Computer Graphics 12:
Spline Representations

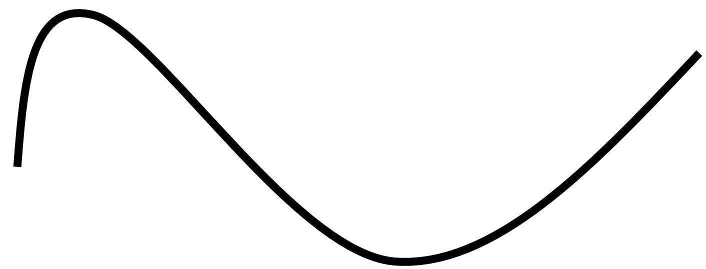
Contents

Today we are going to look at Bézier spline curves

- Introduction to splines
- Bézier curves
- Bézier cubic splines

Spline Representations

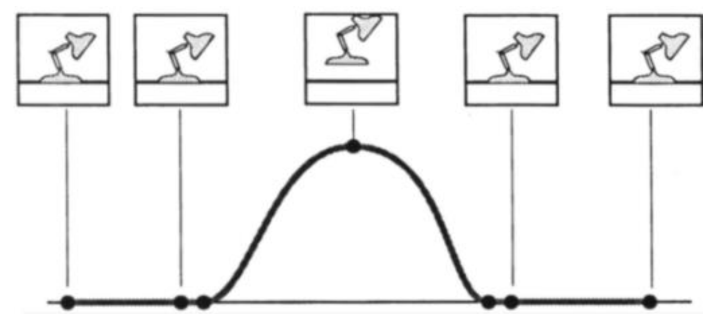
A spline is a smooth curve defined mathematically using a set of constraints



Splines have many uses:

- 2D illustration
- Fonts
- 3D Modelling
- Animation

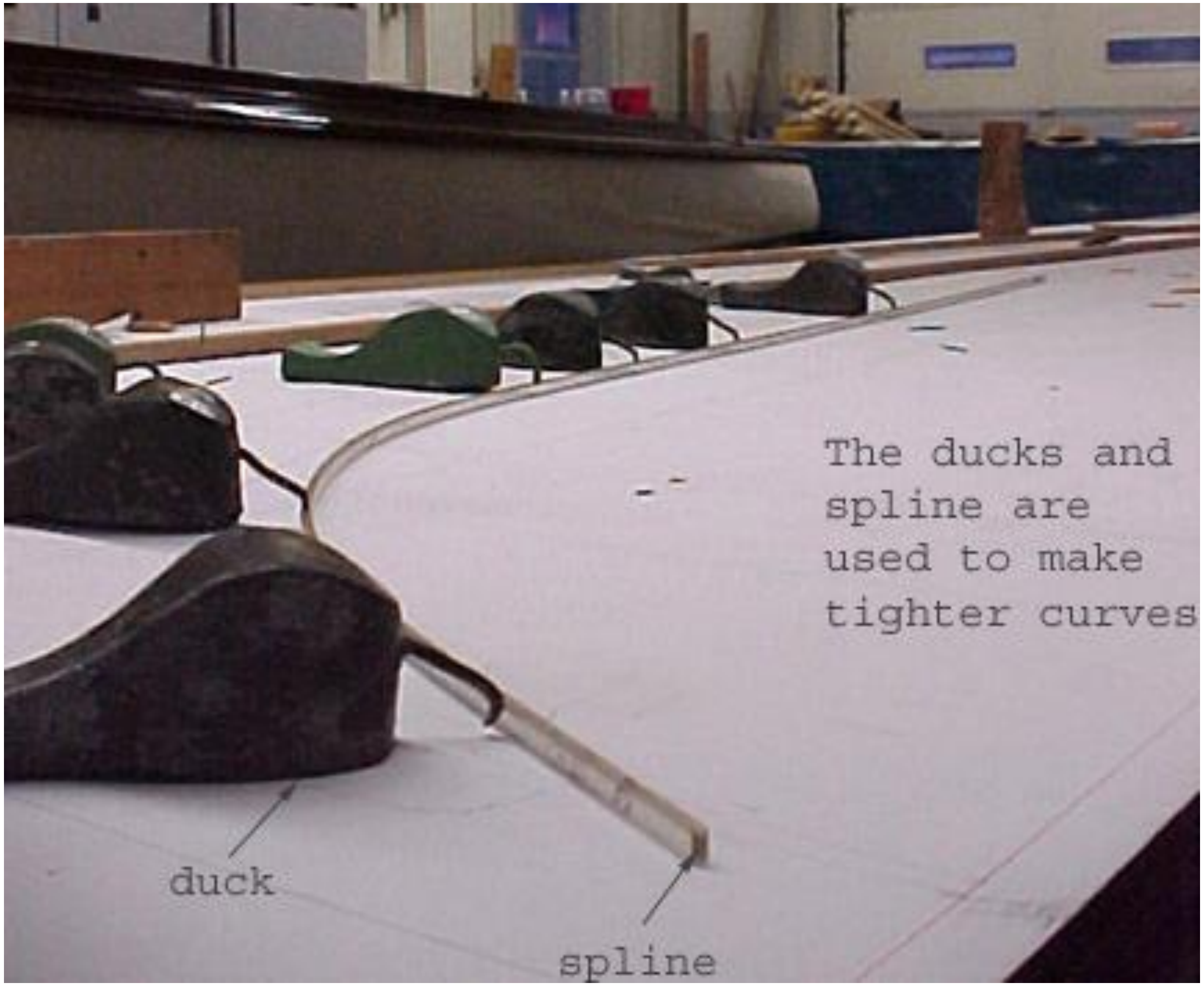
“Manifold Splines”, X. Gu, Y. He & H. Qin, Solid and Physics Modeling 2005.



