Section 2-6 : Quadratic Equations - Part II

The topic of solving quadratic equations has been broken into two sections for the benefit of those viewing this on the web. As a single section the load time for the page would have been quite long. This is the second section on solving quadratic equations.

In the previous section we looked at using factoring and the square root property to solve quadratic equations. The problem is that both of these solution methods will not always work. Not every quadratic is factorable and not every quadratic is in the form required for the square root property.

It is now time to start looking into methods that will work for all quadratic equations. So, in this section we will look at completing the square and the quadratic formula for solving the quadratic equation,

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

Completing the Square

The first method we'll look at in this section is completing the square. It is called this because it uses a process called completing the square in the solution process. So, we should first define just what completing the square is.

Let's start with

$$x^2 + bx$$

and notice that the x^2 has a coefficient of one. That is required in order to do this. Now, to this lets add $\left(\frac{b}{2}\right)^2$. Doing this gives the following **factorable** quadratic equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

This process is called **completing the square** and if we do all the arithmetic correctly we can guarantee that the quadratic will factor as a perfect square.

Let's do a couple of examples for just completing the square before looking at how we use this to solve quadratic equations.

Example 1 Complete the square on each of the following.

(a) $x^2 - 16x$

(b) $y^2 + 7y$

Solution

(a) $x^2 - 16x$

Here's the number that we'll add to the equation.

$$\left(\frac{-16}{2}\right)^2 = (-8)^2 = 64$$

Notice that we kept the minus sign here even though it will always drop out after we square things. The reason for this will be apparent in a second. Let's now complete the square.

$$x^2 - 16x + 64 = (x - 8)^2$$

Now, this is a quadratic that hopefully you can factor fairly quickly. However notice that it will always factor as *x* plus the blue number we computed above that is in the parenthesis (in our case that is -8). This is the reason for leaving the minus sign. It makes sure that we don't make any mistakes in the factoring process.

(b) $y^2 + 7y$

Here's the number we'll need this time.

$$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

It's a fraction and that will happen fairly often with these so don't get excited about it. Also, leave it as a fraction. Don't convert to a decimal. Now complete the square.

$$y^2 + 7y + \frac{49}{4} = \left(y + \frac{7}{2}\right)^2$$

This one is not so easy to factor. However, if you again recall that this will ALWAYS factor as *y* plus the blue number above we don't have to worry about the factoring process.

It's now time to see how we use completing the square to solve a quadratic equation. The process is best seen as we work an example so let's do that.

Example 2 Use completing the square to solve each of the following quadratic equations.

(a) $x^2 - 6x + 1 = 0$

(b) $2x^2 + 6x + 7 = 0$

(c) $3x^2 - 2x - 1 = 0$

Solution

We will do the first problem in detail explicitly giving each step. In the remaining problems we will just do the work without as much explanation.

(a) $x^2 - 6x + 1 = 0$

So, let's get started.

Step 1 : Divide the equation by the coefficient of the x^2 term. Recall that completing the square required a coefficient of one on this term and this will guarantee that we will get that. We don't need to do that for this equation however.

Step 2 : Set the equation up so that the *x*'s are on the left side and the constant is on the right side.

$$x^2 - 6x = -1$$

Step 3 : Complete the square on the left side. However, this time we will need to add the number to both sides of the equal sign instead of just the left side. This is because we have to remember the rule that what we do to one side of an equation we need to do to the other side of the equation.

First, here is the number we add to both sides.

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Now, complete the square.

$$x^2 - 6x + 9 = -1 + 9$$
$$(x-3)^2 = 8$$

Step 4 : Now, at this point notice that we can use the square root property on this equation. That was the purpose of the first three steps. Doing this will give us the solution to the equation.

$$x - 3 = \pm\sqrt{8} \qquad \Rightarrow \qquad x = 3 \pm \sqrt{8}$$

And that is the process. Let's do the remaining parts now.

(b) $2x^2 + 6x + 7 = 0$

We will not explicitly put in the steps this time nor will we put in a lot of explanation for this equation. This that being said, notice that we will have to do the first step this time. We don't have a coefficient of one on the x^2 term and so we will need to divide the equation by that first.

