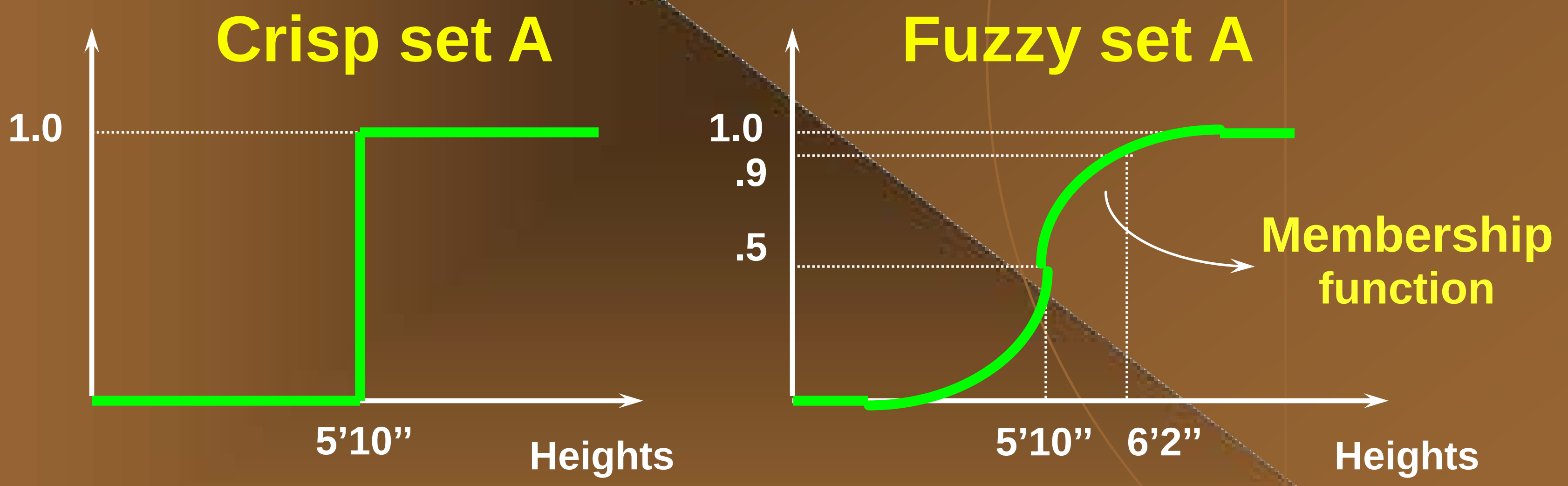
# FUZZY SETS

# Introduction (2.1)

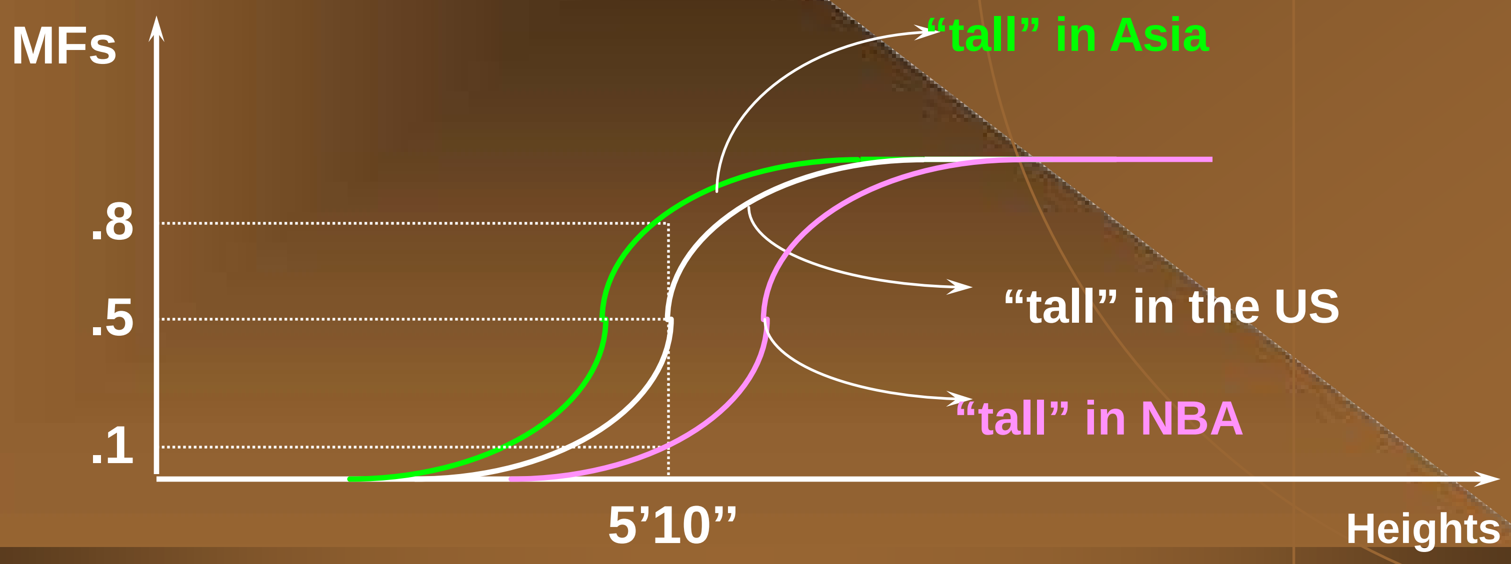


- Sets with fuzzy boundaries

A = Set of tall people



- Membership Functions (MFs)
  - Characteristics of MFs:
    - Subjective measures
    - Not probability functions