ACM © 1987 “Principles of traditional animation applied to 3D computer animation”

Physical Splines

Physical splines are used in car/boat design

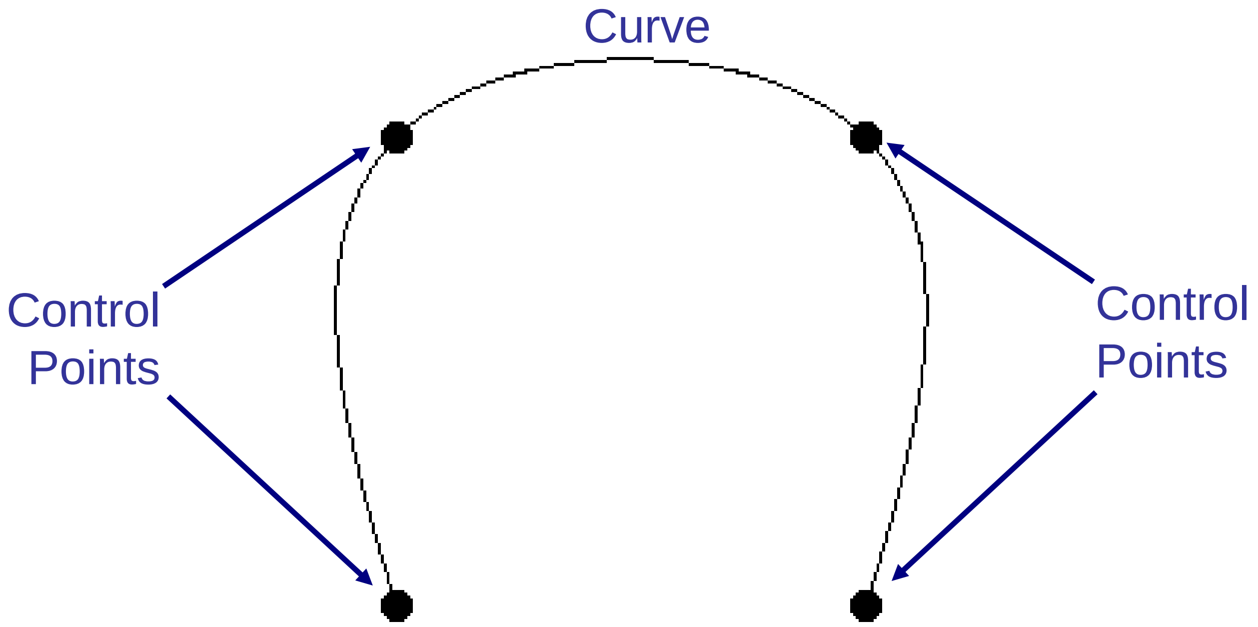


Pierre Bézier

Big Idea

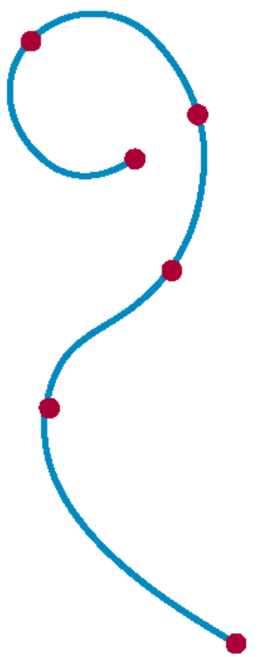
User specifies control points

Defines a smooth curve



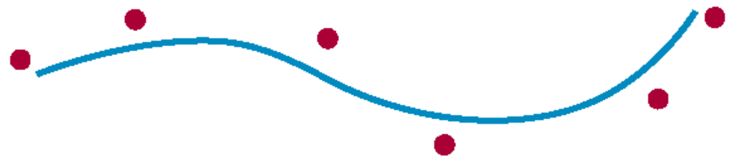
Interpolation Vs Approximation

A spline curve is specified using a set of **control points**



