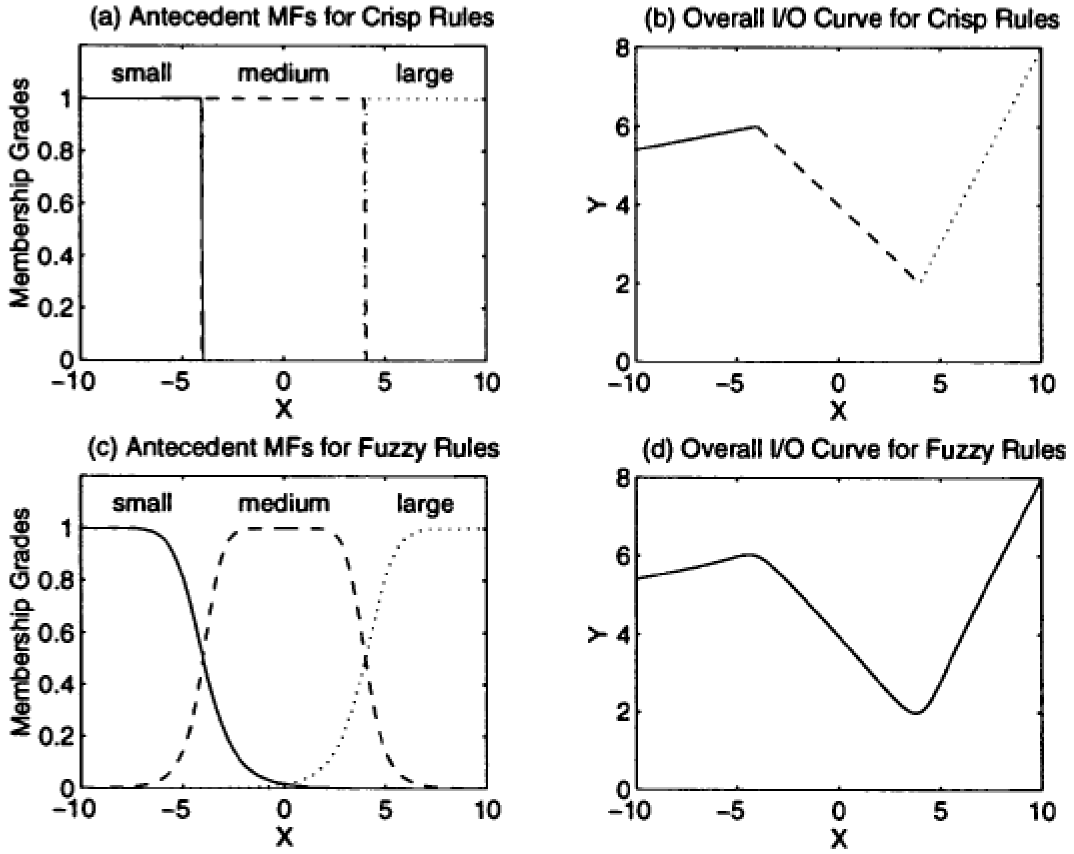


Figure 4.9. *Comparison between fuzzy and nonfuzzy rules in Example 4.3: (a) Antecedent MFs and (b) input-output curve for nonfuzzy rules; (c) Antecedent MFs and (d) input-output curve for fuzzy rules.* (MATLAB file: `sug1.m`)

2 – input single output Sugeno Fuzzy model

$$\begin{cases} \text{If } X \text{ is small and } Y \text{ is small then } z = -x + y + 1. \\ \text{If } X \text{ is small and } Y \text{ is large then } z = -y + 3. \\ \text{If } X \text{ is large and } Y \text{ is small then } z = -x + 3. \\ \text{If } X \text{ is large and } Y \text{ is large then } z = x + y + 2. \end{cases}$$

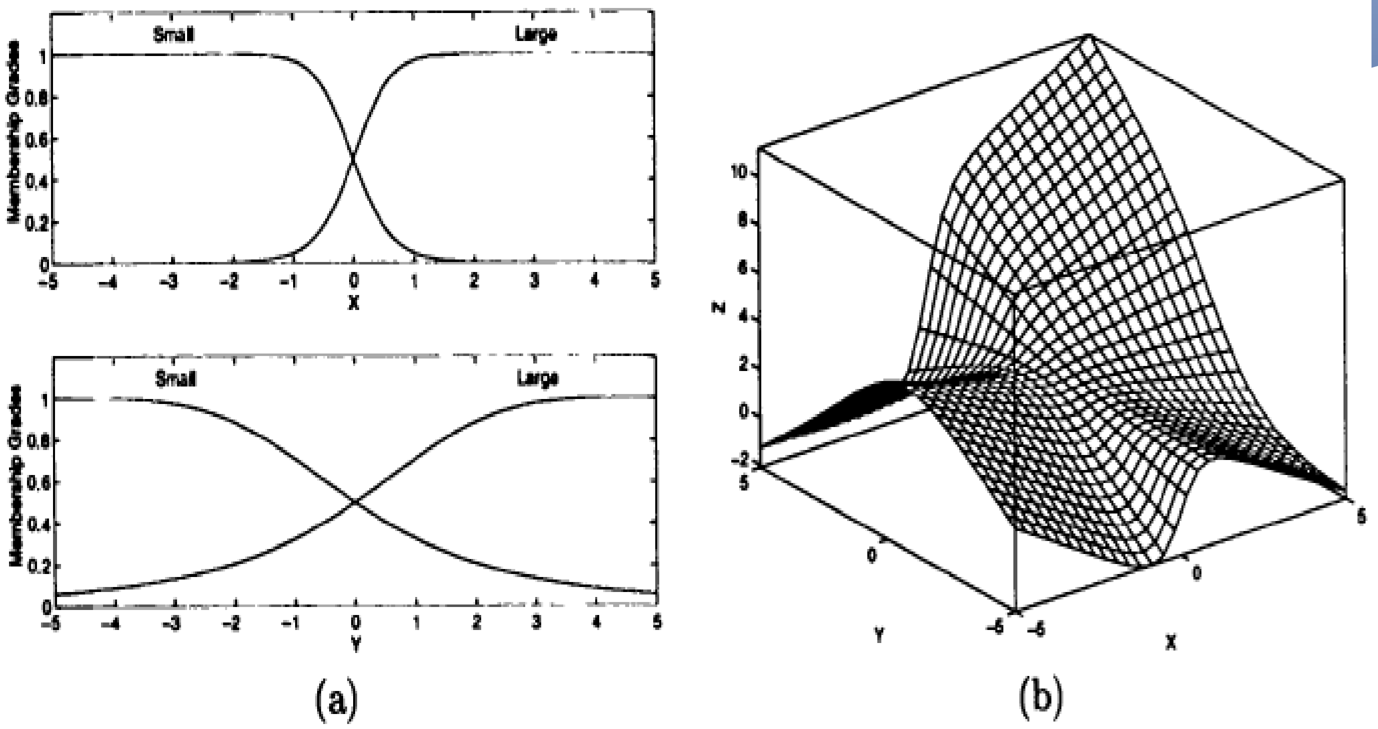


Figure 4.10. *Two-input single-output Sugeno fuzzy model in Example 4.4: (a) antecedent and consequent MFs; (b) overall input-output surface.* (MATLAB file: `sug2.m`)

Sugeno Fuzzy Inference

- Mamdani-style inference, as we have just seen, requires us to find the centroid of a two-dimensional shape by integrating across a continuously varying function. In general, this process is not computationally efficient.

- Michio Sugeno suggested to use a single spike, a singleton, as the membership function of the rule consequent.

- A singleton, or more precisely a fuzzy singleton, is a fuzzy set with a membership function that is unity at a single particular point on the universe of discourse and zero everywhere else.

- Fuzzy set whose support is a single point in X with: $\mu_A(x) = 1$ is called fuzzy singleton.

Sugeno Fuzzy Inference

- Sugeno-style fuzzy inference is very similar to the Mamdani method.
- Sugeno changed only a rule consequent: instead of a fuzzy set, he used a mathematical function of the input variable.
- The format of the **Sugeno-style fuzzy rule** is

IF ***x* is *A*** **AND** ***y* is *B*** **THEN** ***z* is *f*(*x*, *y*)**

where:

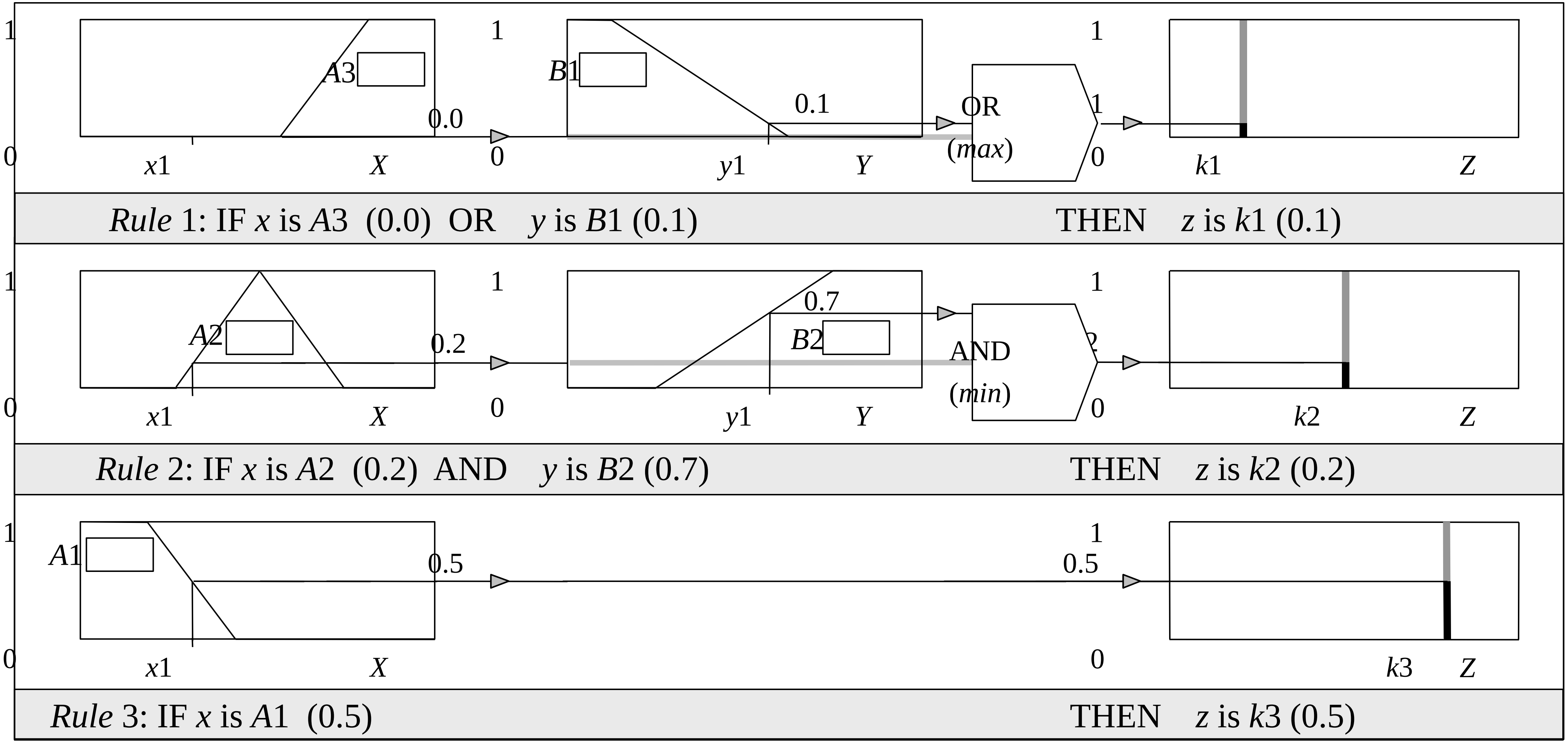
- *x*, *y* and *z* are linguistic variables;
- A and *B* are fuzzy sets on universe of discourses *X* and *Y*, respectively;
- *f* (*x*, *y*) is a mathematical function.

