EE3900 : Assignment-3

Nelakuditi Rahul Naga - AI20BTECH11029

Download all python codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.py

and latex-tikz codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.tex

1 Question: Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.0.1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{1.0.2}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.0.3}$$

2 Solution

The general equation of a circle can be expressed as:

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.2}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.3}$$

Let the circles of the form (2.0.1) touch the lines (1.0.1) and (1.0.2) (which are **parallel**) at general points $\mathbf{M}$ and $\mathbf{N}$ given by:

$$\mathbf{M} = \begin{pmatrix} m \\ 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{N} = \begin{pmatrix} n \\ 4 \end{pmatrix} \tag{2.0.5}$$

respectively.
We note that the common normal of (1.0.1) and (1.0.2) passing through the centre of the circle is of the form :

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{2.0.6}$$

where k is a constant.
Now $\mathbf{M}$ and $\mathbf{N}$ lie on the aforementioned common normal. Hence we have:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} = k \tag{2.0.7}$$

$$\implies m = k \tag{2.0.8}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} n \\ 4 \end{pmatrix} = k \tag{2.0.9}$$

$$\implies n = k \tag{2.0.10}$$

The centre $\mathbf{c}$ of the circle must be the mid-point of $\mathbf{M}$ and $\mathbf{N}$ as $\mathbf{M}$ and $\mathbf{N}$ are the touch points of parallel tangents to a circle. Therefore we have:

$$\mathbf{c} = \frac{\begin{pmatrix} m \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 4 \end{pmatrix}}{2} \tag{2.0.11}$$

$$\mathbf{c} = \frac{\begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} k \\ 4 \end{pmatrix}}{2} \tag{2.0.12}$$

$$\mathbf{c} = \begin{pmatrix} k \\ 2 \end{pmatrix} \tag{2.0.13}$$

Also the radius r of the circles is given by:

$$r = \frac{\|\mathbf{M} - \mathbf{N}\|}{2} = \frac{\sqrt{(k-k)^2 + (0-4)^2}}{2} \tag{2.0.14}$$

$$r = 2 \tag{2.0.15}$$

From (2.0.2) we have:

$$f = \begin{pmatrix} -k & -2 \end{pmatrix} \begin{pmatrix} -k \\ -2 \end{pmatrix} - r^2 \tag{2.0.16}$$

$$f = k^2 + 2^2 - 2^2 \tag{2.0.17}$$

$$f = k^2 \tag{2.0.18}$$

The equation of the remaining line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.0.19}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.20}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics in (2.0.20) are given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{2.0.21}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.22}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.23}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.24}$$

$$\mathbf{IX} = \mathbf{X} \tag{2.0.25}$$

From (2.0.2), (2.0.19) and (2.0.23) we have:

$$\kappa = \pm\sqrt{\frac{r^2}{\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}} \tag{2.0.26}$$

$$= \pm\sqrt{\frac{4}{5}} \tag{2.0.27}$$

$$= \pm\frac{2}{\sqrt{5}} \tag{2.0.28}$$

Therefore from (2.0.21) we have:

$$\mathbf{q} = \pm\begin{pmatrix}\frac{4}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix} - \begin{pmatrix}-k\\-2\end{pmatrix} \tag{2.0.29}$$

$$= \begin{pmatrix}\frac{\pm4}{\sqrt{5}} + k\\\frac{\pm2}{\sqrt{5}} + 2\end{pmatrix} \tag{2.0.30}$$

Now $\mathbf{q}$ lies on the line (2.0.19) therefore,

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}\frac{\pm4}{\sqrt{5}} + k\\\frac{\pm2}{\sqrt{5}} + 2\end{pmatrix} = 2 \tag{2.0.31}$$

$$2\left(\frac{\pm4}{\sqrt{5}} + k\right) + \left(\frac{\pm2}{\sqrt{5}} + 2\right) = 2 \tag{2.0.32}$$

$$2k = \pm\frac{10}{\sqrt{5}} \tag{2.0.33}$$

$$\implies k = \pm\sqrt{5} \tag{2.0.34}$$

$$\implies f = k^2 = 5 \tag{2.0.35}$$

Hence the tangent circles are given by the equations:

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + \begin{pmatrix}2\sqrt{5} & -4\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.0.36}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + \begin{pmatrix}-2\sqrt{5} & -4\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.0.37}$$

The illustration of the circles and the lines is shown below :

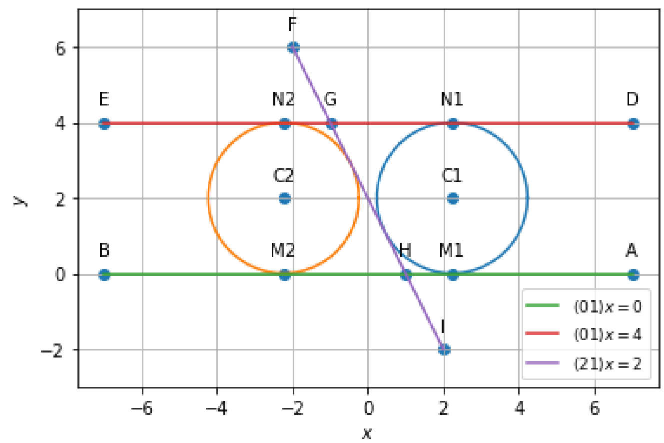

Fig. 0: Circles touching given lines with centres C1,C2