★ GRAPH THEORY

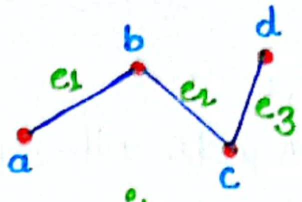
Graph $G = (V, E)$

Set of vertices and ~~eg~~ edges

$V \Rightarrow$ a non-empty set of vertices.

$E \Rightarrow$ a set of edges.

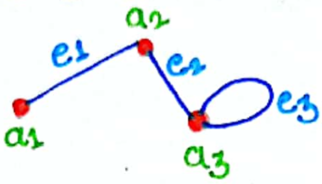
$e_1 = (a, b)$



(i)

★ END POINTS:

Each edge has either one or two vertices associated with it, called its endpoints.



★ FINITE GRAPH:

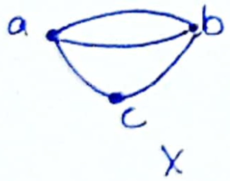
A graph with a finite vertex set is called finite graph.

★ INFINITE GRAPH

A graph with an infinite vertex set is called infinite graph.

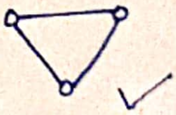
★ SIMPLE GRAPH

A graph in which each edge connects two different vertices and no two edges connect the same pair of vertices is called a simple graph. (# NO LOOP) (# NO Multi Edges)



(a)

(b)

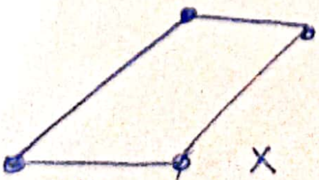


(c)

★ MULTI GRAPH

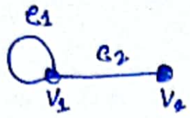
A graph having multiple edges connecting the same vertices is called multigraph.

(a) ✓



(b)

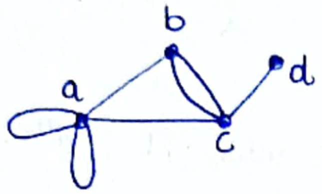
★ **Loop** : An edge that connects a vertex to itself is called loop.

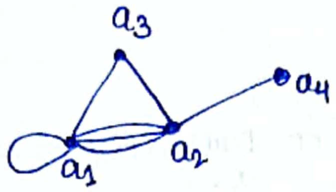


e_1 is a loop

★ **Pseudograph** :

A graph with loops and multiple edges is called a pseudograph.

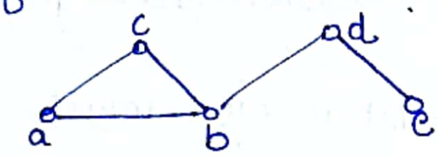




★ **Undirected Graph**

A graph in which each edge is associated with an unordered pair of vertices is called undirected graph.

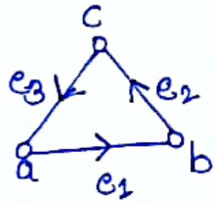
{No dirn. in the edges}

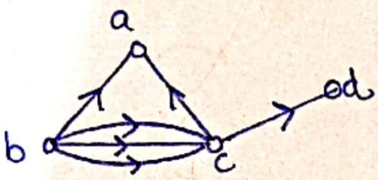


★ **Directed Graph** :

A graph in which each directed edge is associated with an ordered pair of vertices is called directed graph or digraph.

{Dirn.}





$e_1 = (a,b)$ is an directed Edge

Note: If $e = (u,v)$ is a directed edge, u is the initial vertex & v is the terminal vertex.

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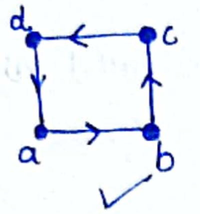
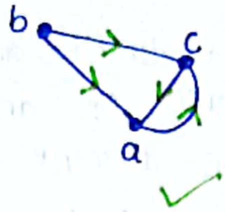
★ Simple Directed Graph :

When a directed graph has no loops and no multiple directed edges, it is called a simple directed graph.



