Section 2-5 : Quadratic Equations - Part I

Before proceeding with this section we should note that the topic of solving quadratic equations will be covered in two sections. This is done for the benefit of those viewing the material on the web. This is a long topic and to keep page load times down to a minimum the material was split into two sections.

So, we are now going to solve quadratic equations. First, the **standard form** of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

The only requirement here is that we have an x^2 in the equation. We guarantee that this term will be present in the equation by requiring $a \neq 0$. Note however, that it is okay if *b* and/or *c* are zero.

There are many ways to solve quadratic equations. We will look at four of them over the course of the next two sections. The first two methods won't always work yet are probably a little simpler to use when they work. This section will cover these two methods. The last two methods will always work, but often require a little more work or attention to get correct. We will cover these methods in the next section.

So, let's get started.

Solving by Factoring

As the heading suggests we will be solving quadratic equations here by factoring them. To do this we will need the following fact.

$$\text{If } \ ab = 0 \ \text{ then either } \ a = 0 \ \text{ and/or } \ b = 0$$

This fact is called the **zero factor property** or **zero factor principle**. All the fact says is that if a product of two terms is zero then at least one of the terms had to be zero to start off with.

Notice that this fact will ONLY work if the product is equal to zero. Consider the following product.

$$ab = 6$$

In this case there is no reason to believe that either *a* or *b* will be 6. We could have $a = 2$ and $b = 3$ for instance. So, do not misuse this fact!

To solve a quadratic equation by factoring we first must move all the terms over to one side of the equation. Doing this serves two purposes. First, it puts the quadratics into a form that can be factored. Secondly, and probably more importantly, in order to use the zero factor property we MUST have a zero on one side of the equation. If we don't have a zero on one side of the equation we won't be able to use the zero factor property.

Let's take a look at a couple of examples. Note that it is assumed that you can do the factoring at this point and so we won't be giving any details on the factoring. If you need a review of factoring you should go back and take a look at the Factoring section of the previous chapter.

Example 1 Solve each of the following equations by factoring.

(a) $x^2 - x = 12$

(b) $x^2 + 40 = -14x$

(c) $y^2 + 12y + 36 = 0$

(d) $4m^2 - 1 = 0$

(e) $3x^2 = 2x + 8$

(f) $10z^2 + 19z + 6 = 0$

(g) $5x^2 = 2x$

Solution

Now, as noted earlier, we won't be putting any detail into the factoring process, so make sure that you can do the factoring here.

(a) $x^2 - x = 12$

First, get everything on side of the equation and then factor.

$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$

Now at this point we've got a product of two terms that is equal to zero. This means that at least one of the following must be true.

$$x - 4 = 0 \qquad \text{OR} \qquad x + 3 = 0$$
$$x = 4 \qquad \text{OR} \qquad x = -3$$

Note that each of these is a linear equation that is easy enough to solve. What this tell us is that we have two solutions to the equation, $x = 4$ and $x = -3$. As with linear equations we can always check our solutions by plugging the solution back into the equation. We will check $x = -3$ and leave the other to you to check.

$$(-3)^2 - (-3) \stackrel{?}{=} 12$$
$$9 + 3 \stackrel{?}{=} 12$$
$$12 = 12 \qquad \text{OK}$$

So, this was in fact a solution.

(b) $x^2 + 40 = -14x$

As with the first one we first get everything on side of the equal sign and then factor.

$$x^2 + 40 + 14x = 0$$
$$(x+4)(x+10) = 0$$

Now, we once again have a product of two terms that equals zero so we know that one or both of them have to be zero. So, technically we need to set each one equal to zero and solve. However, this is usually easy enough to do in our heads and so from now on we will be doing this solving in our head.

The solutions to this equation are,

$$x = -4 \qquad \text{AND} \qquad x = -10$$

To save space we won't be checking any more of the solutions here, but you should do so to make sure we didn't make any mistakes.

(c) $y^2 + 12y + 36 = 0$

In this case we already have zero on one side and so we don't need to do any manipulation to the equation all that we need to do is factor. Also, don't get excited about the fact that we now have *y*'s in the equation. We won't always be dealing with *x*'s so don't expect to always see them.

So, let's factor this equation.

$$\begin{aligned} y^2 + 12y + 36 &= 0 \\ (y+6)^2 &= 0 \\ (y+6)(y+6) &= 0 \end{aligned}$$

In this case we've got a perfect square. We broke up the square to denote that we really do have an application of the zero factor property. However, we usually don't do that. We usually will go straight to the answer from the squared part.

The solution to the equation in this case is,

$$y = -6$$

We only have a single value here as opposed to the two solutions we've been getting to this point. We will often call this solution a **double root** or say that it has **multiplicity of 2** because it came from a term that was squared.