- The most commonly used **zero-order Sugeno fuzzy model** applies fuzzy rules in the following form:

IF ***x* is *A*** **AND** ***y* is *B*** **THEN** ***z* is *k***

- where *k* is a constant.
- In this case, the output of each fuzzy rule is constant and all consequent **membership functions** are represented by **singleton spikes.**

Sugeno Rule Evaluation



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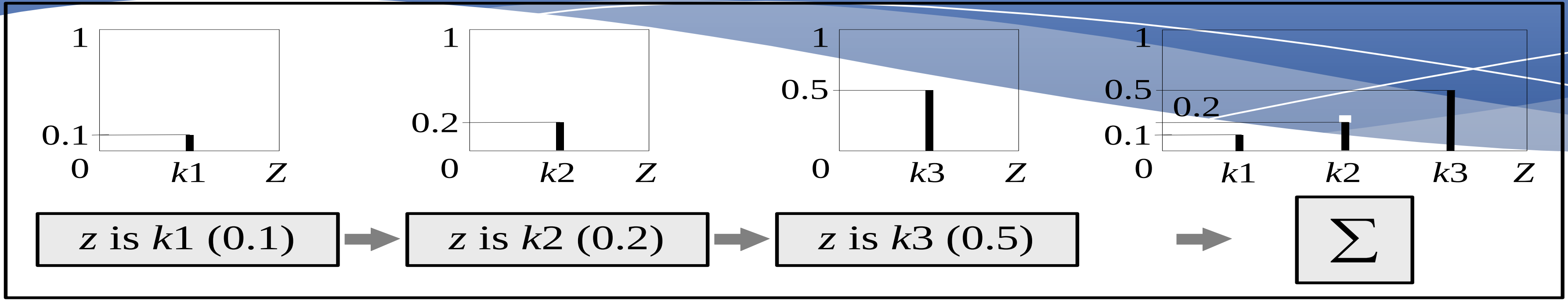
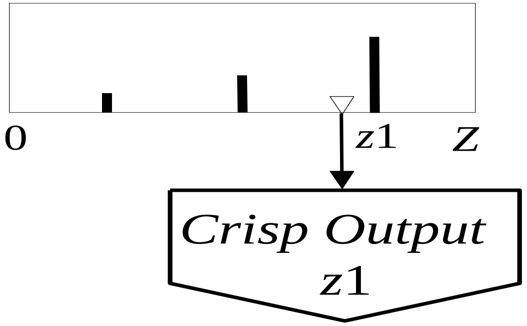
Sugeno Aggregation and Defuzzification



COG becomes Weighted Average (WA)



$$WA = \frac{\mu(k1)\times k1 + \mu(k2)\times k2 + \mu(k3)\times k3}{\mu(k1)+\mu(k2)+\mu(k3)} = \frac{0.1\times 20 + 0.2\times 50 + 0.5\times 80}{0.1+0.2+0.5} = 65$$

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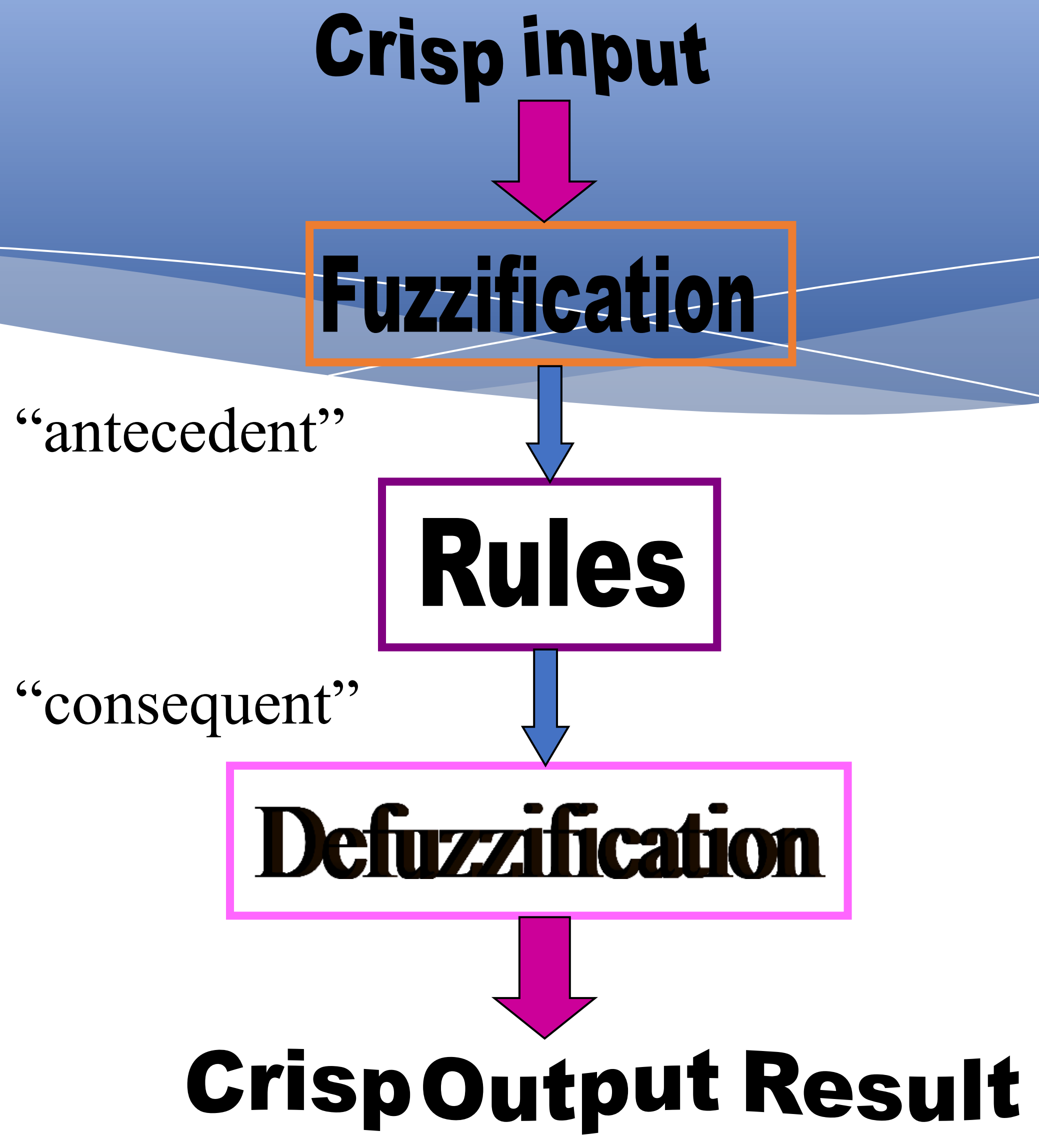
Mamdani or Sugeno?

- Mamdani method is widely accepted for capturing expert knowledge. It allows us to describe the expertise in more intuitive, more human-like manner. However, Mamdani-type fuzzy inference entails a substantial computational burden.

