Section 2-2 : Linear Equations

We'll start off the solving portion of this chapter by solving linear equations. A **linear equation** is any equation that can be written in the form

$$ax + b = 0$$

where *a* and *b* are real numbers and *x* is a variable. This form is sometimes called the **standard form** of a linear equation. Note that most linear equations will not start off in this form. Also, the variable may or may not be an *x* so don't get too locked into always seeing an *x* there.

To solve linear equations we will make heavy use of the following facts.

1. If $a = b$ then $a + c = b + c$ for any *c*. All this is saying is that we can add a number, *c*, to both sides of the equation and not change the equation.

2. If $a = b$ then $a - c = b - c$ for any *c*. As with the last property we can subtract a number, *c*, from both sides of an equation.

3. If $a = b$ then $ac = bc$ for any *c*. Like addition and subtraction, we can multiply both sides of an equation by a number, *c*, without changing the equation.

4. If $a = b$ then $\frac{a}{c} = \frac{b}{c}$ for any non-zero *c*. We can divide both sides of an equation by a non-zero number, *c*, without changing the equation.

These facts form the basis of almost all the solving techniques that we'll be looking at in this chapter so it's very important that you know them and don't forget about them. One way to think of these rules is the following. What we do to one side of an equation we have to do to the other side of the equation. If you remember that then you will always get these facts correct.

In this section we will be solving linear equations and there is a nice simple process for solving linear equations. Let's first summarize the process and then we will work some examples.

Process for Solving Linear Equations

1. If the equation contains any fractions use the least common denominator to clear the fractions. We will do this by multiplying both sides of the equation by the LCD.

   Also, if there are variables in the denominators of the fractions identify values of the variable which will give division by zero as we will need to avoid these values in our solution.

2. Simplify both sides of the equation. This means clearing out any parenthesis and combining like terms.

3. Use the first two facts above to get all terms with the variable in them on one side of the equations (combining into a single term of course) and all constants on the other side.

4. If the coefficient of the variable is not a one use the third or fourth fact above (this will depend on just what the number is) to make the coefficient a one.

   Note that we usually just divide both sides of the equation by the coefficient if it is an integer or multiply both sides of the equation by the reciprocal of the coefficient if it is a fraction.

5. **VERIFY YOUR ANSWER!** This is the final step and the most often skipped step, yet it is probably the most important step in the process. With this step you can know whether or not you got the correct answer long before your instructor ever looks at it. We verify the answer by plugging the results from the previous steps into the **original** equation. It is very important to plug into the original equation since you may have made a mistake in the very first step that led you to an incorrect answer.

   Also, if there were fractions in the problem and there were values of the variable that give division by zero (recall the first step…) it is important to make sure that one of these values didn't end up in the solution set. It is possible, as we'll see in an example, to have these values show up in the solution set.

Let's take a look at some examples.

Example 1 Solve each of the following equations.

(a) $3(x+5)=2(-6-x)-2x$

(b) $\dfrac{m-2}{3}+1=\dfrac{2m}{7}$

(c) $\dfrac{5}{2y-6}=\dfrac{10-y}{y^2-6y+9}$

(d) $\dfrac{2z}{z+3}=\dfrac{3}{z-10}+2$

Solution

In the following problems we will describe in detail the first problem and the leave most of the explanation out of the following problems.

(a) $3(x+5)=2(-6-x)-2x$

For this problem there are no fractions so we don't need to worry about the first step in the process. The next step tells to simplify both sides. So, we will clear out any parenthesis by multiplying the numbers through and then combine like terms.

$$3(x+5)=2(-6-x)-2x$$
$$3x+15=-12-2x-2x$$
$$3x+15=-12-4x$$

The next step is to get all the *x*'s on one side and all the numbers on the other side. Which side the *x*'s go on is up to you and will probably vary with the problem. As a rule of thumb, we will usually put the variables on the side that will give a positive coefficient. This is done simply because it is often easy to lose track of the minus sign on the coefficient and so if we make sure it is positive we won't need to worry about it.

So, for our case this will mean adding 4*x* to both sides and subtracting 15 from both sides. Note as well that while we will actually put those operations in this time we normally do these operations in our head.

$$\begin{aligned} 3x+15&=-12-4x \\ 3x+15-15+4x&=-12-4x+4x-15 \\ 7x&=-27 \end{aligned}$$

The next step says to get a coefficient of 1 in front of the *x*. In this case we can do this by dividing both sides by a 7.

$$\begin{aligned} \frac{7x}{7}&=\frac{-27}{7} \\ x&=-\frac{27}{7} \end{aligned}$$

Now, if we've done all of our work correct $x=-\frac{27}{7}$ is the solution to the equation.

