CSIR-UGC NET-June 2018-Problem(50)

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Question

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Let X and Y be two independent and identically distributed (I.I.D) random variables uniformly distributed in (0,1). Let $Z = max(X, Y)$ and $W = min(X, Y)$, then the probability that $[Z - W > \frac{1}{2}]$ is

1. $\frac{1}{2}$
2. $\frac{3}{4}$
3. $\frac{1}{4}$
4. $\frac{2}{3}$

Convolution of two Random variables

Let X and Y be two independent continuous random variables. Let

$$f_X(x) = \Pr(X = x) \tag{1}$$
$$f_Y(y) = \Pr(Y = y) \tag{2}$$
$$f_V(v) = \Pr(V = v) \tag{3}$$

be the probability densities of random variables X ,Y and V=X+Y.

Convolution of two Random variables

The density of $V = X + Y$ is given by the convolution of $f_X(v)$ with $f_Y(v)$.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v - y) f_Y(y)\, dy \tag{4}$$

Convolution of two Random variables

Similarly if $V = X - Y$ then $f_V(v)$ is given by the convolution of $f_X(-v)$ with $f_Y(v)$.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v+y) f_Y(y)\, dy \tag{5}$$

Solution

X and Y are independent and uniformly distributed random variables in (0,1). Let

$$f_X(x) = \Pr(X = x) \tag{6}$$
$$f_Y(y) = \Pr(Y = y) \tag{7}$$
$$f_V(v) = \Pr(V = v) \tag{8}$$

be the probability densities of random variables X ,Y and V=X-Y.
The density for X is

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{9}$$

and the density of Y is,

$$f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{10}$$

Solution Contd.

We have ,

$$V = X - Y \iff v = x - y \iff x = v + y \tag{11}$$

Therefore the density of X can also be represented as,

$$f_X(v+y) = \begin{cases} 1 & 0 \leq v+y \leq 1 \\ 0 & otherwise \end{cases} \tag{12}$$

The density of V is given by the convolution of $f_X(-v)$ with $f_Y(v)$

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v+y) f_Y(y)\, dy \tag{13}$$

Solution Contd.

From (10) and (12) ,the integrand in (13) is 1 when :

$$0 \leq y \leq 1 \tag{14}$$

$$0 \leq v + y \leq 1 \tag{15}$$

$$-v \leq y \leq 1 - v \tag{16}$$

and zero, otherwise.
Now when $-1 \leq v \leq 0$ we have,

$$f_V(v) = \int_{-v}^{1} dy \tag{17}$$

$$= (1 - (-v)) \tag{18}$$

$$= 1 + v \tag{19}$$

Solution Contd.

For $0 \leq v \leq 1$ we have,

$$f_V(v) = \int_0^{1-v} dy \tag{20}$$
$$= (1 - v - (0)) \tag{21}$$
$$= 1 - v \tag{22}$$

Therefore the density of V is given by

$$f_V(v) = \begin{cases} 1 + v & -1 \leq v \leq 0 \\ 1 - v & 0 < v \leq 1 \\ 0 & otherwise \end{cases} \tag{23}$$

Solution Contd.

The plot for PDF of V can be observed below.

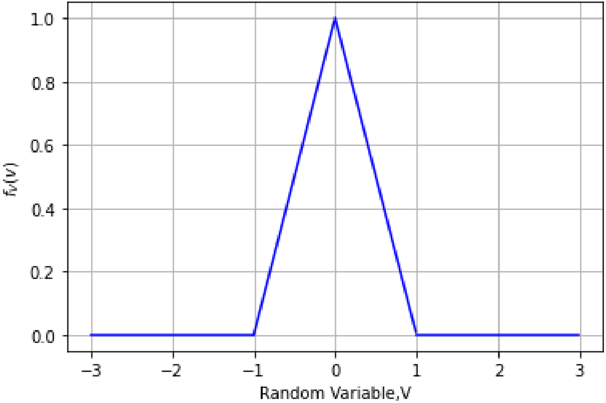

Figure: The PDF of V

Solution Contd.

The CDF of V is defined as,

$$F_V(v) = \Pr(V \le v) \tag{24}$$

Now for $v \le 0$,

$$\Pr(V \le v) = \int_{-\infty}^{v} f_V(v)\, dv \tag{25}$$

$$= \int_{-1}^{v} (1+v)\, dv \tag{26}$$

$$= \left(\frac{v^2}{2} + v\right)\Bigg|_{-1}^{v} \tag{27}$$

$$= \left(\left(\frac{v^2}{2} + v\right) - \left(\frac{1}{2} - 1\right)\right) \tag{28}$$

$$= \frac{v^2 + 2v + 1}{2} \tag{29}$$

Solution Contd.

Similarly for $v \leq 1$,

$$\Pr\left(V \leq v\right) = \int_{-\infty}^{v} f_V(v)\, dv \tag{30}$$

$$= \frac{1}{2} + \int_{0}^{v} (1 - v)\, dv \tag{31}$$

$$= \frac{-v^2 + 2v + 1}{2} \tag{32}$$

The CDF is as below:

$$F_V(v) = \begin{cases} 0 & v < -1 \\ \frac{v^2 + 2v + 1}{2} & v \leq 0 \\ \frac{-v^2 + 2v + 1}{2} & v \leq 1 \\ 1 & v > 1 \end{cases} \tag{33}$$

Solution Contd.

The plot for CDF of V can be observed below.

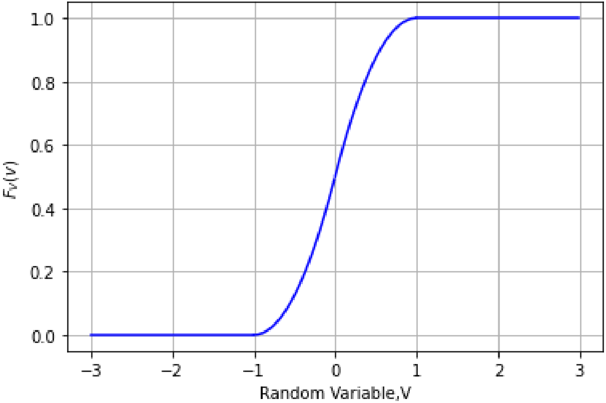

Figure: The CDF of V

Solution Contd.

We need $\Pr\left(Z - W > \frac{1}{2}\right)$ where $Z = max(X, Y)$ and $W = min(X, Y)$.
Now,

$$Z - W = \begin{cases} X - Y & \text{for } X \geq Y \\ Y - X & \text{for } X < Y \end{cases} \tag{34}$$

Solution Contd.

Therefore,

$$\Pr\left(Z - W > \frac{1}{2}\right) = \Pr\left(X - Y > \frac{1}{2}, X \geq Y\right) + \Pr\left(Y - X > \frac{1}{2}, X < Y\right) \tag{35}$$

$$= \Pr\left(X - Y > \frac{1}{2}\right) + \Pr\left(Y - X > \frac{1}{2}\right) \tag{36}$$

$$= \Pr\left(V > \frac{1}{2}\right) + \Pr\left(-V > \frac{1}{2}\right) \tag{37}$$

$$= 1 - \Pr\left(V \leq \frac{1}{2}\right) + \Pr\left(V < \frac{-1}{2}\right) \tag{38}$$

$$= 1 - F_V(\frac{1}{2}) + F_V(-\frac{1}{2}) \tag{39}$$

$$= 1 - \frac{7}{8} + \frac{1}{8} \tag{40}$$

$= \frac{1}{4} \implies$ Option (3) is correct.