There are two ways to fit a curve to these points:

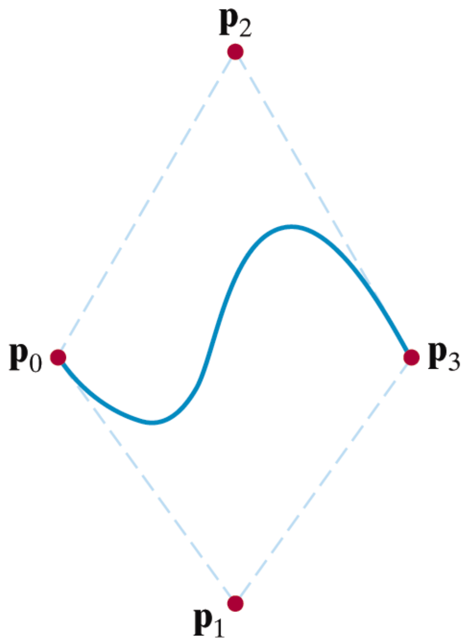
- **Interpolation -** the curve passes through all of the control points
- **Approximation -** the curve does not pass through all of the control points

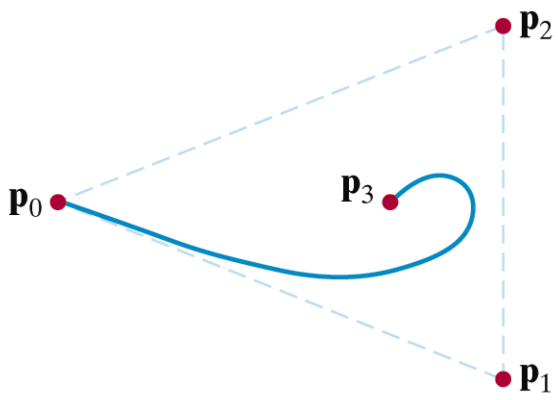


Convex Hulls

The boundary formed by the set of control points for a spline is known as a **convex hull**

