GATE-EC 2016 Q.33

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Question

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The Discrete Fourier Transform (DFT) of the 4 point sequence $x[n] = \{3, 2, 3, 4\}$ is given by $X[k] = \{12, 2j, 0, -2j\}$. If $X_1[k]$ is the DFT of the 12 point sequence $x_1[n] = \{3, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0\}$, find the value of $\left|\frac{X_1[8]}{X_1[11]}\right|$.

Solution

4-point DFT matrix

The 4-point DFT matrix is given by:

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \tag{2}$$

where $\omega = e^{\dfrac{-2\pi j}{4}} = -j$.

Now from the given information we can write:

$$X = W_4 x \tag{3}$$

where

$$x = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{4}$$

and

$$X = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \tag{5}$$

Now we need to find X_1 satisfying the relation :

$$X_1 = W_{12} x_1 \tag{6}$$

where

$$x_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

and W_{12} is the 12-point DFT matrix.

12-point DFT matrix

$$W_{12} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \Omega & \Omega^2 & -j & -j\Omega & -j\Omega^2 & -1 & -\Omega & -\Omega^2 & j & j\Omega & j\Omega^2 \\
1 & \Omega^2 & -j\Omega & -1 & -\Omega^2 & j\Omega & 1 & \Omega^2 & -j\Omega & -1 & -\Omega^2 & j\Omega \\
1 & -j & -1 & j & 1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 \\
1 & -j\Omega^2 & j\Omega & -j & -\Omega^2 & \Omega & -1 & j\Omega^2 & -j\Omega & j & \Omega^2 & -\Omega \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\Omega & \Omega^2 & j & -j\Omega & j\Omega^2 & -1 & \Omega & -\Omega^2 & -j & j\Omega & -j\Omega^2 \\
1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega \\
1 & j & -1 & -j & 1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & j\Omega & -\Omega^2 & -1 & -j\Omega & \Omega^2 & 1 & j\Omega & -\Omega^2 & -1 & -j\Omega & \Omega^2 \\
1 & j\Omega^2 & j\Omega & j & -\Omega^2 & -\Omega & -1 & -j\Omega^2 & -j\Omega & -j & \Omega^2 & \Omega
\end{pmatrix}$$

where

$$\Omega = e^{\frac{-2\pi j}{12}} = \frac{\sqrt{3} - j}{2} \tag{8}$$

We can express x_1 in terms of x as follows :

$$x_1 = Ax \tag{9}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$\implies A = \begin{pmatrix} e_1 & e_4 & e_7 & e_{10} \end{pmatrix} \tag{11}$$

where e_1, e_4, e_7, e_{10} are the unit basis vectors.

Now from (6) have :

$$X_1 = (W_{12}A)x \tag{12}$$

Only the red coloured columns in W_{12} give non-zero output when multiplied with A. We can express the matrix W_{12} as a block matrix in the following way:

$$W_{12} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} \end{pmatrix} \tag{13}$$

where c_i is the i^{th} column matrix of W_{12}. Now we have:

$$X_1 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} \end{pmatrix} Ax$$
$$\implies X_1 = \begin{pmatrix} c_1 & c_4 & c_7 & c_{10} \end{pmatrix} x \tag{14}$$

We can also express W_4 as block matrix as follows:

$$W_4 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \tag{15}$$

where w_i is the i^{th} column matrix of W_4.

We can importantly note that :

$$c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes w_1 = \begin{pmatrix} w_1 \\ w_1 \\ w_1 \end{pmatrix} \tag{16}$$

$$c_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes w_2 = \begin{pmatrix} w_2 \\ w_2 \\ w_2 \end{pmatrix} \tag{17}$$

$$c_7 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes w_3 = \begin{pmatrix} w_3 \\ w_3 \\ w_3 \end{pmatrix} \tag{18}$$

$$c_{10} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes w_4 = \begin{pmatrix} w_4 \\ w_4 \\ w_4 \end{pmatrix} \tag{19}$$

where $\otimes$ represents the **Kronecker Product**.

Therefore from (14) we have:

$$X_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} x = \begin{pmatrix} X \\ X \\ X \end{pmatrix} = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \\ 12 \\ 2j \\ 0 \\ -2j \\ 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \tag{20}$$

Therefore we have:

$$\left|\frac{X_1[8]}{X_1[11]}\right| = \left|\frac{12}{-2j}\right| = \left|6j\right| = 6 \tag{21}$$

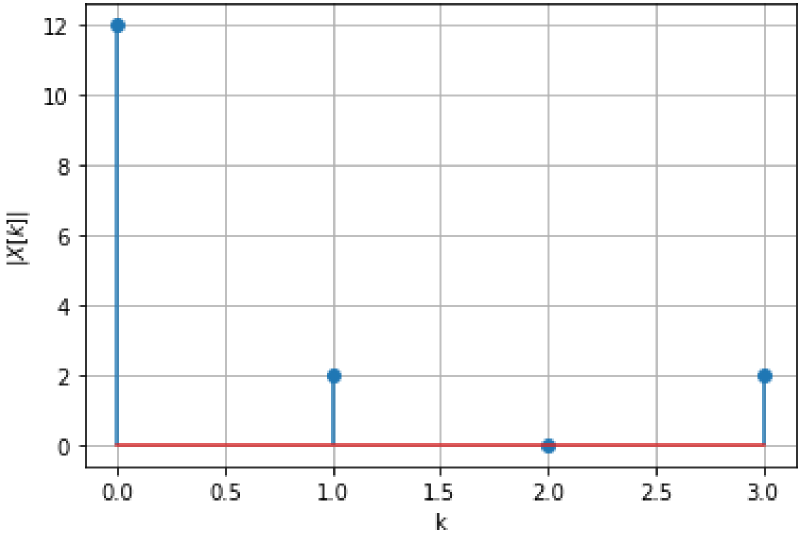

Figure: Magnitude of X[k] vs k

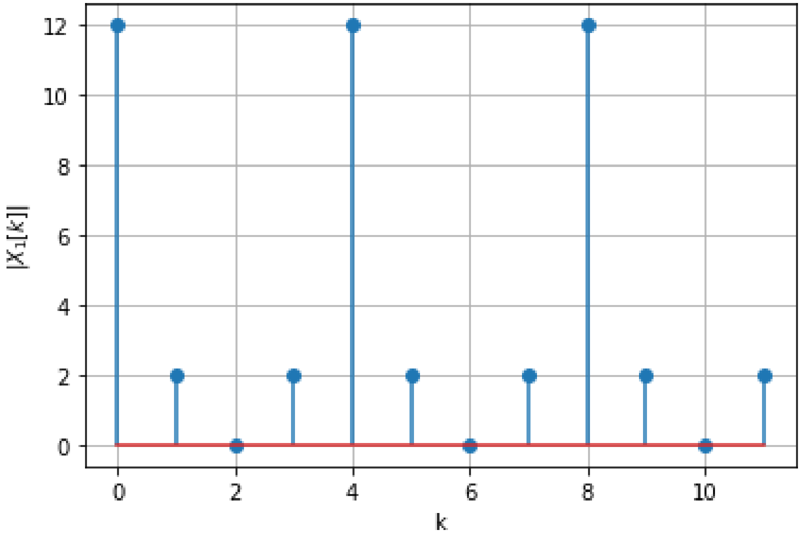

Figure: Magnitude of $X_1[k]$ vs k