The last and final step is to then check the solution. As pointed out in the process outline we need to check the solution in the **original** equation. This is important, because we may have made a mistake in the very first step and if we did and then checked the answer in the results from that step it may seem to indicate that the solution is correct when the reality will be that we don't have the correct answer because of the mistake that we originally made.

The problem of course is that, with this solution, that checking might be a little messy. Let's do it anyway.

$$\begin{aligned} 3\left(-\frac{27}{7}+5\right)&\overset{?}{=}2\left(-6-\left(-\frac{27}{7}\right)\right)-2\left(-\frac{27}{7}\right) \\ 3\left(\frac{8}{7}\right)&\overset{?}{=}2\left(-\frac{15}{7}\right)+\frac{54}{7} \\ \frac{24}{7}&=\frac{24}{7} \qquad \text{OK} \end{aligned}$$

So, we did our work correctly and the solution to the equation is,

$$x = -\frac{27}{7}$$

Note that we didn't use the solution set notation here. For single solutions we will rarely do that in this class. However, if we had wanted to the solution set notation for this problem would be,

$$\left\{-\frac{27}{7}\right\}$$

Before proceeding to the next problem let's first make a quick comment about the "messiness' of this answer. Do NOT expect all answers to be nice simple integers. While we do try to keep most answer simple often they won't be so do NOT get so locked into the idea that an answer must be a simple integer that you immediately assume that you've made a mistake because of the "messiness" of the answer.

(b) $\frac{m-2}{3} + 1 = \frac{2m}{7}$

Okay, with this one we won't be putting quite as much explanation into the problem.

In this case we have fractions so to make our life easier we will multiply both sides by the LCD, which is 21 in this case. After doing that the problem will be very similar to the previous problem. Note as well that the denominators are only numbers and so we won't need to worry about division by zero issues.

Let's first multiply both sides by the LCD.

$$\begin{aligned} 21\left(\frac{m-2}{3} + 1\right) &= \left(\frac{2m}{7}\right)21 \\ 21\left(\frac{m-2}{3}\right) + 21(1) &= \left(\frac{2m}{7}\right)21 \\ 7(m-2) + 21 &= (2m)(3) \end{aligned}$$

Be careful to correctly distribute the 21 through the parenthesis on the left side. Everything inside the parenthesis needs to be multiplied by the 21 before we simplify. At this point we've got a problem that is similar the previous problem and we won't bother with all the explanation this time.

$$\begin{aligned} 7(m-2) + 21 &= (2m)(3) \\ 7m - 14 + 21 &= 6m \\ 7m + 7 &= 6m \\ m &= -7 \end{aligned}$$

So, it looks like $m = -7$ is the solution. Let's verify it to make sure.

$$\frac{-7-2}{3}+1\stackrel{?}{=}\frac{2(-7)}{7}$$

$$\frac{-9}{3}+1\stackrel{?}{=}-\frac{14}{7}$$

$$-3+1\stackrel{?}{=}-2$$

$$-2=-2 \qquad \text{OK}$$

So, it is the solution.

(c) $\frac{5}{2y-6}=\frac{10-y}{y^2-6y+9}$

This one is similar to the previous one except now we've got variables in the denominator. So, to get the LCD we'll first need to completely factor the denominators of each rational expression.

$$\frac{5}{2(y-3)}=\frac{10-y}{(y-3)^2}$$

So, it looks like the LCD is $2(y-3)^2$. Also note that we will need to avoid $y=3$ since if we plugged that into the equation we would get division by zero.

Now, outside of the *y*'s in the denominator this problem works identical to the previous one so let's do the work.

$$(2)(y-3)^2\left(\frac{5}{2(y-3)}\right)=\left(\frac{10-y}{(y-3)^2}\right)(2)(y-3)^2$$

$$5(y-3)=2(10-y)$$

$$5y-15=20-2y$$

$$7y=35$$

$$y=5$$

Now the solution is not $y=3$ so we won't get division by zero with the solution which is a good thing. Finally, let's do a quick verification.

$$\frac{5}{2(5)-6}\stackrel{?}{=}\frac{10-5}{5^2-6(5)+9}$$

$$\frac{5}{4}=\frac{5}{4} \qquad \text{OK}$$

So we did the work correctly.

