Design of Arithmetic circuits

5/8/14

Adders

Fast addition of 2-n bit unsigned numbers

* use 2-level combinational ckt expressed in SOP & POS of n-I/p variables

C(n) — # gates

f(n) — fan-in

Practical circuits < multilevel combinational / sequential ckts

Half adder

$$S_i = x_i \oplus y_i$$

$$C_i = x_i y_i$$

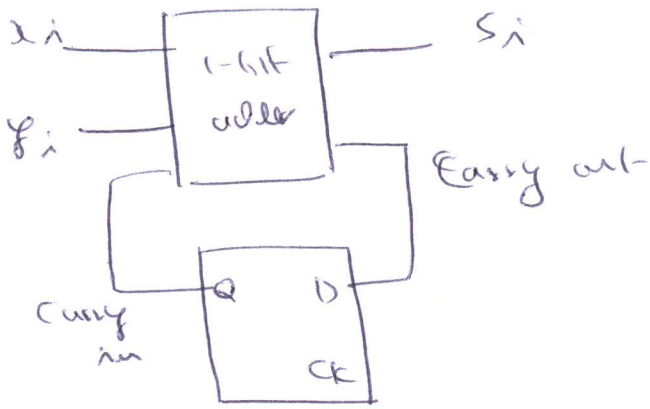
Full adder

$$S_i = x_i \oplus y_i \oplus c_{i-1}$$

$$C_i = x_i y_i + x_i c_{i-1} + y_i c_{i-1}$$

x_i	y_i	c_{i-1}	S_i	c_i
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	10	1

Serial adder

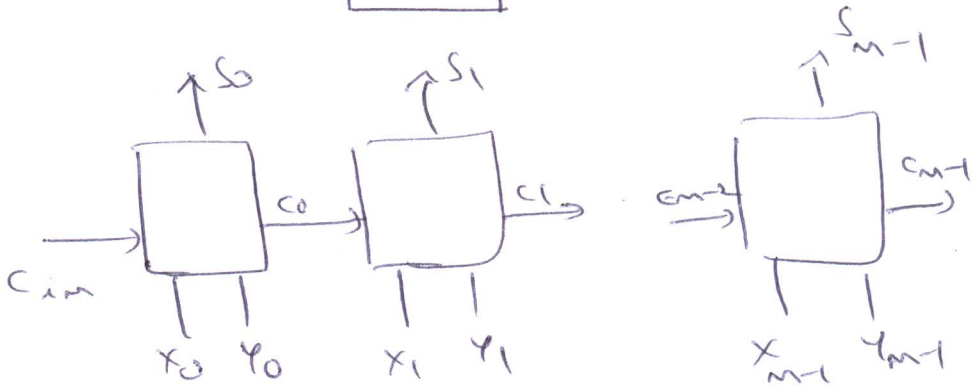


For n-bit addition

n - clock cycles

Parallel adder

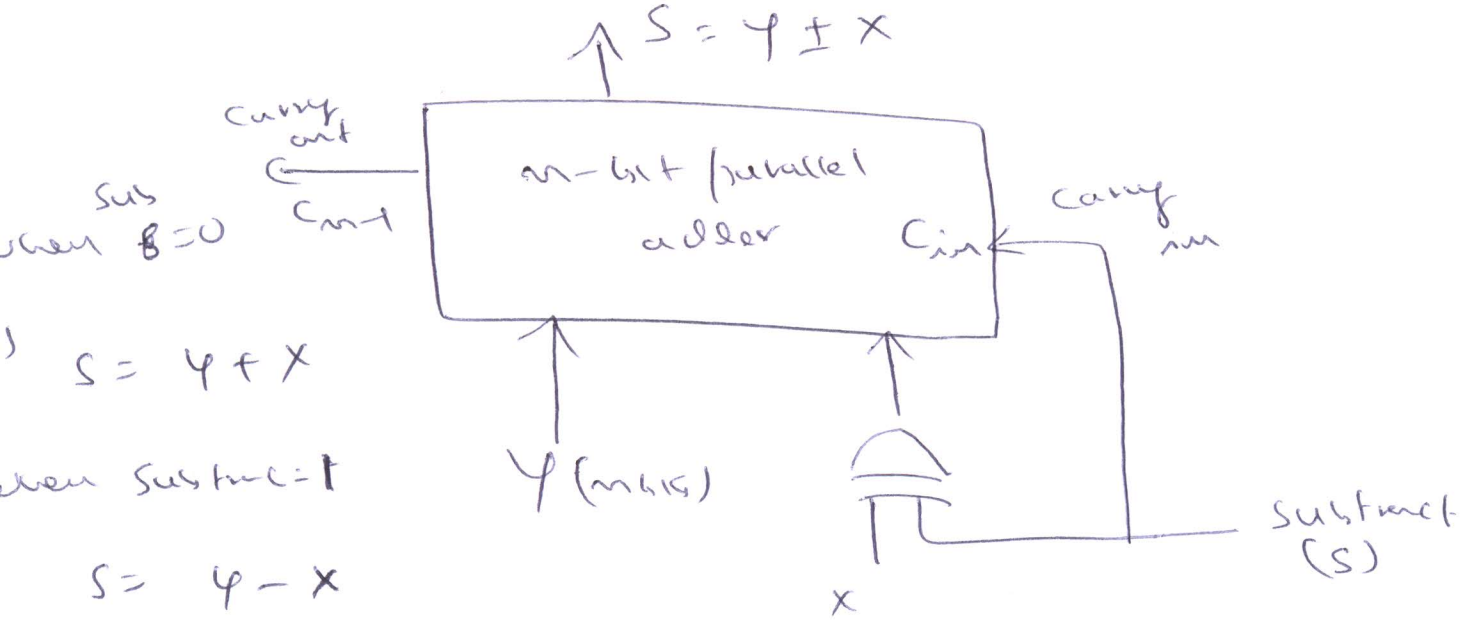
n-bit ripple carry adder →



Time delay = nd

↳ delay in FA stage

n-bit 2's complement adder - Subtractor



Binary Subtractor

S_i = difference bit

b_{i-1} = borrow-in

b_i = borrow out

$$S_i = x_i \oplus y_i \oplus b_{i-1}$$

$$b_i = \bar{x}_i y_i + \bar{x}_i b_{i-1} + y_i b_{i-1}$$

Over flow

unsigned numbers => $c_{n-1} = 1$

Signed Numbers — both +ve => $c_{n-2} = 1$

both -ve => $c_{n-2} = 0$

$$V = x_{n-1} y_{n-1} \bar{c}_{n-2} + \bar{x}_{n-1} \bar{y}_{n-1} c_{n-2}$$

$$c_{n-1} = x_{n-1} y_{n-1} + x_{n-1} c_{n-2} + y_{n-1} c_{n-2}$$

$$\Rightarrow V = c_{n-1} \oplus c_{n-2}$$

High Speed Adders

Delay = Sum of logic gate delays along the longest signal propagation path through N/w

n-bit ripple adder

c_{n-1} → 2(n-1) gate delays

S_{n-1} → 2(n-1) + 1 xor delay

c_n → 2n gate delays

overflow → 2n+2 gate delays

Reducing delay 1) Fastest Electronic technology

2) modify the logic gate structure (Fast propagation of carry)

Carry - lookahead - adder

$$S_i = x_i \oplus y_i \oplus c_i$$

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

$$= x_i y_i + c_i (x_i + y_i)$$

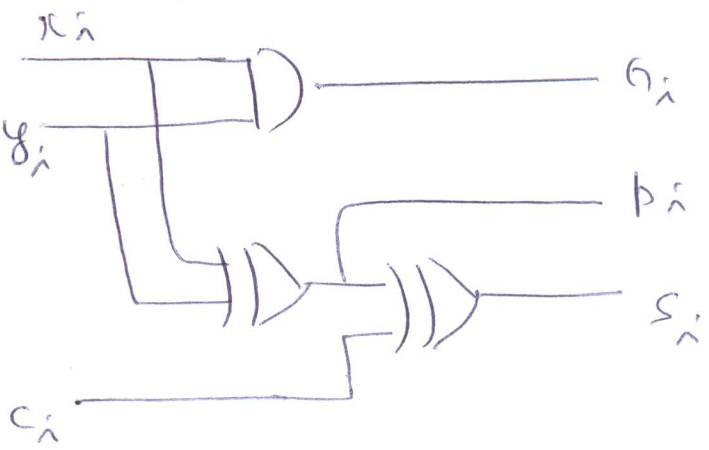
$$= G_i + c_i P_i$$

$G_i = x_i y_i$

$P_i = x_i + y_i$

$G \rightarrow$ generate fn.

$P \Rightarrow$ propagate fn



$$c_{i+1} = G_i + P_i (G_{i-1} + c_{i-1} P_{i-1})$$

$$= G_i + P_i G_{i-1} + P_i P_{i-1} c_{i-1}$$

$$= G_i + P_i G_{i-1} + P_i P_{i-1} G_{i-2} + \cdots + P_i P_{i-1} \cdots P_1 G_0 + P_i P_{i-1} \cdots P_0 c_0$$

1 Gate delay $\Rightarrow G_i$ & P_i

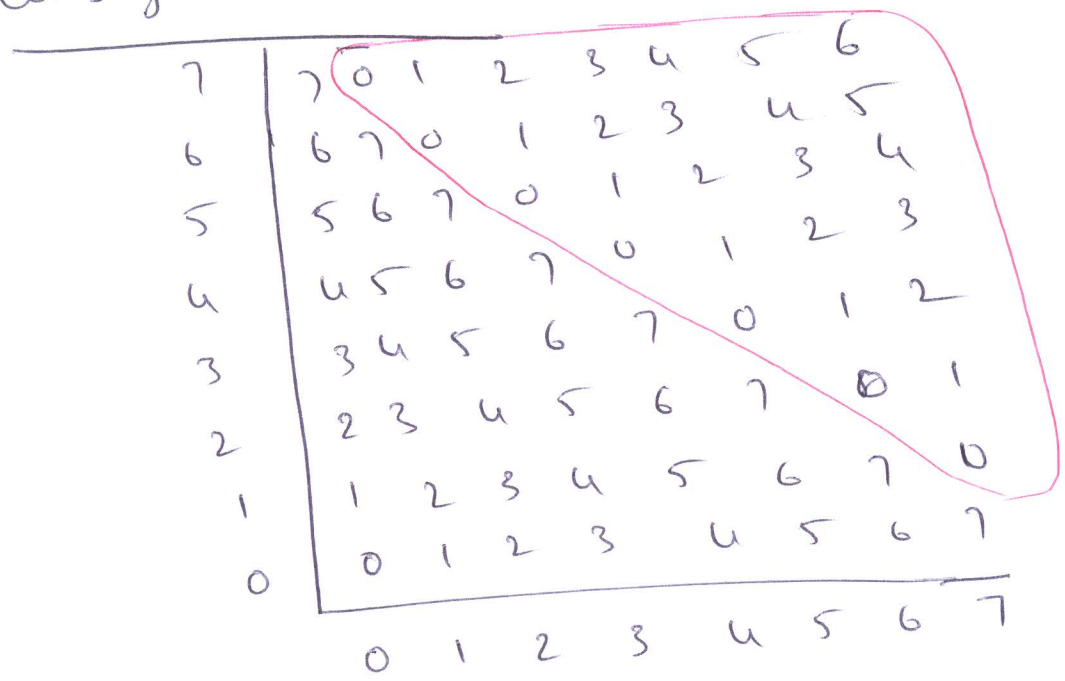
2 Gate delays $\Rightarrow c_{i+1}$ (AND-OR)