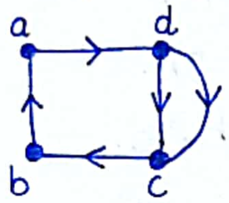


$e_1 = (a, c)$
$e_2 = (c, a)$

★ Directed Multigraph :

A directed graph with multiple directed edges is called directed multigraph.

Loops are also allowed in this graph.



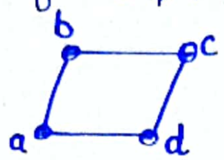
Type	Edges	Multiple Edges Allowed!	Loops Allowed
Simple Graph	Undirected	NO	NO
Multigraph	Undirected	Yes	NO
Pseudo Graph	Undirected	Yes	Yes
Simple Directed Graph	Directed	NO	NO
Directed Multigraph	Directed	Yes	Yes

Degree of a Vertex

★ Adjacent Vertices:

Two vertices u & v in an undirected graph G are called adjacent in G if u & v are endpoints of an edge of G.

Ex:



a & b are adjacent vertices

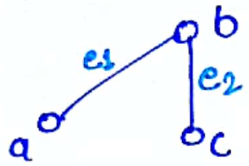
a & c are not adjacent vertices.

★ Incident Edge:

If an edge e is associated with $\{u, v\}$, then e is called incident with u & v.

∴ the vertices u & v are called endpoints of e.

Ex



e_1 is incident with a & b.

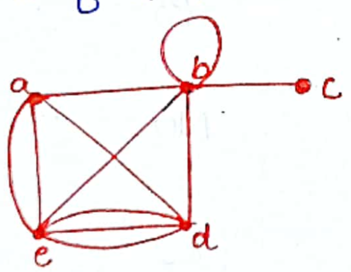
e_2 is incident with b & c.

★ Degree of a Vertex:

The degree of a vertex in an undirected graph is the no. of edges incident with it.

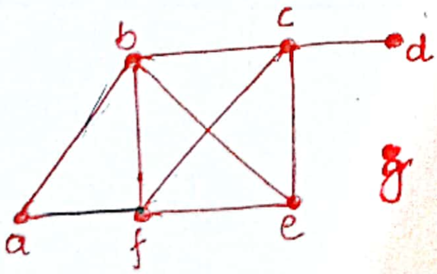
- A loop at a vertex contributes twice to the degree of that vertex.
- Degree of a vertex v is denoted by $\boxed{deg(v)}$.

Ex 1



$deg(a) = 4$
$deg(b) = 6$
$deg(c) = 1$
$deg(d) = 5$
$deg(e) = 6$

Ex 2

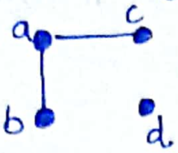


$deg(a) = 2$
$deg(b) = 4$
$deg(c) = 4$
$deg(d) = 1$
$deg(e) = \cancel{4}3$
$deg(f) = 4$

$deg(g) = 0$

★ Isolated Vertex:

A vertex of degree zero is called isolated vertex.

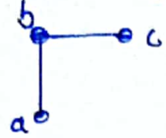
Ex:


deg(d) = 0

⇒ deg d is an isolated vertex.

★ Pendant Vertex:

A vertex of degree one is called pendant vertex.

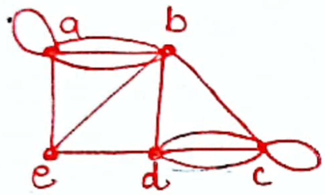
Ex:


deg(a) = 1
deg(b) = 2
deg(c) = 1

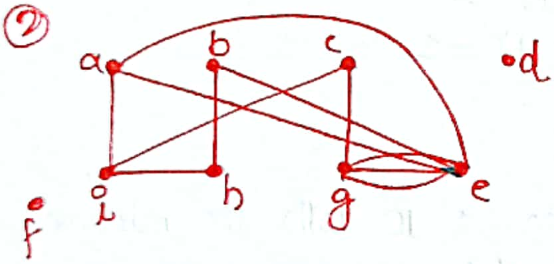
a & c are pendant vertices.