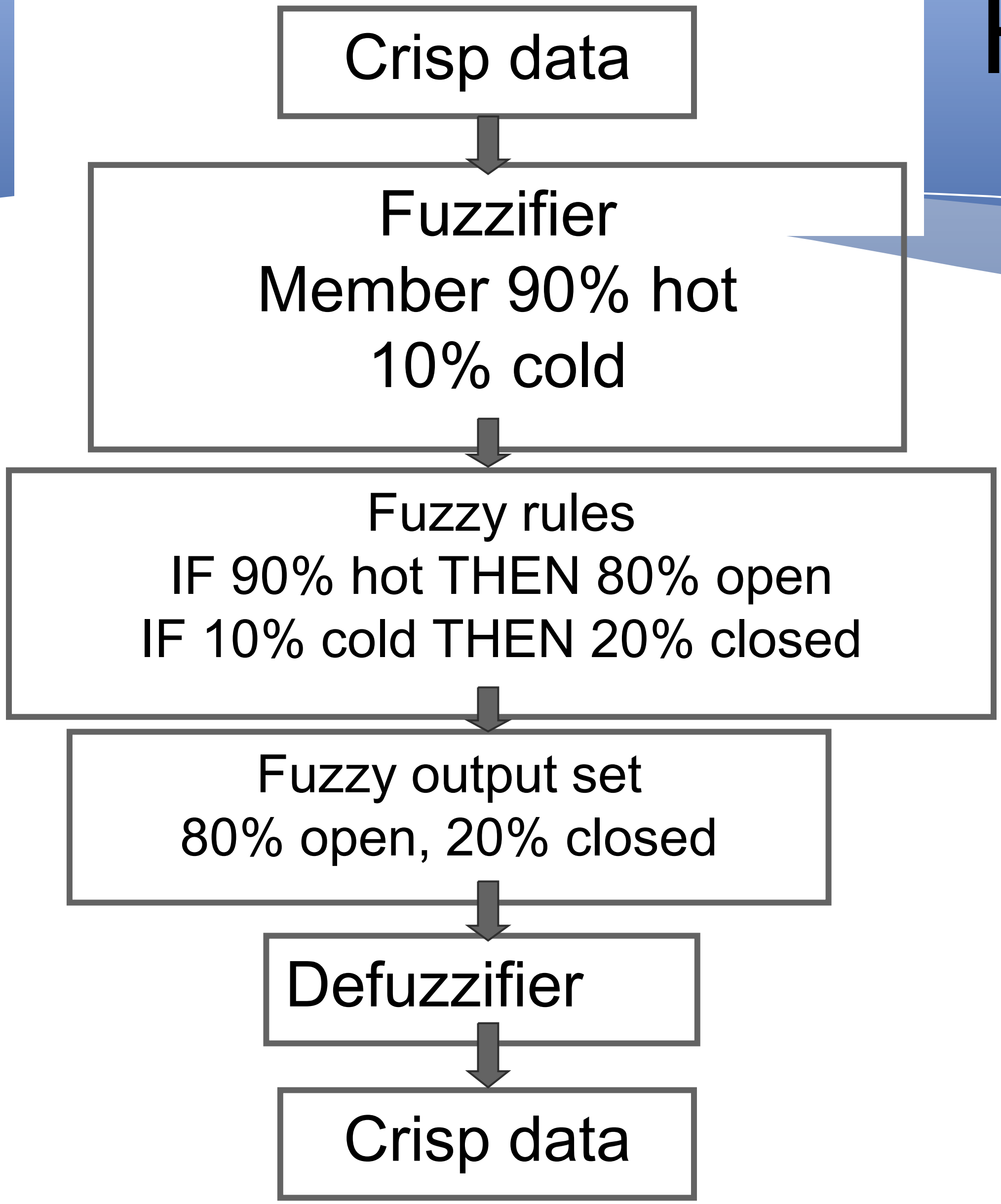
- On the other hand, Sugeno method is computationally effective and works well with optimization and adaptive techniques, which makes it very attractive in control problems, particularly for dynamic nonlinear systems.

Fuzzy System Applications

Fuzzy Inference



How the models work



Inputs converted to degrees of membership of fuzzy sets.

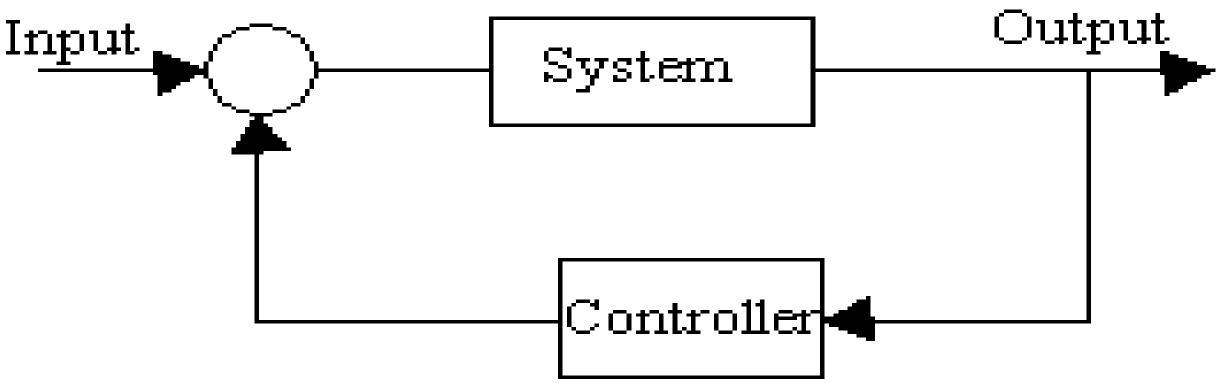
Fuzzy rules applied to get new sets of members.

These sets are then converted back to real numbers.

http://www.powershow.com/view/11df5c-NGJhN/Introduction_to_Fuzzy_Set_Theory_powerpoint_ppt_presentation

Fuzzy Controllers

- Used to control a physical system



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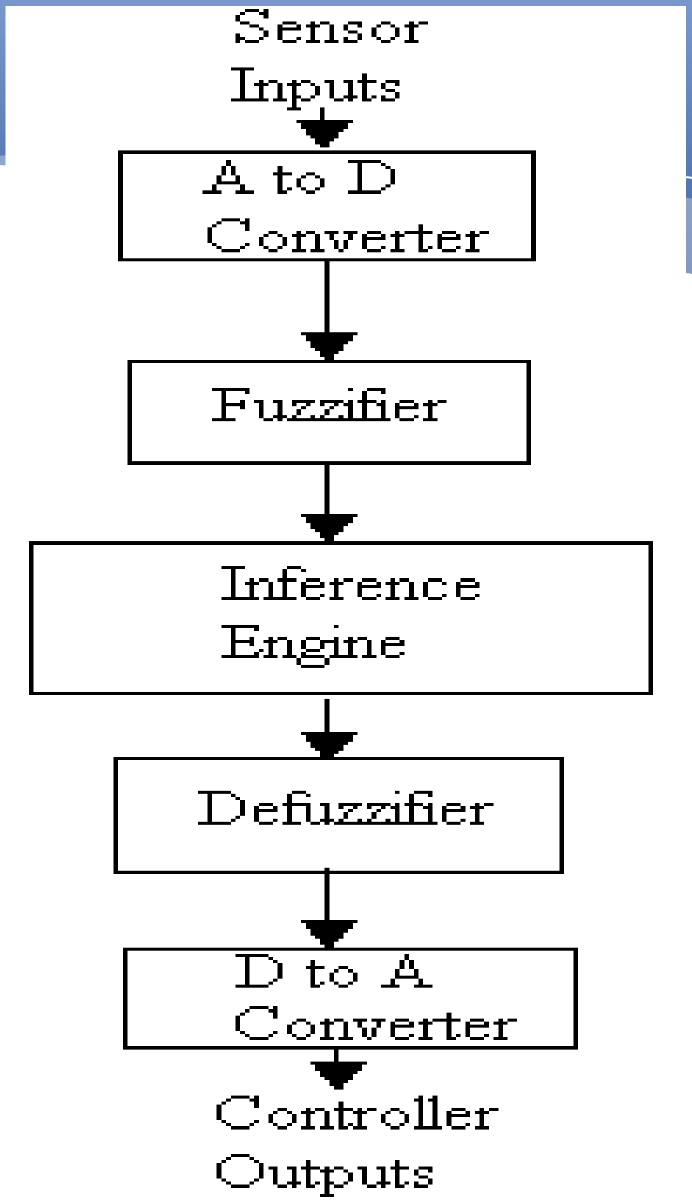
Structure of a Fuzzy Controller



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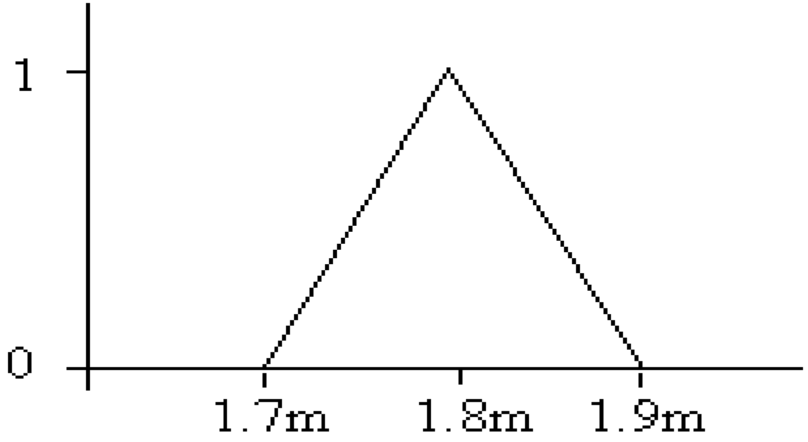
Fuzzification

- Conversion of real input to fuzzy set values
- e.g. Medium (x) = {
  - 0 if x >= 1.90 or x < 1.70,
  - (1.90 - x)/0.1 if x >= 1.80 and x < 1.90,
  - (x- 1.70)/0.1 if x >= 1.70 and x < 1.80 }



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Inference Engine

- Fuzzy rules
  - based on fuzzy premises and fuzzy consequences

- e.g.
  - If height is Short and weight is Light then feet are Small
  - Short(height) AND Light(weight) => Small(feet)

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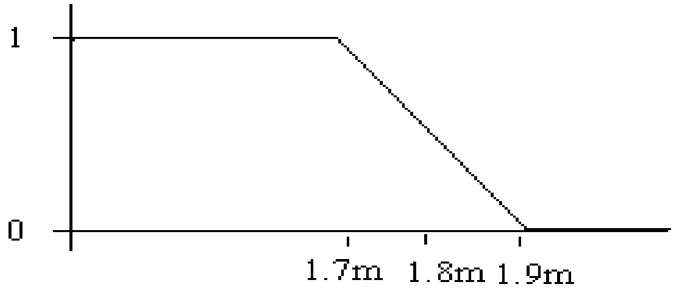
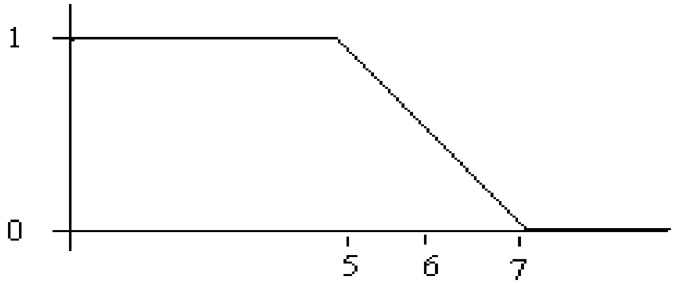
Fuzzification & Inference Example

- If height is 1.7m and weight is 55kg
  - what is the value of Size(feet)





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Defuzzification

- Rule base has many rules
  - so some of the output fuzzy sets will have membership value > 0

  - Defuzzify to get a real value from the fuzzy outputs
    - One approach is to use a centre of gravity method

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Fuzzy Inference Example

Assume that we need to evaluate student applicants based on their ***GPA and GRE scores.***

For simplicity, let us have ***three categories*** *for each score [****High (H), Medium (M), and Low(L)]***

Let us assume that the ***decision*** *should be* ***Excellent (E), Very Good (VG), Good (G), Fair (F) or Poor (P)***

An expert will associate the decisions to the GPA and GRE score. They are then Tabulated.