# Basic definitions & Terminology (2.2)



- Formal definition:

  A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$\underbrace{A}_{\text{Fuzzy set}} = \{(x, \underbrace{\mu_A(x)}_{\text{Membership function (MF)}}) \mid x \in \underbrace{X}_{\text{Universe or universe of discourse}}\}$$

***A fuzzy set is totally characterized by a membership function (MF).***

# Fuzzy Sets with Discrete Universes

- Fuzzy set C = “desirable city to live in”

  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

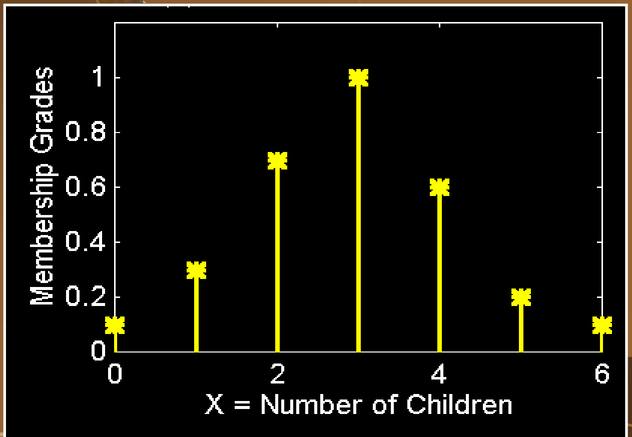
  (subjective membership values!)

- Fuzzy set A = “sensible number of children”

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

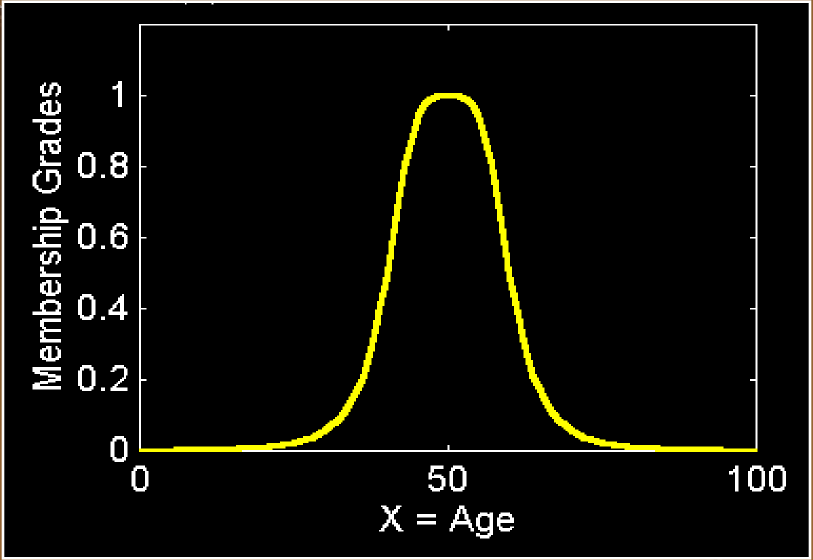
  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}

  (subjective membership values!)



- Fuzzy Sets with Cont. Universes
  - Fuzzy set B = “about 50 years old”
    - X = Set of positive real numbers (continuous)
    - B = {(x, $\mu_B$(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



## Alternative Notation

- A fuzzy set A can be alternatively denoted as follows:

**X is discrete** →

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

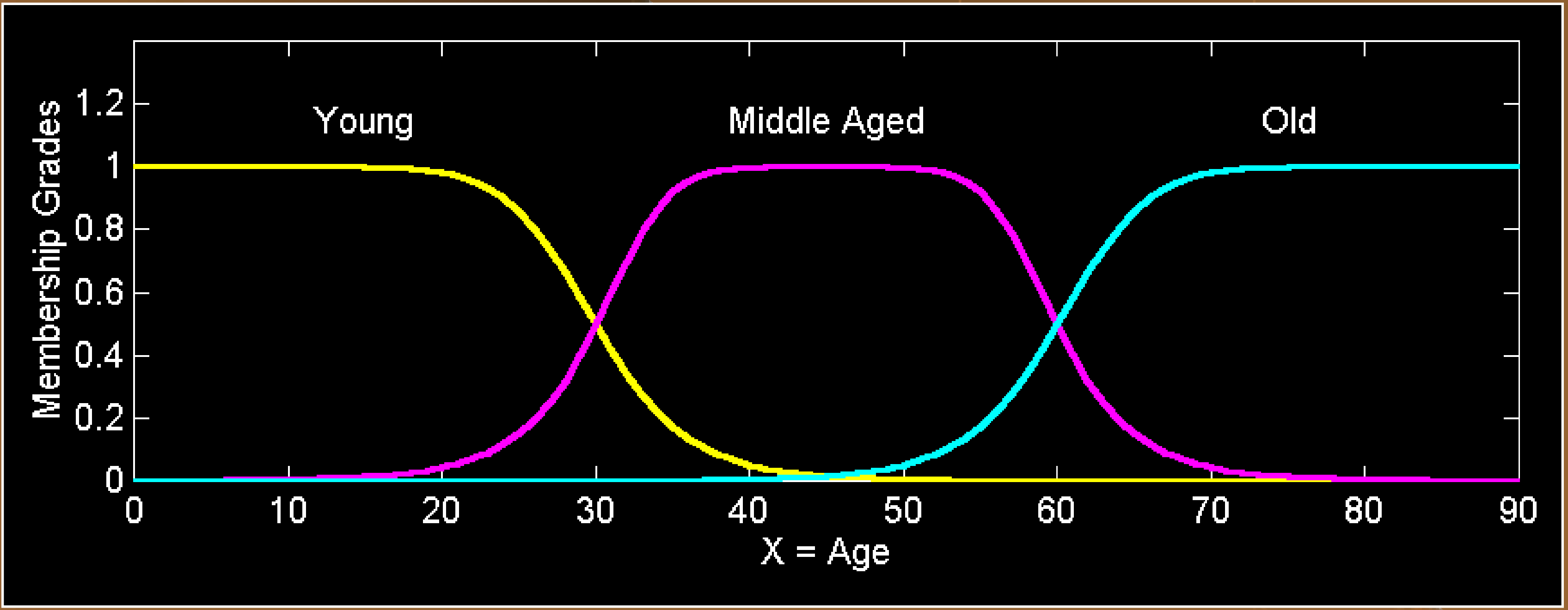
**X is continuous** →

$$A = \int_X \mu_A(x) / x$$

**Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.**

# Fuzzy Partition

- Fuzzy partitions formed by the linguistic values
- “young”, “middle aged”, and “old”:

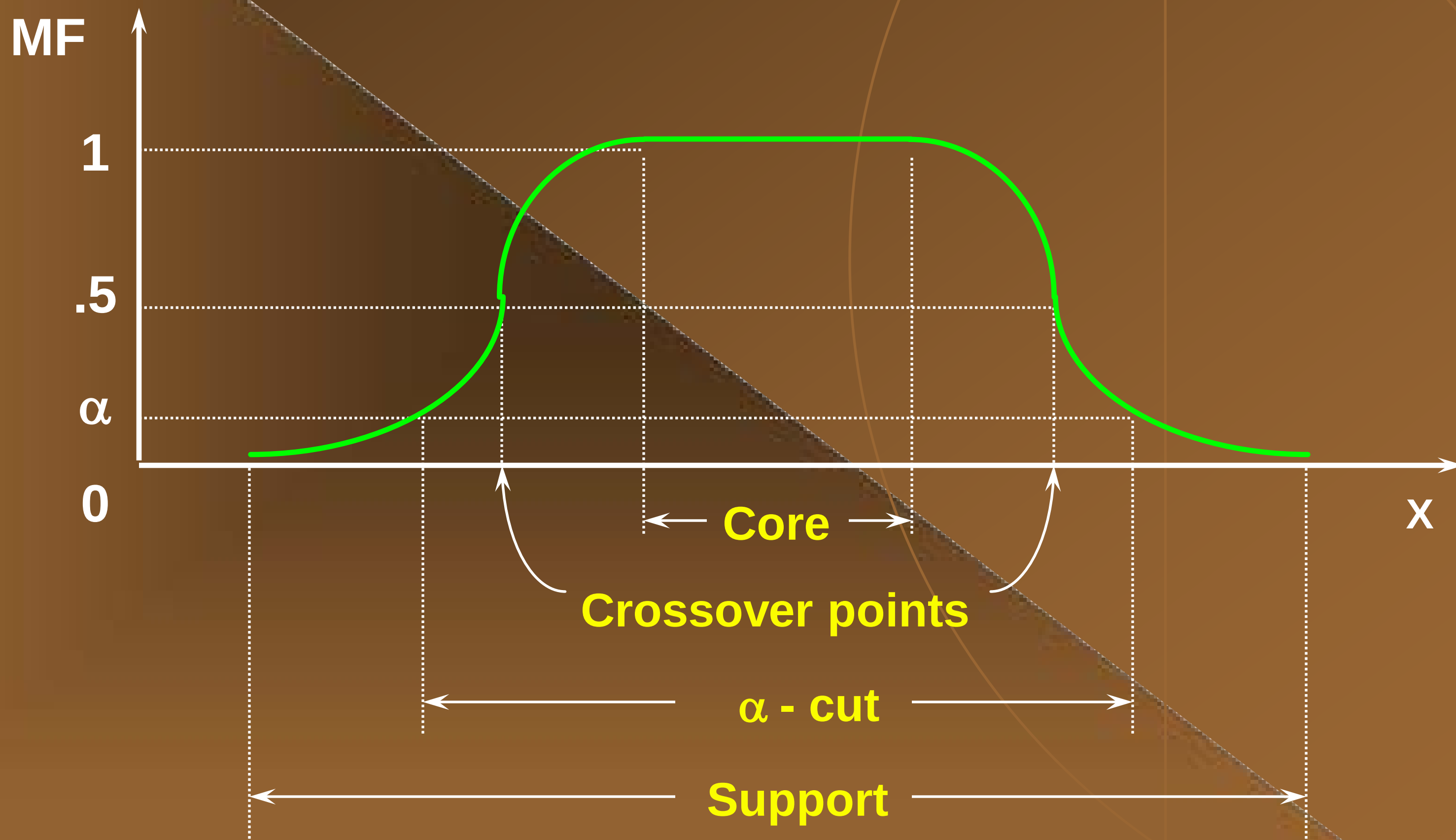


# Basic definitions & Terminology (2.2) (cont.)



- Support(A) = $\{x \in X \mid \mu_A(x) > 0\}$
- Core(A) = $\{x \in X \mid \mu_A(x) = 1\}$
- Normality: core(A) $\neq \emptyset \Rightarrow$ A is a normal fuzzy set
- Crossover(A) = $\{x \in X \mid \mu_A(x) = 0.5\}$
- $\alpha$ - cut: $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$
- Strong $\alpha$ - cut: $A'_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$