Question

Identify all isolated & pendant vertices. Also find degree of each vertex.

①


deg(a) = 6
deg(b) = 6
deg(c) = 6
deg(d) = 5
deg(e) = 3

②


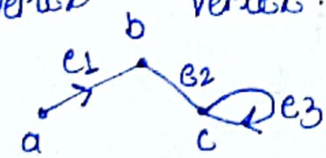
deg(a) = 3
deg(b) = 2
deg(c) = 2
deg(d) = 0
deg(i) = 3
deg(e) = 6
deg(f) = 0
deg(g) = 4
deg(h) = 2

★ Indegree & Outdegree ★

★ Initial Vertex & Terminal Vertex:

(u, v) → Edge of directed graph G.

u → Initial Vertex
v → Terminal Vertex.

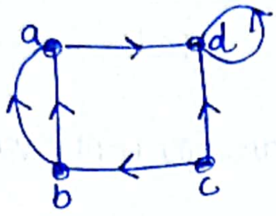


edge	IV	TV
$e_1 = (a,b)$	a	b
$e_2 = (b,c)$	b	c
$e_3 = (c,c)$	c	c

☆ Indegree of a vertex

The no. of edges with v as their terminal vertex is in-degree of v.

- It is denoted by $deg^-(v)$

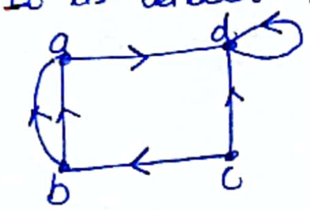


$deg^-(a) = 2$
$deg^-(b) = 1$
$deg^-(c) = 0$
$deg^-(d) = 3$

★ Out-Degree of a vertex:

The no. of edges with v as their initial vertex is out-degree of v.

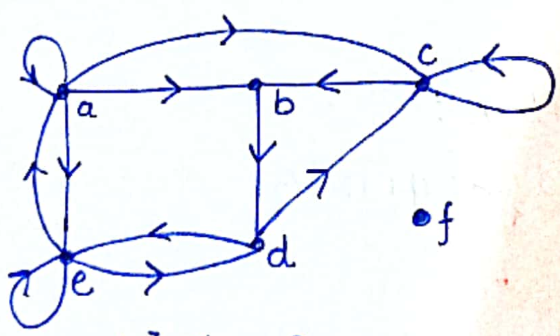
- It is denoted by $deg^+(v)$.



$deg^+(a) = 1$
$deg^+(b) = 2$
$deg^+(c) = 2$
$deg^+(d) = 1$

NOTE: A loop at a vertex contributes 1 to both the indegree and the outdegree of the vertex.

Ex:



Find Indegree & Outdegree of each vertex

$deg^-(a) = 2$	$deg^+(a) = 4$
$deg^-(b) = 2$	$deg^+(b) = 1$
$deg^-(c) = 3$	$deg^+(c) = 2$
$deg^-(d) = 2$	$deg^+(d) = 2$
$deg^-(e) = 3$	$deg^+(e) = 3$
$deg^-(f) = 0$	$deg^+(f) = 0$

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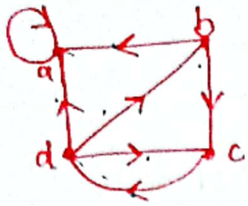
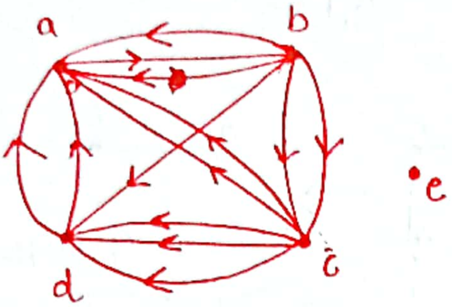
Q1

a) 