Fuzzy Rule Table

		GRE		
		H	M	L
GPA	H	E	VG	F
	M	G	G	P
	L	F	P	P

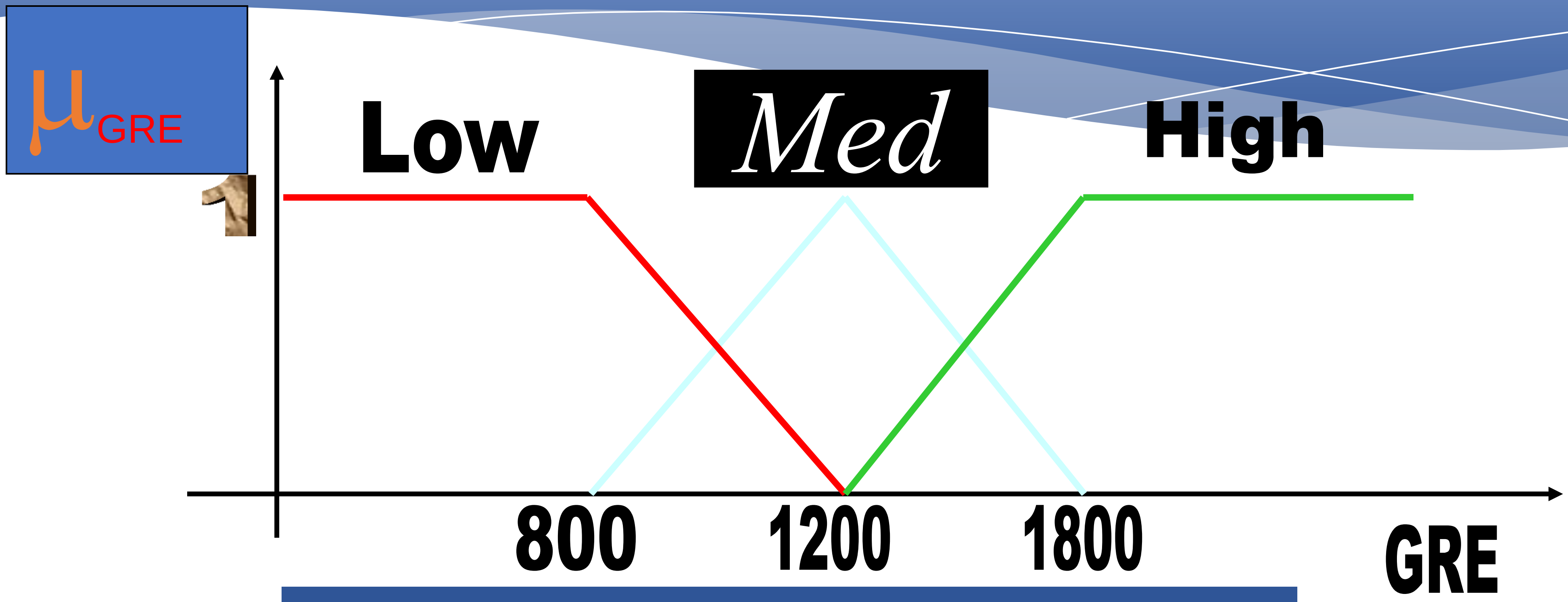
Fuzzification

- **Fuzzifier converts a crisp input into a vector of fuzzy membership values.**

- **The membership functions**
  - reflects the designer's knowledge
  - provides smooth transition between fuzzy sets
  - are simple to calculate

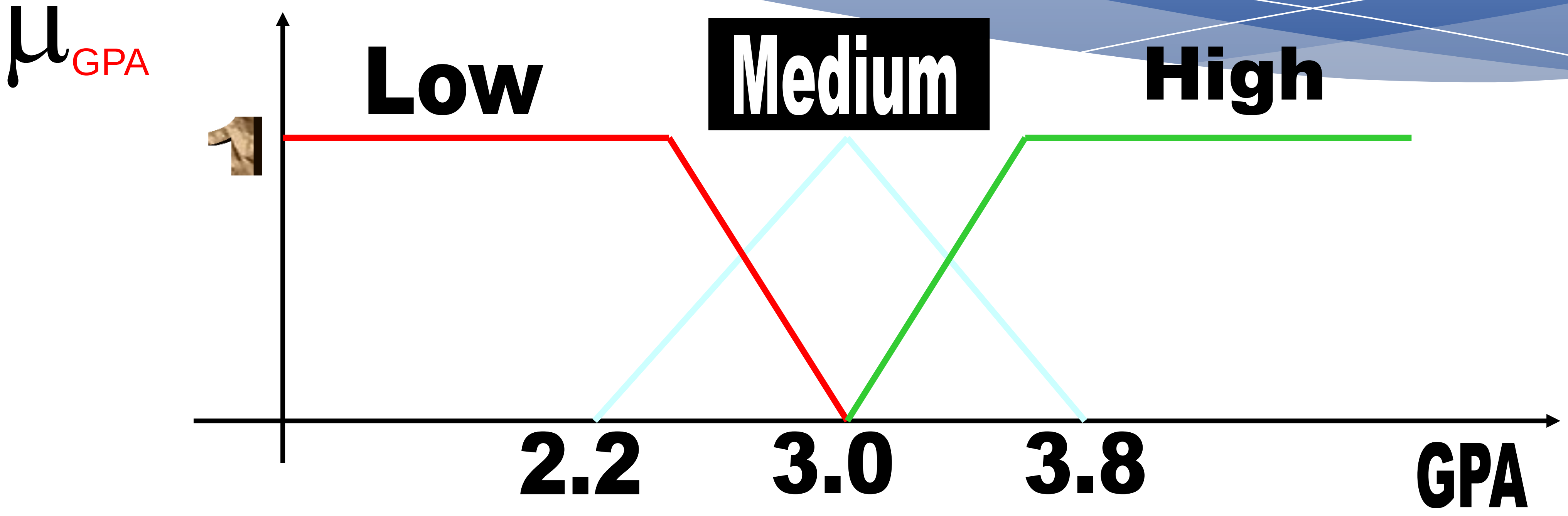
- **Typical shapes of the membership function are Gaussian, trapezoidal and triangular.**

Membership Functions for GRE



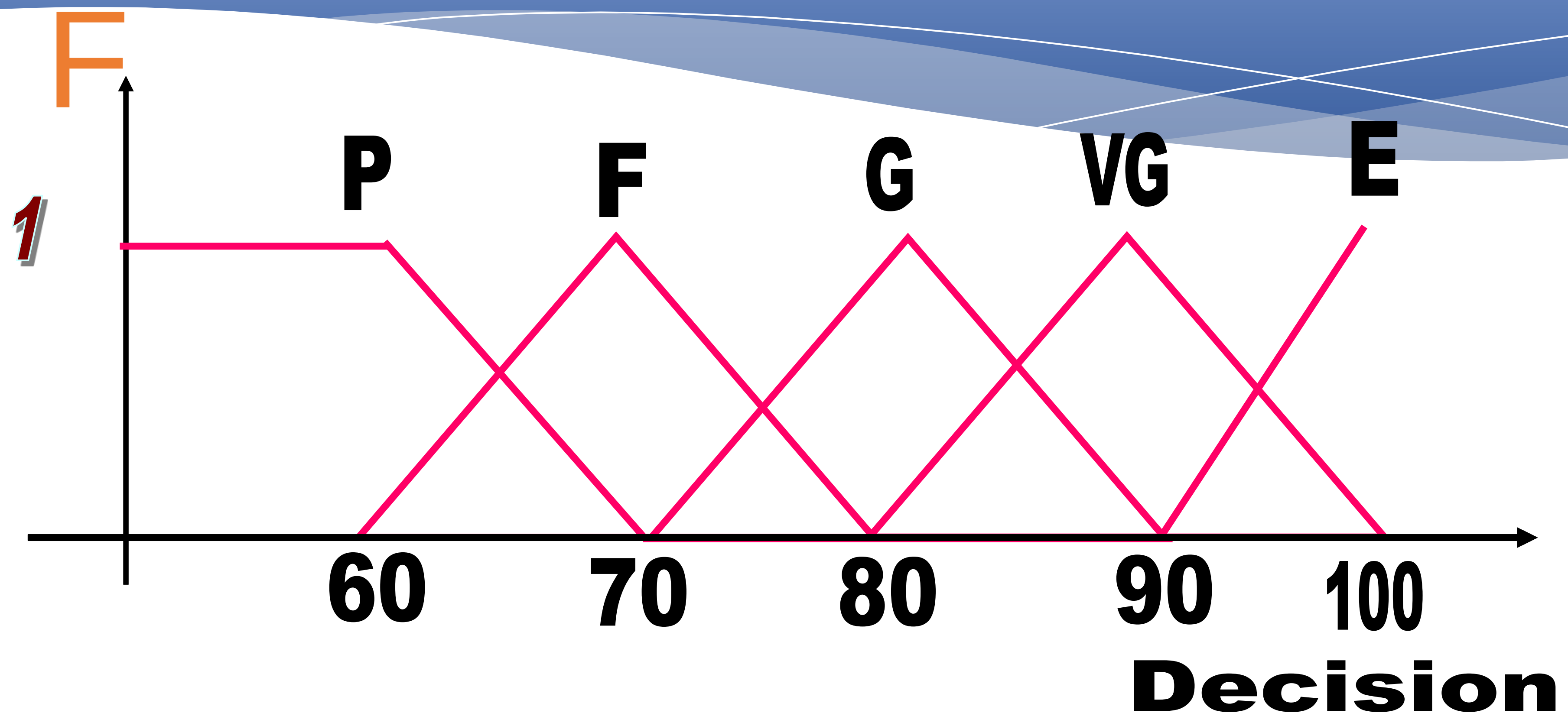
$$\mu_{GRE} = \{\mu_L , \mu_M , \mu_H\}$$

Membership Functions for the GPA



$$\mu_{GPA} = \{\mu_L , \mu_M , \mu_H\}$$

Membership Function for the Consequent



Fuzzification

Transform the crisp antecedents into a vector of fuzzy membership values.