## MF Terminology

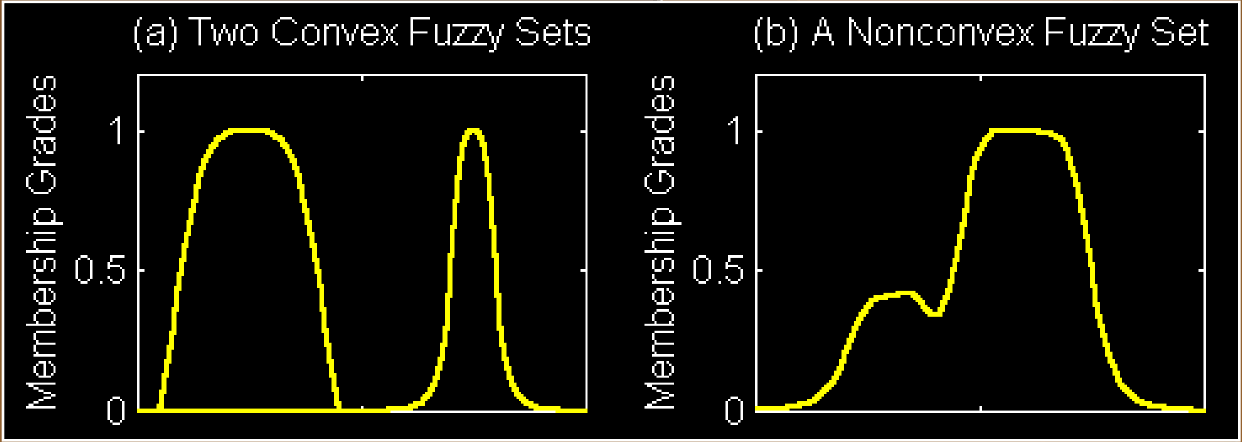


## Convexity of Fuzzy Sets

- A fuzzy set *A* is convex if for any $\lambda$ in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, *A* is convex if all its $\alpha$-cuts are convex.



# Basic definitions & Terminology (2.2) (cont.)

- **Fuzzy numbers:** a fuzzy number A is a fuzzy set in IR that satisfies normality & convexity

- **Bandwidths:** for a normal & convex set, the bandwidth is the distance between two unique crossover points

$$\text{Width}(A) = |x_2 - x_1|$$

$$\text{With } \mu_A(x1) = \mu_A(x2) = 0.5$$

- **Symmetry:** a fuzzy set A is symmetric if its MF is symmetric around a certain point x = c, namely

$$\mu_A(x + c) = \mu_A(c - x) \quad \forall x \in X$$

- Open left, open right, closed:

$$\text{open left fuzzy set } A \Leftrightarrow \lim_{x \to -\infty} \mu_A(x) = 1 \text{ and } \lim_{x \to +\infty} \mu_A(x) = 0$$

$$\text{open right fuzzy set } A \Leftrightarrow \lim_{x \to -\infty} \mu_A(x) = 0 \text{ and } \lim_{x \to +\infty} \mu_A(x) = 1$$

$$\text{closed fuzzy set } A \Leftrightarrow \lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$$

# Set-Theoretic Operations (2.3)



- Subset:

$$A \subseteq B \Leftrightarrow \mu_A \leq \mu_B$$

- Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

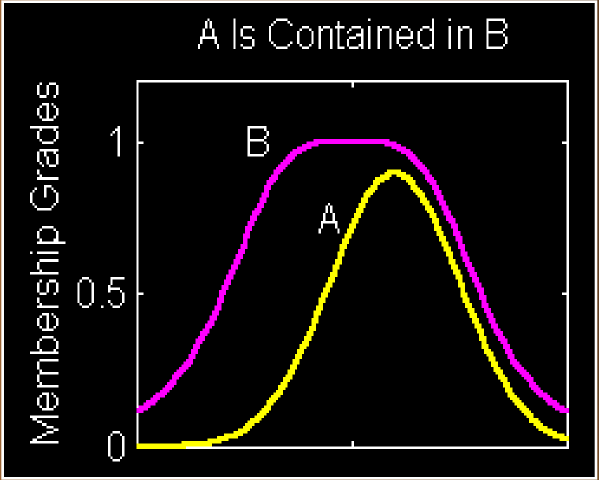
- Union:

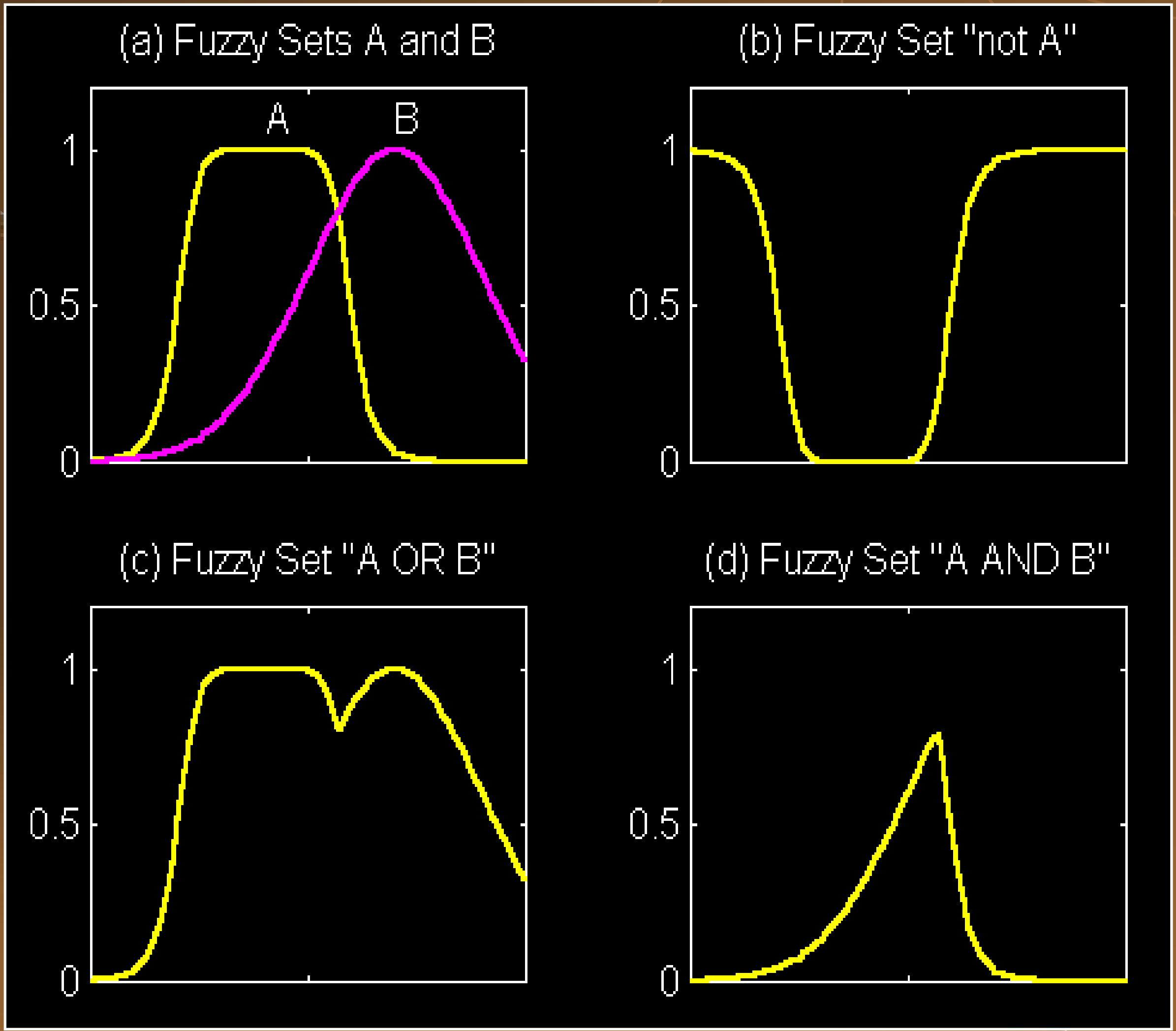
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

- Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

# Set-Theoretic Operations (2.3) (cont.)



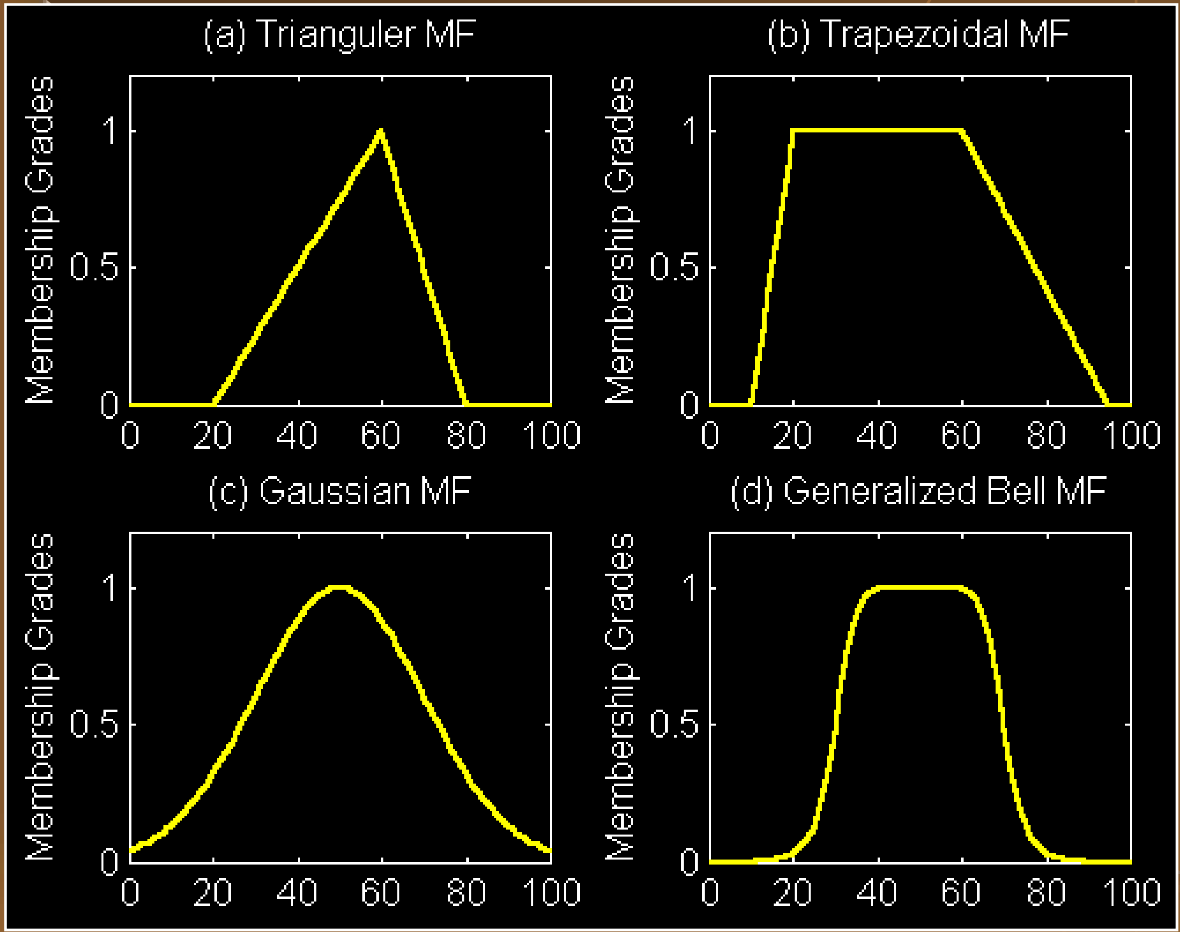
# MF Formulation & Parameterization (2)

