CS 201: Discrete Mathematical Structures: Topics 5,6,7,8

- Topic 5: Functions *Grimaldi*: Chapter 5, *Rosen*: 2.3, 2.5 (and to some extent Sections 6.1, 6.2)
- Topic 6: Graphs and Trees *Grimaldi*: Chapters 11-13, *Rosen*: Chapters 10, 11
- Topic 7: Recursion *Grimaldi*: Chapters 9-10, *Rosen:* chapters 5, 8
- Topic 8: Efficiency of Algorithms *Rosen*: chapter 3 (pages 201-250)

Topic 5: Functions

- 5.1 Definitions and Properties
- 5.2 Inverse and Composition
- 5.3 Pigeonhole Principle

5.1 Definitions and Properties

- Introduction
- One-to-One Functions
- Onto Functions
- Lot of Examples
- References:
  - *Grimaldi* : *Chapter 5, Rosen* : 2.3

Introduction

Definition 1:

Let A and B be sets.

A **function** f from A to B is an assignment of **exactly** one element of B to each element of A.

We write $\boldsymbol{f(a) = b}$ if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write $\boldsymbol{f: A \to B}$.

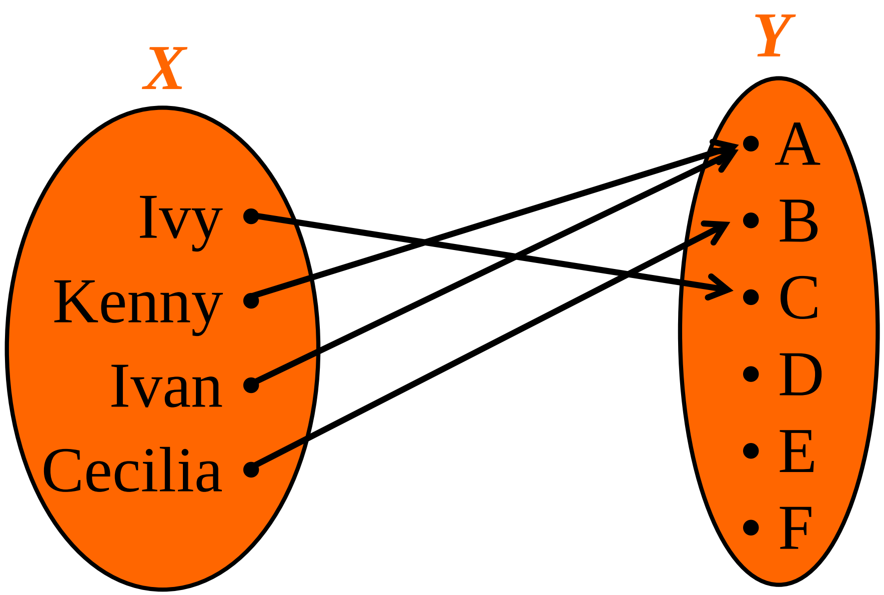
Introduction

<u>Example 1:</u> Function that assigns a grade to a student in our discrete mathematics class.

X = {Ivy, Kenny, Ivan, Cecilia}, Y = {A, B, C, D, E, F}

$f: X \rightarrow Y$, f(Ivy)=C, f(Kenny)=A, f(Ivan)=A, f(Cecilia)=B

Arrow Diagram of f:



Introduction

Definition 2:

If f is a function from A to B, we say that A is the **domain** of f and B is the **co-domain** of f.