1 Gate delay $\Rightarrow S_i$ (X-OR Gate)

$\Rightarrow$ 4-Gate delays

overflow

unsigned addition

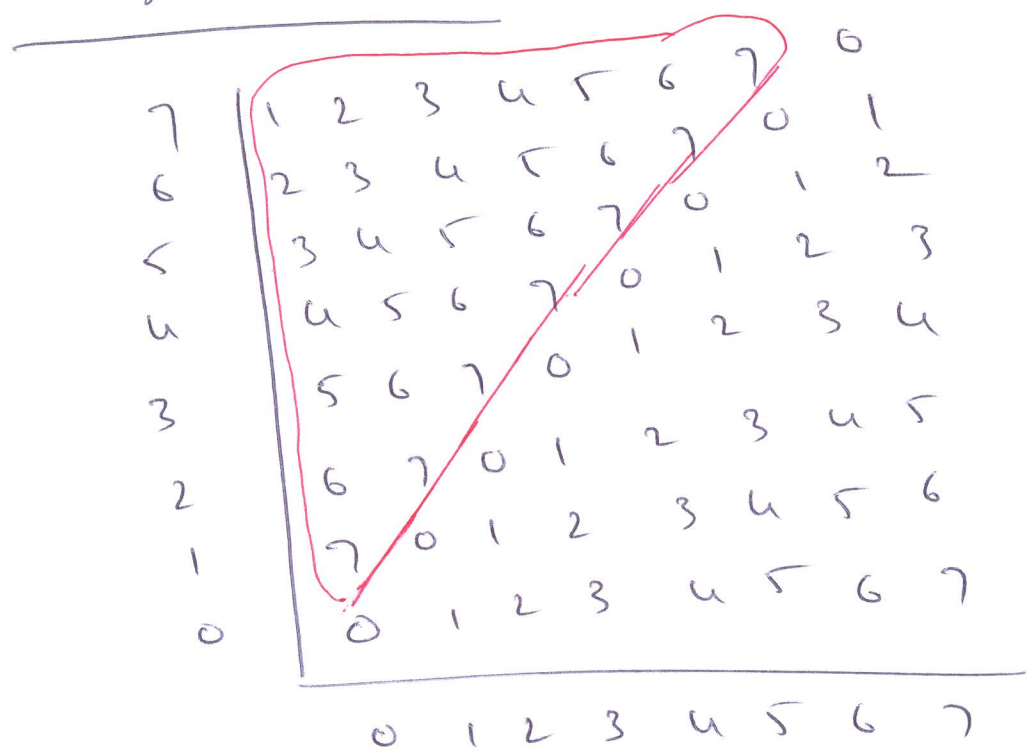


$C_n = 1$

$s < a$

$s < b$

unsigned substract



$C_n = 0$

$a < s$

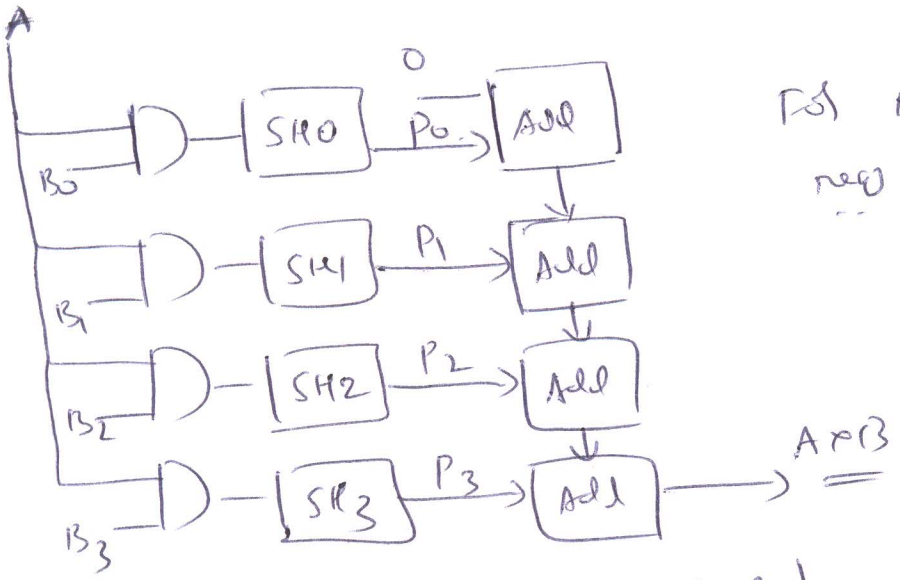
$a < b$

Multiplier

Simple Method

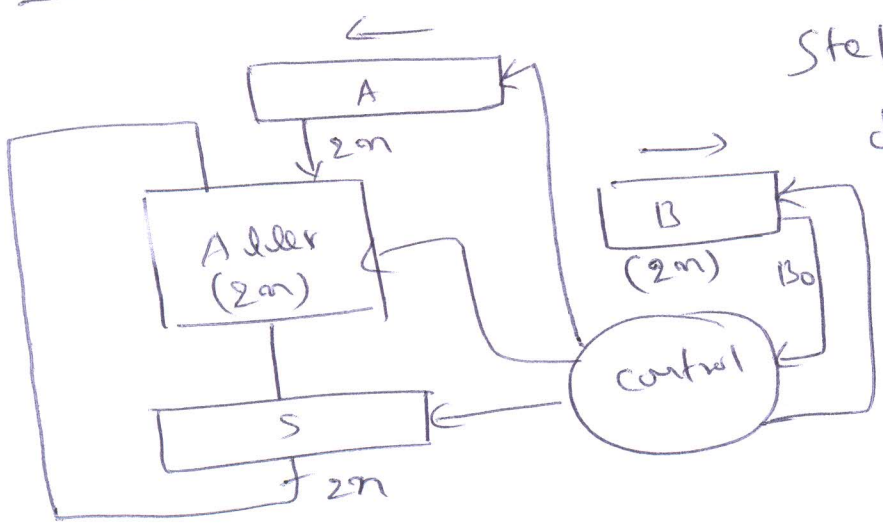
```
A    0 1 1
B    1 0 1
-----------
     0 1 1
   0 0 0 x
 0 1 1 x x
-----------
 0 1 1 1 1   =>  Σ (i=0 to n-1) A · B_i · 2^i
-----------
```

```
P0 -> 0 0 0 0 0
P1 -> 0 0 0 1 1
P2 -> 0 0 0 1 1
P3 -> 0 1 1 1 1
```



For N×N multiplier
req N adders.

Shift add Multiplier (sequential)



Step 1; i=0; S=0

do {

Step 2:

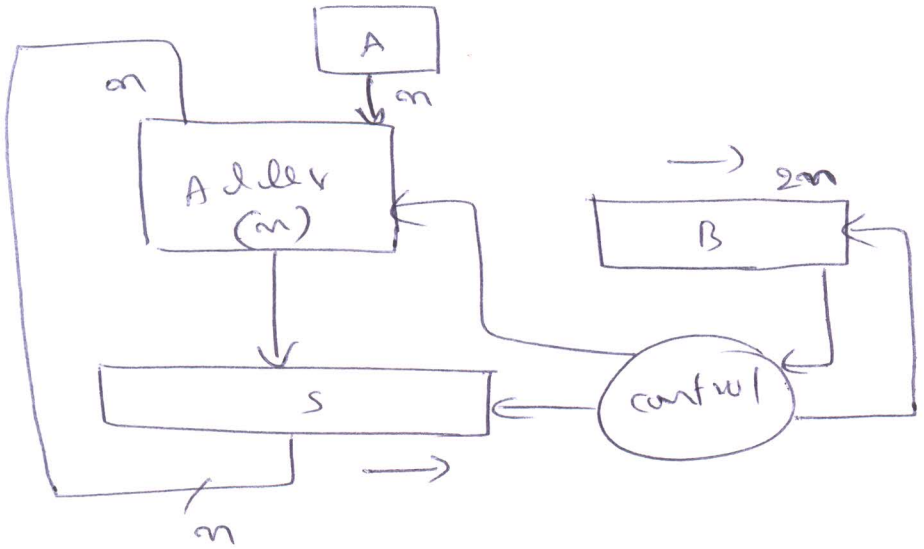
if (B0) S += A

A = 2×A

B = B/2; i++

} while (i<n)

Sequential Shift add multiplier 2



Step 1 : i = 0, S = 0

do

Step 2 :

if (B_0) S_H += A

S = S/2

B = B/2

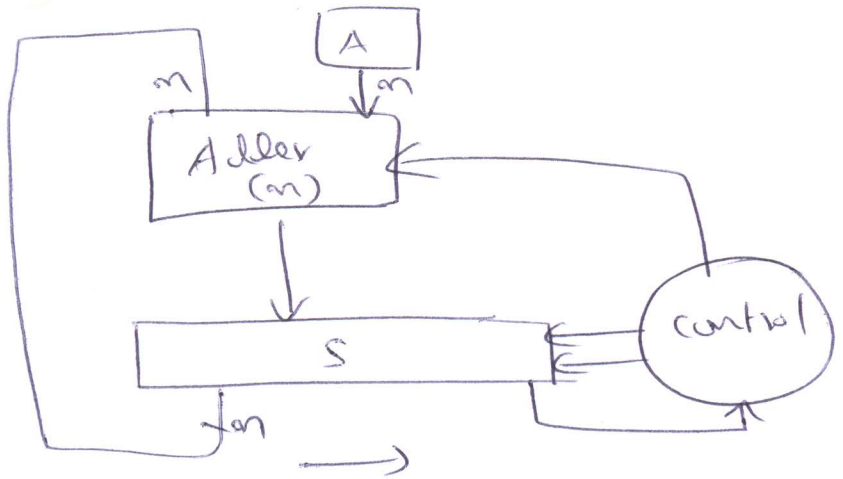
i = i + 1

} while i < M

✓ Reducing size of A from 2n to n

✓ Reducing the size of adder for 2n to n

Sequential Shift add multiplier 3



Step 1 : i = 0; S = 0|B

do

{

Step 2

if (S_0) S_H += A

S = S/2 ; i++

} while (i < M)

✓ Avoid the use of separate Reg for B

H, L Register for holding 64 bits

Signed Multiplication

Common expression to represent both positive as well as negative values.

$$B = -B_{n-1}2^{n-1} + \sum_{i=0}^{n-2} B_i 2^i$$

for $B \geq 0$ $\quad B = \sum_{i=0}^{n-1} B_i 2^i = -B_{n-1}2^{n-1} + \sum_{i=0}^{n-2} B_i 2^i$

$(\because B_{n-1} = 0)$

for $B < 0$, $\quad B = -|B|$

$$|B| = 2^n - \sum_{i=0}^{n-1} B_i 2^i = 2^n - B_{n-1}2^{n-1} - \sum_{i=0}^{n-2} B_i 2^i$$

$$= B_{n-1}2^{n-1} - \sum_{i=0}^{n-2} B_i 2^i$$

$$B = -|B| = -B_{n-1}2^{n-1} + \sum_{i=0}^{n-2} B_i 2^i$$

$$= -B_{n-1}2^{n-1} + B_{n-2}2^{n-2} + \cdots + B_0$$

$$= -B_{n-1}2^{n-1} + B_{n-2}2^{n-2} + \cdots + B_1 2 + B_0$$

$$= -B_{n-1}2^{n-1} + B_{n-2}2^{n-1} - B_{n-2}2^{n-2} + \cdots + B_1 2^1 - B_1 2 + B_0 2 - B_0 2^0$$

$$= \sum_{i=0}^{n-1} (B_{i-1} - B_i) 2^i \quad \text{where } B_{-1} = 0$$

$$\therefore A \times B = \sum_{i=0}^{n-1} A \cdot (B_{i-1} - B_i) 2^i$$

Comparison bet signed and unsigned multiply

Unsigned		Signed		
B_i	operation	B_i	B_{i-1}	operation
0	no addition	0	0	no addition
1	add A	0	1	Add A
		1	0	Sub A
		1	1	no addition

Motivation of Booth's Algorithm

0 0 0 | 1 1 1 1 1 1 | 0 0 0 0

add sub

additions will come down

Range of Multiplication o/p

on-int unsigned multiplication $0 \leq \text{result} \leq (2^n - 1)^2$

$\leq 2^{2n} - 2 \cdot 2^n + 1$

1 1 - - - - 1 | 0 0 - - 0 1

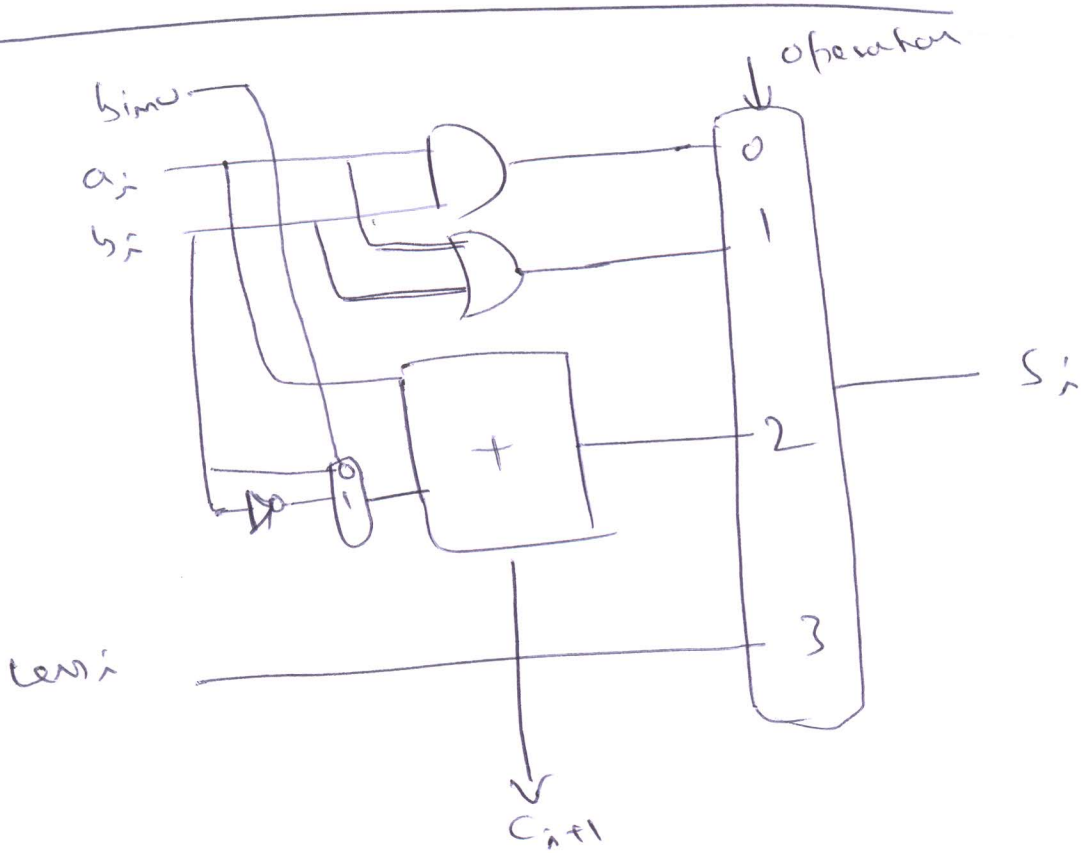
n-2 n 1

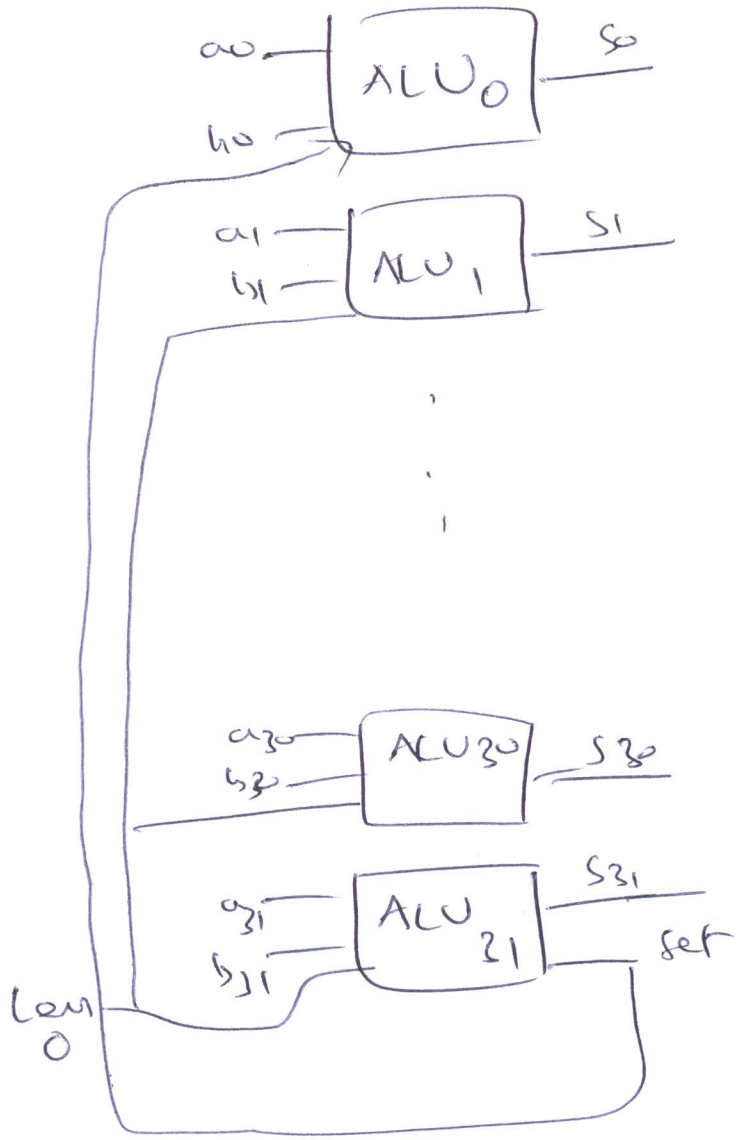
Sign multiplication

$-2^{2n-2} \leq \text{result} \leq 2^{2n-2}$

No overflow

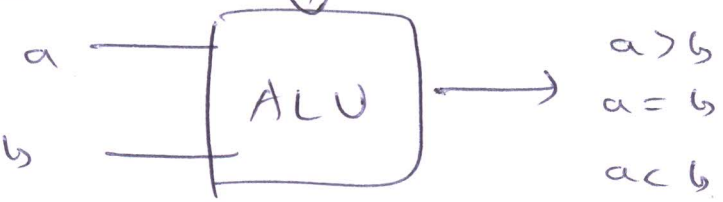
ALU for Add, Sub, AND, OR, SLT





Comparison

(1)