Here is the work for this equation.

$$x^2 + 3x + \frac{7}{2} = 0$$
$$x^2 + 3x = -\frac{7}{2} \qquad \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$
$$x^2 + 3x + \frac{9}{4} = -\frac{7}{2} + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = -\frac{5}{4}$$
$$x + \frac{3}{2} = \pm\sqrt{-\frac{5}{4}} \qquad \Rightarrow \qquad x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2} i$$

Don't forget to convert square roots of negative numbers to complex numbers!

(c) $3x^2 - 2x - 1 = 0$

Again, we won't put a lot of explanation for this problem.

$$x^2 - \frac{2}{3}x - \frac{1}{3} = 0$$
$$x^2 - \frac{2}{3}x = \frac{1}{3}$$

At this point we should be careful about computing the number for completing the square since *b* is now a fraction for the first time.

$$\left(\frac{-\frac{2}{3}}{2}\right)^2 = \left(-\frac{2}{3}\cdot\frac{1}{2}\right)^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$$

Now finish the problem.

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{3} + \frac{1}{9}$$
$$\left(x - \frac{1}{3}\right)^2 = \frac{4}{9}$$
$$x - \frac{1}{3} = \pm\sqrt{\frac{4}{9}} \qquad \Rightarrow \qquad x = \frac{1}{3} \pm \frac{2}{3}$$

In this case notice that we can actually do the arithmetic here to get two integer and/or fractional solutions. We should always do this when there are only integers and/or fractions in our solution. Here are the two solutions.

$$x = \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1 \qquad \text{and} \qquad x = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

A quick comment about the last equation that we solved in the previous example is in order. Since we received integer and fractions as solutions, we could have just factored this equation from the start rather than used completing the square. In cases like this we could use either method and we will get the same result.

Now, the reality is that completing the square is a fairly long process and it's easy to make mistakes. So, we rarely actually use it to solve equations. That doesn't mean that it isn't important to know the process however. We will be using it in several sections in later chapters and is often used in other classes.

Quadratic Formula

This is the final method for solving quadratic equations and will always work. Not only that, but if you can remember the formula it's a fairly simple process as well.

We can derive the quadratic formula by completing the square on the general quadratic formula in standard form. Let's do that and we'll take it kind of slow to make sure all the steps are clear.

First, we MUST have the quadratic equation in standard form as already noted. Next, we need to divide both sides by a to get a coefficient of one on the x^2 term.

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Next, move the constant to the right side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now, we need to compute the number we'll need to complete the square. Again, this is one-half the coefficient of x, squared.

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Now, add this to both sides, complete the square and get common denominators on the right side to simplify things up a little.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now we can use the square root property on this.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Solve for x and we'll also simplify the square root a little.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

As a last step we will notice that we've got common denominators on the two terms and so we'll add them up. Doing this gives,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, summarizing up, provided that we start off in standard form,

$$ax^2 + bx + c = 0$$

and that's very important, then the solution to any quadratic equation is,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's work a couple of examples.

Example 3 Use the quadratic formula to solve each of the following equations.

(a) $x^2 + 2x = 7$

(b) $3q^2 + 11 = 5q$

(c) $7t^2 = 6 - 19t$

(d) $\dfrac{3}{y-2} = \dfrac{1}{y} + 1$

(e) $16x - x^2 = 0$

Solution

The important part here is to make sure that before we start using the quadratic formula that we have the equation in standard form first.

(a) $x^2 + 2x = 7$

So, the first thing that we need to do here is to put the equation in standard form.

$$x^2 + 2x - 7 = 0$$

At this point we can identify the values for use in the quadratic formula. For this equation we have.

$$a = 1 \qquad b = 2 \qquad c = -7$$

Notice the "-" with *c*. It is important to make sure that we carry any minus signs along with the constants.

At this point there really isn't anything more to do other than plug into the formula.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-7)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{32}}{2} \end{aligned}$$

There are the two solutions for this equation. There is also some simplification that we can do. We need to be careful however. One of the larger mistakes at this point is to "cancel" two 2's in the numerator and denominator. Remember that in order to cancel anything from the numerator or denominator then it must be multiplied by the whole numerator or denominator. Since the 2 in the numerator isn't multiplied by the whole denominator it can't be canceled.

In order to do any simplification here we will first need to reduce the square root. At that point we can do some canceling.

$$x = \frac{-2 \pm \sqrt{(16)2}}{2} = \frac{-2 \pm 4\sqrt{2}}{2} = \frac{2\left(-1 \pm 2\sqrt{2}\right)}{2} = -1 \pm 2\sqrt{2}$$

That's a much nicer answer to deal with and so we will almost always do this kind of simplification when it can be done.