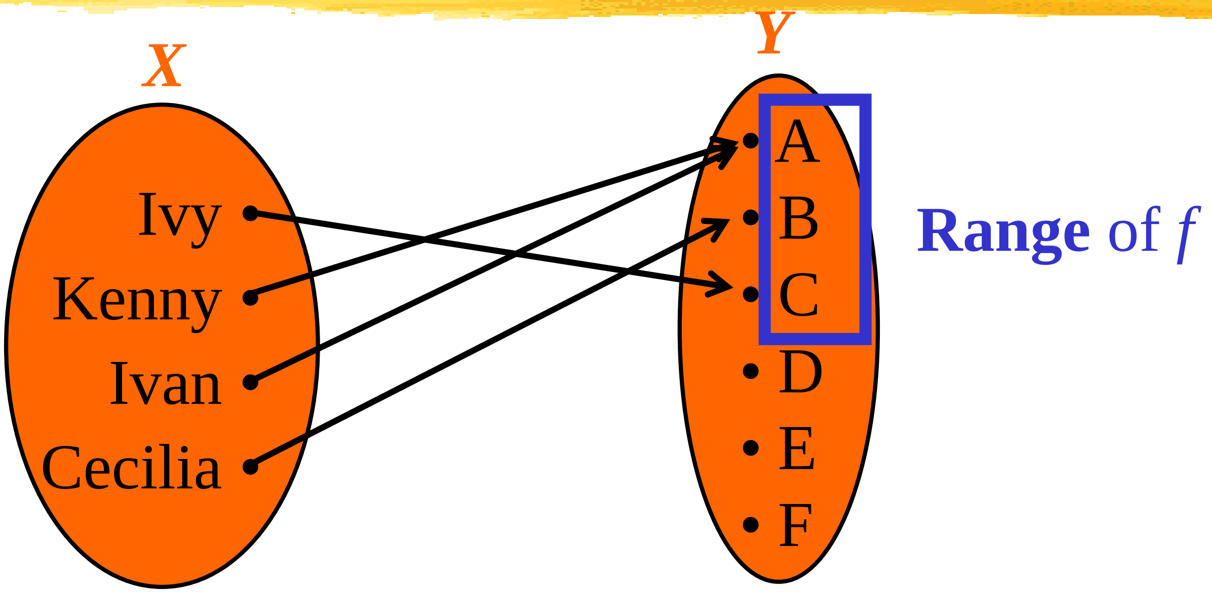
if $f(a) = b$, we say that b is the **image** of a and a is a **pre-image** of b.

The **range** of f is the set of all images of elements of A.

Also, if f is a function from A to B, we say f **maps** A to B.

Introduction

Example 1:



X = domain of f Y = co-domain of f

- f(Ivan) = A: A is the image of Ivan
- Ivan is a pre-image of A, Kenny is also a pre-image of A
- E has no pre-image
- Range(f) = {A, B, C}

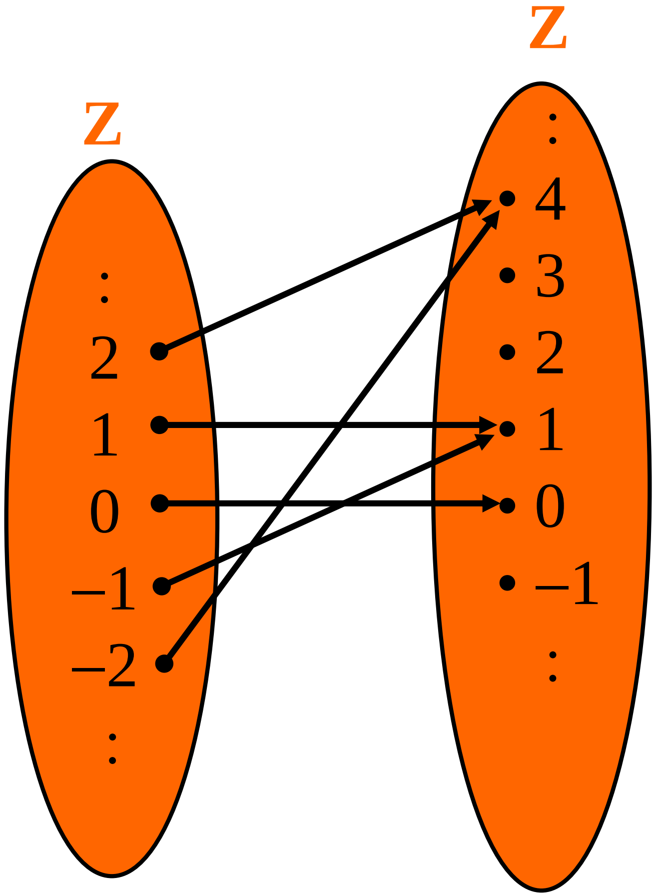
Introduction

Example 2:

Let f be the function from **Z** to **Z** that assigns the square of an integer to this integer,

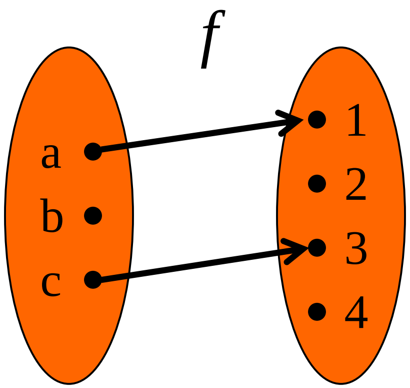
Then, f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = x^2$