Assume a student with GRE=900 and GPA=3.6. Examining the membership function gives

$$\mu_{GRE} = \{\mu_L = 0.8\,,\ \mu_M = 0.2\,,\ \mu_H = 0\}$$

$$\mu_{GPA} = \{\mu_L = 0\,,\ \mu_M = 0.6\,,\ \mu_H = 0.4\}$$

Activated Rules

		GRE		
		H	M	L
GPA	H	E	VG	F
	M	G	G	P
	L	F	P	P

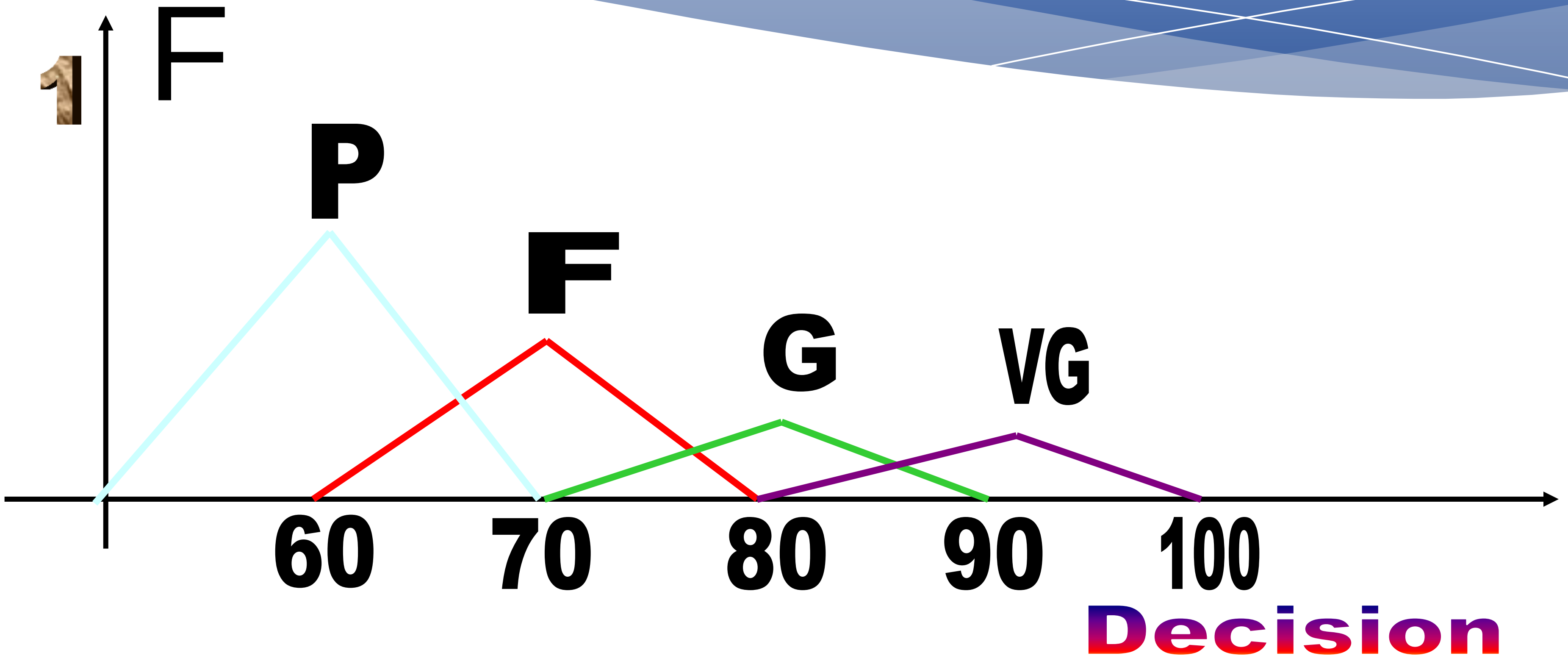
Memberships of Activated Rules

GRE

GPA	0	0.2	0.8
0.4	0	0.2	0.4
0.6	0	0.2	0.6
0	0	0	0

$$F = \{P, F, G, VG, E\}$$
$$F = \{0.6, 0.4, 0.2, 0.2, 0\}$$

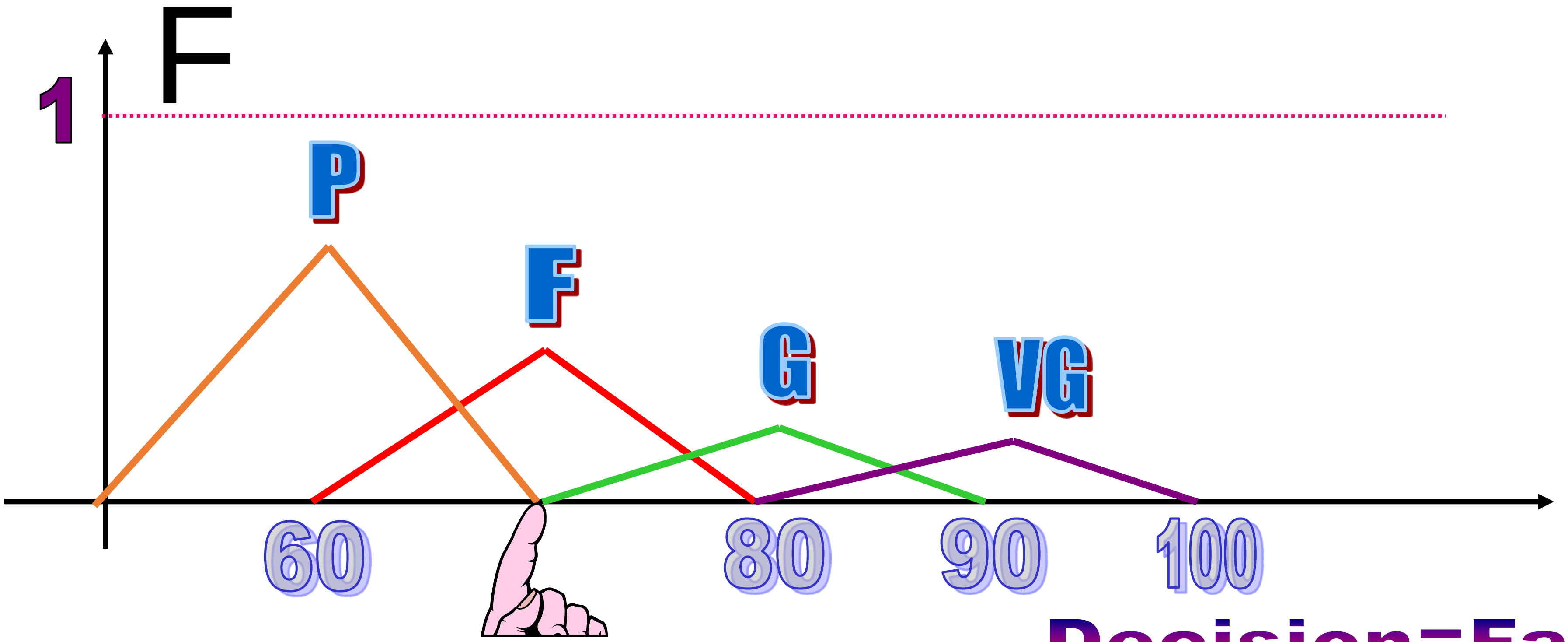
Weight Consequent Memberships



Defuzzification

Converts the output fuzzy numbers into a unique (crisp) number

Method: Add all weighted curves and find the center of mass