	In	Out
a	3	1
b	1	2
c	2	1
d	1	3

e=4
V=7

Q2 

	In	Out
a	6	1
b	1	5
c	2	5
d	4	2
e	0	0

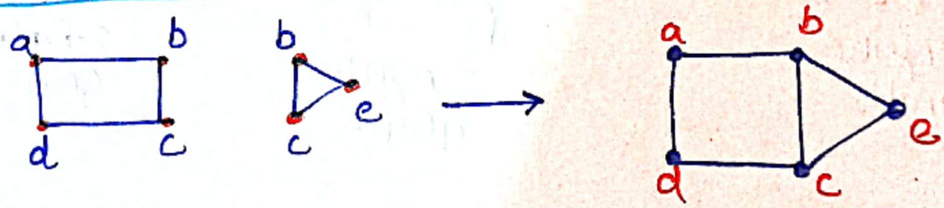
e= 13
V= ~~4~~ 5

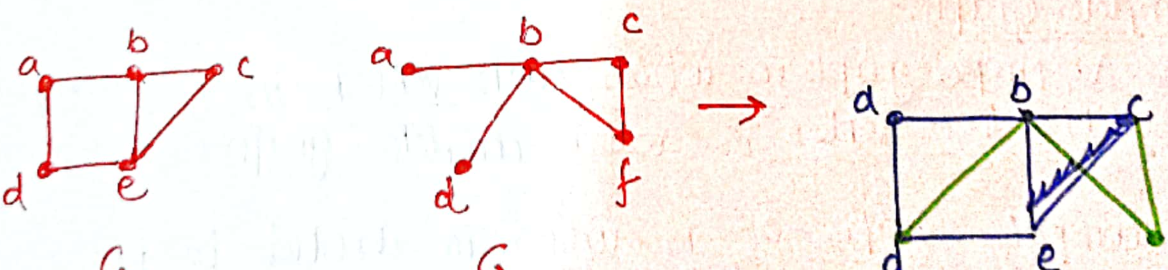
Union & Intersection of two graphs

Union of Two Graphs

$G_1 = (V_1, E_1)$
$G_2 = (V_2, E_2)$

$$\boxed{G_1 \cup G_2 = (V_1 \cup V_2 \;,\; E_1 \cup E_2)}$$

Ex: 

Ex 

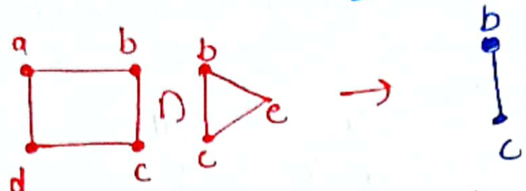
G_1 G_2

Intersection of two simple graphs

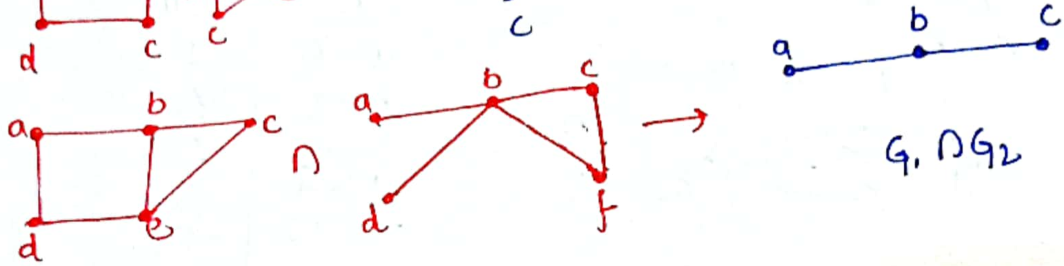
$G_1 = (V_1, E_1)$ $\quad$ $G_2 = (V_2, E_2)$

$\Rightarrow \boxed{G_1 \cap G_2 = (V_1 \cap V_2 \,,\, E_1 \cap E_2)}$

Ex.



$G_1 \cap G_2$



$G_1 \cap G_2$

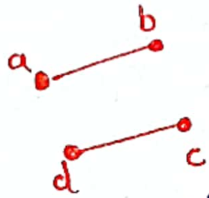
Regular Graph,

$\Rightarrow$ A graph in which degree of each vertex is same is called regular graph.