Range(f) = {0, 1, 4, 9, 16, 25, ….}

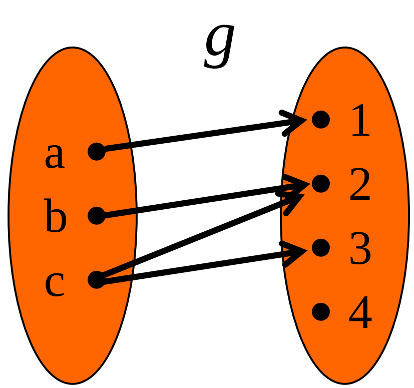


Introduction

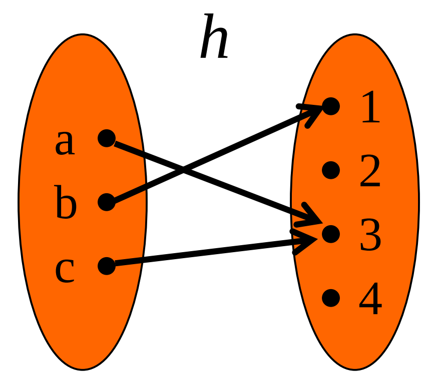
Functions and Nonfunctions: Which of the arrow diagrams defines a function from X = {a,b,c} to Y = {1,2,3,4} ?



No,
b has no image



No,
c has two images



Yes,
each element of X has exactly one image

Introduction

Definition 3:

Let $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ be functions from A to $\boldsymbol{R}$

Then $\boldsymbol{f_1} + \boldsymbol{f_2}$ and $\boldsymbol{f_1} \cdot \boldsymbol{f_2}$ are also functions from A to $\boldsymbol{R}$ defined by

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$,

$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$

Introduction

Example 3:

Let $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ be functions from $\boldsymbol{R}$ to $\boldsymbol{R}$ such that

$f_1(x) = x^2$ and $f_2(x) = x - x^2$;

What are the functions $f_1 + f_2$ and $f_1 \cdot f_2$?

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$

$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = x^2 \cdot (x - x^2) = x^3 - x^4$

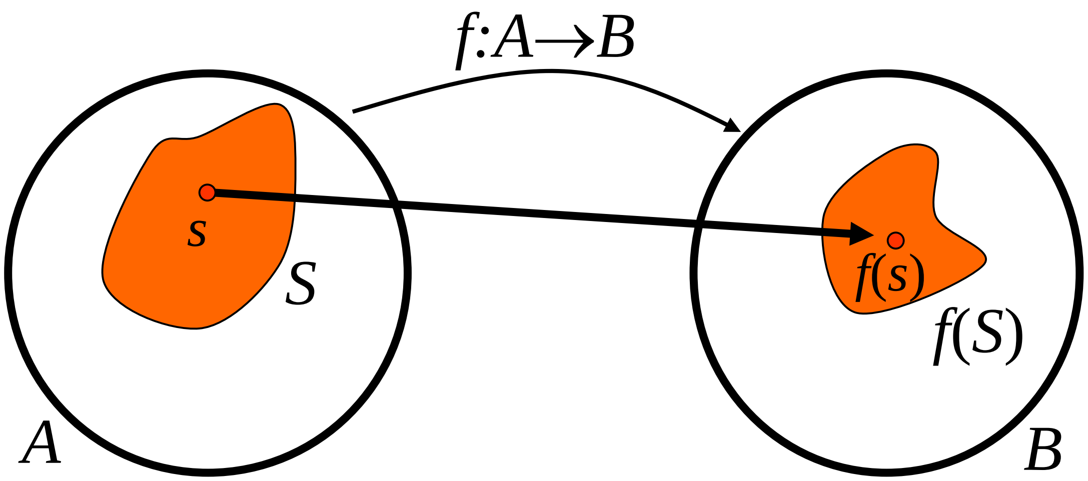
Introduction

Definition 4:

Let $\boldsymbol{f}$ be a functions from A to B and $S \subseteq A$.

The **image of S** is the subset of B that consists of the images of the elements of S.

