INDIAN INSTITUTE OF TECHNOLOGY INDORE

MA 203 Complex Analysis and Differential Equations-II

Autumn Semester

Assignment − CA 5

1. Find the zeros and their orders for the following functions:
   (a) $z^2(e^{z^2} - 1)$ Ans: $z = 0$, order 4
   (b) $z^2 + 9$ Ans:$z = \pm 3i$ are zeros of order 1
   (c) $\frac{z^2+9}{z^4}$ Ans:$z = \pm 3i$ are zeros of order 1
   (d) $z \sin z$ $z = 0$ (order 2), $z = k\pi$ (order 1)
2. Let $f(z)$ be analytic function on $N(0;1) = \{z \in \mathbb{C} : |z| < 1\}$ and $f(x) = 0$ for all $x \in (0,1)$. Then show that $f(z) = 0$ for all $z \in N(0;1)$.
3. Show that there does not exists any analytic function $f(z)$ on $N(0;1)$ such that $f(x) = |x|^3$ for $x \in (-1,1)$.
4. Let D be a bounded domain that is enclosed by the closed contour C. Let f be analytic in D and continuous on C. Let $U = \text{Re } f$. Then show that U cannot attain its maximum value in D, unless f is a constant function.
5. Show that a function u which is harmonic function in a simply connected bounded domain cannot have a maximum in the domain.
6. At what point in the region $|z - z_0| \leq 1$ does $|e^z|$ attain its maximum value? Find this maximum value. Ans: At $z = z_0 + 1$, maximum value $= e^{Re(z_0)+1}$
7. Let $f(z)$ be analytic at z_0 and $f(z_0) \neq 0$. Then show that
   $$g(z) = \frac{f(z)}{(z - z_0)^m}$$
   has a pole of order m at z_0.
8. Let z_0 be a pole of order m. Then
   $$\lim_{z \to z_0} (z - z_0)^k f(z) = \begin{cases} l, & k = m, \\ 0, & k > m, \\ \infty, & k < m. \end{cases}$$
   for some $l \neq 0$.
9. Which of the following singularities are removable/pole.
   (a) $\frac{\sin z}{z^2 - \pi^2}$, $z = \pi$
   (b) $\frac{\sin z}{(z-\pi)^2}$, $z = \pi$
10. Discuss the singularity of the functions $\frac{1}{\sin z}$ and $\frac{1}{\sin \frac{1}{z}}$ at $z = 0$.
11. Discuss the singularities of the following functions:
    (a) $\frac{1}{z - z^3}$ Ans: $z = 0, \pm 1$, poles of order 1
    (b) $\frac{z^4}{1+z^4}$ Ans: $z = \frac{1 \pm i}{\sqrt{2}}, \frac{-1 \pm i}{\sqrt{2}}$, poles of order 1

(c) $\frac{z^5}{(1-z)^2}$ Ans: $z = 1$, poles of order 2

(d) $\frac{e^z}{1+z^2}$ Ans: $z = i, -1$, poles of order 1

(e) $e^{-\frac{1}{z^2}}$ Ans: $z = 0$, essential

(f) $\frac{\cos z}{z^2}$ $z = 0$ pole of order 2

(g) $\frac{\sin z}{z^2}$ $z = 0$ pole of order 1

(h) $\frac{\sin z}{z}$ $z = 0$, removal

(i) $\tan z$ $z = (2k+1)\pi/2$, pole of order 1

(j) $\sin \frac{1}{z}$ $z = 1$, essential

12. Prove that if the function f has zero of order k, $k \geq 2$ at $z = z_0$, then f' has zero of the order $k-1$ at $z = z_0$.

13. Let D_1 and D_2 be two domains such that $D_1 \subseteq D_2$. Let f be an analytic function on D_2 such that $f(z) = K$ for all $z \in D_1$. Then show that $f(z) = K$ for all $z \in D_2$.

14. Let f be a non-constant analytic function in $|z| < R$ and $M(r) = \max_{|z|=r} |f(z)|$. Then show that $M(r)$ is a strictly increasing function of r in $[0, R]$.

15. Let D be a bounded domain that is enclosed by the closed contour C. Let f be a non-constant analytic function in D and continuous on C. Suppose $|f(z)| = \textit{constant}$ on C. Then f has at least one zero in D.

16. Is there a polynomial $P(z)$ such that $P(z)e^{\frac{1}{z}}$ is an entire function? Justify your answer!

17. Prove that if the function f has pole of the order k at $z = z_0$, then f' has pole of the order $k+1$ at $z = z_0$.

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