(d) $4m^2 - 1 = 0$

As always let's first factor the equation.

$$\begin{aligned} 4m^2 - 1 &= 0 \\ (2m-1)(2m+1) &= 0 \end{aligned}$$

Now apply the zero factor property. The zero factor property tells us that,

$$\begin{array}{ccc} 2m - 1 = 0 & \text{OR} & 2m + 1 = 0 \\ 2m = 1 & \text{OR} & 2m = -1 \\ m = \dfrac{1}{2} & \text{OR} & m = -\dfrac{1}{2} \end{array}$$

Again, we will typically solve these in our head, but we needed to do at least one in complete detail. So, we have two solutions to the equation.

$$m = \frac{1}{2} \qquad \text{AND} \qquad m = -\frac{1}{2}$$

(e) $3x^2 = 2x + 8$

Now that we've done quite a few of these, we won't be putting in as much detail for the next two problems. Here is the work for this equation.

$$3x^2 - 2x - 8 = 0$$

$$\left(3x+4\right)\left(x-2\right) = 0 \qquad \Rightarrow \qquad x = -\frac{4}{3} \quad \text{and} \quad x = 2$$

(f) $10z^2 + 19z + 6 = 0$

Again, factor and use the zero factor property for this one.

$$10z^2 + 19z + 6 = 0$$

$$\left(5z+2\right)\left(2z+3\right) = 0 \qquad \Rightarrow \qquad z = -\frac{2}{5} \quad \text{and} \quad z = -\frac{3}{2}$$

(g) $5x^2 = 2x$

This one always seems to cause trouble for students even though it's really not too bad.

First off. DO NOT CANCEL AN *x* FROM BOTH SIDES!!!! Do you get the idea that might be bad? It is. If you cancel an *x* from both sides, you WILL miss a solution so don't do it. Remember we are solving by factoring here so let's first get everything on one side of the equal sign.

$$5x^2 - 2x = 0$$

Now, notice that all we can do for factoring is to factor an *x* out of everything. Doing this gives,

$$x\left(5x-2\right) = 0$$

From the first factor we get that $x = 0$ and from the second we get that $x = \frac{2}{5}$. These are the two solutions to this equation. Note that if we'd canceled an *x* in the first step we would NOT have gotten $x = 0$ as an answer!

Let's work another type of problem here. We saw some of these back in the Solving Linear Equations section and since they can also occur with quadratic equations we should go ahead and work on to make sure that we can do them here as well.

Example 2 Solve each of the following equations.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

(b) $x+3+\dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

Solution

Okay, just like with the linear equations the first thing that we're going to need to do here is to clear the denominators out by multiplying by the LCD. Recall that we will also need to note value(s) of *x* that will give division by zero so that we can make sure that these aren't included in the solution.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

The LCD for this problem is $(x+1)(2x-4)$ and we will need to avoid $x=-1$ and $x=2$ to make sure we don't get division by zero. Here is the work for this equation.

$$\begin{aligned}(x+1)(2x-4)\left(\frac{1}{x+1}\right) &= (x+1)(2x-4)\left(1-\frac{5}{2x-4}\right)\\ 2x-4 &= (x+1)(2x-4)-5(x+1)\\ 2x-4 &= 2x^2-2x-4-5x-5\\ 0 &= 2x^2-9x-5\\ 0 &= (2x+1)(x-5)\end{aligned}$$

So, it looks like the two solutions to this equation are,

$$x=-\frac{1}{2} \qquad \text{and} \qquad x=5$$

Notice as well that neither of these are the values of *x* that we needed to avoid and so both are solutions.

(b) $x+3+\dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

In this case the LCD is $x-1$ and we will need to avoid $x=1$ so we don't get division by zero. Here is the work for this problem.

$$\begin{aligned}(x-1)\left(x+3+\frac{3}{x-1}\right) &= \left(\frac{4-x}{x-1}\right)(x-1)\\ (x-1)(x+3)+3 &= 4-x\\ x^2+2x-3+3 &= 4-x\\ x^2+3x-4 &= 0\\ (x-1)(x+4) &= 0\end{aligned}$$

So, the quadratic that we factored and solved has two solutions, $x=1$ and $x=-4$. However, when we found the LCD we also saw that we needed to avoid $x=1$ so we didn't get division by zero. Therefore, this equation has a single solution,

$$x = -4$$

Before proceeding to the next topic we should address that this idea of factoring can be used to solve equations with degree larger than two as well. Consider the following example.

Example 3 Solve $5x^3 - 5x^2 - 10x = 0$.

Solution
The first thing to do is factor this equation as much as possible. In this case that means factoring out the greatest common factor first. Here is the factored form of this equation.

$$5x\left(x^2 - x - 2\right) = 0$$
$$5x\left(x-2\right)\left(x+1\right) = 0$$

Now, the zero factor property will still hold here. In this case we have a product of three terms that is zero. The only way this product can be zero is if one of the terms is zero. This means that,

$$\begin{array}{lcl} 5x = 0 & \Rightarrow & x = 0 \\ x - 2 = 0 & \Rightarrow & x = 2 \\ x + 1 = 0 & \Rightarrow & x = -1 \end{array}$$

So, we have three solutions to this equation.

