Section 2-9 : Equations Reducible to Quadratic in Form

In this section we are going to look at equations that are called **quadratic in form** or **reducible to quadratic in form**. What this means is that we will be looking at equations that if we look at them in the correct light we can make them look like quadratic equations. At that point we can use the techniques we developed for quadratic equations to help us with the solution of the actual equation.

It is usually best with these to show the process with an example so let's do that.

Example 1 Solve $x^4 - 7x^2 + 12 = 0$

Solution
Now, let's start off here by noticing that

$$x^4 = \left(x^2\right)^2$$

In other words, we can notice here that the variable portion of the first term (*i.e.* ignore the coefficient) is nothing more than the variable portion of the second term squared. Note as well that all we really needed to notice here is that the exponent on the first term was twice the exponent on the second term.

This, along with the fact that third term is a constant, means that this equation is reducible to quadratic in form. We will solve this by first defining,

$$u = x^2$$

Now, this means that

$$u^2 = \left(x^2\right)^2 = x^4$$

Therefore, we can write the equation in terms of u's instead of x's as follows,

$$x^4 - 7x^2 + 12 = 0 \qquad \Rightarrow \qquad u^2 - 7u + 12 = 0$$

The new equation (the one with the u's) is a quadratic equation and we can solve that. In fact, this equation is factorable, so the solution is,

$$u^2 - 7u + 12 = (u-4)(u-3) = 0 \qquad \Rightarrow \qquad u = 3, \, u = 4$$

So, we get the two solutions shown above. These aren't the solutions that we're looking for. We want values of x, not values of u. That isn't really a problem once we recall that we've defined

$$u = x^2$$

To get values of x for the solution all we need to do is plug in u into this equation and solve that for x. Let's do that.

$$u = 3: \qquad 3 = x^2 \qquad \Rightarrow \qquad x = \pm\sqrt{3}$$
$$u = 4: \qquad 4 = x^2 \qquad \Rightarrow \qquad x = \pm\sqrt{4} = \pm 2$$

So, we have four solutions to the original equation, $x = \pm 2$ and $x = \pm\sqrt{3}$.

So, the basic process is to check that the equation is reducible to quadratic in form then make a quick substitution to turn it into a quadratic equation. We solve the new equation for u, the variable from the substitution, and then use these solutions and the substitution definition to get the solutions to the equation that we really want.

In most cases to make the check that it's reducible to quadratic in form all that we really need to do is to check that one of the exponents is twice the other. There is one exception to this that we'll see here once we get into a set of examples.

Also, once you get "good" at these you often don't really need to do the substitution either. We will do them to make sure that the work is clear. However, these problems can be done without the substitution in many cases.

Example 2 Solve each of the following equations.

    (a) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

    (b) $y^{-6} - 9y^{-3} + 8 = 0$

    (c) $z - 9\sqrt{z} + 14 = 0$

    (d) $t^4 - 4 = 0$

Solution

(a) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

Okay, in this case we can see that,

$$\frac{2}{3} = 2\left(\frac{1}{3}\right)$$

and so one of the exponents is twice the other so it looks like we've got an equation that is reducible to quadratic in form. The substitution will then be,

$$u = x^{\frac{1}{3}} \qquad\qquad u^2 = \left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$$

Substituting this into the equation gives,

$$u^2 - 2u - 15 = 0$$
$$(u - 5)(u + 3) = 0 \qquad\qquad \Rightarrow \qquad\qquad u = -3, \ u = 5$$

Now that we've gotten the solutions for u we can find values of x.

$$u = -3: \qquad\qquad x^{\frac{1}{3}} = -3 \qquad \Rightarrow \qquad x = (-3)^3 = -27$$

$$u = 5: \qquad\qquad x^{\frac{1}{3}} = 5 \qquad \Rightarrow \qquad x = (5)^3 = 125$$

So, we have two solutions here $x = -27$ and $x = 125$.