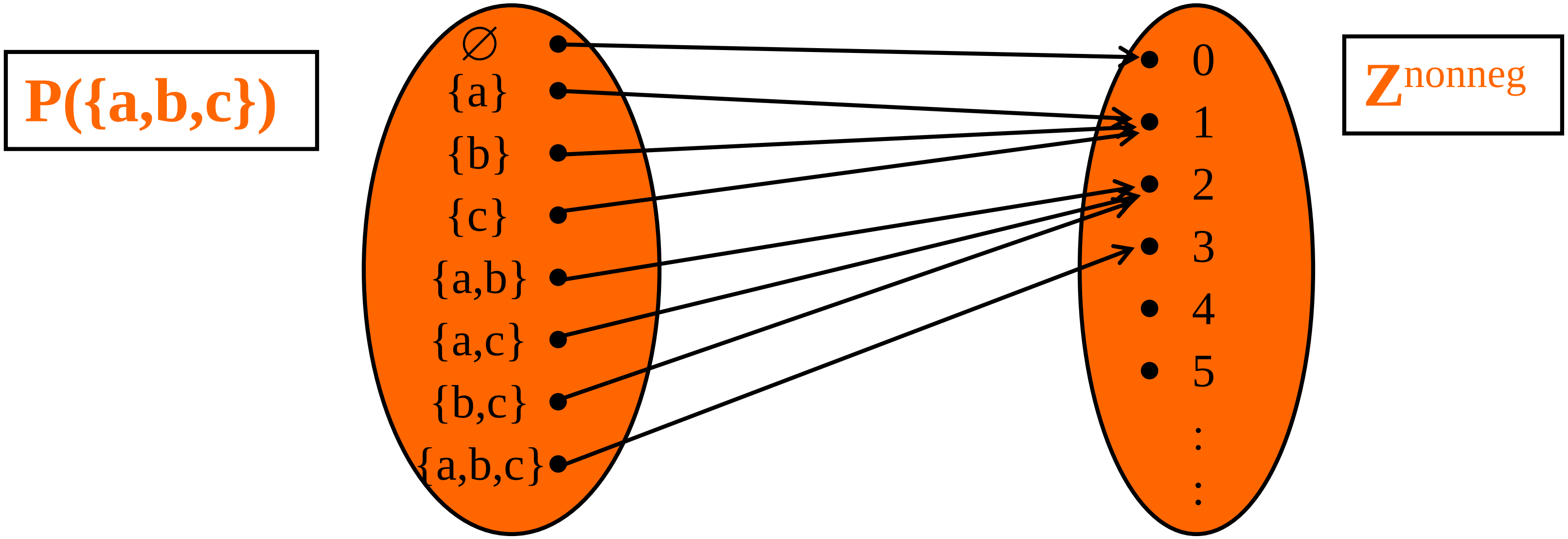
We denote the image of S by $f(S)$: $\boldsymbol{f(S) = \{f(s) \mid s \in S\}}$



Introduction

Example 4:

Define a function F:P({a,b,c}) $\rightarrow$ $\mathbf{Z}^{\text{nonneg}}$ as follows:
for each $X \in P(\{a,b,c\})$, $F(X) = |X|$ (number of elements in X)



Let $S = \{\emptyset,\{a\},\{b\},\{a,b,c\}\}$. What is $F(S)$? $\boldsymbol{F(S) = \{0,1,3\}}$

One-to-One Functions

<u>Definition 5:</u> A function f is **one-to-one** (or **injective**), if and only if $f(x) = f(y)$ implies $x = y$ for all x and y in the domain of f.

In other words:
"*All elements in the domain of f have different images*" or
"*No element in the co-domain of f has two (or more) pre-images*"
$f{:}A \rightarrow B$ is **one-to-one** $\Leftrightarrow \forall x,y \in A\ (f(x)=f(y) \rightarrow x=y)$

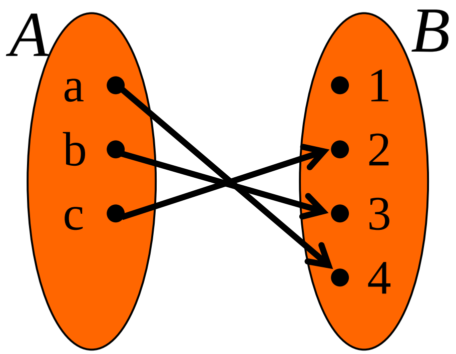
Contrapositive:
$f{:}A \rightarrow B$ is **one-to-one** $\Leftrightarrow \forall x,y \in A\ (x \neq y \rightarrow f(x) \neq f(y))$