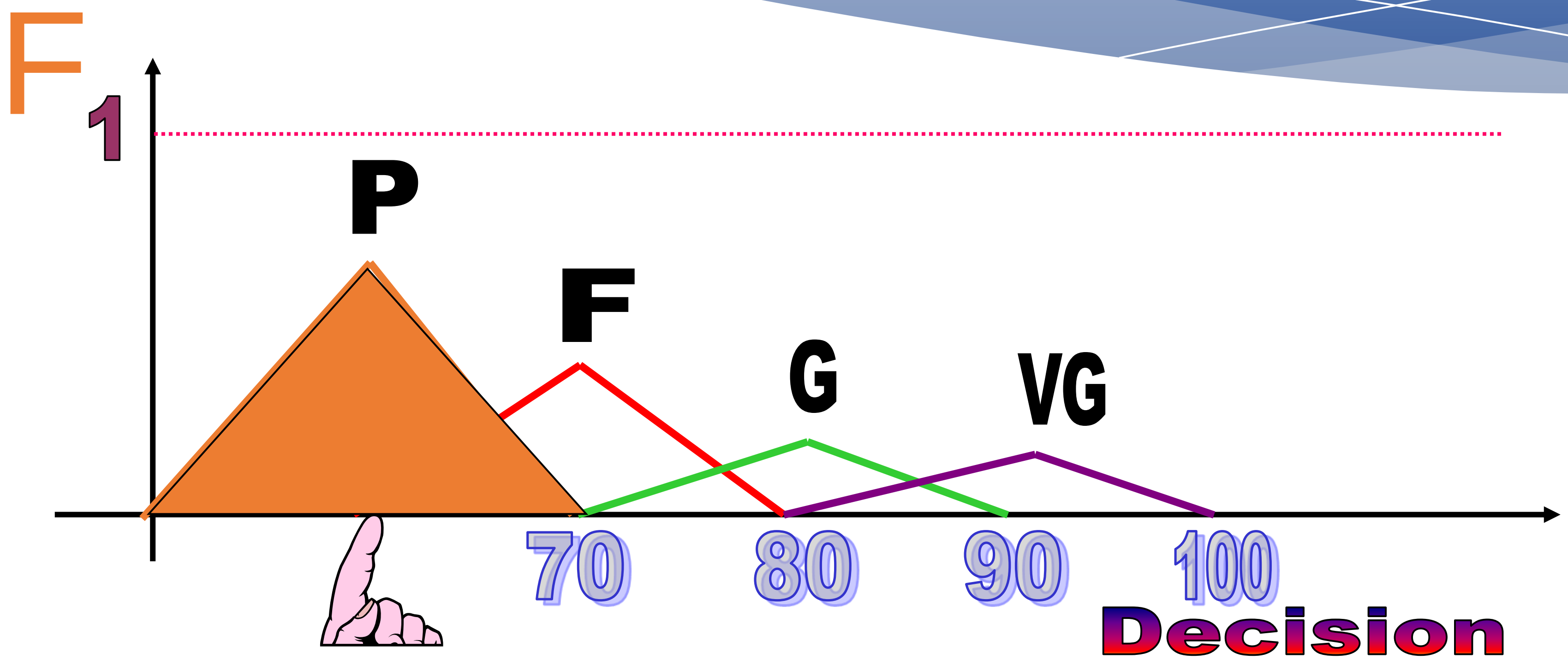


Decision=Fair Student

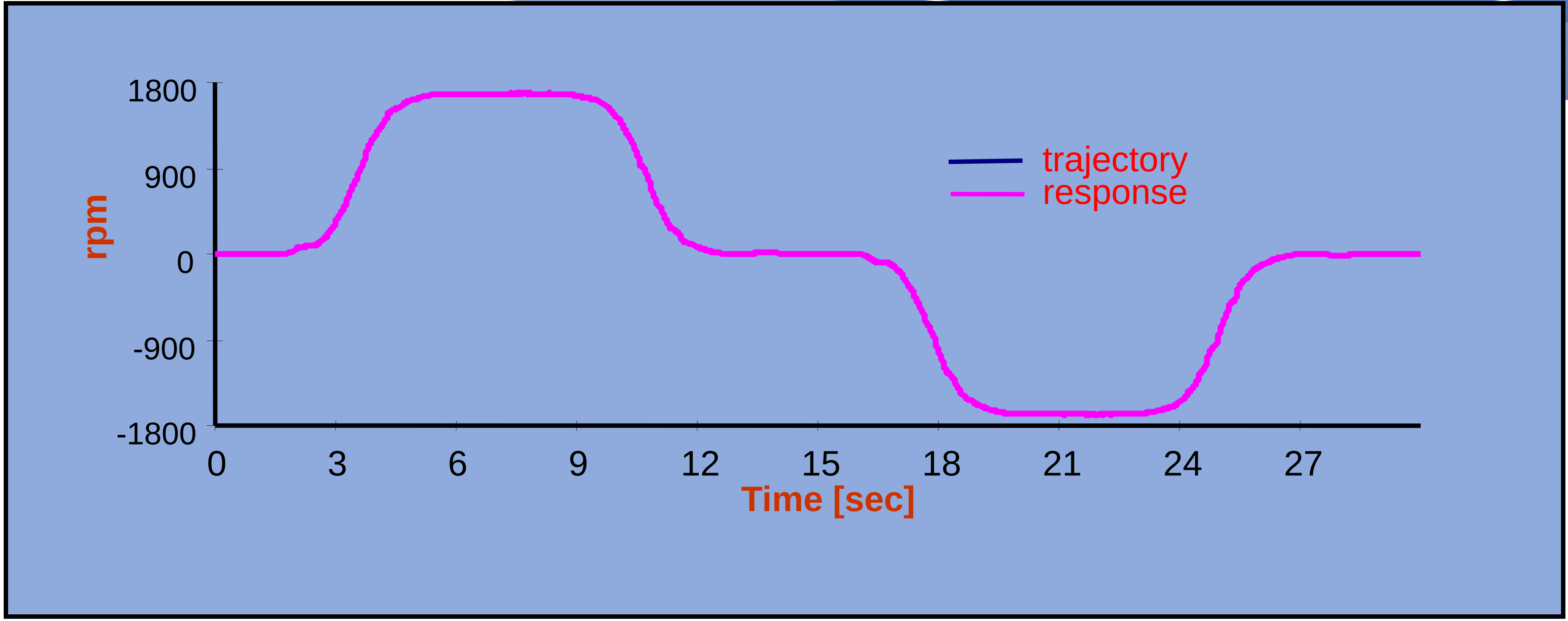
Max Method

- Fuzzy set with the largest membership value is selected.
- Fuzzy decision: F = {P, F, G,VG, E}
  F = {0.6, 0.4, 0.2, 0.2, 0}
- Final Decision (FD) = Poor Student
- If two decisions have same membership max, use the average of the two.

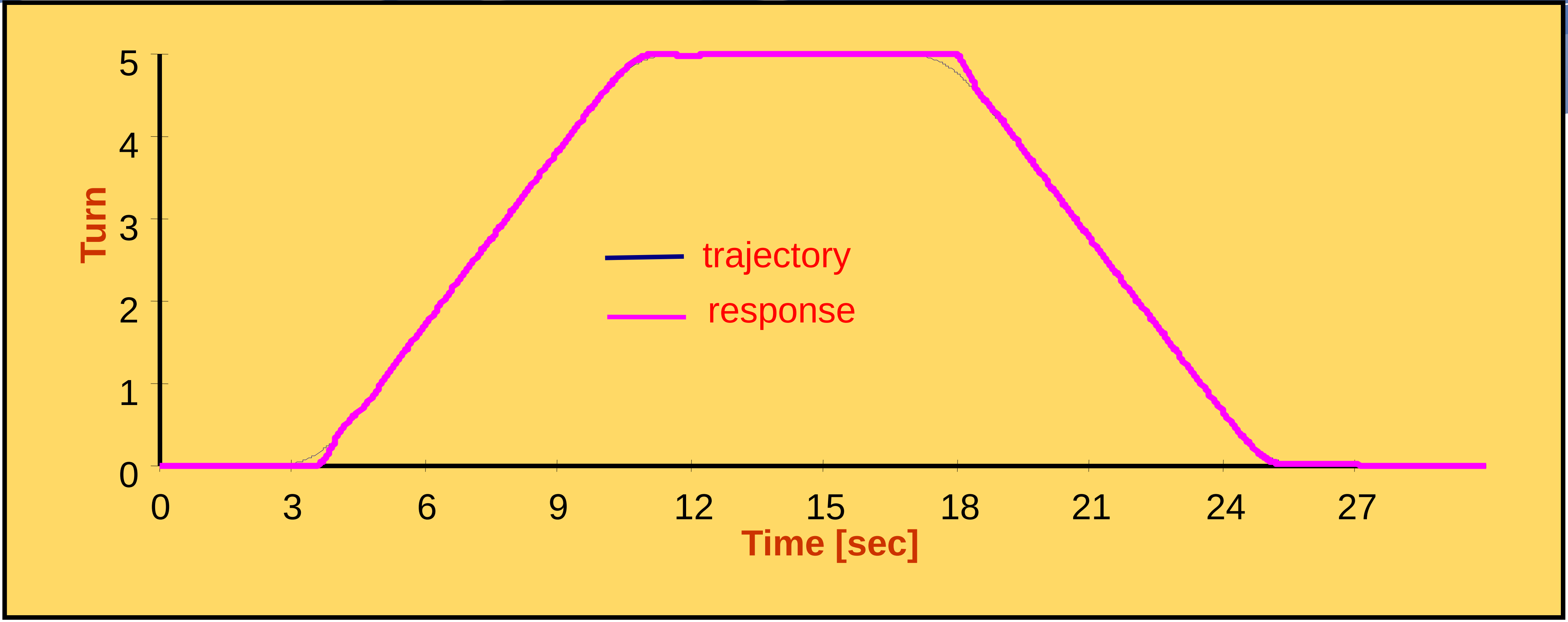
Decision: Max Method



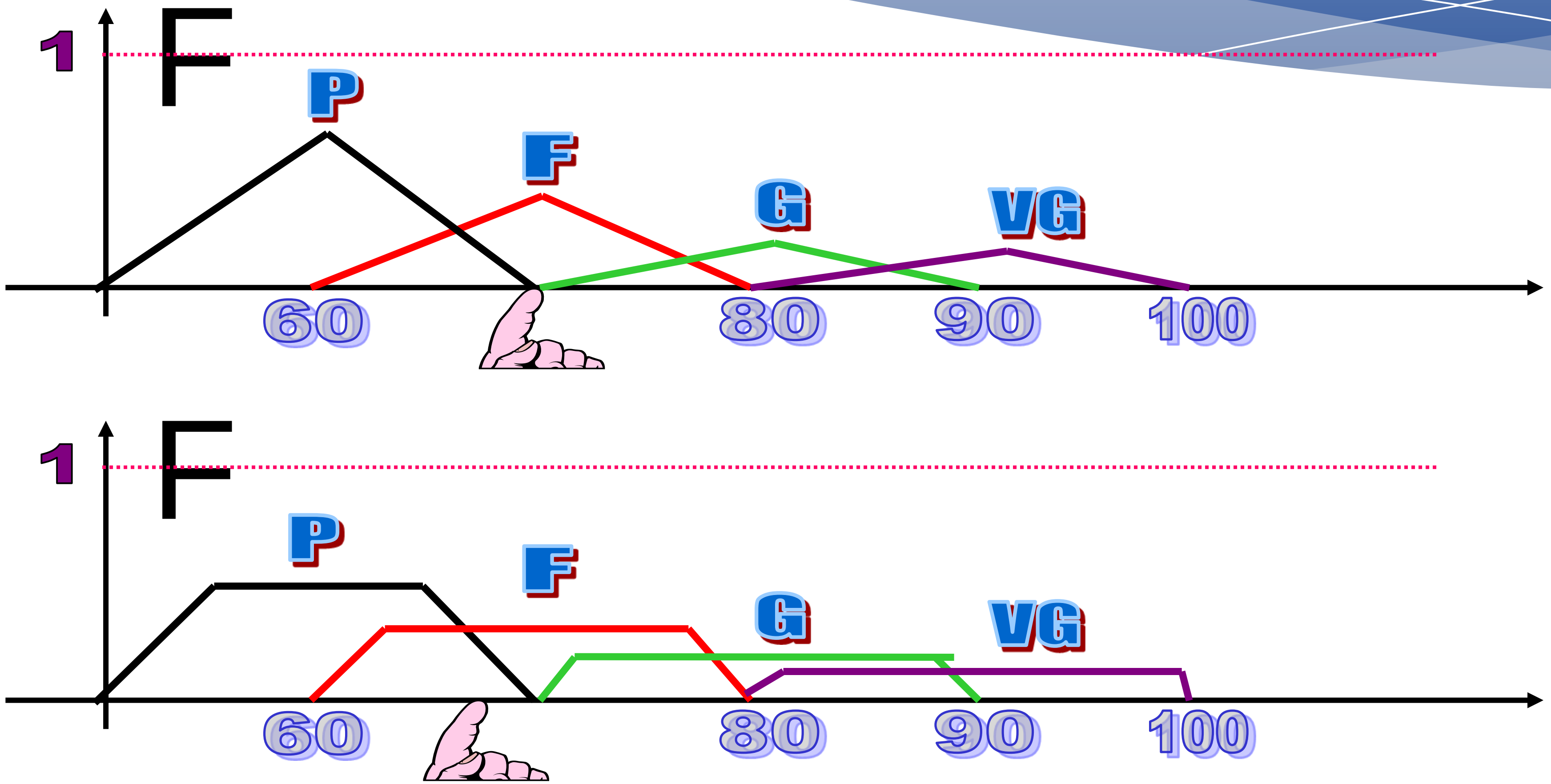
Lab Test: Speed Tracking of IM



Lab Test: Prercision Position Tracking of IM



Commonly Used Variations



Commonly Used Variations

Instead of min(x,y) for fuzzy AND...