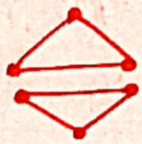
- If the degree of each vertex is k, then it is called k-regular graph.

Ex:

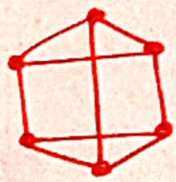
0-regular graph



1 regular graph



2-Regular Graph



3-Regular Graph

Complete Graph

A simple graph in which each vertex is connected to every other vertex is called complete graph.

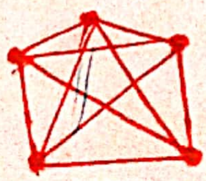
- A complete graph of n-vertices is denoted by K_n.

K_1

K_2

K_3

K_4



K_5

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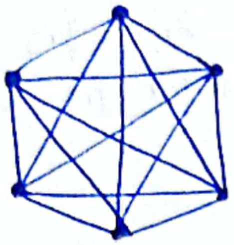
- It is the simple graph that contains exactly one edge b/w each pair of distinct vertices.

K_6

- In K_n, degree of each vertex is ~~K~~ $(n-1)$.

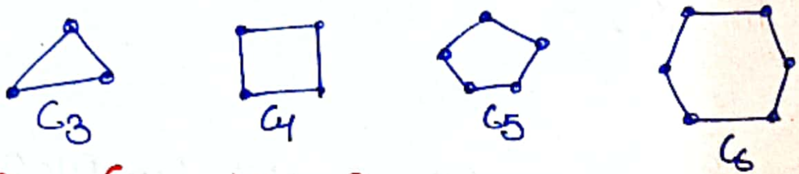


K6

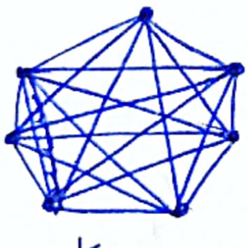
* Cycle Graph.

* A graph that consists of a single cycle is called cycle graph.
* It is denoted by C_n, (n - no. of vertices), $n \geq 3$.
* In C_n, $\boxed{\text{No. of Edges} = \text{No. of vertices}}$

Ex



Ques C_7, K_7 & 4-regular Graph.



K7

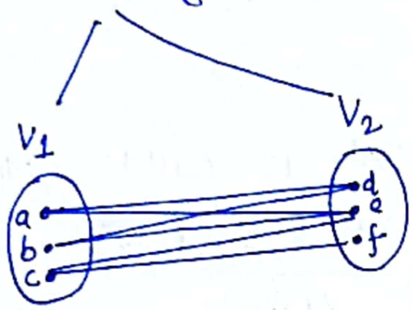
C7

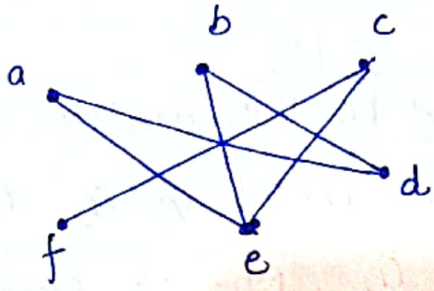
Bipartite & Complete Bipartite Graphs

★ Bipartite Graph

A simple graph is bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that each edge of graph connects a vertex of V_1 to a vertex in V_2.

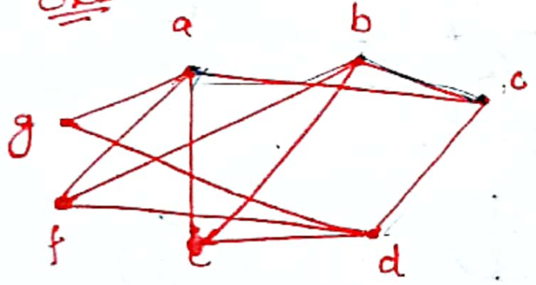
$V = \{a, b, c, d, e, f\}$



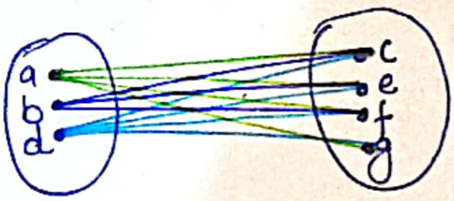


Note : No edge connects either two vertices in V_1 or two vertices in V_2.

