AI1103 : Assignment-4

Nelakuditi Rahul Naga - AI20BTECH11029/EE20BTECH11036

Download all python codes from

```
https://github.com/Rahul27n/Assignment_4/blob/
    main/Assignment_4.py
```

and latex-tikz codes from

```
https://github.com/Rahul27n/Assignment_4/blob/
    main/Assignment_4.tex
```

1 QUESTION: Q.61 GATE-EC-2003

Let X and Y be two statistically independent random variables uniformly distributed in the range (-1, 1) and (-2, 1) respectively. Let $Z = X + Y$, then the probability that $[Z \leq -2]$ is

(A) 0

(B) $\frac{1}{6}$

(C) $\frac{1}{3}$

(D) $\frac{1}{12}$

2 SOLUTION

X and Y are two independent random variables. Let

$$f_X(x) = \Pr(X = x) \tag{2.0.1}$$
$$f_Y(y) = \Pr(Y = y) \tag{2.0.2}$$
$$f_Z(z) = \Pr(Z = z) \tag{2.0.3}$$

be the probability densities of random variables X ,Y and Z.
X lies in range(-1,1), therefore,

$$\int_{-1}^{1} f_X(x)\ dx = 1 \tag{2.0.4}$$
$$2 \times f_X(x) = 1 \tag{2.0.5}$$
$$f_X(x) = 1/2 \tag{2.0.6}$$

Similarly for Y we have,

$$\int_{-2}^{1} f_Y(y)\ dy = 1 \tag{2.0.7}$$
$$3 \times f_Y(y) = 1 \tag{2.0.8}$$
$$f_Y(y) = 1/3 \tag{2.0.9}$$

The density for X is

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{2.0.10}$$

We have ,

$$Z = X + Y \iff z = x + y \iff x = z - y \tag{2.0.11}$$

The density of X can also be represented as,

$$f_X(z-y) = \begin{cases} \frac{1}{2} & -1 \leq z - y \leq 1 \\ 0 & otherwise \end{cases} \tag{2.0.12}$$

and the density of Y is,

$$f_Y(y) = \begin{cases} \frac{1}{3} & -2 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{2.0.13}$$

The density of Z i.e. $Z = X + Y$ is given by the convolution of the densities of X and Y

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy \tag{2.0.14}$$

From 2.0.12 and 2.0.13 we have,
The integrand is $\dfrac{1}{6}$ when,

$$2 \leq y \leq 1 \tag{2.0.15}$$
$$-1 \leq z - y \leq 1 \tag{2.0.16}$$
$$z - 1 \leq y \leq z + 1 \tag{2.0.17}$$

and zero, otherwise.

Now when $-3 \le z \le -2$ we have,

$$f_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{2.0.18}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{2.0.19}$$
$$= \frac{1}{6}(z+3) \tag{2.0.20}$$

For $-2 < z \le -1$,

$$f_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{2.0.21}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{2.0.22}$$
$$= \frac{1}{6}(z+3) \tag{2.0.23}$$

For $-1 < z \le 0$,

$$f_Z(z) = \int_{z-1}^{z+1} \frac{1}{6}\, dy \tag{2.0.24}$$
$$= \frac{1}{6} \times (z + 1 - (z-1)) \tag{2.0.25}$$
$$= \frac{1}{3} \tag{2.0.26}$$

For $0 < z \le 2$,

$$f_Z(z) = \int_{z-1}^{1} \frac{1}{6}\, dy \tag{2.0.27}$$
$$= \frac{1}{6} \times (1 - (z-1)) \tag{2.0.28}$$
$$= \frac{1}{6}(2-z) \tag{2.0.29}$$

Therefore the density of Z is given by

$$f_Z(z) = \begin{cases} \frac{1}{6}(z+3) & -3 \le z \le -2 \\ \frac{1}{6}(z+3) & -2 < z \le -1 \\ \frac{1}{3} & -1 < z \le 0 \\ \frac{1}{6}(2-z) & 0 < z \le 2 \\ 0 & otherwise \end{cases} \tag{2.0.30}$$