Think of an elastic band stretched around the control points

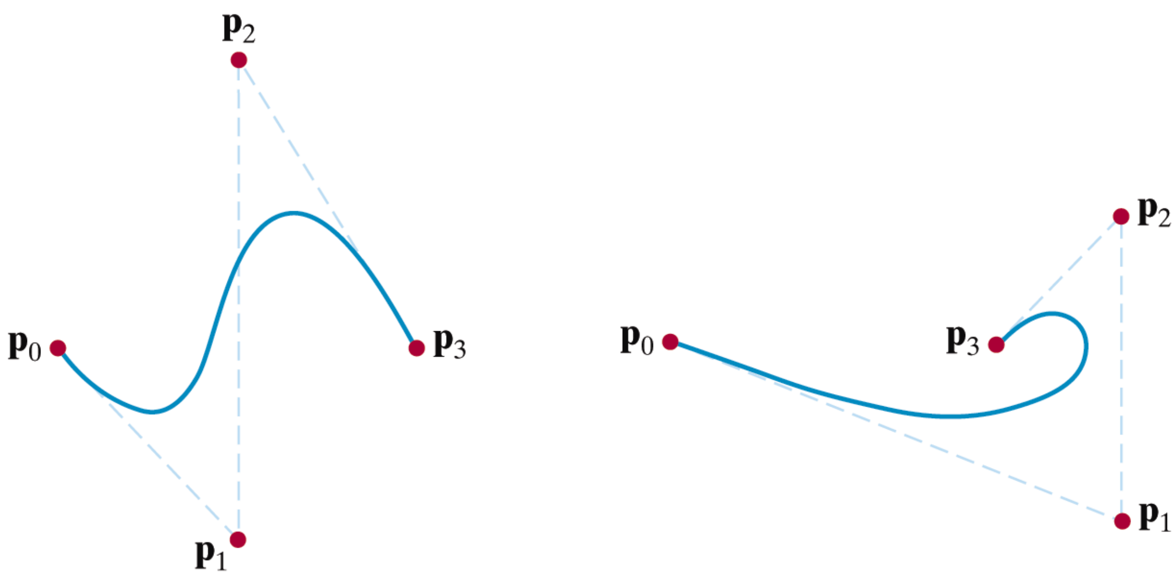




Control Graphs

A polyline connecting the control points in order is known as a **control graph**

Usually displayed to help designers keep track of their splines



Bézier Spline Curves

A spline approximation method developed by the French engineer Pierre Bézier for use in the design of Renault car bodies

A Bézier curve can be fitted to any number of control points – although usually 4 are used

Bézier Spline Curves (cont…)

Consider the case of $n+1$ control points denoted as $p_k=(x_k, y_k, z_k)$ where k varies from 0 to n

The coordinate positions are blended to produce the position vector $P(u)$ which describes the path of the Bézier polynomial function between p_0 and p_n

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

Bézier Spline Curves (cont…)

The Bézier blending functions $BEZ_{k,n}(u)$ are the *Bernstein polynomials*

$$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$$

where parameters $C(n,k)$ are the binomial coefficients

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Bézier Spline Curves (cont…)