Ex 1



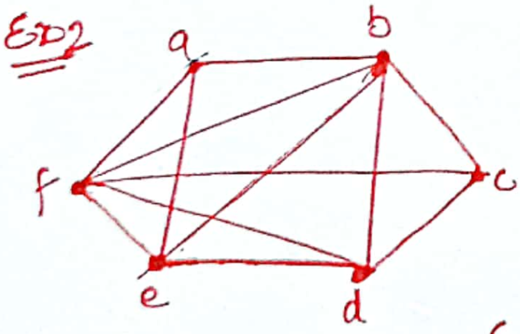
Is this graph bipartite?

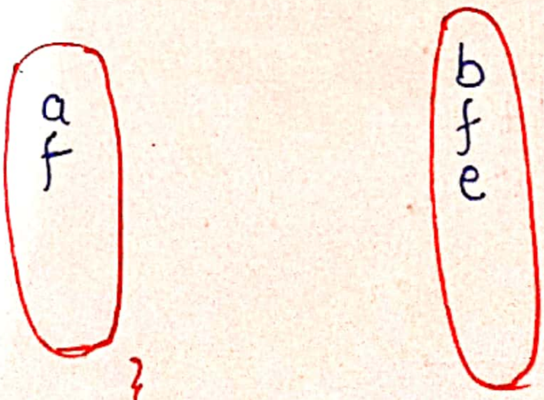


$V_1 = \{a, b, d\}$

$V_2 = \{c, e, f, g\}$

Ex 2





$V_1 = \{$ $\}$

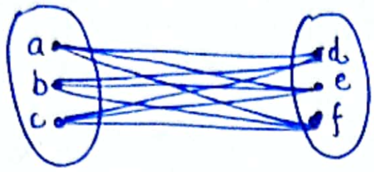
$V_2 = \{$ Not a bipartite $\}$

{ ∵ its vertex set cannot be partitioned into two disjoint vertex sets }

* Complete Bipartite Graph

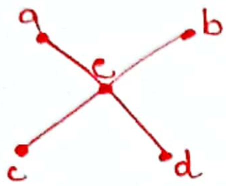
A bipartite graph, in which every vertex of one set is connected with every vertex of another set.

It is denoted by $K_{m,n}$.

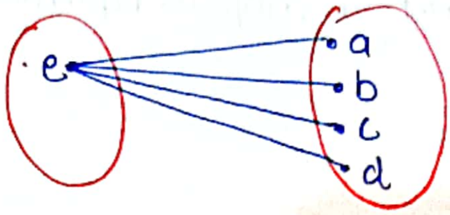


$K_{3,3}$

Ex



Is the graph complete bipartite?



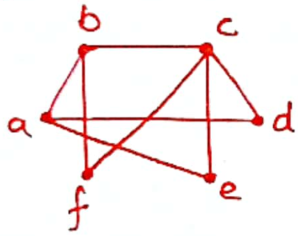
$V_1 = \{e\}$

$V_2 = \{a, b, c, d\}$

$K_{1,4}$

∴ Complete Bipartite

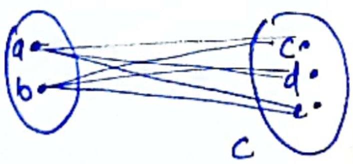
Ex

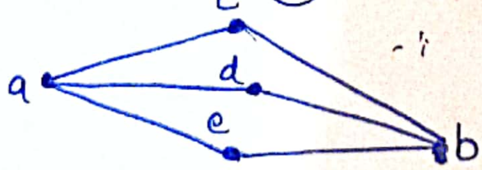


Not a bipartite {f creating problem}

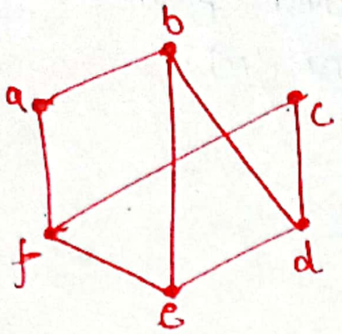
∴ Not a complete Bipartite.