Negation: "*There are two different elements in the domain of f with the same image*"
$f{:}A \rightarrow B$ is **<u>not</u> one-to-one** $\Leftrightarrow \exists x,y \in A\ (f(x)=f(y) \wedge x \neq y)$

One-to-One Functions

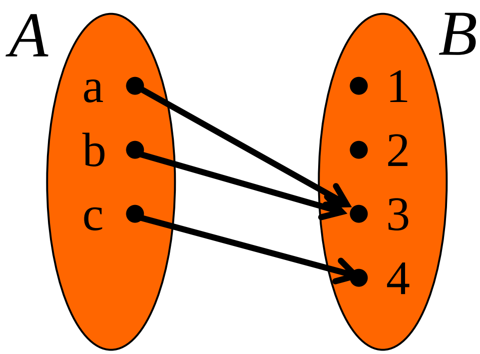
Example 5:

Which of the following functions is one-to-one?



one-to-one

(all elements in *A* have a different image)



not one-to-one

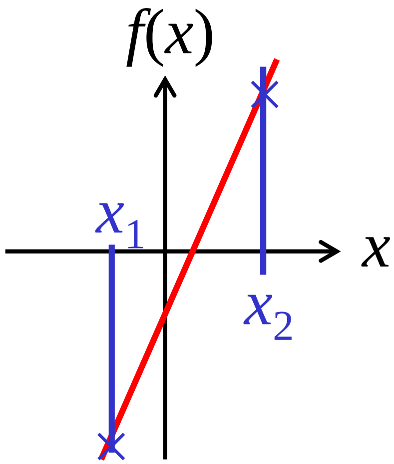
(*a* and *b* have the same image)

One-to-One Functions

Example 6:

Is f: **R** $\rightarrow$ **R**, $f(x) = 4x-1$ one-to-one?

(“Do all elements in **R** have a different image ?”)



Yes !

To show: $\forall x_1, x_2 \in \mathbf{R}\ (f(x_1)=f(x_2) \rightarrow x_1= x_2)$

Suppose arbitrary $x_1, x_2 \in$ R with $f(x_1) = f(x_2)$.

Then $4x_1-1 = 4x_2-1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1=x_2$

One-to-One Functions

Example 7:

Is g: $\mathbf{R} \to \mathbf{R}$, $g(x) = x^2$ one-to-one?

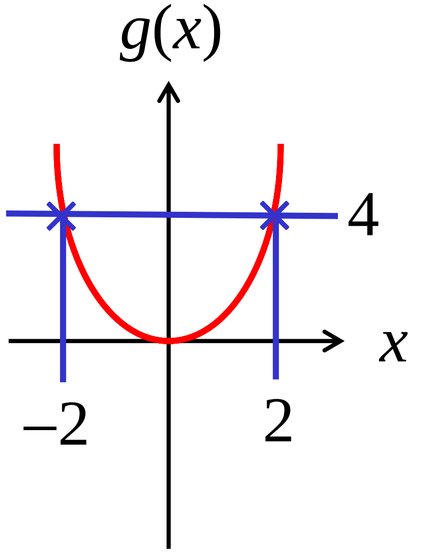
("Do all elements in **R** have a different image ?")

No !

To show: $\exists\, x_1, x_2 \in \mathbf{R}\ (g(x_1)=g(x_2) \land x_1 \neq x_2)$

Take $x_1 = 2$ and $x_2 = -2$.

Then $g(x_1) = 2^2 = 4 = -2^2 = g(x_2)$ and $x_1 \neq x_2$



Onto Functions

Definition 6:

A function f from X to Y is **onto** (or **surjective**), if and only if for every element $y \in Y$ there is an element $x \in X$ with $f(x)=y$.

