AI1103 : Assignment-3

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Download all python codes from

https://github.com/Rahul27n/Assignment_3/blob/main/Assignment_3.py

and latex-tikz codes from

https://github.com/Rahul27n/Assignment_3/blob/main/Assignment_3.tex

1 QUESTION: Q.50 UGC 2018(June math set-a)

Let X and Y be two independent and identically distributed (I.I.D) random variables uniformly distributed in (0,1). Let $Z = max(X, Y)$ and $W = min(X, Y)$, then the probability that $[Z - W > \frac{1}{2}]$ is

(A) $\frac{1}{2}$

(B) $\frac{3}{4}$

(C) $\frac{1}{4}$

(D) $\frac{2}{3}$

2 SOLUTION

X and Y are two independent random variables. Let

$$f_X(x) = \Pr(X = x) \tag{2.0.1}$$
$$f_Y(y) = \Pr(Y = y) \tag{2.0.2}$$
$$f_V(v) = \Pr(V = v) \tag{2.0.3}$$

be the probability densities of random variables X ,Y and V=X-Y.
The density for X is

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{2.0.4}$$

We have ,

$$V = X - Y \iff v = x - y \iff x = v + y \tag{2.0.5}$$

The density of X can also be represented as,

$$f_X(v + y) = \begin{cases} 1 & 0 \le v + y \le 1 \\ 0 & otherwise \end{cases} \tag{2.0.6}$$

and the density of Y is,

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{2.0.7}$$

The density of V i.e. $V = X - Y$ is given by the convolution of $f_X(-v)$ with $f_Y(v)$.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v + y) f_Y(y)\, dy \tag{2.0.8}$$

From 2.0.6 and 2.0.7 we have,
The integrand is 1 when,

$$0 \le y \le 1 \tag{2.0.9}$$
$$0 \le v + y \le 1 \tag{2.0.10}$$
$$-v \le y \le 1 - v \tag{2.0.11}$$

and zero, otherwise.
Now when $-1 \le v \le 0$ we have,

$$f_V(v) = \int_{-v}^{1} dy \tag{2.0.12}$$
$$= (1 - (-v)) \tag{2.0.13}$$
$$= 1 + v \tag{2.0.14}$$

For $0 \le v \le 1$ we have,

$$f_V(v) = \int_{0}^{1-v} dy \tag{2.0.15}$$
$$= (1 - v - (0)) \tag{2.0.16}$$
$$= 1 - v \tag{2.0.17}$$

Therefore the density of V is given by

$$f_V(v) = \begin{cases} 1 + v & -1 \le v \le 0 \\ 1 - v & 0 < v \le 1 \\ 0 & otherwise \end{cases} \tag{2.0.18}$$

The plot for PDF of V can be observed at figure 0

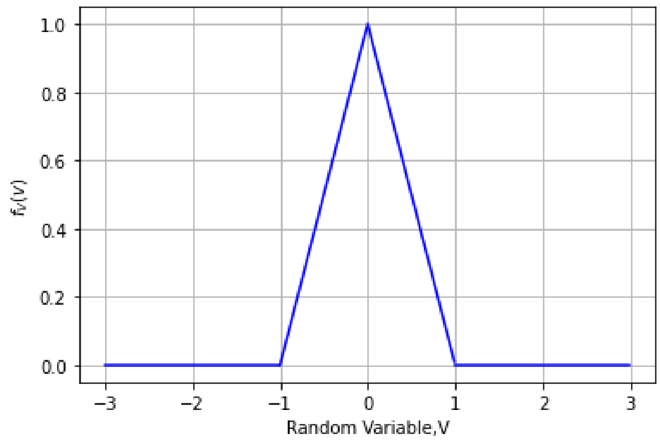

Fig. 0: The PDF of V

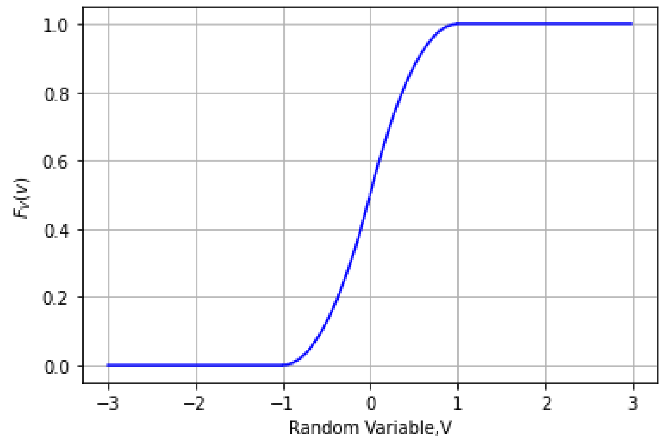

Fig. 0: The CDF of V

The CDF of V is defined as,

$$F_V(v) = \Pr(V \le v) \tag{2.0.19}$$

Now for $v \le 0$,

$$\Pr(V \le v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{2.0.20}$$
$$= \int_{-1}^{v} (1+v)\,dv \tag{2.0.21}$$
$$= \left(\frac{v^2}{2} + v\right)\Bigg|_{-1}^{v} \tag{2.0.22}$$
$$= \left(\left(\frac{v^2}{2} + v\right) - \left(\frac{1}{2} - 1\right)\right) \tag{2.0.23}$$
$$= \frac{v^2 + 2v + 1}{2} \tag{2.0.24}$$

Similarly for $v \le 1$,

$$\Pr(V \le v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{2.0.25}$$
$$= \frac{1}{2} + \int_{0}^{v} (1-v)\,dz \tag{2.0.26}$$
$$= \frac{-v^2 + 2v + 1}{2} \tag{2.0.27}$$

The CDF is as below:

$$F_V(v) = \begin{cases} 0 & v < -1 \\ \dfrac{v^2+2v+1}{2} & v \le 0 \\ \dfrac{-v^2+2v+1}{2} & v \le 1 \\ 1 & v > 1 \end{cases} \tag{2.0.28}$$

The plot for CDF of V can be observed at figure 0

We need $\Pr\left(Z - W > \frac{1}{2}\right)$ where $Z = max(X,Y)$ and $W = min(X,Y)$. Now,

$$Z - W = \begin{cases} X - Y & \text{for } X \ge Y \\ Y - X & \text{for } X < Y \end{cases} \tag{2.0.29}$$

Therefore,

$$\Pr\left(Z - W > \frac{1}{2}\right) = \Pr\left(X - Y > \frac{1}{2}, X \ge Y\right) + \Pr\left(Y - X > \frac{1}{2}, X < Y\right) \tag{2.0.30}$$
$$= \Pr\left(X - Y > \frac{1}{2}\right) + \Pr\left(Y - X > \frac{1}{2}\right) \tag{2.0.31}$$
$$= \Pr\left(V > \frac{1}{2}\right) + \Pr\left(-V > \frac{1}{2}\right) \tag{2.0.32}$$
$$= 1 - \Pr\left(V \le \frac{1}{2}\right) + \Pr\left(V < \frac{-1}{2}\right) \tag{2.0.33}$$
$$= 1 - F_V(\frac{1}{2}) + F_V(-\frac{1}{2}) \tag{2.0.34}$$
$$= 1 - \frac{7}{8} + \frac{1}{8} \tag{2.0.35}$$
$$= \frac{1}{4} \tag{2.0.36}$$

Hence the correct answer is option (C).