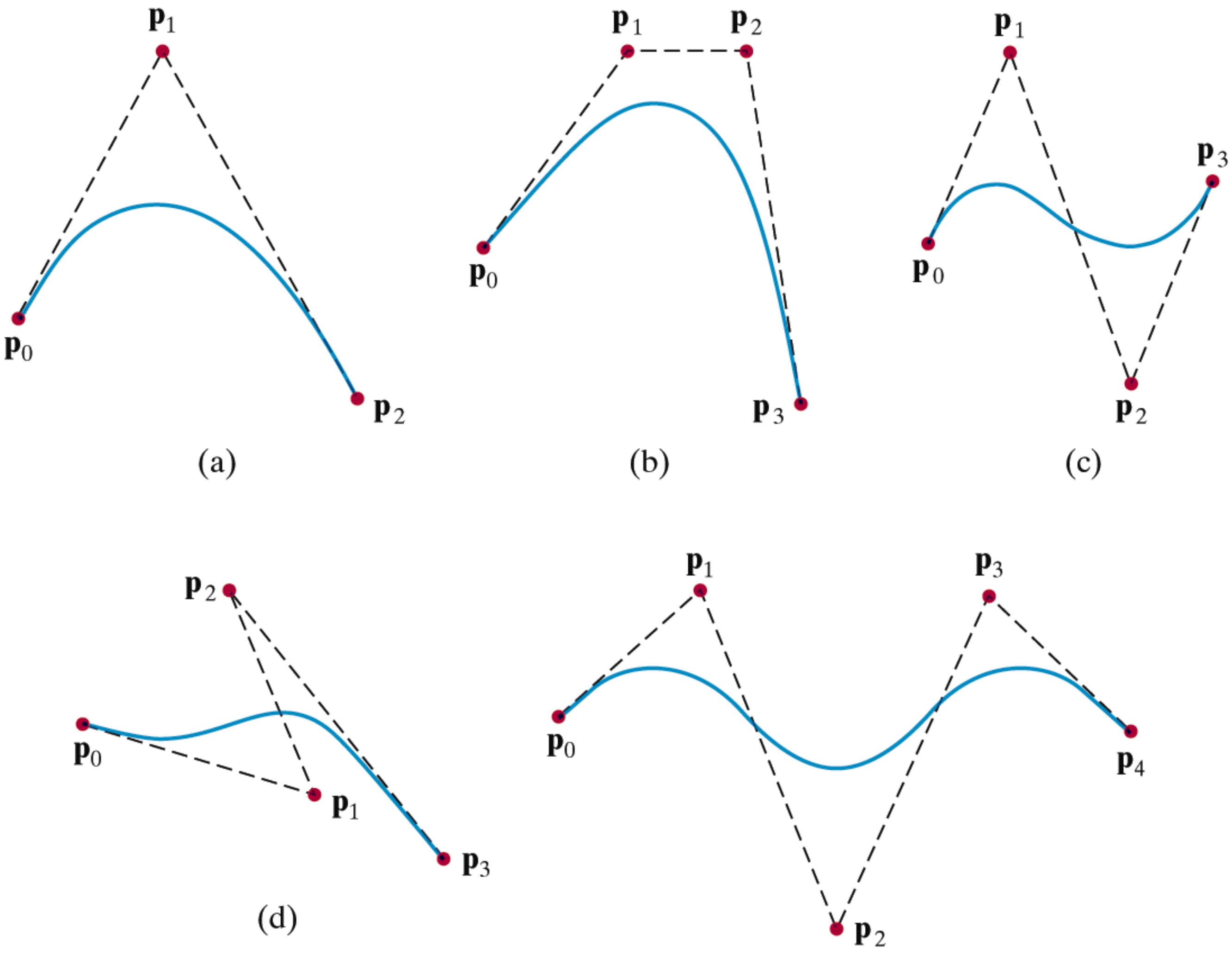
So, the individual curve coordinates can be given as follows

$$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$

Bézier Spline Curves (cont…)



(a) (b) (c) (d) (e)

Important Properties Of Bézier Curves

- The first and last control points are the first and last point on the curve
  - $P(0) = p_0$
  - $P(1) = p_n$
- The curve lies within the convex hull as the Bézier blending functions are all positive and sum to 1

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

The slope at the beginning and end of the curve are along the along the first two and the last two points respectively

Cubic Bézier Curve

Many graphics packages restrict Bézier curves to have only 4 control points (i.e. n = 3)

The blending functions when n = 3 are simplified as follows:

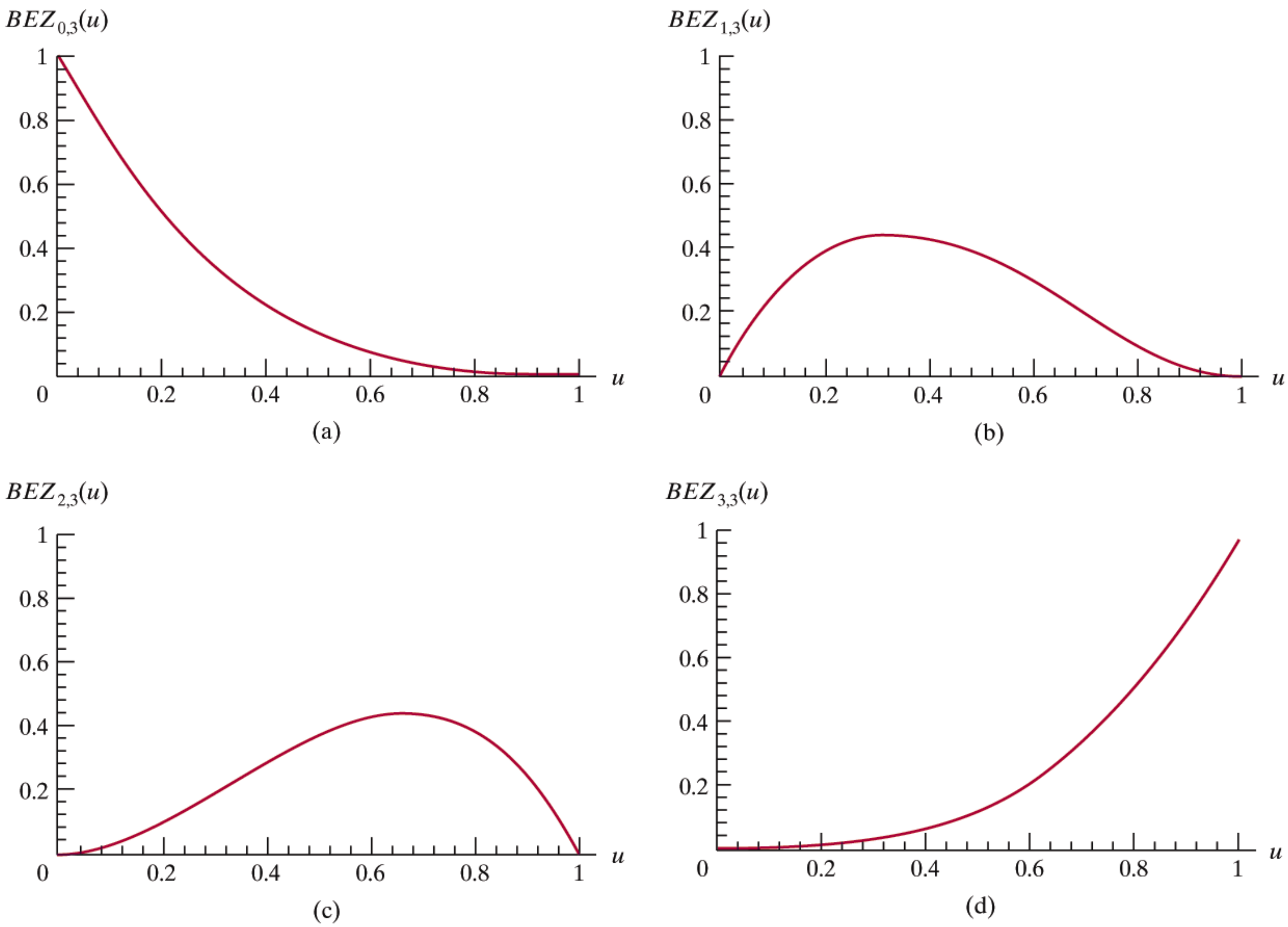
$$BEZ_{0,3} = (1-u)^3$$

$$BEZ_{1,3} = 3u(1-u)^2$$

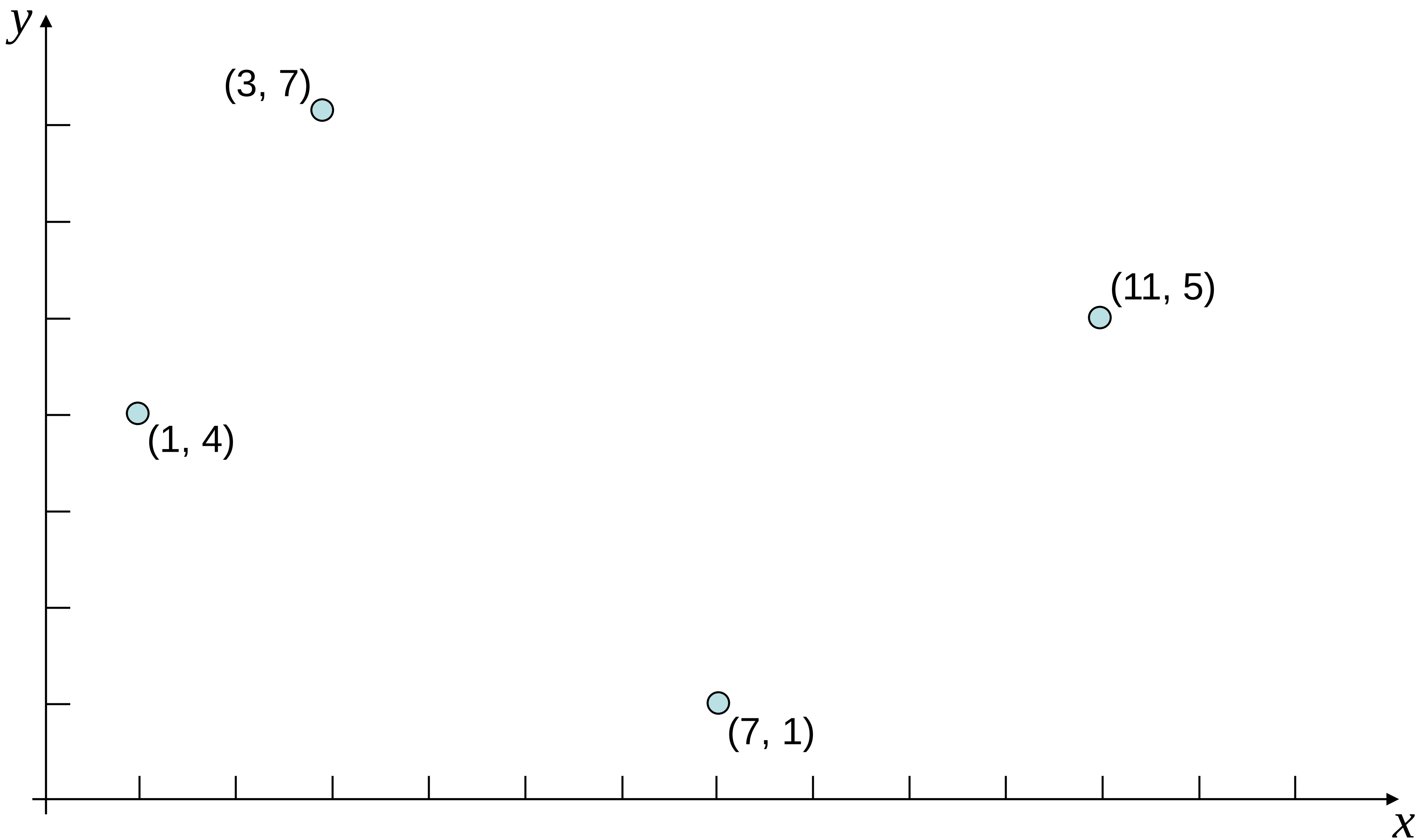
$$BEZ_{2,3} = 3u^2(1-u)$$

$$BEZ_{3,3} = u^3$$

Cubic Bézier Blending Functions



Bézier Spline Curve Exercise



Summary

Today we had a look at spline curves and in particular Bézier curves

The whole point is that the spline functions give us an approximation to a smooth curve