## MFs of One Dimension

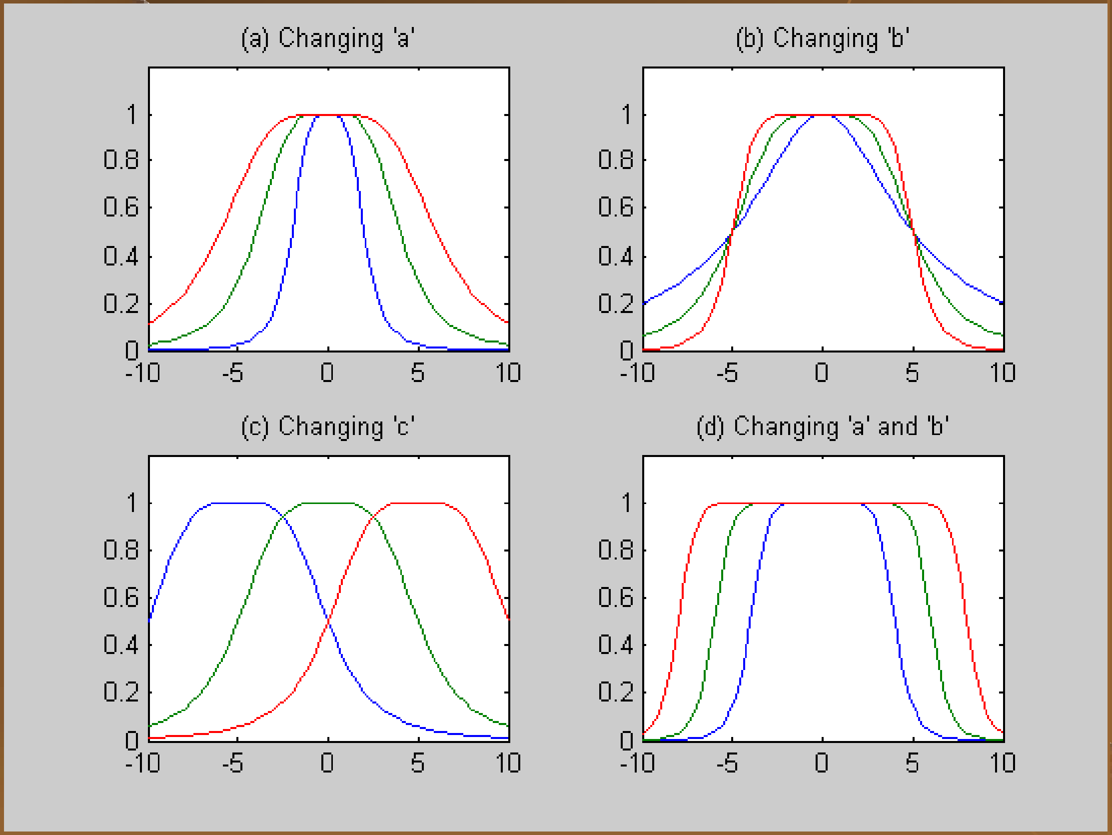
- Triangular MF: $$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$
- Trapezoidal MF: $$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$
- Gaussian MF: $$\mathbf{gaussmf(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}}$$
- Generalized bell MF: $$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