The plot for PDF of Z can be observed at figure 0

The CDF of Z is defined as,

$$F_Z(z) = \Pr(Z \le z) \tag{2.0.31}$$

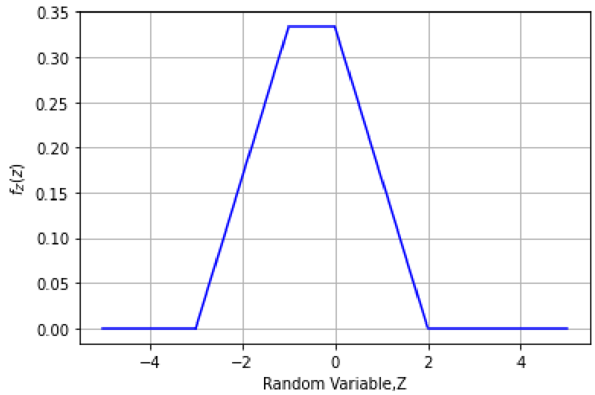

Fig. 0: The PDF of Z

Now for $z \le -1$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} f_Z(z)\, dz \tag{2.0.32}$$
$$= \int_{-3}^{z} \frac{1}{6}(z+3)\, dz \tag{2.0.33}$$
$$= \frac{1}{6}\left(\frac{z^2}{2} + 3z\right)\Bigg|_{-3}^{z} \tag{2.0.34}$$
$$= \frac{1}{6} \times \left(\left(\frac{z^2}{2} + 3z\right) - \left(\frac{9}{2} - 9\right)\right) \tag{2.0.35}$$
$$= \frac{z^2 + 6z + 9}{12} \tag{2.0.36}$$

Similarly for $z \le 0$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} f_Z(z)\, dz \tag{2.0.37}$$
$$= \frac{1}{3} + \int_{-1}^{z} \frac{1}{3}\, dz \tag{2.0.38}$$
$$= \frac{z+2}{3} \tag{2.0.39}$$

Finally for $z \le 2$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} f_Z(z)\, dz \tag{2.0.40}$$
$$= \frac{2}{3} + \int_{0}^{z} \frac{1}{6}(2-z)\, dz \tag{2.0.41}$$
$$= \frac{2}{3} + \frac{4z - z^2}{12} \tag{2.0.42}$$
$$= \frac{8 + 4z - z^2}{12} \tag{2.0.43}$$

The CDF is as below:

$$F_Z(z) = \begin{cases} 0 & z < -3 \\ \frac{z^2+6z+9}{12} & z \leq -1 \\ \frac{z+2}{3} & z \leq 0 \\ \frac{8+4z-z^2}{12} & z \leq 2 \\ 1 & z > 2 \end{cases} \tag{2.0.44}$$

The plot for CDF of Z can be observed at figure 0

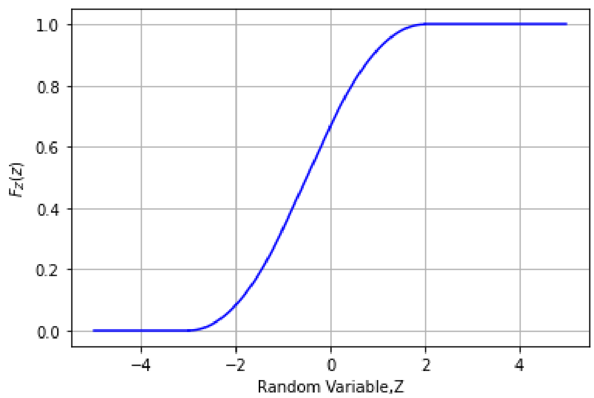

Fig. 0: The CDF of Z

Therefore,

$$\Pr(Z \leq -2) = F_Z(-2) \tag{2.0.45}$$

$$= \frac{1}{12} \tag{2.0.46}$$

Hence the correct answer is option (D).