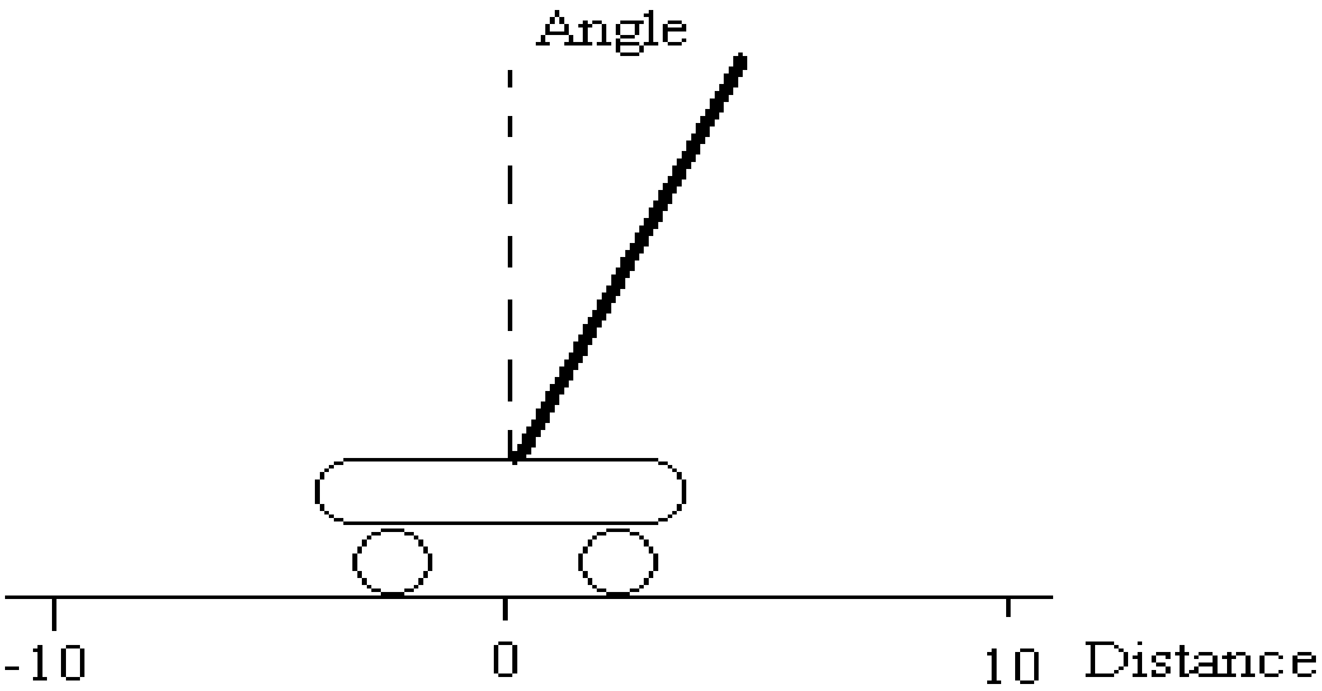
Use $\Rightarrow$ $x \bullet y$

Instead of max(x,y) for fuzzy OR...

Use $\Rightarrow$ $\min(1, x + y)$

Why?

Fuzzy Control Example



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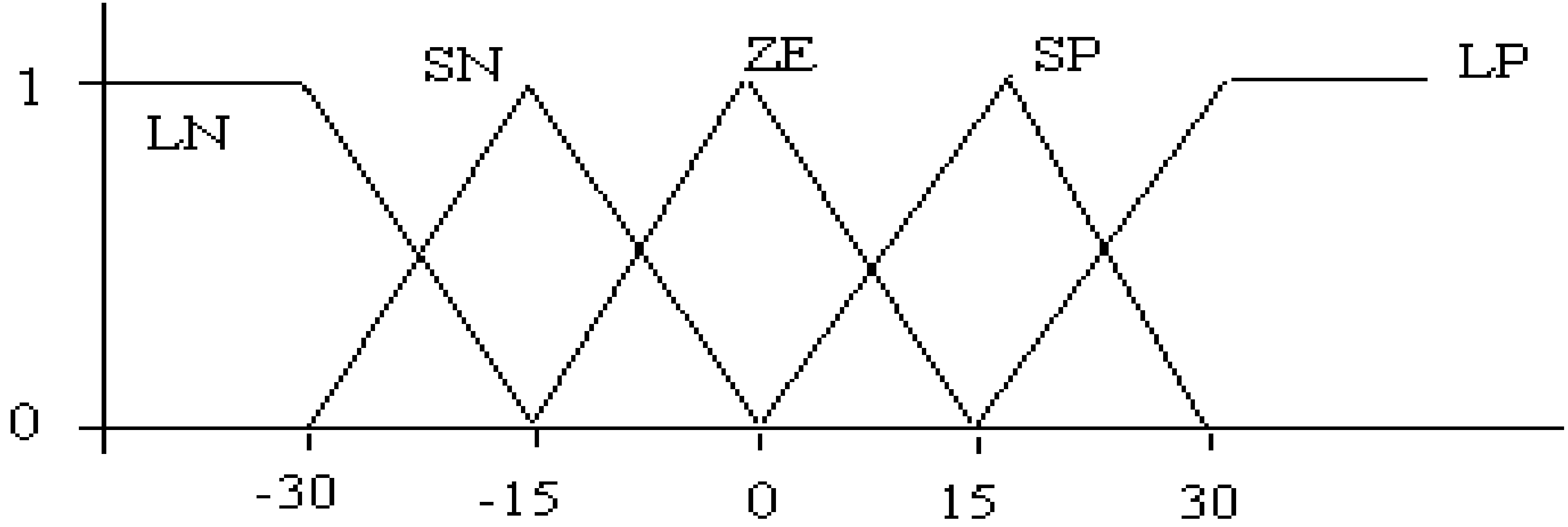
Input Fuzzy Sets

- Angle:- -30 to 30 degrees



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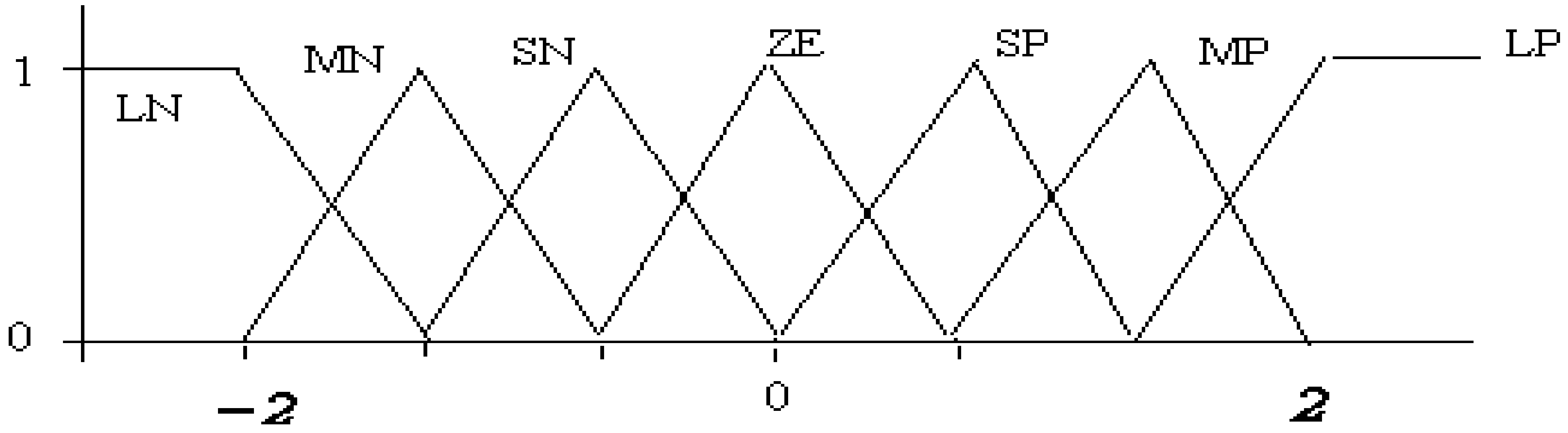
Output Fuzzy Sets

- Car velocity:- -2.0 to 2.0 meters per second



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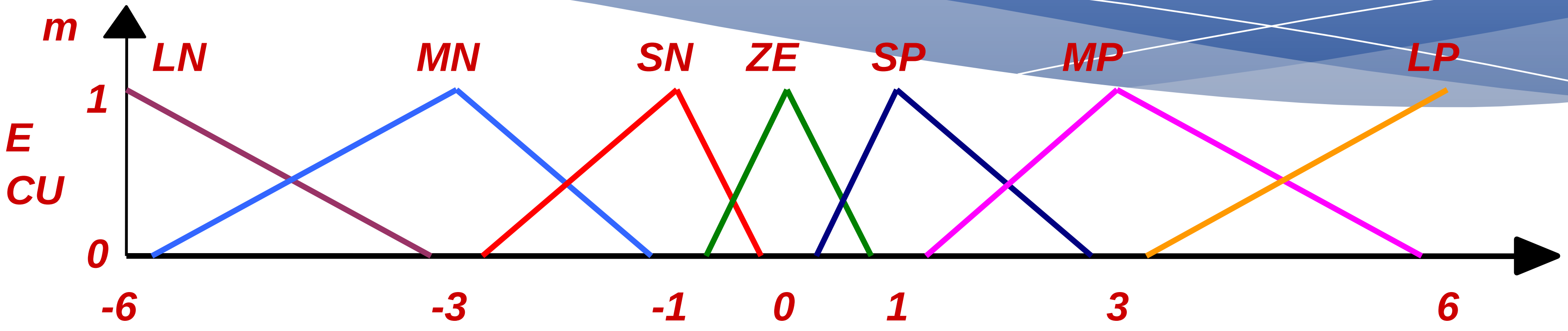
Fuzzy Rules

