EE3900 : Assignment-3

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Download all python codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.py

and latex-tikz codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.tex

1 Question: Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.0.1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{1.0.2}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.0.3}$$

2 Solution

The general equation of a circle can be expressed as:

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.2}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.3}$$

The standard basis vectors in 2D plane are given by:

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.5}$$

We note that the lines (1.0.1) and (1.0.2) are **parallel** with a common normal along $\mathbf{e_2}$.

Thus the common normal passing through the centre of the circle is of the form :

$$\mathbf{e_1}^T\mathbf{x} = k \tag{2.0.6}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{2.0.7}$$

where k is a constant.

Let the circles of the form (2.0.1) touch the lines (1.0.1) and (1.0.2) at $\mathbf{M}$ and $\mathbf{N}$.

$\mathbf{M}$ is the point of intersection of the following lines:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{2.0.8}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.9}$$

The above equations can be expressed as the matrix equation :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} k \\ 0 \end{pmatrix} \tag{2.0.10}$$

The augmented matrix for the above equation is given by :

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.11}$$

As the left part is already a identity matrix ,the intersection point $\mathbf{M}$ is given by :

$$\mathbf{M} = \begin{pmatrix} k \\ 0 \end{pmatrix} \tag{2.0.12}$$

$\mathbf{N}$ is the point of intersection of the following lines:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{2.0.13}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{2.0.14}$$

The above equations can be expressed as the matrix equation :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} k \\ 4 \end{pmatrix} \tag{2.0.15}$$

The augmented matrix for the above equation is given by :

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 4 \end{pmatrix} \tag{2.0.16}$$

As the left part is already a identity matrix ,the intersection point $\mathbf{N}$ is given by :

$$\mathbf{N} = \begin{pmatrix} k \\ 4 \end{pmatrix} \tag{2.0.17}$$

The centre $\mathbf{c}$ of the circle must be the mid-point of $\mathbf{M}$ and $\mathbf{N}$ as $\mathbf{M}$ and $\mathbf{N}$ are the touch points of parallel tangents to a circle. Therefore we have:

$$\mathbf{c} = \frac{\begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} k \\ 4 \end{pmatrix}}{2} \tag{2.0.18}$$

$$\mathbf{c} = \begin{pmatrix} k \\ 2 \end{pmatrix} \tag{2.0.19}$$

Also the radius r of the circles is given by:

$$r = \frac{\|\mathbf{M} - \mathbf{N}\|}{2} = \frac{\sqrt{(k-k)^2 + (0-4)^2}}{2} \tag{2.0.20}$$

$$r = 2 \tag{2.0.21}$$

From (2.0.2) we have:

$$f = \begin{pmatrix} -k & -2 \end{pmatrix} \begin{pmatrix} -k \\ -2 \end{pmatrix} - r^2 \tag{2.0.22}$$

$$f = k^2 + 2^2 - 2^2 \tag{2.0.23}$$

$$f = k^2 \tag{2.0.24}$$

The equation of the remaining line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.0.25}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.0.26}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics in (2.0.26) are given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \tag{2.0.27}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{2.0.28}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.29}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.30}$$

$$\mathbf{IX} = \mathbf{X} \tag{2.0.31}$$

From (2.0.2), (2.0.25) and (2.0.29) we have:

$$\kappa = \pm \sqrt{\frac{r^2}{\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}} \tag{2.0.32}$$

$$= \pm \sqrt{\frac{4}{5}} \tag{2.0.33}$$

$$= \pm \frac{2}{\sqrt{5}} \tag{2.0.34}$$

Therefore from (2.0.27) we have:

$$\mathbf{q} = \pm \begin{pmatrix} \frac{4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} -k \\ -2 \end{pmatrix} \tag{2.0.35}$$

$$= \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \frac{\pm 2}{\sqrt{5}} + 2 \end{pmatrix} \tag{2.0.36}$$

Now $\mathbf{q}$ lies on the line (2.0.25) therefore,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \frac{\pm 2}{\sqrt{5}} + 2 \end{pmatrix} = 2 \tag{2.0.37}$$

$$\implies k = \pm \sqrt{5} \tag{2.0.38}$$

$$\implies f = k^2 = 5 \tag{2.0.39}$$

Hence the tangent circles are given by the equations:

$$\mathbf{x}^{\mathbf{T}} \mathbf{x} + \begin{pmatrix} 2\sqrt{5} & -4 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.0.40}$$

$$\mathbf{x}^{\mathbf{T}} \mathbf{x} + \begin{pmatrix} -2\sqrt{5} & -4 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.0.41}$$

The illustration of the circles and the lines is shown below :

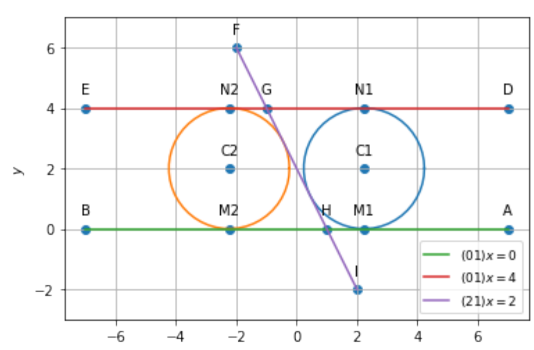

Fig. 0: Circles touching given lines with centres C1,C2