(2) Direct comparison

greater than

$$a_{0\cdots i+1} > b_{0\cdots i+1} = (a_{0\cdots i} > b_{0\cdots i})(a_{i+1}\bar{b}_{i+1}) + (a_{0\cdots i} = b_{0\cdots i})a_{i+1}\bar{b}_{i+1}$$

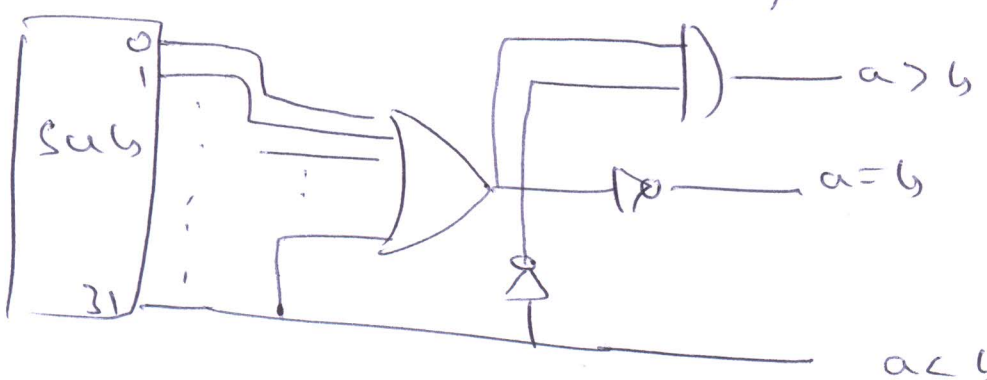
from LSB

from MSB

$$a_{i+1\cdots 31} > b_{i+1\cdots 31} = (a_{i\cdots 31} > b_{i\cdots 31}) + (a_{i\cdots 31} = b_{i\cdots 31})a_{i-1}\bar{b}_{i-1}$$

Equal

$$a = b \Rightarrow \prod_{i=0}^{31} (a_i \oplus b_i)$$



Division

$A \div B = Q \times B + R / B$

$A = Q \times B + R$

```
0100 )00001101( 0011
     01000000     ---- 2^4 x B
     ---------
     0000 1101
      0100 --     ---- 2^3 x B
     ---------
     0000 1101
       0100 --   ---- 2^2 x B
     ---------
     0000 1101
        0100 --  ---- 2^1 x B
     ---------
          0101
          0100   ---- 2^0 x B
     ---------
             1
```

Step 1 : i = 0, R = A, Q = 0, D = B

do {

Step 2

if ($D \times 2^{n-i-1} \leq R$) then

$R = R - D \times 2^{n-i-1}$

$Q_{n-i-1} = 1$

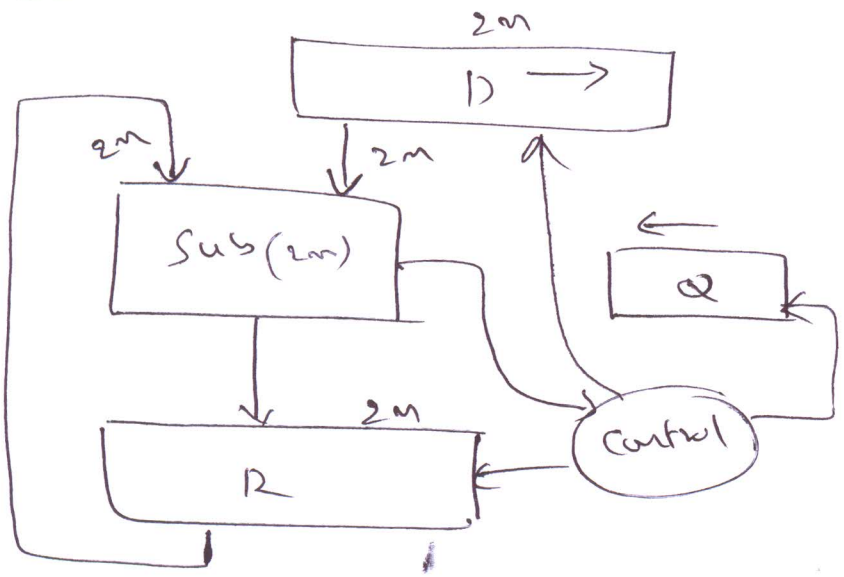
else

$Q_{n-i-1} = 0$

} while $i < n$

Loop invariant $A = Q \times D + R$ & $0 \leq R \leq D \times 2^{n-i}$

Design-1



Step-1 : i = 0, R = A, Q = 0, $D = B \times 2^{n-1}$

do {

Step 2 :

if ($D \leq R$)

then $R = R - D$; $Q = 2Q + 1$

else

$Q = 2Q$

$D = D/2$; i++

} while ($i < n$)

Reducing Substracter Size

$A = Q \times B + R$

Step 1: $i = 0$; $R = 2 \times A$; $Q = 0$; $D = B$

do {

Step 2:

if ($D \leq R_H$) $R_H = R_H - D$; $Q = 2Q + 1$

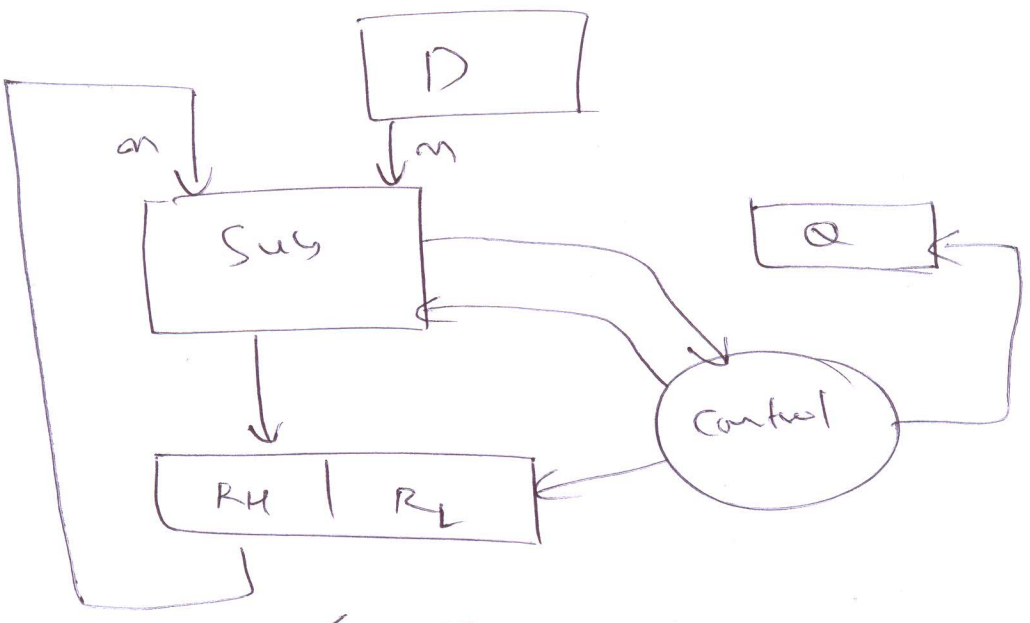
else

$Q = 2Q$

$R = 2R$; $i++$

} while ($i < m$)

Design-2



Reducing Registers

$A = Q \times B + R$

Step 1: $i = 0$; $R = 2 \times A$; $D = B$

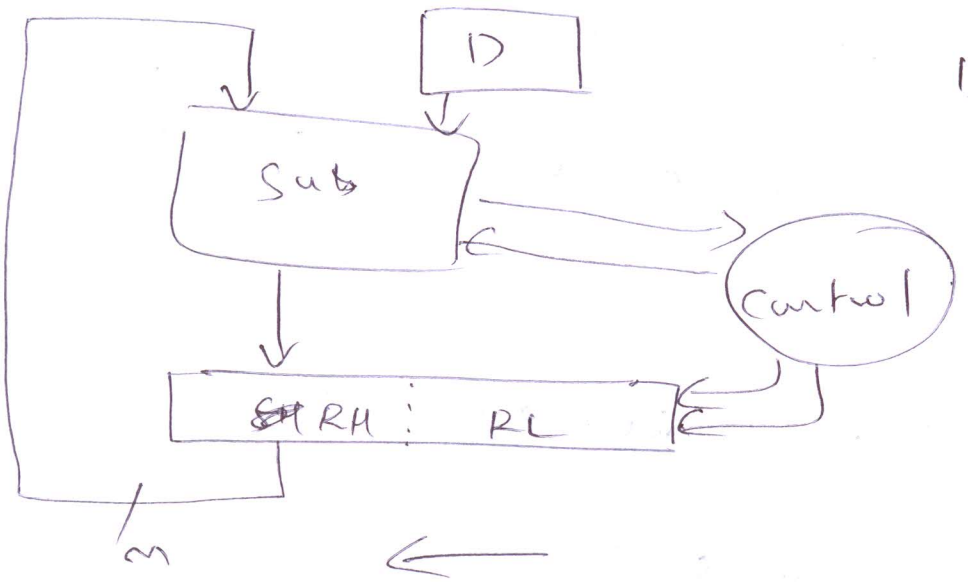
do Step 2: if ($D \leq R_H$) $R_H = R_H - D$; $R = 2R + 1$

else

$R = 2R$

$i++$

} while ($i < m$)



Design-3

Restoring Division

Non-restoring algorithm

```
if (D ≤ R) R = R-D ; D = D/2 ; Q = 2Q+1
    ELSE  D = D/2   Q = 2Q
          itf
```

Restored algorithm

$R = R - D$

```
if(R<0) R = R+D ;  Q = 2Q
    ELSE  Q = 2Q+1
    D = D/2 ;  itf
```

Delaying Restoration

$R = R+D$ & $R = R - D/2$ $\Rightarrow$ $R = R + D/2$

Improved restoring division

Step 1: $i = 0$; $R = A$; $Q = 0$; $D = B \times 2^{n-1}$

do {

Step 2A:

if ($R < 0$) $R = R + D$ else $R = R - D$

Step 2B:

if ($R < 0$) $Q = 2Q$ else $Q = 2Q + 1$

$D = D/2$; i++

} while $i < n$

if ($R < 0$) $R = R + B$

Further improvement

Step 1 $i = 0$; $R = A$; $Q = 0$; $D = B \times 2^{n-1}$

do {

Step 2 if ($R < 0$)

$R = R + D$; $Q = 2 \times Q - 1$

ELSE

$R = R - D$; $Q = 2 \times Q + 1$

$D = D/2$; i++

} while ($i < n$)

if ($R < 0$) $R = R + B$; $Q = Q - 1$

Extending to Signed division

Step 1: $i = 0$ $R = A$, $Q = 0$; $D = B \times 2^{n-1}$

do {

Step 2

if ($R_{n-1} \neq D_{n-1}$)