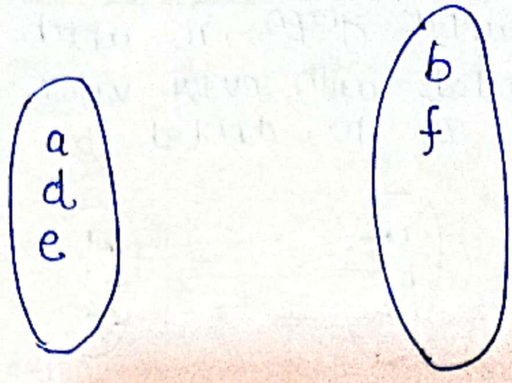
Q Draw $K_{2,3}$.





Ex Complete Bipartite?

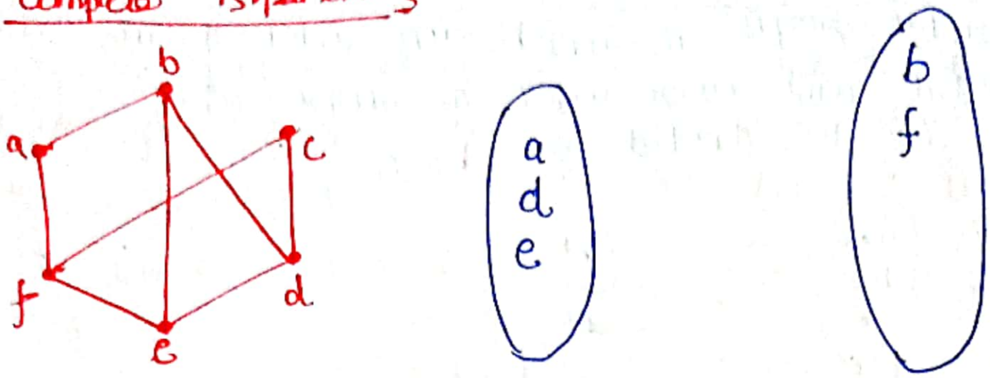




Not a Bipartite.

∴ Not a Complete Bipartite.

Ex Complete Bipartite?



Not a Bipartite.

$\therefore$ Not a Complete Bipartite.

THE HANDSHAKING THEOREM

Let $G=(V,E)$ be an undirected graph with e edges.
Then

$$2e = \sum_{v \in V} \deg(v)$$

Example

How many edges are there in a graph with 10 vertices each of degree six?

Because the sum of degrees of the vertices is $6 \times 10 = 60$, it follows that $2e = 60$

Therefore, $e = 30$.

Theorem:

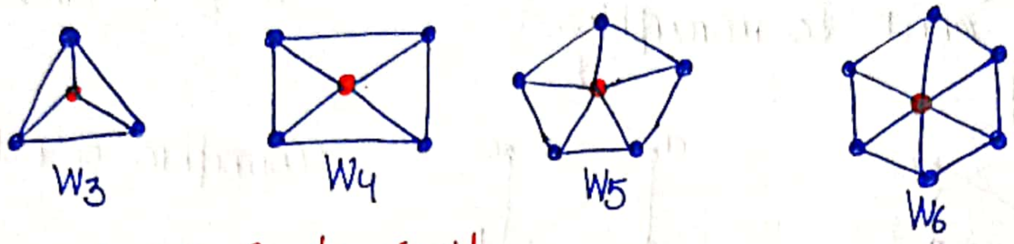
- Let $G=(V,E)$ be a graph with directed edges.
  Then,

$$\boxed{\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|}$$

The sum of the in-degrees and the sum of the outdegree of all vertices in a graph with directed edges are the same. Both of those sums are the number of edges in the graph.