(d) $\frac{2z}{z+3}=\frac{3}{z-10}+2$

In this case it looks like the LCD is $(z+3)(z-10)$ and it also looks like we will need to avoid $z=-3$ and $z=10$ to make sure that we don't get division by zero.

Let's get started on the work for this problem.

$$(z+3)(z-10)\left(\frac{2z}{z+3}\right)=\left(\frac{3}{z-10}+2\right)(z+3)(z-10)$$
$$2z(z-10)=3(z+3)+2(z+3)(z-10)$$
$$2z^2-20z=3z+9+2\left(z^2-7z-30\right)$$

At this point let's pause and acknowledge that we've got a z^2 in the work here. Do not get excited about that. Sometimes these will show up temporarily in these problems. You should only worry about it if it is still there after we finish the simplification work.

So, let's finish the problem.

$$\cancel{2z^2}-20z=3z+9+\cancel{2z^2}-14z-60$$
$$-20z=-11z-51$$
$$51=9z$$
$$\frac{51}{9}=z$$
$$\frac{17}{3}=z$$

Notice that the z^2 did in fact cancel out. Now, if we did our work correctly $z=\frac{17}{3}$ should be the solution since it is not either of the two values that will give division by zero. Let's verify this.

$$\frac{2\left(\frac{17}{3}\right)}{\frac{17}{3}+3}\stackrel{?}{=}\frac{3}{\frac{17}{3}-10}+2$$
$$\frac{\frac{34}{3}}{\frac{26}{3}}\stackrel{?}{=}\frac{3}{-\frac{13}{3}}+2$$
$$\frac{34}{3}\left(\frac{3}{26}\right)\stackrel{?}{=}3\left(-\frac{3}{13}\right)+2$$
$$\frac{17}{13}=\frac{17}{13} \qquad \text{OK}$$

The checking can be a little messy at times, but it does mean that we KNOW the solution is correct.

Okay, in the last couple of parts of the previous example we kept going on about watching out for division by zero problems and yet we never did get a solution where that was an issue. So, we should now do a couple of those problems to see how they work.

Example 2 Solve each of the following equations.

(a) $\dfrac{2}{x+2} = \dfrac{-x}{x^2+5x+6}$

(b) $\dfrac{2}{x+1} = 4 - \dfrac{2x}{x+1}$

Solution

(a) $\dfrac{2}{x+2} = \dfrac{-x}{x^2+5x+6}$

The first step is to factor the denominators to get the LCD.

$$\frac{2}{x+2} = \frac{-x}{(x+2)(x+3)}$$

So, the LCD is $(x+2)(x+3)$ and we will need to avoid $x=-2$ and $x=-3$ so we don't get division by zero.

Here is the work for this problem.

$$\begin{aligned} (x+2)(x+3)\left(\frac{2}{x+2}\right) &= \left(\frac{-x}{(x+2)(x+3)}\right)(x+2)(x+3) \\ 2(x+3) &= -x \\ 2x+6 &= -x \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

So, we get a "solution" that is in the list of numbers that we need to avoid so we don't get division by zero and so we can't use it as a solution. However, this is also the only possible solution. That is okay. This just means that this equation has **no solution**.

(b) $\dfrac{2}{x+1} = 4 - \dfrac{2x}{x+1}$

The LCD for this equation is $x+1$ and we will need to avoid $x=-1$ so we don't get division by zero. Here is the work for this equation.

$$\begin{aligned}\left(\frac{2}{x+1}\right)(x+1) &= \left(4-\frac{2x}{x+1}\right)(x+1)\\ 2 &= 4(x+1)-2x\\ 2 &= 4x+4-2x\\ 2 &= 2x+4\\ -2 &= 2x\\ -1 &= x\end{aligned}$$

So, we once again arrive at the single value of *x* that we needed to avoid so we didn't get division by zero. Therefore, this equation has **no solution**.

So, as we've seen we do need to be careful with division by zero issues when we start off with equations that contain rational expressions.

At this point we should probably also acknowledge that provided we don't have any division by zero issues (such as those in the last set of examples) linear equations will have exactly one solution. We will never get more than one solution and the only time that we won't get any solutions is if we run across a division by zero problems with the "solution".

Before leaving this section we should note that many of the techniques for solving linear equations will show up time and again as we cover different kinds of equations so it very important that you understand this process.