$R = R + D$; $Q = 2Q - 1$

ELSE

$R = R - D$; $Q = 2Q + 1$

$D = D/2$; $i = i + 1$

} while ($i < n$)

if ($R_{n-1} \neq D_{n-1}$) $R = R + B$; $Q = Q - 1$

Signs in signed division

Dividend	Divisor	Quotient	Remainder
+	+	+	+
+	−	−	+
−	+	−	−
−	−	+	−

Carry-lookahead adder

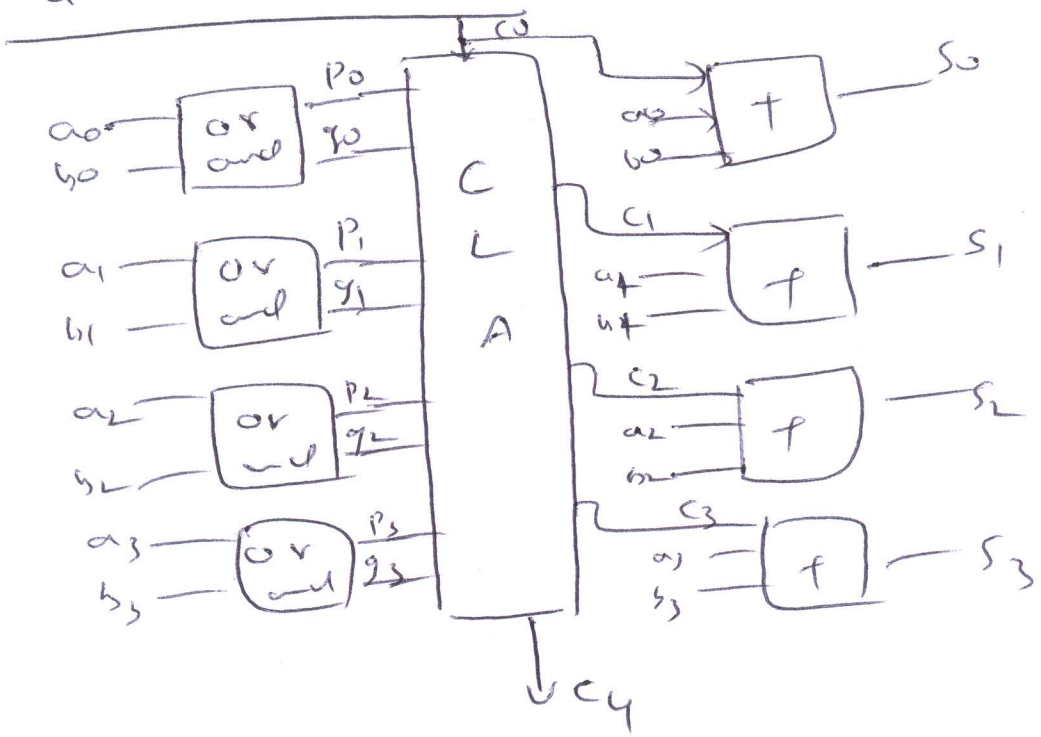
$$C_1 = P_0 C_0 + g_0$$

$$C_2 = P_1 C_1 + g_1 = P_1 P_0 C_0 + P_1 g_0 + g_1$$

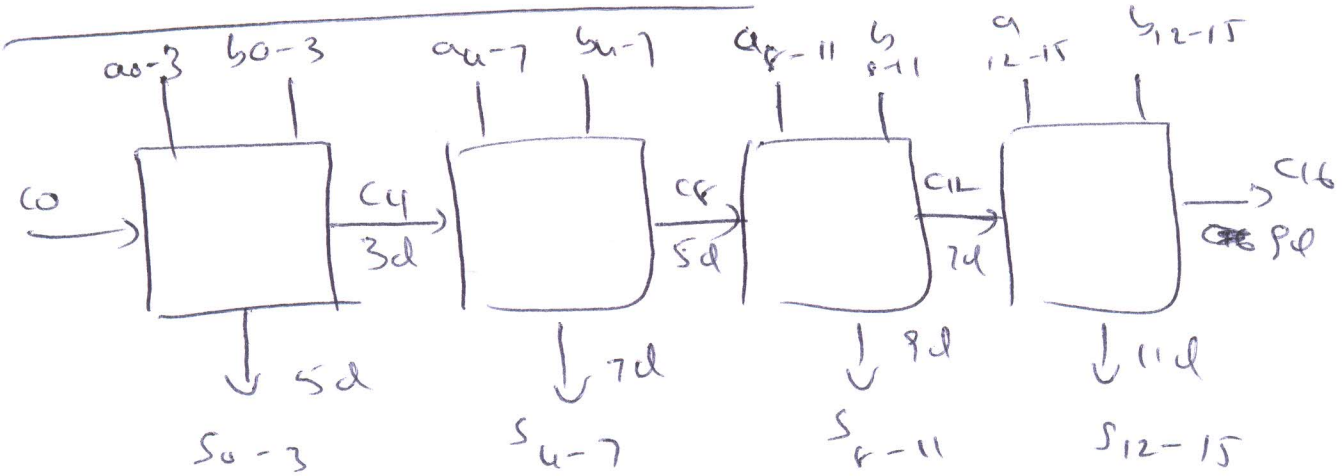
$$C_3 = P_2 C_2 + g_2 = P_2 P_1 P_0 C_0 + P_2 P_1 g_0 + P_2 g_1 + g_2$$

$$C_4 = P_3 C_3 + g_3 = P_3 P_2 P_1 P_0 C_0 + P_3 P_2 P_1 g_0 + P_3 P_2 g_1 + P_3 g_2 + g_3$$

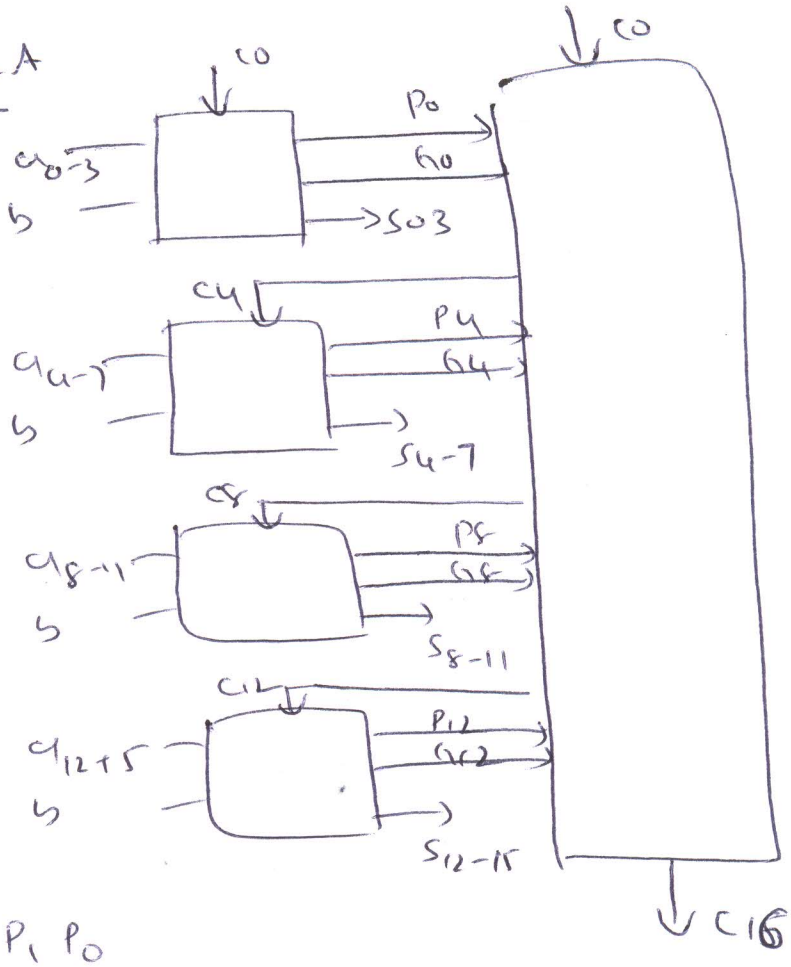
4-bit CLA adder



16-bit adder with 4-bit CLA



2-Level of CLA



$$P_0 = P_3 P_2 P_1 P_0$$

$$G_0 = P_3 P_2 P_1 g_0 + P_3 P_2 g_1 + P_3 g_2 + g_3$$

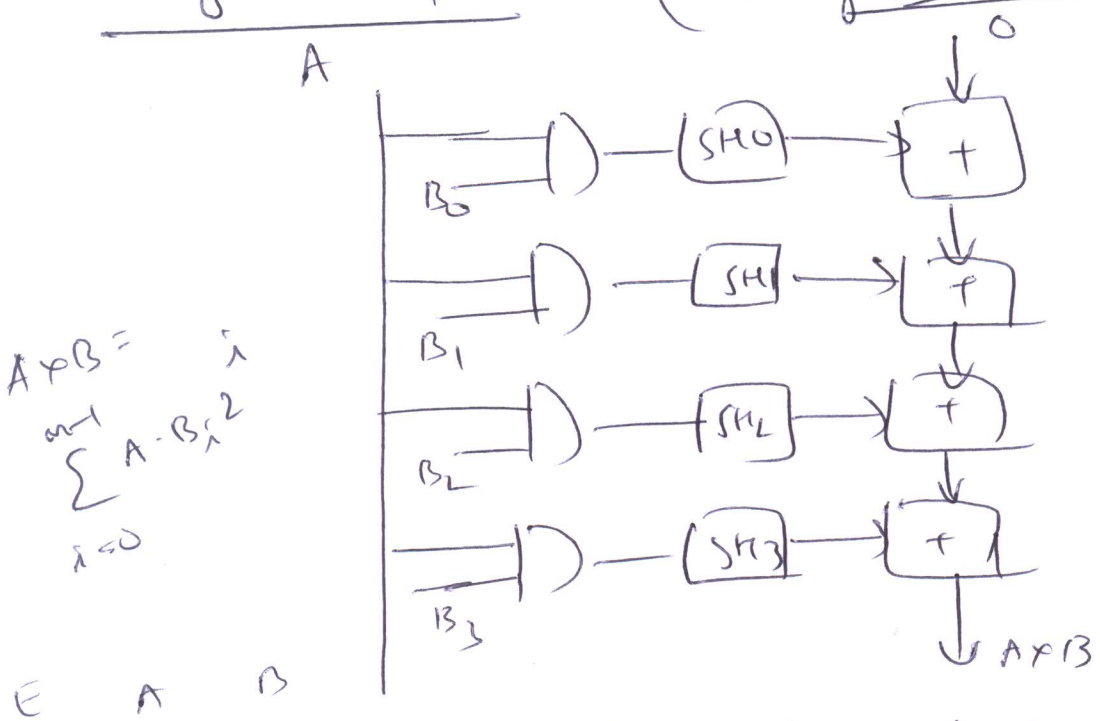
$$C_4 = P_0 C_0 + G_0$$

$$C_8 = P_4 P_0 C_0 + P_4 G_0 + G_4$$

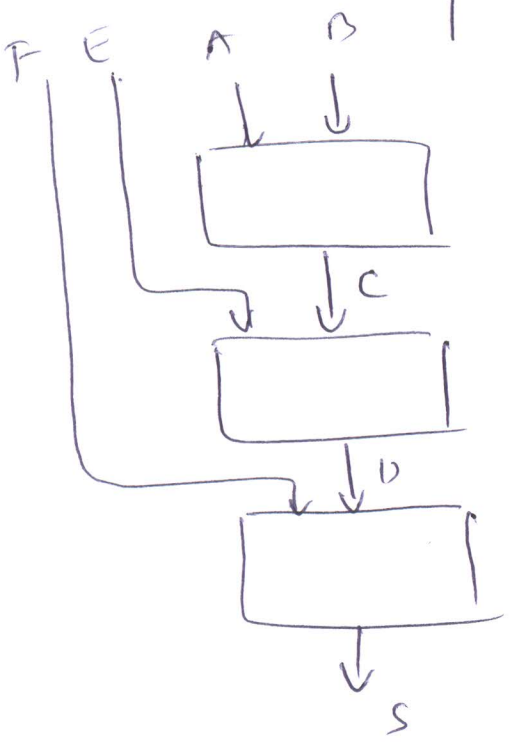
$$C_{12} = P_8 P_4 P_0 C_0 + P_8 P_4 G_0 + P_8 G_4 + G_8$$

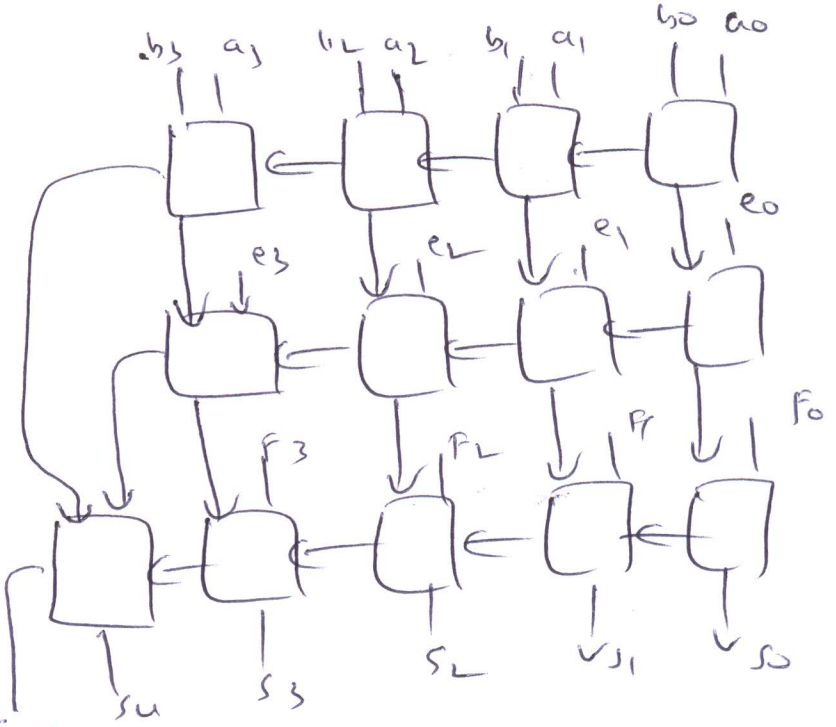
$$C_{16} = P_{12} P_8 P_4 P_0 C_0 + P_{12} P_8 P_4 G_0 + P_{12} P_8 G_4 + P_{12} G_8 + G_{12}$$

Array Multiplier (Carry Save adder)

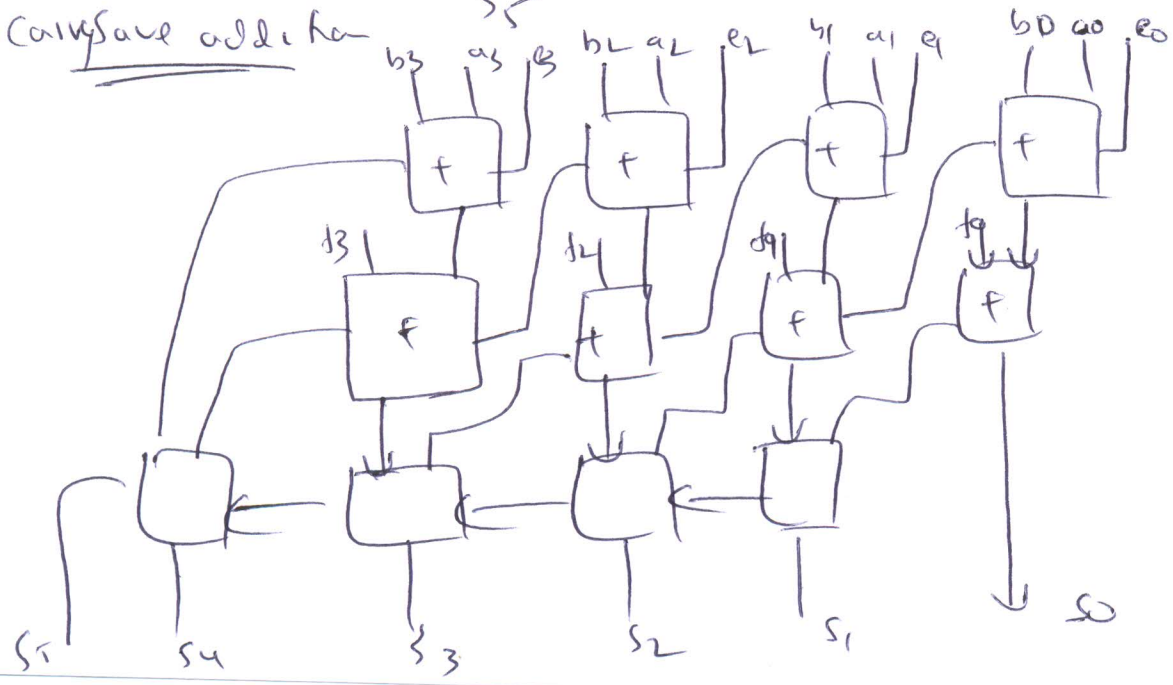


$$A \times B = \sum_{i=0}^{i} A \cdot B_i^{2}$$





Carry save adder



Floating Point Numbers & System Design

12/8/14

Datatypes: Integer, rational, real & Complex

Scientific Applications < Astronomy (distance bet Sun & Pluto) 5.9×10^{12} m
weight of an electron 9.1×10^{-28} gms

Representing Fractions

Integer pairs (for rational numbers)

[5] [8] => 5/8

Strings with explicit decimal point

[-][2][4][7][.][0][1] => -247.01

Fixed point rep (implicit pt in a fixed pos)

01001110.01001

Floating point

Fraction x Base ^{power}

Precision & accuracy → Mantissa

Base & power → range

0.6 = 0.10011001100110011...