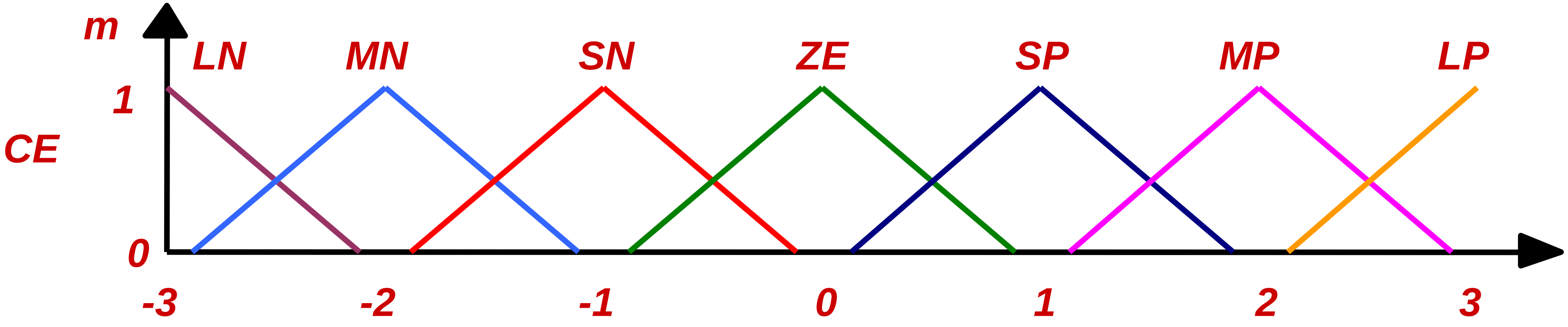
- If Angle is Zero then output ?
- If Angle is SP then output ?
- If Angle is SN then output ?
- If Angle is LP then output ?
- If Angle is LN then output ?

Example: Fuzzy Table for Control

		CE						
		LN	MN	SN	ZE	SP	MP	LP
	LN	LN	LN	LN	LN	MN	SN	SN
	MN	LN	LN	LN	MN	SN	ZE	ZE
	SN	LN	LN	MN	SN	ZE	ZE	SP
E	ZE	LN	MN	SN	ZE	SP	MP	LP
	SP	SN	ZE	ZE	SP	MP	LP	LP
	MP	ZE	ZE	SP	MP	LP	LP	LP
	LP	SP	SP	MP	LP	LP	LP	LP

Membership Functions





Rule Aggregation

		CE						
		LN	MN	SN	ZE	SP	MP	LP
E	LN	LN	LN	LN	LN	MN	e. SN 0.2	f. SN 0.0
	MN	LN	LN	LN	MN	d. SN 0.5	ZE	ZE
	SN	LN	LN	MN	c.SN 0.3	ZE	ZE	SP
	ZE	LN	MN	b.SN 0.4	ZE	SP	MP	LP
	SP	a. SN 0.1	ZE	ZE	SP	MP	LP	LP
	MP	ZE	SP	SP	MP	LP	LP	LP
	LP	SP	SP	MP	LP	LP	LP	LP

Consequent is or SN if *a* or *b* or *c* or *d* or *f*.

Rule Aggregation

Consequent is or SN if *a* or *b* or *c* or *d* or *f*.

Consequent Membership = max(a,b,c,d,e,f) = 0.5

More generally:

$$agg_\alpha(\vec{x}) = \left[\frac{1}{N}\sum_{n=1}^{N} x_n^\alpha\right]^{1/\alpha}$$

Rule Aggregation

$$agg_{\alpha}(\vec{x}) = \left[\frac{1}{N}\sum_{n=1}^{N} x_n^{\alpha}\right]^{1/\alpha}$$

Special Cases:

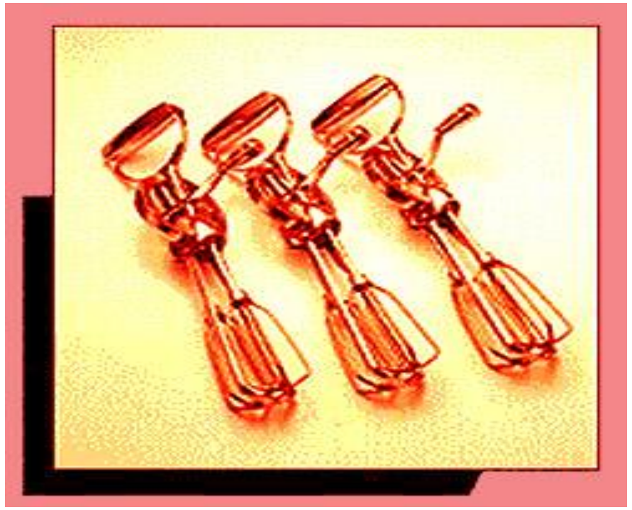
$$agg_{-\infty}(\vec{x}) = \min_n x_n \text{; minimum}$$

$$agg_{-1}(\vec{x}) = \left[\frac{1}{N}\sum_{n=1}^{N}\frac{1}{x_n}\right]^{-1} \text{; harmonic mean}$$

$$agg_{0}(\vec{x}) = \left[\Pi_{n=1}^{N} x_n\right]^{1/N} \text{; geometric mean}$$

$$agg_{1}(\vec{x}) = \frac{1}{N}\sum_{n=1}^{N} x_n \text{; average}$$

$$agg_{2}(\vec{x}) = \left[\frac{1}{N}\sum_{n=1}^{N} x_n^{2}\right]^{1/2} \text{; rms}$$



@Robert Jackson Marks II

Fuzzy Rule Table

Angle	Output Velocity
LN	MP
SN	SP
ZE	ZE
SP	SN
LP	MN

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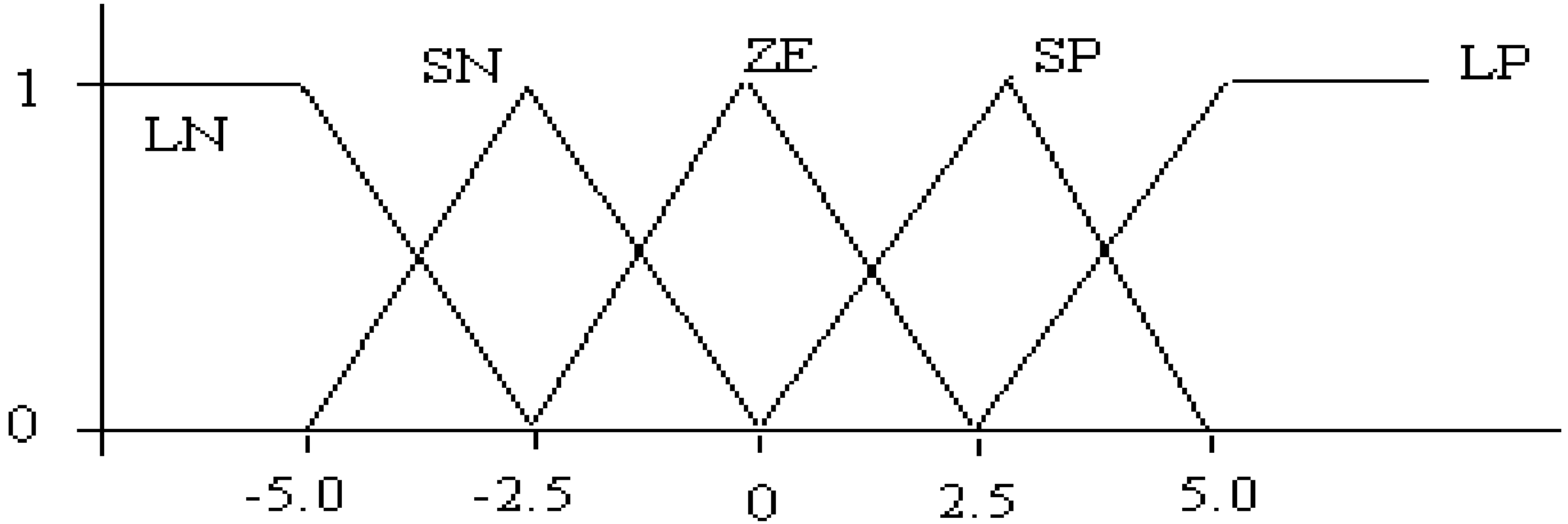
Extended System

- Make use of additional information
  - angular velocity:- -5.0 to 5.0 degrees/ second
- Gives better control



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New Fuzzy Rules

- Make use of old Fuzzy rules for angular velocity Zero
- If Angle is Zero and **Angular vel is Zero**
  - then output Zero velocity
- If Angle is SP and **Angular vel is Zero**
  - then output SN velocity
- If Angle is SN and **Angular vel is Zero**
  - then output SP velocity

Table format

AngleVel / Angle	LN	SN	ZE	SP	LP
LN			MP		
SN			SP		
ZE			ZE		
SP			SN		
LP			MN		

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Complete Table

- When angular velocity is opposite to the angle do nothing
  - System can correct itself
- If Angle is SP and Angular velocity is SN
  - then output ZE velocity
- etc

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Example

- Inputs:10 degrees, -3.5 degrees/sec
- Fuzzified Values
- Inference Rules
- Output Fuzzy Sets
- Defuzzified Values

Commonly Used Variations

- Sugeno inferencing
- Other Norms and co-norms
- Relationship with Neural Networks
- Explanation Facilities
- Teaching a Fuzzy System
- Tuning a Fuzzy System