# MF Formulation & Parameterization (2.4) (cont.)

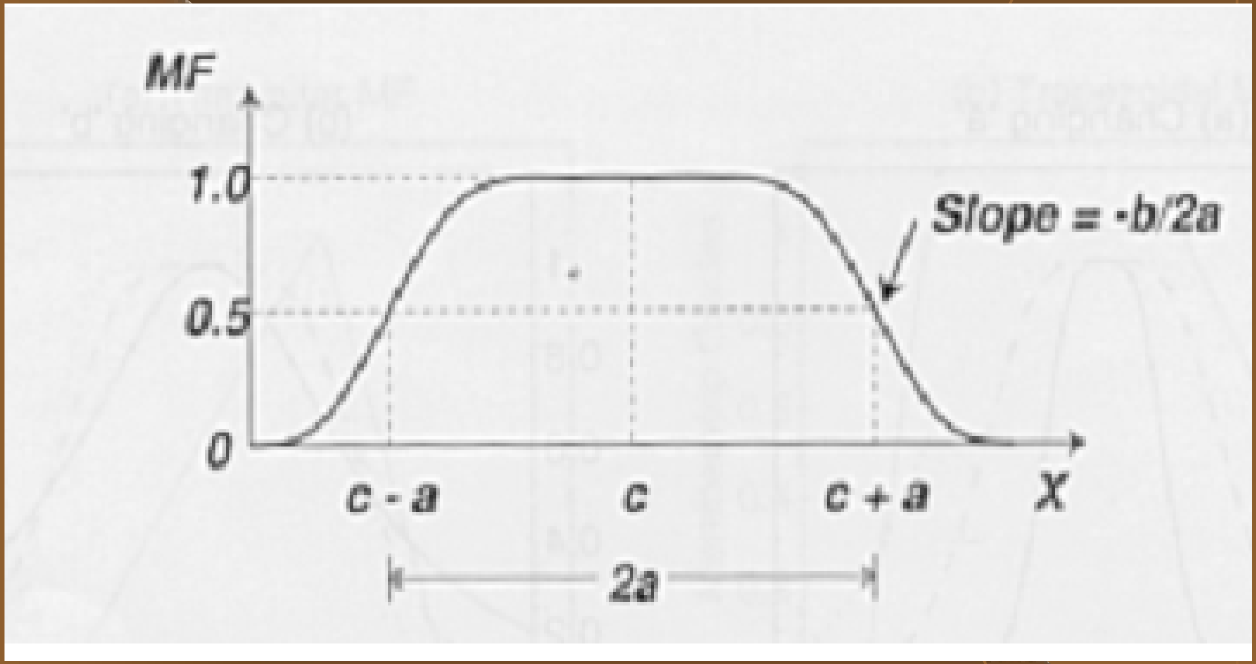




- Change of parameters in the generalized bell MF



# MF Formulation & Parameterization (2.4) (cont.)




Physical meaning of parameters in a generalized bell MF

# MF Formulation & Parameterization (2.4) (cont.)

- Gaussian MFs and bell MFs achieve smoothness, they are unable to specify asymmetric Mfs which are important in many applications

- Asymmetric & close MFs can be synthesized using either the absolute difference or the product of two sigmoidal functions

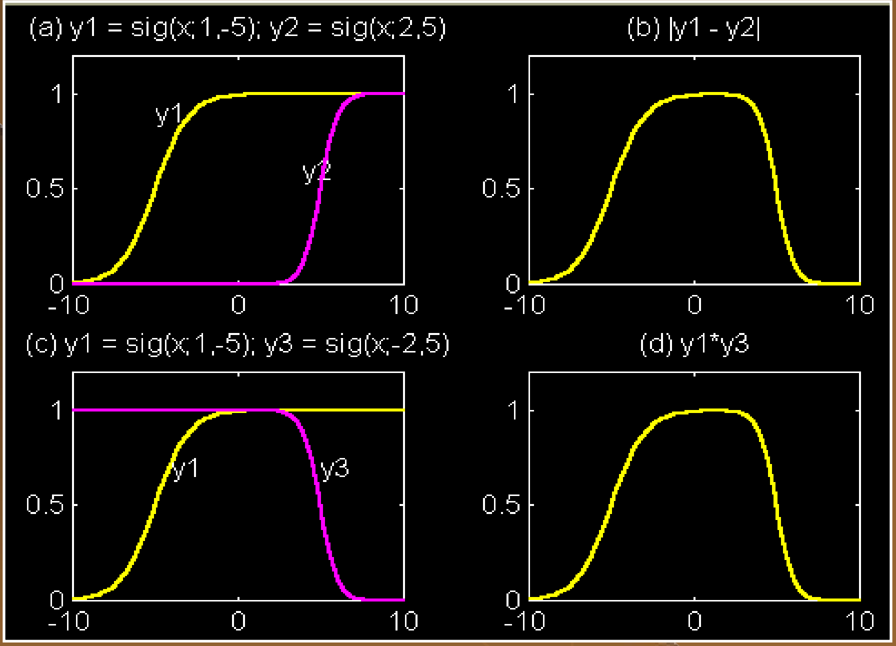
# MF Formulation & Parameterization (2.4) (cont.)

- Sigmoidal MF: $$\mathrm{sigmf}(x; a, c) = \frac{1}{1 + e^{-a(x-c)}}$$

Extensions:

Abs. difference of two sig. MF

Product of two sig. MF



# MF Formulation & Parameterization (2.4) (cont.)



- A sigmoidal MF is inherently open right or left & thus, it is appropriate for representing concepts such as “very large” or “very negative”

- Sigmoidal MF mostly used as activation function of artificial neural networks (NN)

- A NN should synthesize a close MF in order to simulate the behavior of a fuzzy inference system

# MF Formulation & Parameterization (2.4) (cont.)

- The list of MFs introduced in this section is by no means exhaustive
- Other specialized MFs can be created for specific applications if necessary
- Any type of continuous probability distribution functions can be used as an MF

# More on Fuzzy Union, Intersection & Complement (2.5)



- Fuzzy complement
  - Another way to define reasonable & consistent operations on fuzzy sets
    - General requirements:
      - Boundary: $N(0)=1$ and $N(1) = 0$
      - Monotonicity: $N(a) > N(b)$ if $a < b$
      - Involution: $N(N(a) = a$

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- Two types of fuzzy complements:
  - Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$ (s > -1)

(Family of fuzzy complement operators)

  - Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$ (w > 0)