0.6 x 2 = 1 + 0.2
0.2 x 2 = 0 + 0.4
0.4 x 2 = 0 + 0.8
0.8 x 2 = 1 + 0.6
0.6 x 2 = 1 + 0.2
.
.
.

FP Numbers with base 2

$(-1)^S \times F \times 2^E$

S = Sign

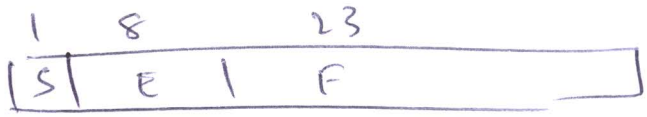
F = Fraction / Mantissa / Significant

E = Exponent (± integer)

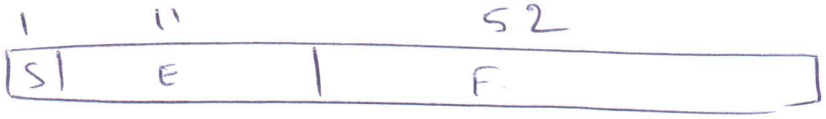
Fixed point

IEEE - 754

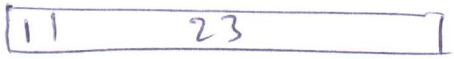
Single precision



Double precision



Rep of F in IEEE 754 (Representation of mantissa)



$1 \leq F \leq 2 - 2^{-23} \approx 2$

$1 \leq F \leq 2 - 2^{-52} \approx 2$

Rep of exponent

Single precision (8) : bias 127

Double precision (11) : bias 1023

$0 - 255 \Rightarrow 1 - 254 \Rightarrow$ 1-127 to 254-127

-126 to 127

Double precision

$0 - 2047 \Rightarrow 1 - 2046 \Rightarrow$ 1-1023 to 2046-1023

-1022 to 1023

Rep of zero

All bits (S, F, E) are zeros

Testing & Comparing

zero, +ve, -ve in same as integers

magnitude comparison in same as integers

Overflow & underflow

Largest +ve or -ve Number (SP) = $\pm (2 - 2^{-23}) \times 2^{127} \approx \pm 2 \times 10^{38}$

" " " (DP) = $\pm (2 - 2^{-52}) \times 2^{1023} \approx \pm 2 \times 10^{308}$

Smallest +ve or -ve " (SP) = $\pm 1 \times 2^{-126} \approx 0 \pm 2 \times 10^{-38}$

" " (DP) = $\pm 1 \times 10^{-1022} \approx \pm 2 \times 10^{-308}$

Precision $\Rightarrow$ SP $\rightarrow \frac{23}{\log_2 10} \approx 8$ DP $\rightarrow \frac{52}{\log_2 10} \approx 17$

Floating Point operations

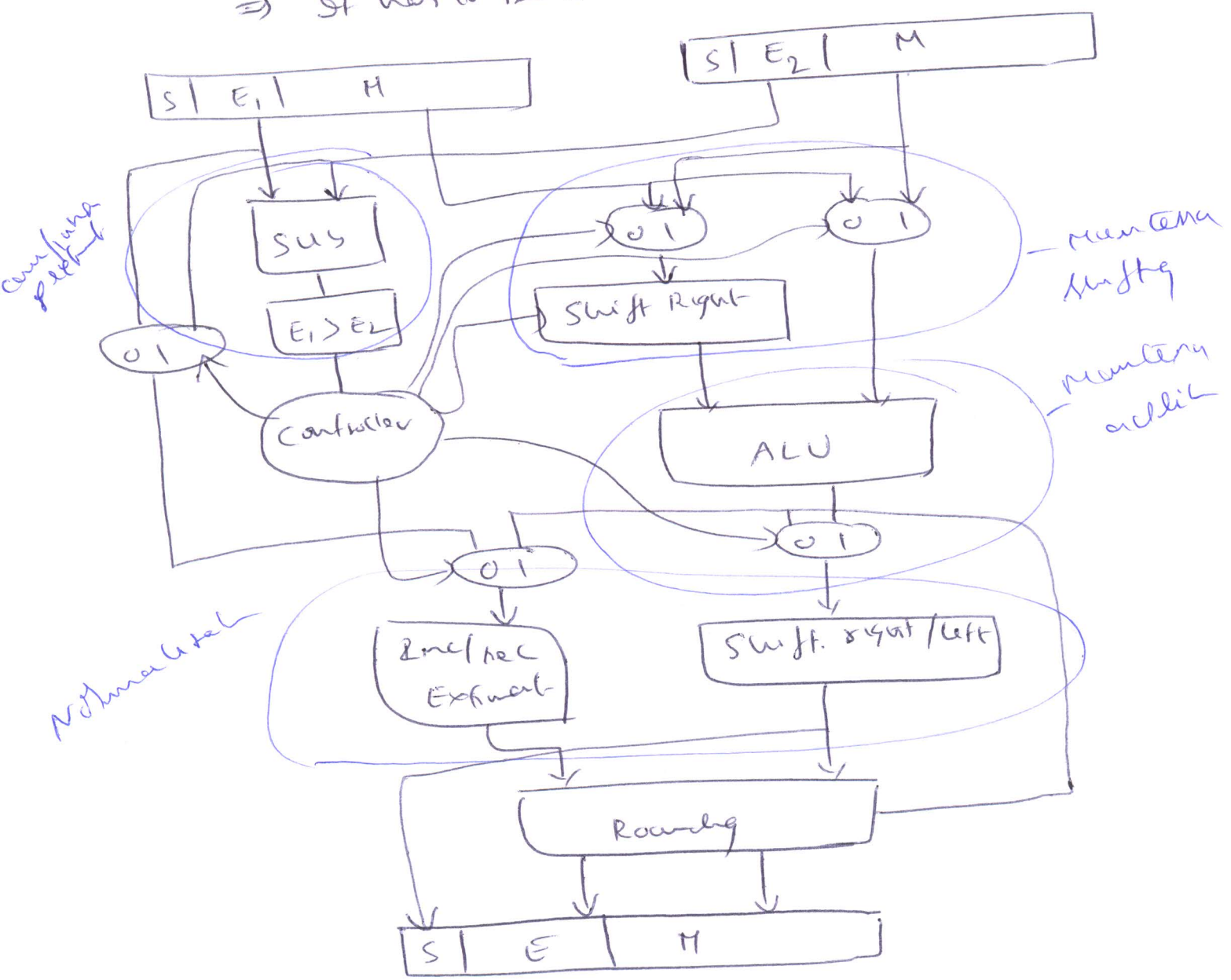
Add / Subtract $\left[(-1)^{S_1} \times F_1 \times 2^{E_1}\right] \pm \left[(-1)^{S_2} \times F_2 \times 2^{E_2}\right]$

if $E_1 > E_2$

$$\left[(-1)^{S_1} \times F_1 \times 2^{E_1}\right] \pm \left[(-1)^{S_2} \times F_2' \times 2^{E_1}\right] ; \quad F_2' = \frac{F_2}{2^{E_1 - E_2}}$$

$$\Rightarrow (-1)^{S_1} \times (F_1 \pm F_2) 2^{E_1}$$

$\Rightarrow$ If has to be normalized



Multiply $\left[(-1)^{S_1} \times F_1 \times 2^{E_1}\right] \times \left[(-1)^{S_2} \times F_2 \times 2^{E_2}\right]$

$\Rightarrow (-1)^{S_1 \oplus S_2} \times (F_1 \times F_2) \times 2^{E_1 + E_2}$

$\therefore \quad 1 \le (F_1 \times F_2) \le 4 \Rightarrow$ the result has to be normalized

Divide $\left[(-1)^{S_1} \times F_1 \times 2^{E_1}\right] \div \left[(-1)^{S_2} \times F_2 \times 2^{E_2}\right]; \quad F_2 \neq 0$

$\Rightarrow (-1)^{S_1 \oplus S_2} \times (F_1 \div F_2) 2^{E_1 - E_2}$

$\therefore \quad 0.5 \le (F_1 \div F_2) < 2 \Rightarrow$ the result has to be normalized

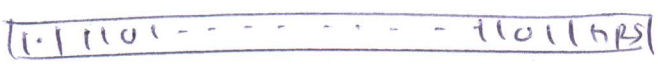
Accuracy & Rounding off

- Precision is lost when bits are shifted to the right of right most bit
- 3 extra bits are used internally

G (guard) R (round) & S (sticky)

GR are 2 bits after LSB

S = 1 iff any bit to the right of R is non-zero



Rounding using G, R & S

if G=1 & R=1 ; add 1 to LSB

if G=0 & R=0 or 1 ; no change

if G=1 & R=0 (look at S)

if S=1 => add 1 to LSB

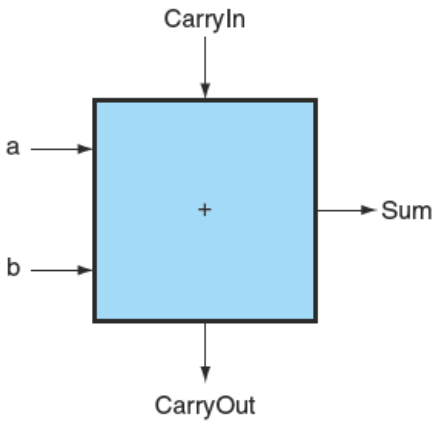
if S=0 => round to nearest "even" (i.e, add 1 to LSB, if LSB=1)

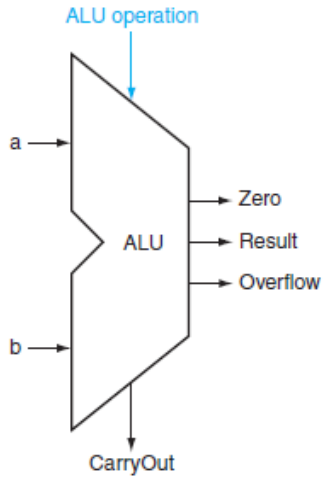
SP		DP		Object
	M	E	M	
0	0	0	0	zero
0	NZ	0	NZ	denormalized
254	any	0-2046	any	normalized
255	0	2047	0	infinity
255	non-zero	2047	non-zero	NaN

Arithmetic Circuits

ALU

- 1-bit Adder





Inputs			Outputs		Comments
a	b	CarryIn	CarryOut	Sum	
0	0	0	0	0	$0 + 0 + 0 = 00_{two}$
0	0	1	0	1	$0 + 0 + 1 = 01_{two}$
0	1	0	0	1	$0 + 1 + 0 = 01_{two}$
0	1	1	1	0	$0 + 1 + 1 = 10_{two}$
1	0	0	0	1	$1 + 0 + 0 = 01_{two}$
1	0	1	1	0	$1 + 0 + 1 = 10_{two}$
1	1	0	1	0	$1 + 1 + 0 = 10_{two}$
1	1	1	1	1	$1 + 1 + 1 = 11_{two}$

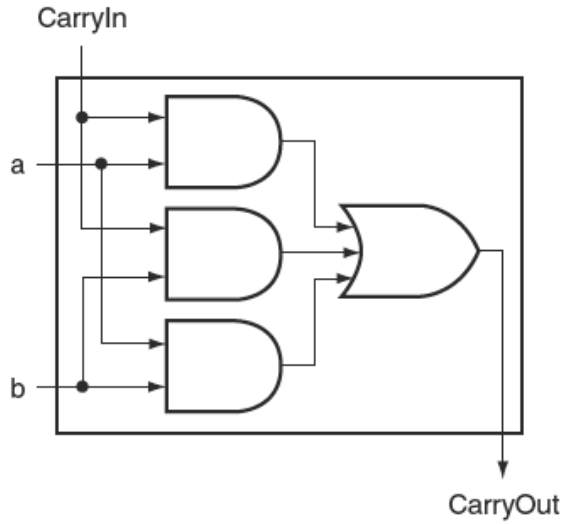
FIGURE B.5.3 Input and output specification for a 1-bit adder.