In other words:

"All elements in the co-domain of f have a pre-image"

$f:X \to Y$ is **onto** $\Leftrightarrow \forall y \in Y\ \exists x \in X$ such that $f(x) = y$

Negation:

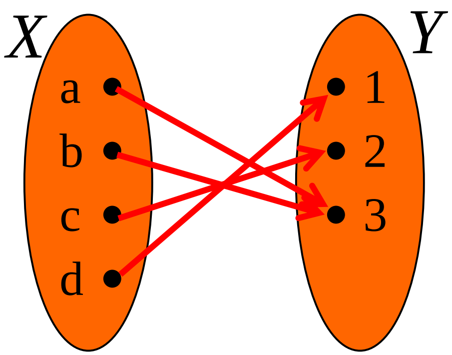
"There is an element in the co-domain of f which has no pre-image"

$f:X \to Y$ is **not onto** $\Leftrightarrow \exists y \in Y$ such that $\forall x \in X\ f(x) \neq y$

Onto Functions

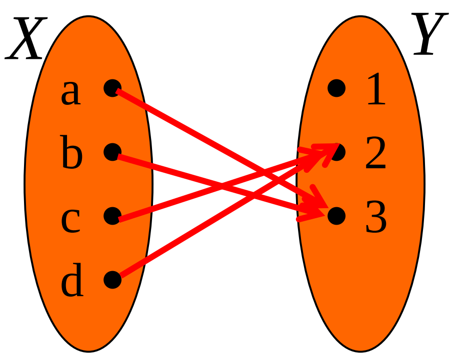
Example 8:

Which of the following functions is onto?



onto

(all elements in Y have a pre-image)



not onto

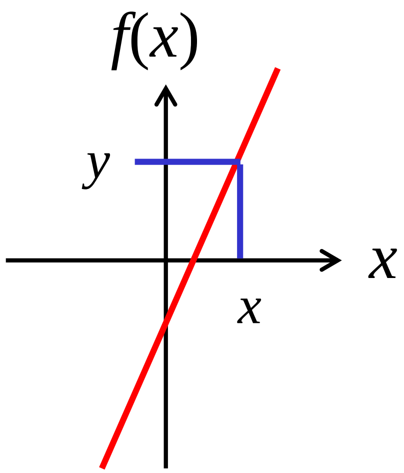
(1 has no pre-image)

Onto Functions

<u>Example 9:</u>

Is f:$\mathbf{R}\rightarrow\mathbf{R}$, $f(x)=4x-1$ onto ?

("Do all elements in **R** have a pre-image ?")



Yes !

To show: $\forall y\in\mathbf{R}\ \exists x\in\mathbf{R}$ such that $f(x) = y$

Let $y\in\mathbf{R}$.

[Calculate x with $f(x)=y$: $y = 4x-1 \Rightarrow (y+1)/4 = x$]

Set $x = (y+1)/4$

Then $f(x) = 4x-1 = 4((y+1)/4) -1 = (y+1) -1 = y$

Onto Functions

Example 10:

Is g:$\mathbf{R}\to\mathbf{R}$, $g(x)=x^2$ onto ?

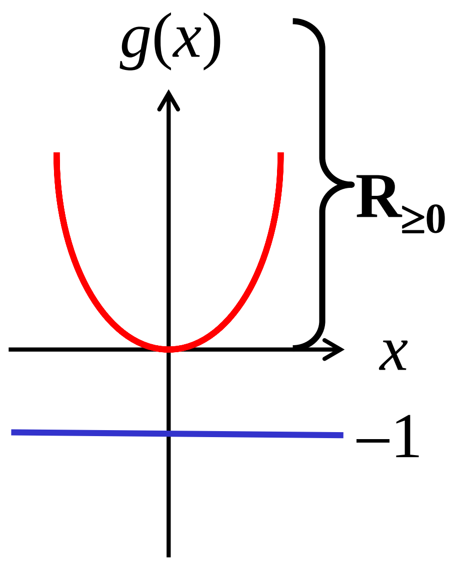
("Do all elements in **R** have a pre-image ?")

No !

To show: $\exists y \in \mathbf{R}$ such that $\forall x \in \mathbf{R}$ $g(x) \neq y$

Take $y = -1$

Then any $x \in \mathbf{R}$ holds $g(x) = x^2 \neq -1 = y$



But g:$\mathbf{R}\to\mathbf{R}_{\geq 0}$, $g(x)=x^2$,(where $\mathbf{R}_{\geq 0}$ denotes the set of non-negative real numbers) is onto !

One-to-one and Onto

Example 11:

Is f:$\mathbf{Z}^{\text{nonneg}}\times\mathbf{Z}^{\text{nonneg}} \rightarrow \mathbf{Z}^{\text{nonneg}}$, $f(l,m)=l + m$ **onto** or **one-to-one** or **both** ?