(b) $3q^2 + 11 = 5q$

Now, in this case don't get excited about the fact that the variable isn't an *x*. Everything works the same regardless of the letter used for the variable. So, let's first get the equation into standard form.

$$3q^2 + 11 - 5q = 0$$

Now, this isn't quite in the typical standard form. However, we need to make a point here so that we don't make a very common mistake that many student make when first learning the quadratic formula.

Many students will just get everything on one side as we've done here and then get the values of *a*, *b*, and *c* based upon position. In other words, often students will just let *a* be the first number listed, *b* be the second number listed and then *c* be the final number listed. This is not correct however. For the quadratic formula *a* is the coefficient of the squared term, *b* is the coefficient of the term with just the variable in it (not squared) and *c* is the constant term. So, to avoid making this mistake we should always put the quadratic equation into the official standard form.

$$3q^2 - 5q + 11 = 0$$

Now we can identify the value of *a*, *b*, and *c*.

$$a = 3 \qquad\qquad b = -5 \qquad\qquad c = 11$$

Again, be careful with minus signs. They need to get carried along with the values.

Finally, plug into the quadratic formula to get the solution.

$$\begin{aligned} q &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(11)}}{2(3)} \\ &= \frac{5 \pm \sqrt{25 - 132}}{6} \\ &= \frac{5 \pm \sqrt{-107}}{6} \\ &= \frac{5 \pm \sqrt{107}\, i}{6} \end{aligned}$$

As with all the other methods we've looked at for solving quadratic equations, don't forget to convert square roots of negative numbers into complex numbers. Also, when *b* is negative be very careful with the substitution. This is particularly true for the squared portion under the radical. Remember that when you square a negative number it will become positive. One of the more common mistakes here is to get in a hurry and forget to drop the minus sign after you square *b*, so be careful.

(c) $7t^2 = 6 - 19t$

We won't put in quite the detail with this one that we've done for the first two. Here is the standard form of this equation.

$$7t^2 + 19t - 6 = 0$$

Here are the values for the quadratic formula as well as the quadratic formula itself.

$$a = 7 \qquad b = 19 \qquad c = -6$$

$$\begin{aligned} t &= \frac{-19 \pm \sqrt{(19)^2 - 4(7)(-6)}}{2(7)} \\ &= \frac{-19 \pm \sqrt{361 + 168}}{14} \\ &= \frac{-19 \pm \sqrt{529}}{14} \\ &= \frac{-19 \pm 23}{14} \end{aligned}$$

Now, recall that when we get solutions like this we need to go the extra step and actually determine the integer and/or fractional solutions. In this case they are,

$$t = \frac{-19 + 23}{14} = \frac{2}{7} \qquad\qquad t = \frac{-19 - 23}{14} = -3$$

Now, as with completing the square, the fact that we got integer and/or fractional solutions means that we could have factored this quadratic equation as well.

(d) $\dfrac{3}{y-2} = \dfrac{1}{y} + 1$

So, an equation with fractions in it. The first step then is to identify the LCD.

$$\text{LCD} : y\left(y-2\right)$$

So, it looks like we'll need to make sure that neither $y = 0$ or $y = 2$ is in our answers so that we don't get division by zero.

Multiply both sides by the LCD and then put the result in standard form.

$$\begin{aligned} \left(y\right)\left(y-2\right)\left(\frac{3}{y-2}\right) &= \left(\frac{1}{y}+1\right)\left(y\right)\left(y-2\right) \\ 3y &= y - 2 + y\left(y-2\right) \\ 3y &= y - 2 + y^2 - 2y \\ 0 &= y^2 - 4y - 2 \end{aligned}$$

Okay, it looks like we've got the following values for the quadratic formula.

$$a = 1 \qquad\qquad b = -4 \qquad\qquad c = -2$$

Plugging into the quadratic formula gives,

$$\begin{aligned} y &= \frac{-\left(-4\right) \pm \sqrt{\left(-4\right)^2 - 4\left(1\right)\left(-2\right)}}{2\left(1\right)} \\ &= \frac{4 \pm \sqrt{24}}{2} \\ &= \frac{4 \pm 2\sqrt{6}}{2} \\ &= 2 \pm \sqrt{6} \end{aligned}$$

Note that both of these are going to be solutions since neither of them are the values that we need to avoid.