ALU (Cont..)

$$\text{CarryOut} = (b \cdot \text{CarryIn}) + (a \cdot \text{CarryIn}) + (a \cdot b)$$

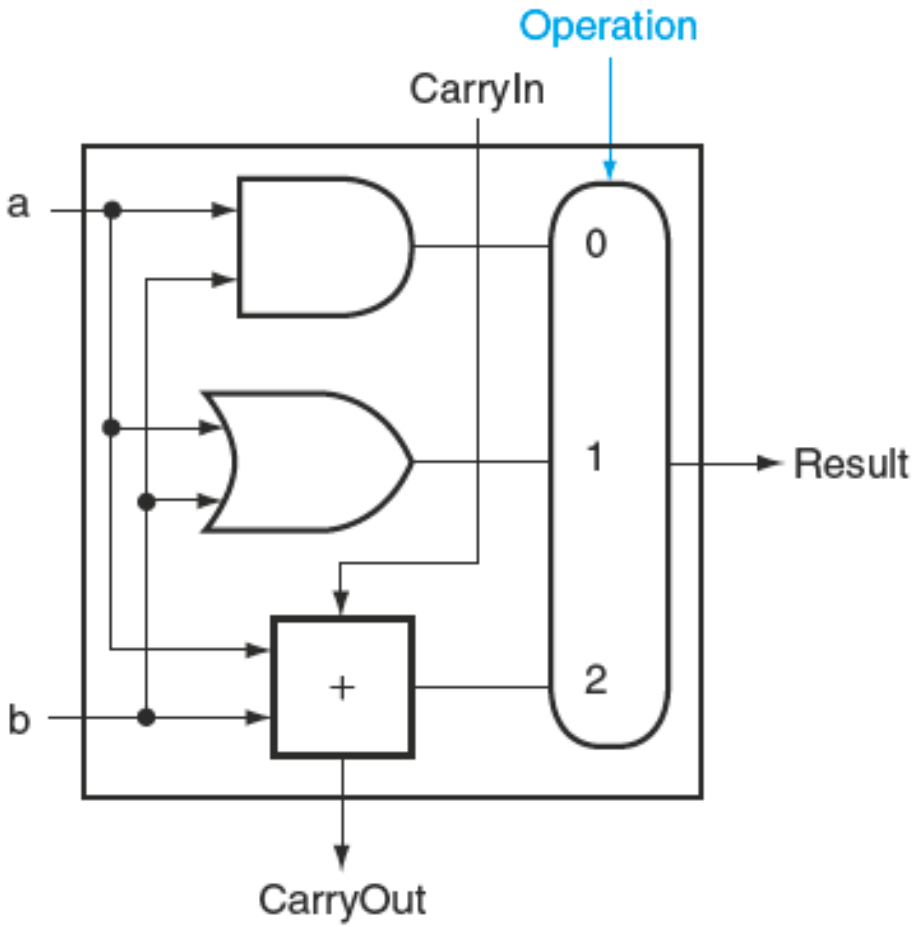
Inputs		
a	b	CarryIn
0	1	1
1	0	1
1	1	0
1	1	1

FIGURE B.5.4 Values of the Inputs when CarryOut Is a 1.

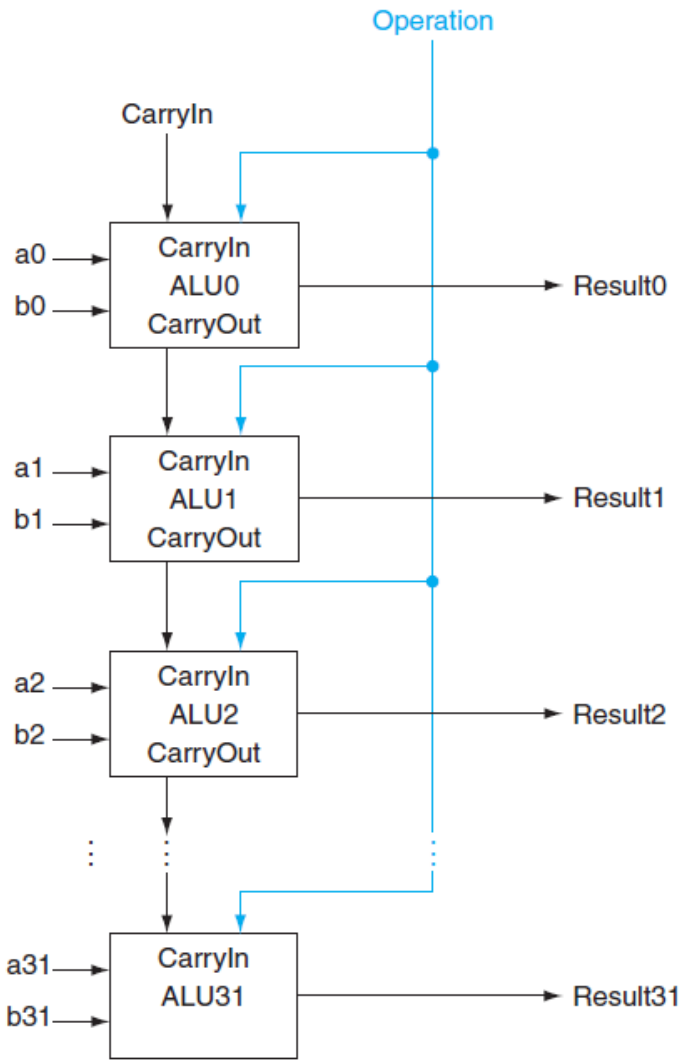


$$\text{Sum} = (a \cdot \overline{b} \cdot \overline{\text{CarryIn}}) + (\overline{a} \cdot b \cdot \overline{\text{CarryIn}}) + (\overline{a} \cdot \overline{b} \cdot \text{CarryIn}) + (a \cdot b \cdot \text{CarryIn})$$

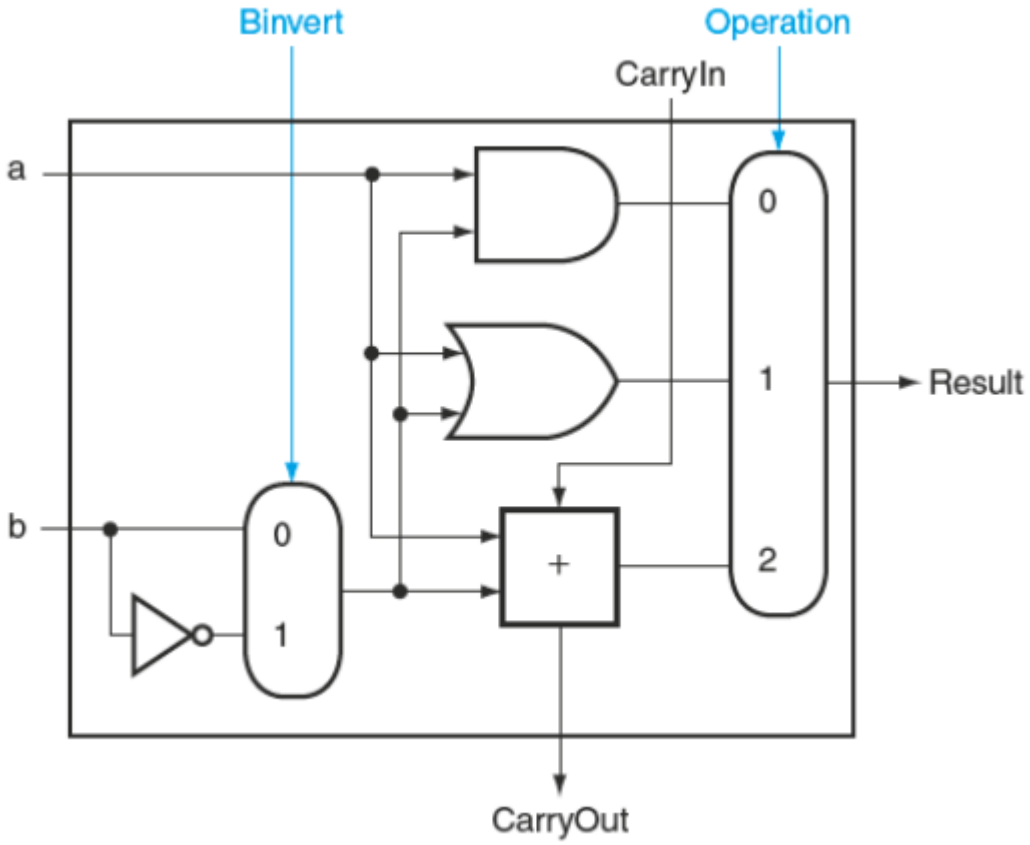
1-bit ALU (ADD, AND & OR)



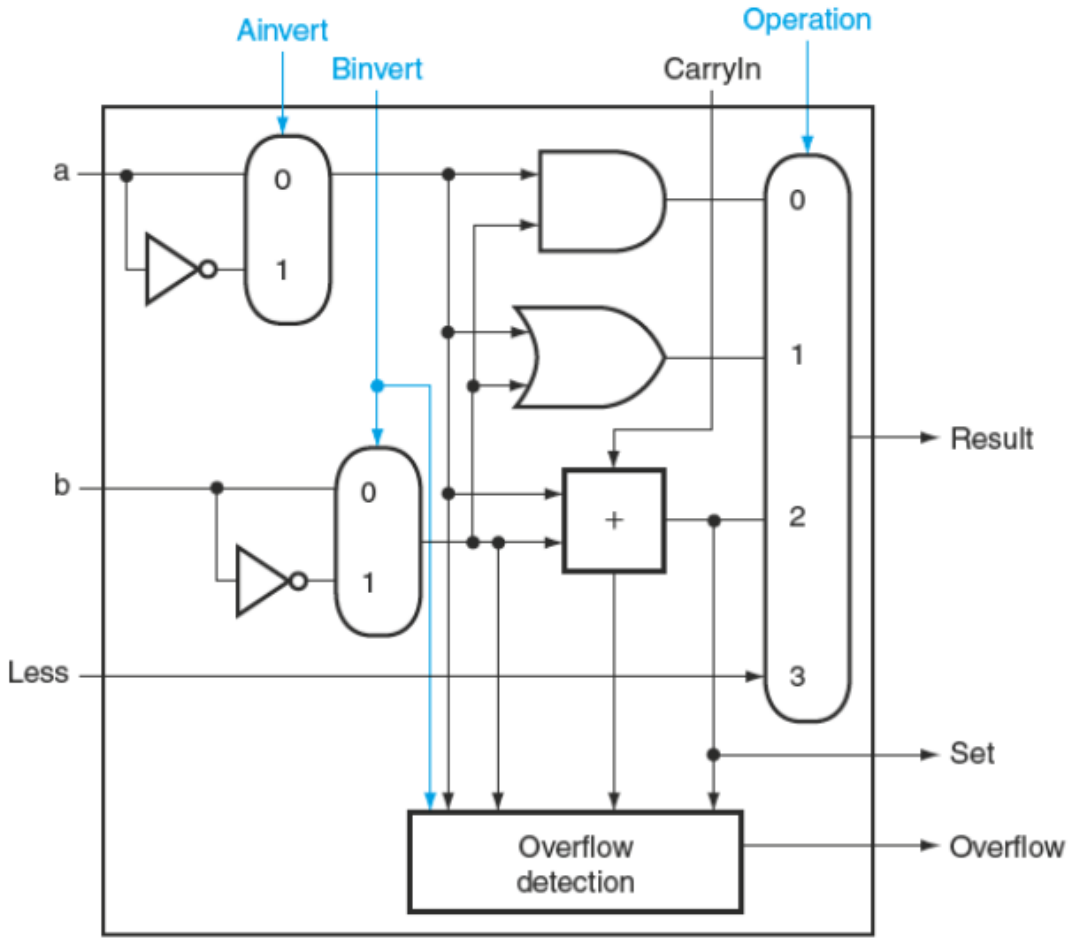
32-bit ALU


Operation
CarryIn
a0
b0
CarryIn
ALU0
CarryOut
Result0
a1
b1
CarryIn
ALU1
CarryOut
Result1
a2
b2
CarryIn
ALU2
CarryOut
Result2
a31
b31
CarryIn
ALU31
Result31

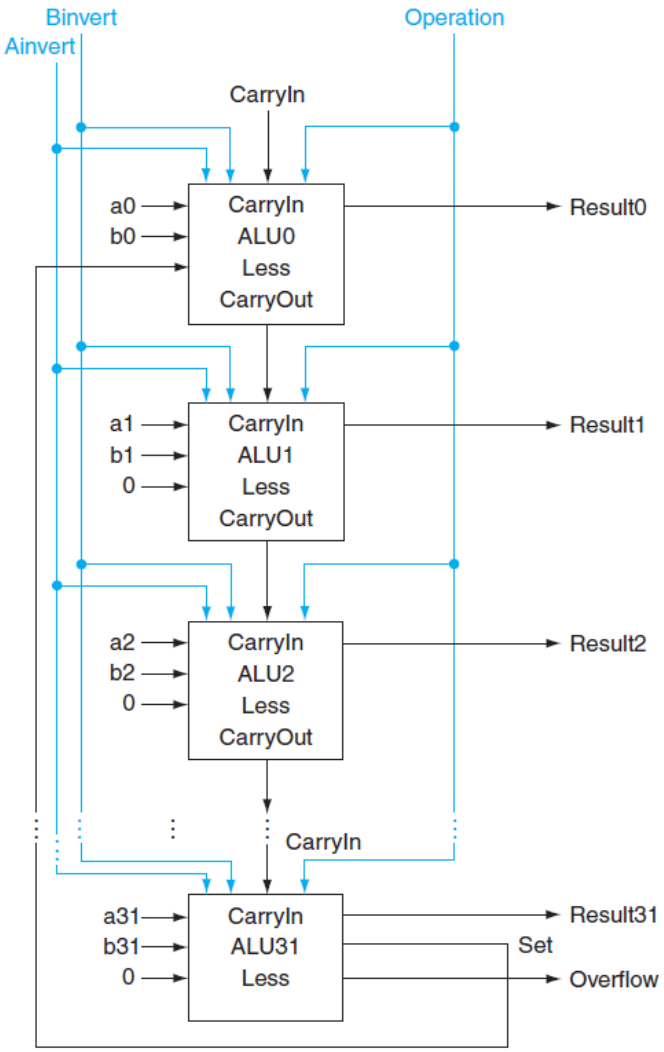
1-bit ALU (ADD, SUB, AND & OR)



