Reference: Advanced Engineering Mathematics (10^{th} Edition) by Erwin Kreyszig

Chapter 5: Series Solutions of ODEs. Special functions.

Linear ODEs with constant coefficients can be solved by algebraic methods, and ~~that~~ their solutions are elementary functions known from calculus. For ODEs with variable coefficients the situation is more complicated, and their solutions may be nonelementary functions. Legendre's, Bessel's and the hypergeometric equations are important ODEs of this kind.

We consider the two standard methods for solving ODEs like Legendre's, Bessel's equations.etc.

1) Power Series method
2) Frobenius Method:-

$\boxed{\text{Power Series Method}}$: The power series method is the standard method for solving linear ODEs with variable coefficients. It gives solutions in the form of power series.

A power series (in power of $x-x_0$) is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \quad (1)$$

Here x is a variable. $a_0, a_1, \ldots$, are constants, called the coefficients of the series. x_0 is a constant, called the centre of the series. In particular, if $x_0=0$, we obtain a power series in powers of x

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots \quad (2)$$

We shall assume that all variables and constants are real.

"Power Series" usually refers to a series of the form (1) or (2) but does not include series of negative or fractional powers of x.

02/11/2021, (Hy2)

$\boxed{\text{Example}}$:- Solve $y'-y=0$ using power series method.

$\boxed{\text{Solution}}$:- We consider the solution of the power series form

$$y=\sum_{m=0}^{\infty} a_m x^m = a_0+a_1x+a_2x^2+\cdots$$

We obtain the series by termwise differentian

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sum_{m=0}^{\infty} a_m x^m\right)=\sum_{m=0}^{\infty}\frac{d}{dx}(a_m x^m)$$

$$=\sum_{m=0}^{\infty} m a_m x^{m-1} = a_1+2a_2x+3a_3x^2+\cdots$$

Substitute the values of y & $\frac{dy}{dx}$ in the ODE, we have

$$(a_1+2a_2x+3a_3x^2+\cdots)-(a_0+a_1x+a_2x^2+\cdots)=0$$

Collecting the like powers of x, we have

$$(a_1-a_0)+(2a_2-a_1)x+(3a_3-a_2)x^2+\cdots+(na_n-a_{n-1})x^{n-1}+\cdots=0$$

Equating the coefficient of each power of x to zero, we have,

$$a_1-a_0=0,\ 2a_2-a_1=0,\ 3a_3-a_2=0,\ \cdots,\ na_n-a_{n-1}=0$$

$$a_1=a_0,\ a_2=\frac{a_1}{2}=\frac{a_0}{2},\ a_3=\frac{a_2}{3}=\frac{a_0}{3!},\ a_4=\frac{a_3}{4}=\frac{a_0}{4!},\ \cdots$$

Substituting these values of coefficients, the series solution is

$$y=a_0+a_0x+\frac{a_0}{2!}x^2+\frac{a_0}{3!}x^3+\cdots=a_0e^x.$$

x ——— x ——— x ——— x

$\boxed{\text{Example}}$:- Solve the ODE

$$(1-x^2)y''-2xy'+2y=0$$

$\boxed{\text{Solution}}$:- Consider the solution in the power series form

$$y=\sum_{m=0}^{\infty} a_m x^m = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$$

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+4a_4x^3+\cdots=\sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$-2xy'=-2[a_1x+2a_2x^2+3a_3x^3+4a_4x^4+\cdots]=-2\sum_{m=1}^{\infty} m a_m x^m$$

02/11/2021 (Pg 3)

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

$$x^2\frac{d^2y}{dx^2} = 2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + 20a_5 x^5 + \cdots$$

$$= \sum_{m=2}^{\infty} m(m-1)a_m x^m$$

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$$y'' - x^2y'' - 2xy' + 2y = 0$$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + 2\sum_{m=0}^{\infty} a_m x^m = 0$$