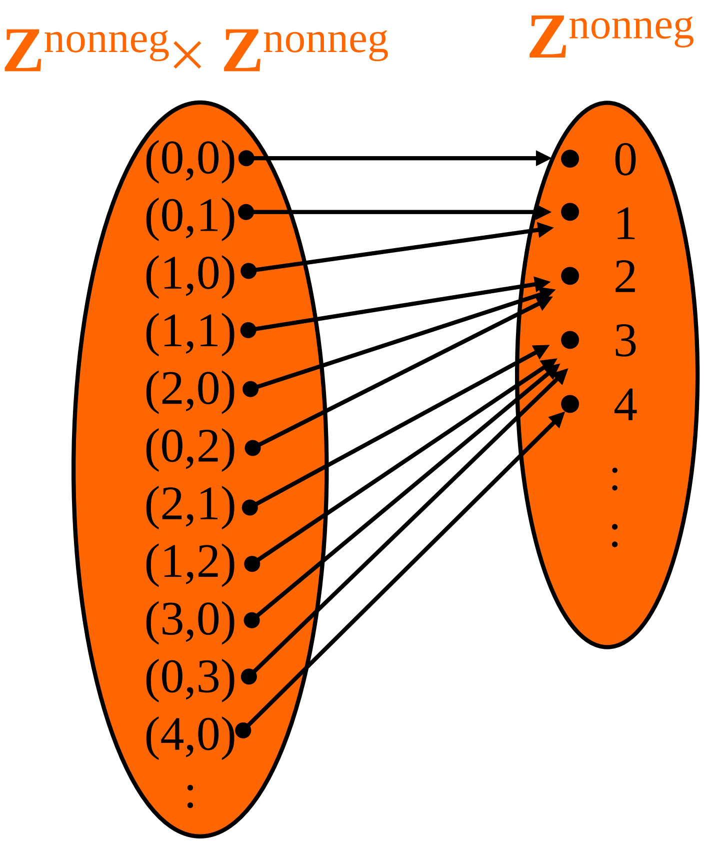
Onto: "Do all elements in $\mathbf{Z}^{\text{nonneg}}$ have a pre-image ?" **Yes !**

Let $n\in \mathbf{Z}^{\text{nonneg}}$.

Then $f((0,n))= 0+n = n$

One-to-one: Do all elements in $\mathbf{Z}^{\text{nonneg}}$ have a different image ? **No !**

$f((1,1))= 1+1 = 2 = 2+0 = f((2,0))$



One-to-one and Onto

<u>Example 12:</u> Let f:$\mathbf{R}\rightarrow\mathbf{R}$, and g:$\mathbf{R}\rightarrow\mathbf{R}$ be functions.

If f and g are both one-to-one, is $f+g$ also one-to-one?

If f and g are both onto, is $f+g$ also onto?

No!

Give a counter-example: $f(x) = x$ and $g(x) = -x$ for all $x\in\mathbf{R}$.

Then f and g are one-to-one and onto.

But $(f+g)(x) = x + (-x) = 0$ for all $x\in\mathbf{R}$.

Thus, $f+g$ is neither one-to-one nor onto.

Tutorial problems and Additional Reading

- Grimaldi, Chapter 5
  - Section 5.2: Functions: Plain and one-to-one
    - Exercises 5.2 (28 problems) page no’s 258-260
  - Section 5.3: Onto Functions: Stirling Numbers of Second kind
    - Exercises 5.3 (19 problems) page no’s 265-267
  - Section 5.4: Special Functions (monary, binary …)
    - Exercises 5.4 (14 problems) page no’s 272-273

Tutorial problems and Additional Reading

- Rosen (8^{th} Edition) Chapter 2
  - Section 2.3: Functions .. page no's 147-164
    - Exercises 2.3 (82 problems) page no's 161-164
  - Section 2.5: Cardinality of Sets .. page no's 179-187
    - Exercises 2.5 (41 problems) page no's 186-187

Summary

- Introduction
- One-to-One Functions: Section 5.2 of Grimaldi (pages 252-260)
- Onto Functions Section 5.3, 5.4 of Grimaldi (pages 260-273)
- References: Grimaldi (5th Edition) Chapter 5; *Rosen* (8th Edition) : Chapter 2 (sections 2.3, 2.5)