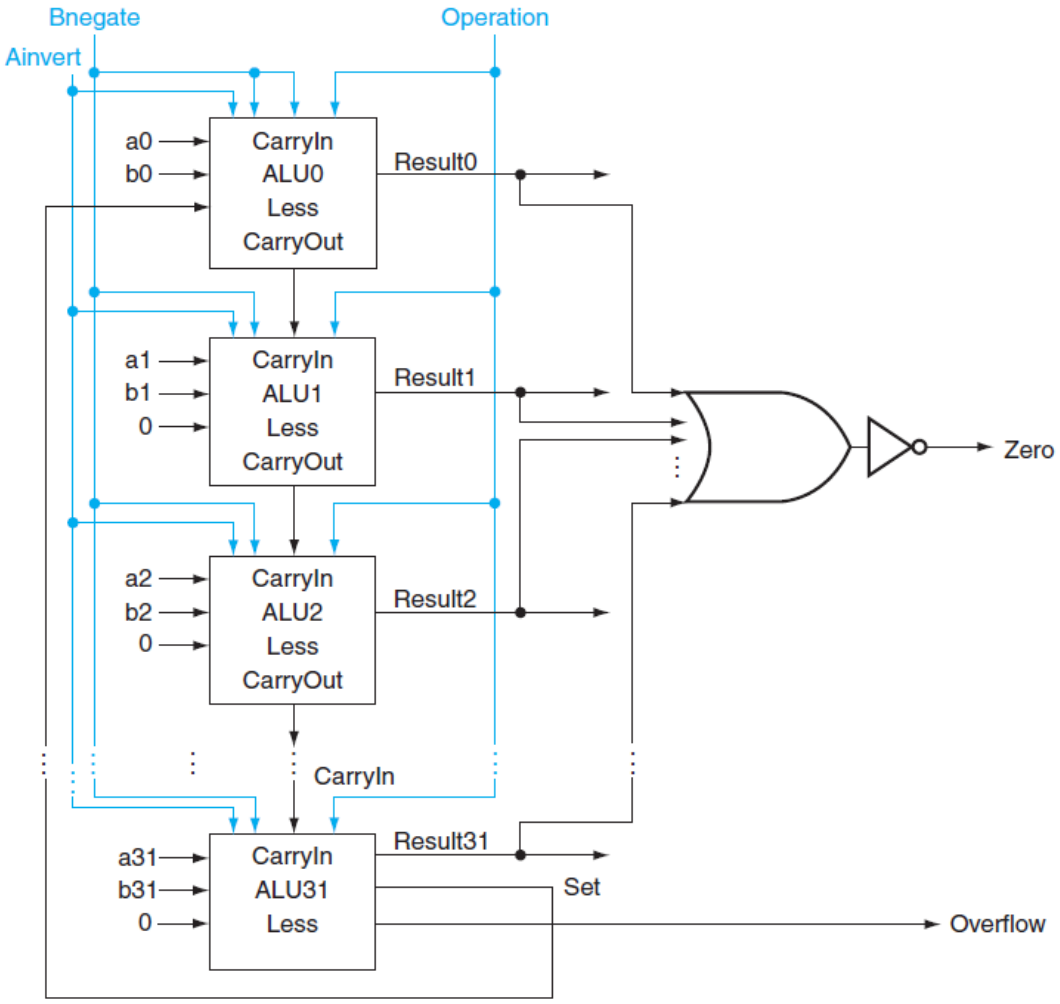
1-bit ALU (ADD, SUB, AND, OR & SLT)



32-bit ALU (ADD, SUB, AND, OR & SLT)



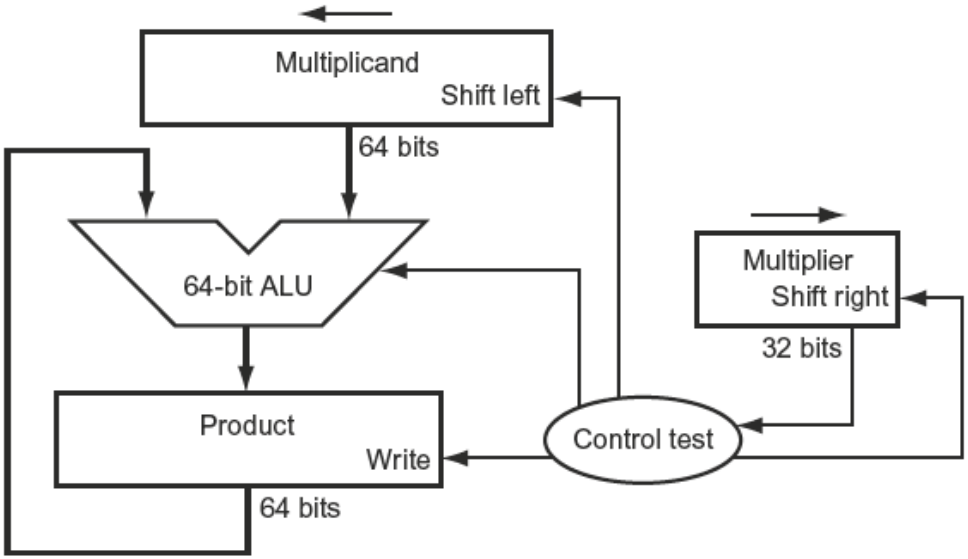
32-bit ALU (ADD, SUB, AND, OR, SLT & EQ)

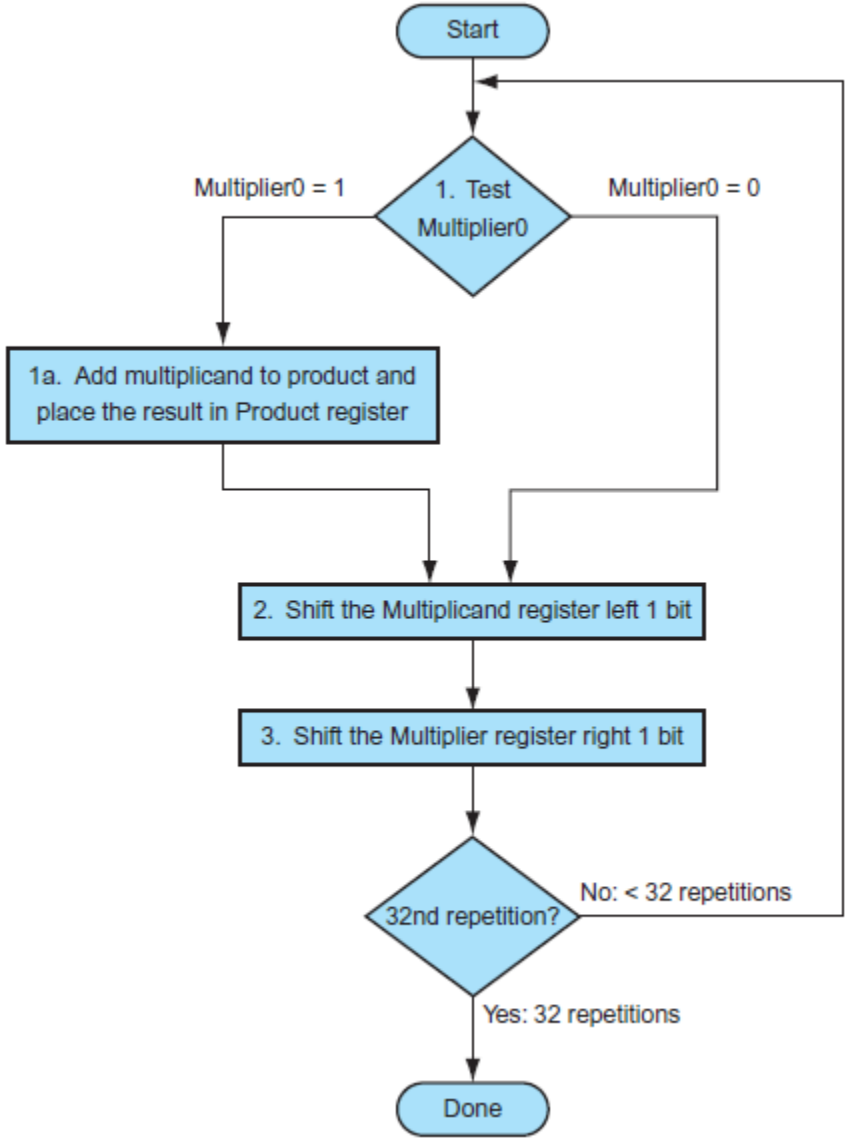


Design of Hardware for Multiplication, Division and Floating point operations

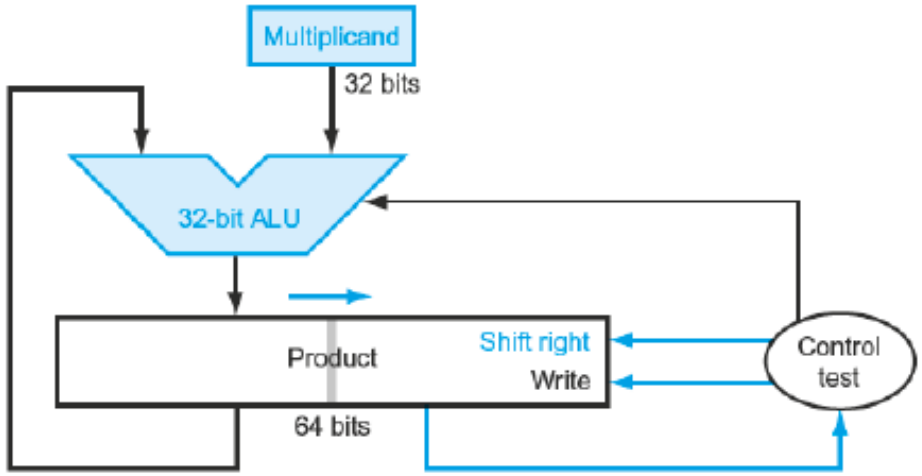
Multiplier

Multiplicand		1000_{ten}
Multiplier	x	1001_{ten}
		1000
		0000
		0000
		1000
Product		1001000_{ten}



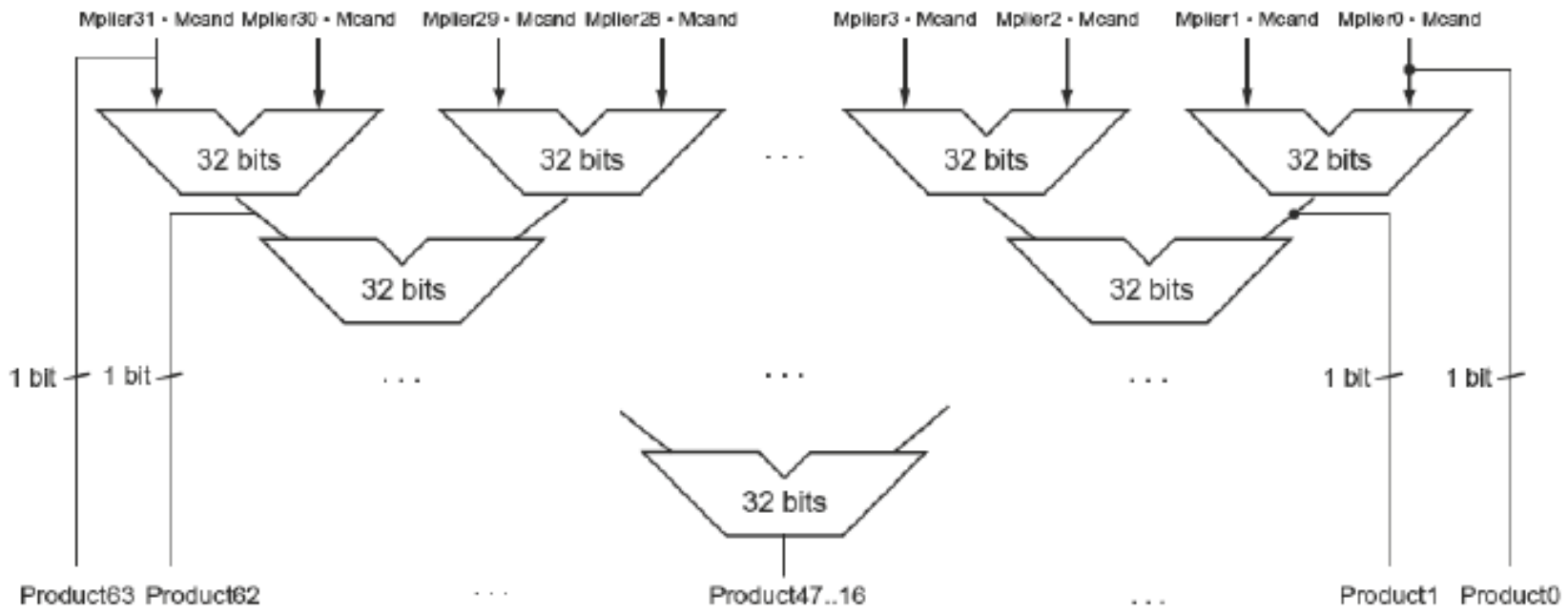


Multiplier (Cont..)