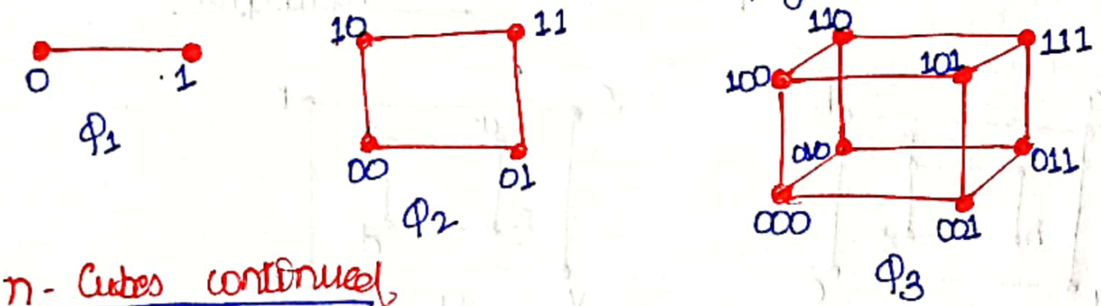
★ Wheels:

We obtain the wheel W_n, when we add an additional vertex to the cycle C_n, for $n>=3$, and connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed as follows:



Some Special Simple Graph

★ **n-Cubes:** The n dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, & Q_3 are displayed as follows:



n-Cubes continued

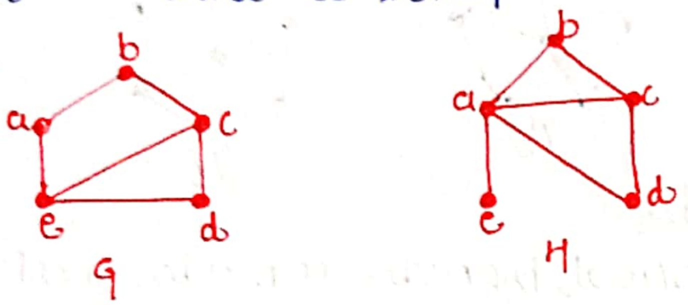
★ Note that you can construct the (n+1) cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n,

★ and adding edges connecting two vertices that have labels differing only in the first bit.

★ Eg. Q_3 is constructed from Q_2 by drawing two copies of Q_2 as the top and bottom faces of Q_3, adding 0 at the beginning of the label of each vertex in the bottom face and 1 at the beginning of the label of each vertex in the top face.

Isomorphism:

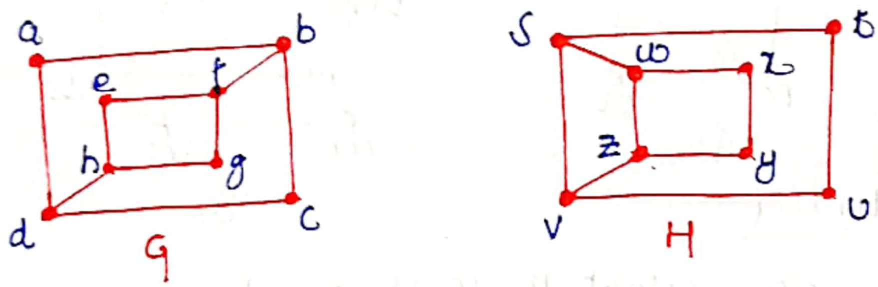
The no. of vertices, the no. of edges, and the number of vertices of each degree are all invariants under isomorphism. If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic.

Ex 

Isomorphic or Not ?

Sol - Both G and H have 5 vertices and 6 edges. However, H has a vertex of degree one, namely e, whereas G has no vertices of degree one.

It follows that G and H are not isomorphic.

Ex 

* Eight vertices, and 10 Edges.
* Four, 2-degree vertex
* four, 3-degree vertex.

However, G and H are not isomorphic.

* to see this, note that because $deg(a) = 2$ in G, a must correspond to either t, u, x or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.