$\uparrow$

Replace m by m+2 ~~Replace m by m+1~~

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \boxed{\cancel{\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^{m+2}}}$$
$$-\sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + 2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\left[2a_2 + 6a_3x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m\right] - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \left[2a_1x + 2\sum_{m=2}^{\infty} m a_m x^m\right]$$
$$+2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$(2a_0+2a_2) + (6a_3 - 2a_1 + 2a_1)x + \sum_{m=2}^{\infty} x^m\left[(m+1)(m+2)a_{m+2} - m(m-1)a_m - 2ma_m + 2a_m\right] = 0$$

$$2(a_0+a_2) + 6a_3x + \sum_{m=2}^{\infty}\left[(m+1)(m+2)a_{m+2} - m(m+1)a_m + 2a_m\right]x^m = 0$$

$$a_2 = -a_0,\ a_3 = 0,\ a_{m+2} = \frac{[m(m+1)-2]a_m}{(m+1)(m+2)} \quad \forall\ m \geq 2. \quad -(*)$$

$$a_5 = a_7 = \cdots = 0 \Rightarrow a_{2m+1} = 0 \quad \forall\ m \geq 1.$$

Put $m=2$ in $(*)$, we have

$$a_4 = \frac{(2\times 3-2)a_2}{3\times 4} = \frac{4a_2}{3\times 4} = \frac{a_2}{3} = \frac{-a_0}{3}$$

Put $m=4$ in ⊛

$$a_6 = \frac{[4\times 5-2]a_4}{5\times 6} = \frac{18a_4}{5\times 6} = \frac{3a_4}{5} = -\frac{a_0}{5}$$

Put $m=6$ in ⊛

$$a_8 = \frac{[6\times 7-2]a_6}{7\times 8} = \frac{40}{7\times 8}\times\left(-\frac{a_0}{5}\right) = -\frac{a_0}{7}.$$

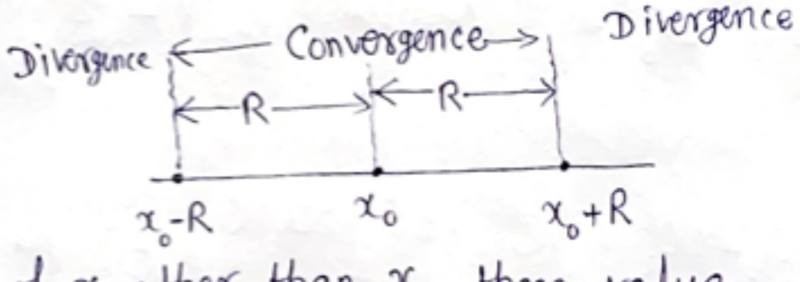
By induction, $a_{2m} = -\frac{a_0}{2m-1}$ $\forall$ $m \geq 1$.

Thus we have solution

$$y = a_1x + (a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots)$$
$$= a_1x + a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} \cdots\cdots\right)$$

where a_1 & a_0 are arbitrary constants. Hence, this is a general solution that consists of two solutions x & $1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} \cdots\cdots$

Radius of Convergence of power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$



If there are values of x other than x_0, ~~these value~~ for which the power series converges, these values form an interval, the convergence interval. This interval may be finite with midpoint x_0. The series converges for all x in the interior of the interval, that is,

$$\forall\ x \text{ for which } |x-x_0| < R$$

and diverges for $|x-x_0| > R$. The interval may also be infinite, that is, the series may converge for all x.

The quantity R is called the radius of convergence.

If series converges for all x, we have $R = \infty$.

If series converges only for $x = x_0$, we have $R = 0$.

The Radius of convergence can be determined from the coefficients of the series by means of each of the formulas:

(a) $R = \dfrac{1}{\lim\limits_{m \to \infty} |a_n|^{\frac{1}{m}}}$ (b) $R = \dfrac{1}{\lim\limits_{m \to \infty} \left| \dfrac{a_{m+1}}{a_m} \right|}$