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

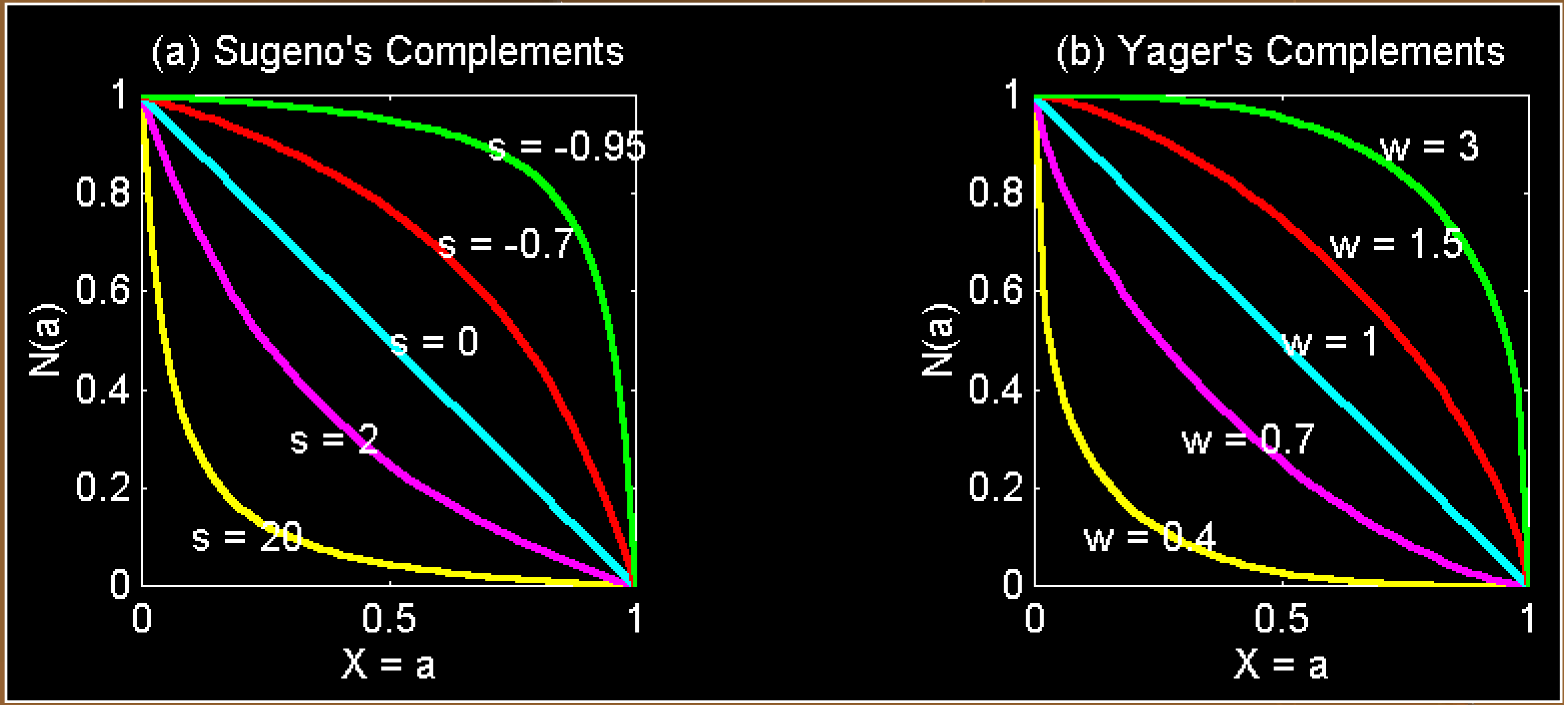


**Sugeno's complement:**

$$N_s(a) = \frac{1-a}{1+sa}$$

**Yager's complement:**

$$N_w(a) = (1-a^w)^{1/w}$$



# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)



- Fuzzy Intersection and Union:
  - The intersection of two fuzzy sets A and B is specified in general by a function

    T: [0,1] * [0,1] $\rightarrow$ [0,1] with

$$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{*} \mu_B(x)$$

where $\tilde{*}$ is a binary operator for the function T.

This class of fuzzy intersection operators are called T-norm (triangular) operators.

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)



- T-norm operators satisfy:
  - Boundary: T(0, 0) = 0, T(a, 1) = T(1, a) = a
    Correct generalization to crisp sets
  - Monotonicity: T(a, b) < T(c, d) if a < c and b < d
    A decrease of membership in A & B cannot increase a membership in $A \cap B$
  - Commutativity: T(a, b) = T(b, a)
    T is indifferent to the order of fuzzy sets to be combined
  - Associativity: T(a, T(b, c)) = T(T(a, b), c)
    Intersection is independent of the order of pairwise groupings

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)



- T-norm (cont.)
  - Four examples:
    - Minimum: $T_m(a, b) = \min(a,b) = a \wedge b$
    - Algebraic product: $T_a(a, b) = ab$
    - Bounded product: $T_b(a, b) = 0 \vee (a + b - 1)$
    - Drastic product: $T_d(a, b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \\ 0, & \text{if } a,b < 1 \end{cases}$

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)



- T-conorm or S-norm

The fuzzy union operator is defined by a function

$$S: [0,1] * [0,1] \rightarrow [0,1]$$

wich aggregates two membership function as:

$$\mu_{A \cup B} = S(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{+} \mu_B(x)$$

where s is called an s-norm satisfying:

- Boundary: S(1, 1) = 1, S(a, 0) = S(0, a) = a
- Monotonicity: S(a, b) < S(c, d) if a < c and b < d
- Commutativity: S(a, b) = S(b, a)
- Associativity: S(a, S(b, c)) = S(S(a, b), c)

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)



- T-conorm or S-norm (cont.)
  - Four examples (page 38):
    - Maximum: $S_m(a, b) = \max(a,b) = a \vee b$
    - Algebraic sum: $S_a(a, b) = a + b - ab$
    - Bounded sum: $S_b(a, b) = 1 \wedge (a + b)$
    - Drastic sum: $S_d(a, b) = \begin{cases} a, & \text{if } b = 0 \\ b, & \text{if } a = 0 \\ 1, & \text{if } a,b > 0 \end{cases}$

# More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- Generalized DeMorgan's Law
  - T-norms and S-norms are duals which support the generalization of DeMorgan's law:
    - $T(a, b) = N(S(N(a), N(b)))$
    - $S(a, b) = N(T(N(a), N(b)))$