(e) $16x - x^2 = 0$

We saw an equation similar to this in the previous section when we were looking at factoring equations and it would definitely be easier to solve this by factoring. However, we are going to use the quadratic formula anyway to make a couple of points.

First, let's rearrange the order a little bit just to make it look more like the standard form.

$$-x^2 + 16x = 0$$

Here are the constants for use in the quadratic formula.

$$a = -1 \qquad\qquad b = 16 \qquad\qquad c = 0$$

There are two things to note about these values. First, we've got a negative *a* for the first time. Not a big deal, but it is the first time we've seen one. Secondly, and more importantly, one of the values is zero. This is fine. It will happen on occasion and in fact, having one of the values zero will make the work much simpler.

Here is the quadratic formula for this equation.

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{(16)^2 - 4(-1)(0)}}{2(-1)} \\ &= \frac{-16 \pm \sqrt{256}}{-2} \\ &= \frac{-16 \pm 16}{-2} \end{aligned}$$

Reducing these to integers/fractions gives,

$$x = \frac{-16+16}{-2} = \frac{0}{-2} = 0 \qquad\qquad x = \frac{-16-16}{-2} = \frac{-32}{-2} = 16$$

So we get the two solutions, $x = 0$ and $x = 16$. These are exactly the solutions we would have gotten by factoring the equation.

To this point in both this section and the previous section we have only looked at equations with integer coefficients. However, this doesn't have to be the case. We could have coefficient that are fractions or decimals. So, let's work a couple of examples so we can say that we've seen something like that as well.

Example 4 Solve each of the following equations.

(a) $\frac{1}{2}x^2 + x - \frac{1}{10} = 0$

(b) $0.04x^2 - 0.23x + 0.09 = 0$

Solution

(a) There are two ways to work this one. We can either leave the fractions in or multiply by the LCD (10 in this case) and solve that equation. Either way will give the same answer. We will only do the fractional case here since that is the point of this problem. You should try the other way to verify that you get the same solution.

In this case here are the values for the quadratic formula as well as the quadratic formula work for this equation.

$$a = \frac{1}{2} \qquad\qquad b = 1 \qquad\qquad c = -\frac{1}{10}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\frac{1}{2}\right)\left(-\frac{1}{10}\right)}}{2\left(\frac{1}{2}\right)} = \frac{-1 \pm \sqrt{1 + \frac{1}{5}}}{1} = -1 \pm \sqrt{\frac{6}{5}}$$

In these cases we usually go the extra step of eliminating the square root from the denominator so let's also do that,

$$x = -1 \pm \frac{\sqrt{6}}{\sqrt{5}} \frac{\sqrt{5}}{\sqrt{5}} = -1 \pm \frac{\sqrt{(6)(5)}}{5} = -1 \pm \frac{\sqrt{30}}{5}$$

If you do clear the fractions out and run through the quadratic formula then you should get exactly the same result. For the practice you really should try that.

(b) In this case do not get excited about the decimals. The quadratic formula works in exactly the same manner. Here are the values and the quadratic formula work for this problem.

$$a = 0.04 \qquad\qquad b = -0.23 \qquad\qquad c = 0.09$$

$$\begin{aligned} x &= \frac{-(-0.23) \pm \sqrt{(-0.23)^2 - 4(0.04)(0.09)}}{2(0.04)} \\ &= \frac{0.23 \pm \sqrt{0.0529 - 0.0144}}{0.08} \\ &= \frac{0.23 \pm \sqrt{0.0385}}{0.08} \end{aligned}$$

Now, to this will be the one difference between these problems and those with integer or fractional coefficients. When we have decimal coefficients we usually go ahead and figure the two individual numbers. So, let's do that,

$$x = \frac{0.23 \pm \sqrt{0.0385}}{0.08} = \frac{0.23 \pm 0.19621}{0.08}$$

$$\begin{aligned} x &= \frac{0.23 + 0.19621}{0.08} \qquad \text{and} \qquad & x &= \frac{0.23 - 0.19621}{0.08} \\ &= 5.327625 \qquad \text{and} \qquad & &= 0.422375 \end{aligned}$$

Notice that we did use some rounding on the square root.

Over the course of the last two sections we've done quite a bit of solving. It is important that you understand most, if not all, of what we did in these sections as you will be asked to do this kind of work in some later sections.