Ramsey 4.2 Tangent and Normal Q.16

Nelakuditi Rahul Naga - AI20BTECH11029

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Question

Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{x} = 0 \tag{1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{x} = 4 \tag{2}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathrm{x} = 2 \tag{3}$$

Solution

General Equation of a Circle

The general equation of a circle can be expressed as:

$$\mathbf{x}^{\top}\mathbf{x} + 2\mathbf{u}^{\top}\mathbf{x} + f = 0 \tag{4}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^{\top}\mathbf{u} - r^2 \tag{5}$$

$$\mathbf{c} = -\mathbf{u} \tag{6}$$

General equation of a 2nd degree conic and point of contact of the tangent

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{7}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics in (7) are given by:

$$\mathbf{q} = \mathbf{V}^{-1} \left(\kappa \mathbf{n} - \mathbf{u}\right) \tag{8}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{9}$$

For a circle we have,

$$\mathbf{V} = \mathbf{I} \tag{10}$$

The touch points of the circles of the form (4) with line (1) are determined by:

$$\kappa_1 = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{11}$$

$$\kappa_1 = \pm \sqrt{\frac{r^2}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} = \pm r \tag{12}$$

Therefore we have:

$$\mathbf{q}_1 = \pm r \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u} \tag{13}$$

Now $\mathbf{q}_1$ lies on the line (1) therefore,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\pm r \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u} \right) = 0 \tag{14}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{u} = \pm r \tag{15}$$

The touch points of the circles of the form (4) with line (2) are determined by:

$$\kappa_2 = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{16}$$

$$\kappa_2 = \pm \sqrt{\frac{r^2}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}} = \pm r \tag{17}$$

Therefore we have:

$$\mathbf{q}_2 = \pm r \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u} \tag{18}$$

Now $\mathbf{q}_2$ lies on the line (2) therefore,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\pm r \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u} \right) = 4 \tag{19}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{u} = \pm r - 4 \tag{20}$$

The touch points of the circles of the form (4) with line (3) are determined by:

$$\kappa_3 = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}} \tag{21}$$

$$\kappa_3 = \pm \sqrt{\frac{r^2}{\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}} = \pm \frac{r}{\sqrt{5}} \tag{22}$$

Therefore we have:

$$\mathbf{q}_3 = \pm \frac{r}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \mathbf{u} \tag{23}$$

Now $\mathbf{q}_3$ lies on the line (3) therefore,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \left(\pm \frac{r}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \mathbf{u} \right) = 2 \tag{24}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{u} = \pm \sqrt{5} r - 2 \tag{25}$$

Now we need to solve the equations (15), (20) and (25) written below to obtain u and r :

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{u} = \pm r$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{u} = \pm r - 4$$
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathrm{u} = \pm\sqrt{5} r - 2$$

The first two equations are consistent and give a positive solution for r only when they are of the form :

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{u} = -r$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{u} = r - 4$$

which upon solving give :

$$r = 2 \tag{26}$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{u} = -2 \tag{27}$$

Now putting $r = 2$ in the third equation we have:

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathrm{u} = \pm 2\sqrt{5} - 2 \tag{28}$$

Let us say $\mathrm{u} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Substituting u in (27) we have:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -2 \tag{29}$$

$$\implies \beta = -2 \tag{30}$$

Substituting u in (28) we have:

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm 2\sqrt{5} - 2 \tag{31}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ -2 \end{pmatrix} = \pm 2\sqrt{5} - 2 \tag{32}$$

$$\implies \alpha = \pm\sqrt{5} \tag{33}$$

Therefore we have:

$$\mathrm{u} = \begin{pmatrix} \pm\sqrt{5} \\ -2 \end{pmatrix} \tag{34}$$

Hence the value of f is given by:

$$f = \mathrm{u}^T\mathrm{u} - r^2 \tag{35}$$

$$f = \begin{pmatrix} \pm\sqrt{5} & -2 \end{pmatrix} \begin{pmatrix} \pm\sqrt{5} \\ -2 \end{pmatrix} - 2^2 \tag{36}$$

$$f = 5 \tag{37}$$

Hence the tangent circles are given by the equations:

$$\mathrm{x}^{\mathrm{T}}\mathrm{x} + \begin{pmatrix} 2\sqrt{5} & -4 \end{pmatrix} \mathrm{x} + 5 = 0 \tag{38}$$

$$\mathrm{x}^{\mathrm{T}}\mathrm{x} + \begin{pmatrix} -2\sqrt{5} & -4 \end{pmatrix} \mathrm{x} + 5 = 0 \tag{39}$$

The illustration of the circles and the lines is shown below :

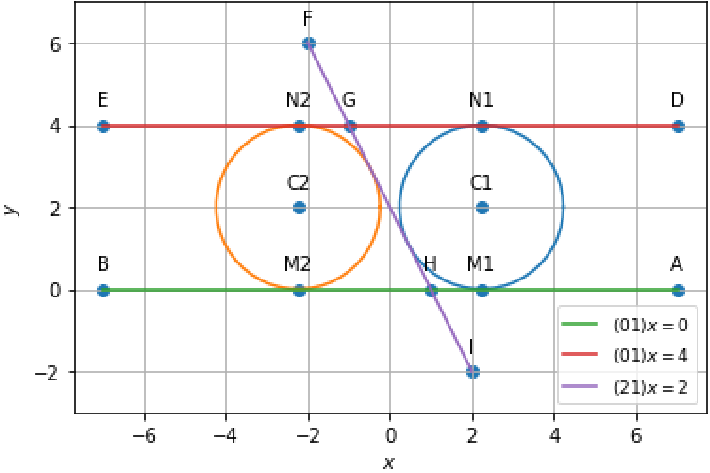

Figure: Circles touching given lines with centres C1,C2