(b) $y^{-6} - 9y^{-3} + 8 = 0$

For this part notice that,

$$-6 = 2(-3)$$

and so we do have an equation that is reducible to quadratic form. The substitution is,

$$u = y^{-3} \qquad\qquad u^2 = \left(y^{-3}\right)^2 = y^{-6}$$

The equation becomes,

$$u^2 - 9u + 8 = 0$$
$$(u-8)(u-1) = 0 \qquad\qquad u = 1,\ u = 8$$

Now, going back to y's is going to take a little more work here, but shouldn't be too bad.

$$u = 1: \quad \Rightarrow \quad y^{-3} = \frac{1}{y^3} = 1 \quad \Rightarrow \quad y^3 = \frac{1}{1} = 1 \quad \Rightarrow \quad y = (1)^{\frac{1}{3}} = 1$$

$$u = 8: \quad \Rightarrow \quad y^{-3} = \frac{1}{y^3} = 8 \quad \Rightarrow \quad y^3 = \frac{1}{8} \quad \Rightarrow \quad y = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$$

The two solutions to this equation are $y = 1$ and $y = \frac{1}{2}$.

(c) $z - 9\sqrt{z} + 14 = 0$

This one is a little trickier to see that it's quadratic in form, yet it is. To see this recall that the exponent on the square root is one-half, then we can notice that the exponent on the first term is twice the exponent on the second term. So, this equation is in fact reducible to quadratic in form.

Here is the substitution.

$$u = \sqrt{z} \qquad\qquad u^2 = \left(\sqrt{z}\right)^2 = z$$

The equation then becomes,

$$u^2 - 9u + 14 = 0$$
$$(u-7)(u-2) = 0 \qquad\qquad u = 2,\ u = 7$$

Now go back to z's.

$$u = 2: \qquad \Rightarrow \qquad \sqrt{z} = 2 \qquad \Rightarrow \qquad z = (2)^2 = 4$$
$$u = 7: \qquad \Rightarrow \qquad \sqrt{z} = 7 \qquad \Rightarrow \qquad z = (7)^2 = 49$$

The two solutions for this equation are $z = 4$ and $z = 49$

(d) $t^4 - 4 = 0$

Now, this part is the exception to the rule that we've been using to identify equations that are reducible to quadratic in form. There is only one term with a *t* in it. However, notice that we can write the equation as,

$$\left(t^2\right)^2 - 4 = 0$$

So, if we use the substitution,

$$u = t^2 \qquad\qquad u^2 = \left(t^2\right)^2 = t^4$$

the equation becomes,

$$u^2 - 4 = 0$$

and so it is reducible to quadratic in form.

Now, we can solve this using the square root property. Doing that gives,

$$u = \pm\sqrt{4} = \pm 2$$

Now, going back to *t*'s gives us,

$$u = 2: \qquad \Rightarrow \qquad t^2 = 2 \qquad \Rightarrow \qquad t = \pm\sqrt{2}$$
$$u = -2: \qquad \Rightarrow \qquad t^2 = -2 \qquad \Rightarrow \qquad t = \pm\sqrt{-2} = \pm\sqrt{2}\, i$$

In this case we get four solutions and two of them are complex solutions. Getting complex solutions out of these are actually more common that this set of examples might suggest. The problem is that to get some of the complex solutions requires knowledge that we haven't (and won't) cover in this course. So, they don't show up all that often.

All of the examples to this point gave quadratic equations that were factorable or in the case of the last part of the previous example was an equation that we could use the square root property on. That need not always be the case however. It is more than possible that we would need the quadratic formula to do some of these. We should do an example of one of these just to make the point.

Example 3 Solve $2x^{10} - x^5 - 4 = 0$.

Solution
In this case we can reduce this to quadratic in form by using the substitution,

$$u = x^5 \qquad\qquad u^2 = x^{10}$$

Using this substitution the equation becomes,

$$2u^2 - u - 4 = 0$$

This doesn't factor and so we'll need to use the quadratic formula on it. From the quadratic formula the solutions are,

$$u = \frac{1 \pm \sqrt{33}}{4}$$

Now, in order to get back to x's we are going to need decimals values for these so,

$$u = \frac{1 + \sqrt{33}}{4} = 1.68614 \qquad\qquad u = \frac{1 - \sqrt{33}}{4} = -1.18614$$

Now, using the substitution to get back to x's gives the following,

$$u = 1.68614 \qquad x^5 = 1.68614 \qquad x = (1.68614)^{\frac{1}{5}} = 1.11014$$

$$u = -1.18614 \qquad x^5 = -1.18614 \qquad x = (-1.18614)^{\frac{1}{5}} = -1.03473$$

We had to use a calculator to get the final answer for these. This is one of the reasons that you don't tend to see too many of these done in an Algebra class. The work and/or answers tend to be a little messy.