Iteration	Step	Multiplier	Multiplicand	Product
0	Initial values	0011	0000 0010	0000 0000
1	1a: 1 ⇒ Prod = Prod + Mcand	0011	0000 0010	0000 0010
	2: Shift left Multiplicand	0011	0000 0100	0000 0010
	3: Shift right Multiplier	0001	0000 0100	0000 0010
2	1a: 1 ⇒ Prod = Prod + Mcand	0001	0000 0100	0000 0110
	2: Shift left Multiplicand	0001	0000 1000	0000 0110
	3: Shift right Multiplier	0000	0000 1000	0000 0110
3	1: 0 ⇒ No operation	0000	0000 1000	0000 0110
	2: Shift left Multiplicand	0000	0001 0000	0000 0110
	3: Shift right Multiplier	0000	0001 0000	0000 0110
4	1: 0 ⇒ No operation	0000	0001 0000	0000 0110
	2: Shift left Multiplicand	0000	0010 0000	0000 0110
	3: Shift right Multiplier	0000	0010 0000	0000 0110

Fast Multiplier



Division

1001_{ten} Quotient

Divisor 1000_{ten} $\overline{)1001010_{ten}}$ Dividend

-1000

10

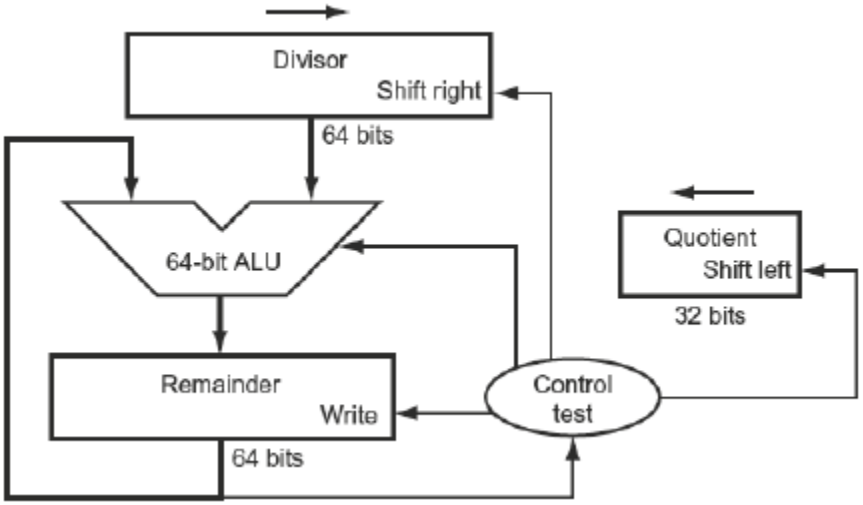
101

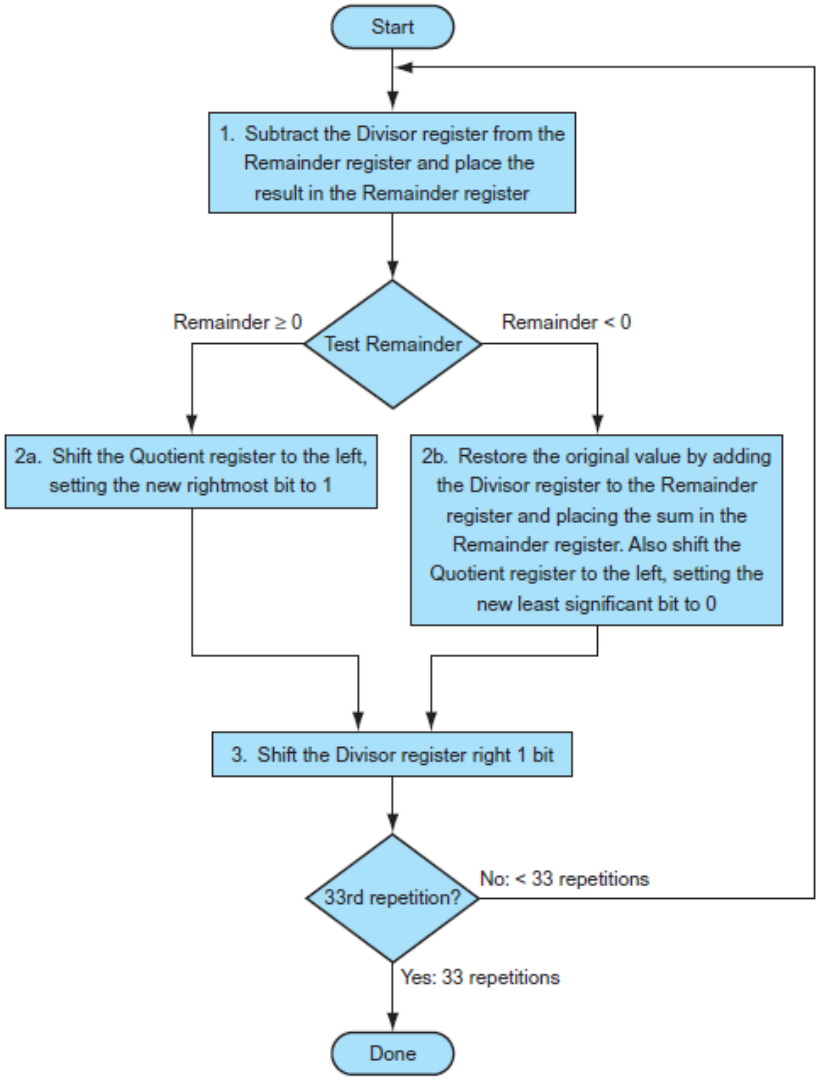
1010

-1000

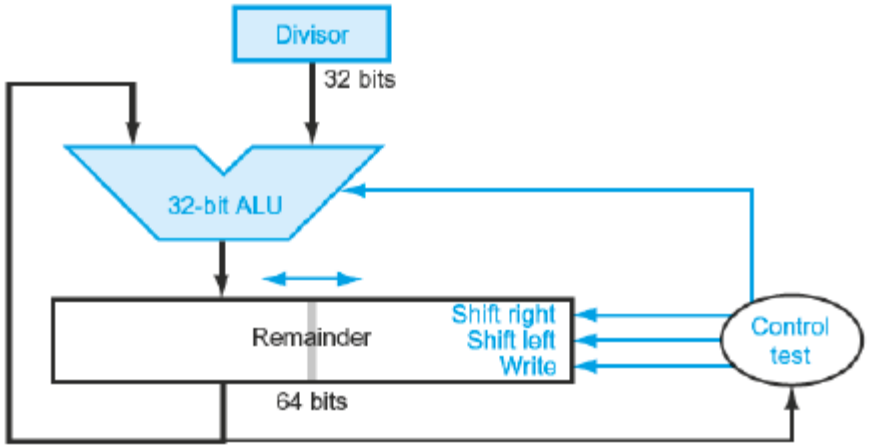
10_{ten} Remainder

Dividend = Quotient × Divisor + Remainder





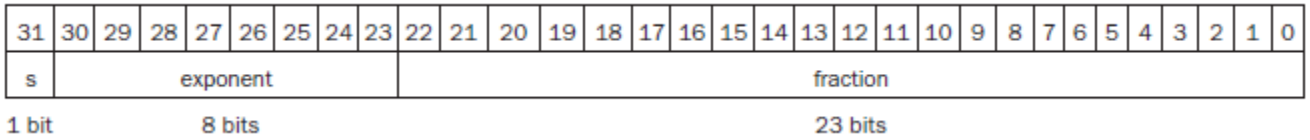
Division (Cont..)



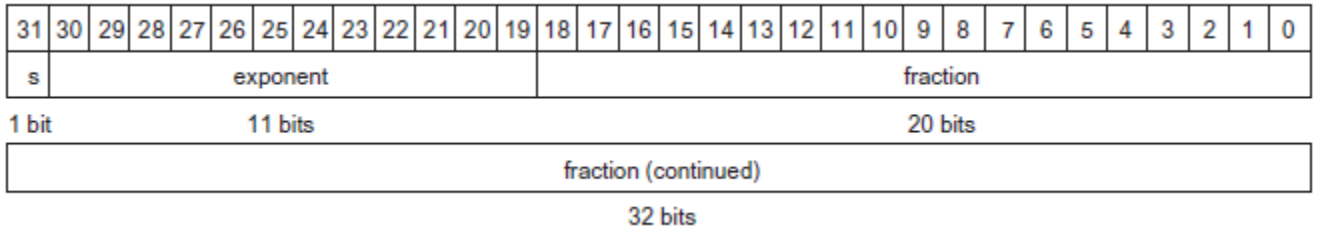
Iteration	Step	Quotient	Divisor	Remainder
0	Initial values	0000	0010 0000	0000 0111
1	1: Rem = Rem – Div	0000	0010 0000	①110 0111
	2b: Rem < 0 ⟹ +Div, sll Q, Q0 = 0	0000	0010 0000	0000 0111
	3: Shift Div right	0000	0001 0000	0000 0111
2	1: Rem = Rem – Div	0000	0001 0000	①111 0111
	2b: Rem < 0 ⟹ +Div, sll Q, Q0 = 0	0000	0001 0000	0000 0111
	3: Shift Div right	0000	0000 1000	0000 0111
3	1: Rem = Rem – Div	0000	0000 1000	①111 1111
	2b: Rem < 0 ⟹ +Div, sll Q, Q0 = 0	0000	0000 1000	0000 0111
	3: Shift Div right	0000	0000 0100	0000 0111
4	1: Rem = Rem – Div	0000	0000 0100	①000 0011
	2a: Rem ≥ 0 ⟹ sll Q, Q0 = 1	0001	0000 0100	0000 0011
	3: Shift Div right	0001	0000 0010	0000 0011
5	1: Rem = Rem – Div	0001	0000 0010	①000 0001
	2a: Rem ≥ 0 ⟹ sll Q, Q0 = 1	0011	0000 0010	0000 0001
	3: Shift Div right	0011	0000 0001	0000 0001

Floating Point Operations

Single Precession FP Representation (IEEE 754)



Double Precession FP Representation (IEEE 754)

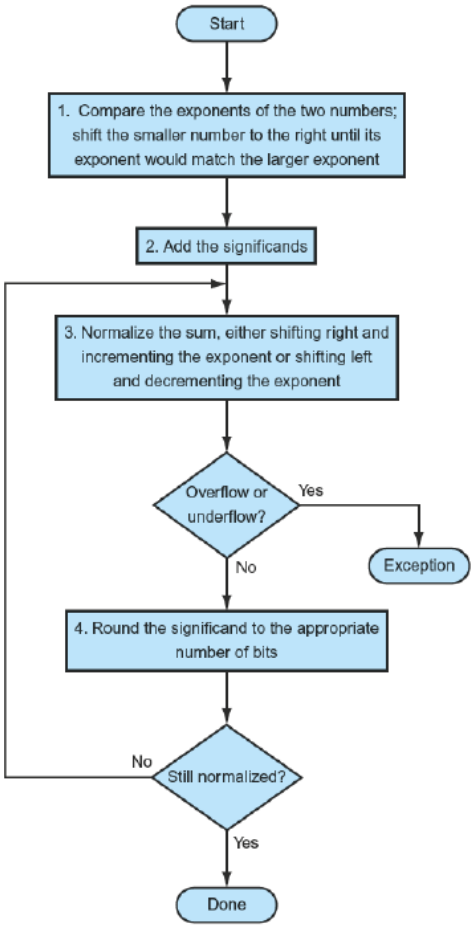


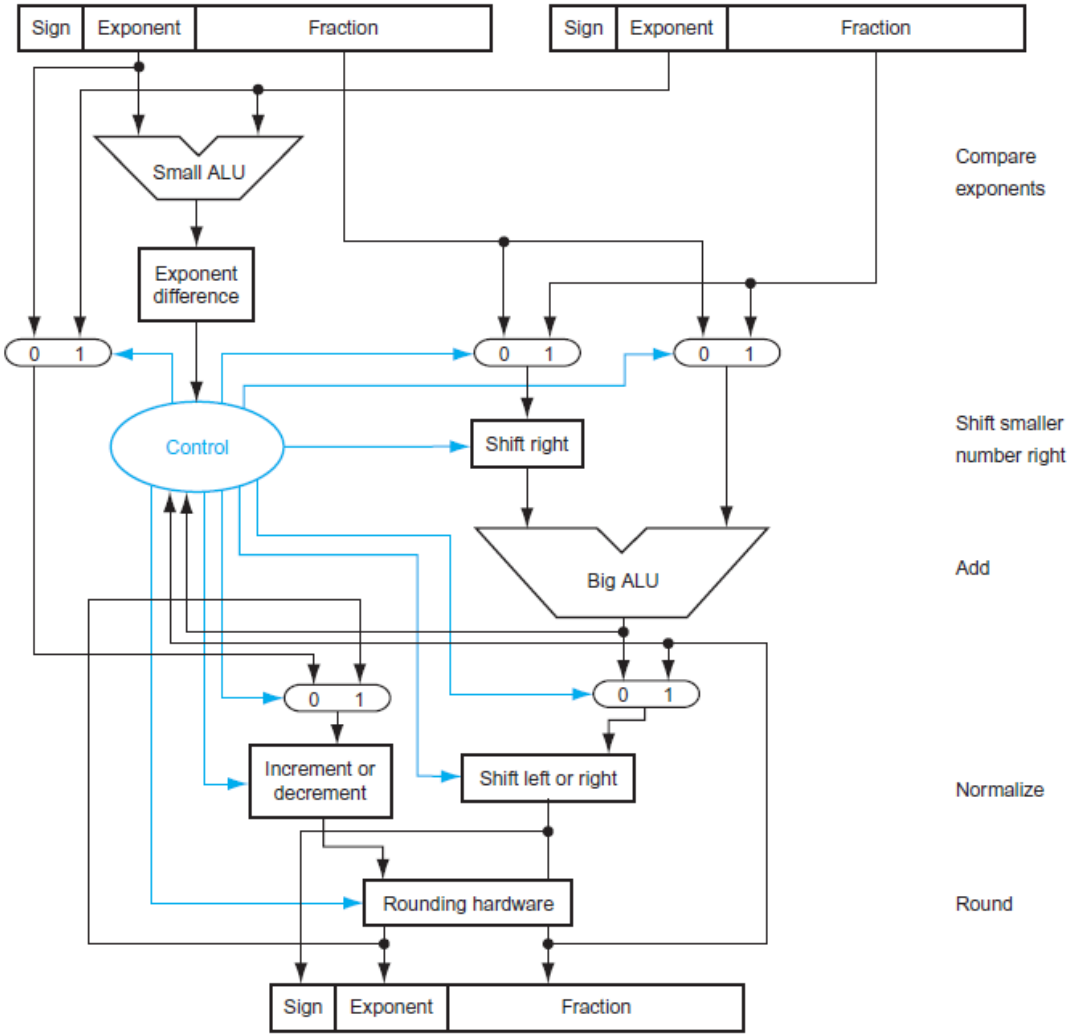
Floating point value

$$(-1)^S \times (1 + \text{Fraction}) \times 2^E$$

Single precision		Double precision		Object represented
Exponent	Fraction	Exponent	Fraction	
0	0	0	0	0
0	Nonzero	0	Nonzero	± denormalized number
1–254	Anything	1–2046	Anything	± floating-point number
255	0	2047	0	± infinity
255	Nonzero	2047	Nonzero	NaN (Not a Number)

Floating point addition





Floating point Multiplication