So, provided we can factor a polynomial we can always use this as a solution technique. The problem is, of course, that it is sometimes not easy to do the factoring.

Square Root Property
The second method of solving quadratics we'll be looking at uses the **square root property**,

$$\text{If } p^2 = d \quad \text{then} \quad p = \pm\sqrt{d}$$

There is a (potentially) new symbol here that we should define first in case you haven't seen it yet. The symbol "$\pm$" is read as : "plus or minus" and that is exactly what it tells us. This symbol is shorthand that tells us that we really have two numbers here. One is $p = \sqrt{d}$ and the other is $p = -\sqrt{d}$. Get used to this notation as it will be used frequently in the next couple of sections as we discuss the remaining solution techniques. It will also arise in other sections of this chapter and even in other chapters.

This is a fairly simple property to use, however it can only be used on a small portion of the equations that we're ever likely to encounter. Let's see some examples of this property.

Example 4 Solve each of the following equations.

(a) $x^2 - 100 = 0$

(b) $25y^2 - 3 = 0$

(c) $4z^2 + 49 = 0$

(d) $(2t - 9)^2 = 5$

(e) $(3x + 10)^2 + 81 = 0$

Solution

There really isn't all that much to these problems. In order to use the square root property all that we need to do is get the squared quantity on the left side by itself with a coefficient of 1 and the number on the other side. Once this is done we can use the square root property.

(a) $x^2 - 100 = 0$

This is a fairly simple problem so here is the work for this equation.

$$x^2 = 100 \qquad\qquad x = \pm\sqrt{100} = \pm 10$$

So, there are two solutions to this equation, $x = \pm 10$. Remember this means that there are really two solutions here, $x = -10$ and $x = 10$.

(b) $25y^2 - 3 = 0$

Okay, the main difference between this one and the previous one is the 25 in front of the squared term. The square root property wants a coefficient of one there. That's easy enough to deal with however; we'll just divide both sides by 25. Here is the work for this equation.

$$25y^2 = 3$$

$$y^2 = \frac{3}{25} \qquad \Rightarrow \qquad y = \pm\sqrt{\frac{3}{25}} = \pm\frac{\sqrt{3}}{5}$$

In this case the solutions are a little messy, but many of these will do so don't worry about that. Also note that since we knew what the square root of 25 was we went ahead and split the square root of the fraction up as shown. Again, remember that there are really two solutions here, one positive and one negative.

(c) $4z^2 + 49 = 0$

This one is nearly identical to the previous part with one difference that we'll see at the end of the example. Here is the work for this equation.

$$4z^2 = -49$$

$$z^2 = -\frac{49}{4} \qquad \Rightarrow \qquad z = \pm\sqrt{-\frac{49}{4}} = \pm i\sqrt{\frac{49}{4}} = \pm\frac{7}{2}i$$

So, there are two solutions to this equation : $z = \pm\frac{7}{2}i$. Notice as well that they are complex solutions. This will happen with the solution to many quadratic equations so make sure that you can deal with them.

(d) $\left(2t-9\right)^2 = 5$

This one looks different from the previous parts, however it works the same way. The square root property can be used anytime we have *something* squared equals a number. That is what we have here. The main difference of course is that the something that is squared isn't a single variable it is something else. So, here is the application of the square root property for this equation.

$$2t-9 = \pm\sqrt{5}$$

Now, we just need to solve for *t* and despite the "plus or minus" in the equation it works the same way we would solve any linear equation. We will add 9 to both sides and then divide by a 2.

$$2t = 9\pm\sqrt{5}$$
$$t = \frac{1}{2}\left(9\pm\sqrt{5}\right) = \frac{9}{2}\pm\frac{\sqrt{5}}{2}$$

Note that we multiplied the fraction through the parenthesis for the final answer. We will usually do this in these problems. Also, do NOT convert these to decimals unless you are asked to. This is the standard form for these answers. With that being said we should convert them to decimals just to make sure that you can. Here are the decimal values of the two solutions.

$$t = \frac{9}{2}+\frac{\sqrt{5}}{2} = 5.61803 \qquad \text{and} \qquad t = \frac{9}{2}-\frac{\sqrt{5}}{2} = 3.38197$$

(e) $\left(3x+10\right)^2 + 81 = 0$

In this final part we'll not put much in the way of details into the work.

$$\left(3x+10\right)^2 = -81$$
$$3x+10 = \pm 9i$$
$$3x = -10\pm 9i$$
$$x = -\frac{10}{3}\pm 3i$$

So we got two complex solutions again and notice as well that with both of the previous part we put the "plus or minus" part last. This is usually the way these are written.

As mentioned at the start of this section we are going to break this topic up into two sections for the benefit of those viewing this on the web. The next two methods of solving quadratic equations, completing the square and quadratic formula, are given in the next section.