EE3900 : Assignment-3

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Download all python codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.py

and latex-tikz codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.tex

1 Question: Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.0.1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 4 \tag{1.0.2}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 2 \tag{1.0.3}$$

2 Solution

The general equation of a circle can be expressed as:

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.2}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.3}$$

As the required circles of the form (2.0.1) are tangent to both lines (1.0.1) and (1.0.2) (which are parallel as they have the same normal vector), the centres of the circles required must lie on the line $\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \frac{4+0}{2} = 2$.

$$\therefore \mathbf{c} = \begin{pmatrix} k \\ 2 \end{pmatrix} \tag{2.0.4}$$

where k is an constant we need to determine. Also the common radius r of the circles is:

$$r = \frac{4+0}{2} = 2 \tag{2.0.5}$$

Also, from (2.0.2) we have:

$$f = \begin{pmatrix} -k & -2 \end{pmatrix}\begin{pmatrix} -k \\ -2 \end{pmatrix} - r^2 \tag{2.0.6}$$

$$f = k^2 + 2^2 - 2^2 \tag{2.0.7}$$

$$f = k^2 \tag{2.0.8}$$

The equation of the remaining line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 2 \tag{2.0.9}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.10}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics in (2.0.10) are given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{2.0.11}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.12}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.13}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.14}$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{2.0.15}$$

From (2.0.2), (2.0.9) and (2.0.13) we have:

$$\kappa = \pm\sqrt{\frac{r^2}{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}} \tag{2.0.16}$$

$$= \pm\sqrt{\frac{4}{5}} \tag{2.0.17}$$

$$= \pm\frac{2}{\sqrt{5}} \tag{2.0.18}$$

Therefore from (2.0.11) we have:

$$\mathbf{q} = \pm \begin{pmatrix} \frac{4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} -k \\ -2 \end{pmatrix} \tag{2.0.19}$$

$$= \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \frac{\pm 2}{\sqrt{5}} + 2 \end{pmatrix} \tag{2.0.20}$$

Now $\mathbf{q}$ lies on the line (2.0.9) therefore,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \frac{\pm 2}{\sqrt{5}} + 2 \end{pmatrix} = 2 \tag{2.0.21}$$

$$2\left(\frac{\pm 4}{\sqrt{5}} + k\right) + \left(\frac{\pm 2}{\sqrt{5}} + 2\right) = 2 \tag{2.0.22}$$

$$2k = \pm \frac{10}{\sqrt{5}} \tag{2.0.23}$$

$$\implies k = \pm \sqrt{5} \tag{2.0.24}$$

$$\implies f = k^2 = 5 \tag{2.0.25}$$

Therefore the tangent circles are given by the equations:

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + \begin{pmatrix} 2\sqrt{5} & -4 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.0.26}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + \begin{pmatrix} -2\sqrt{5} & -4 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.0.27}$$

The illustration of the circles and the lines is shown below :

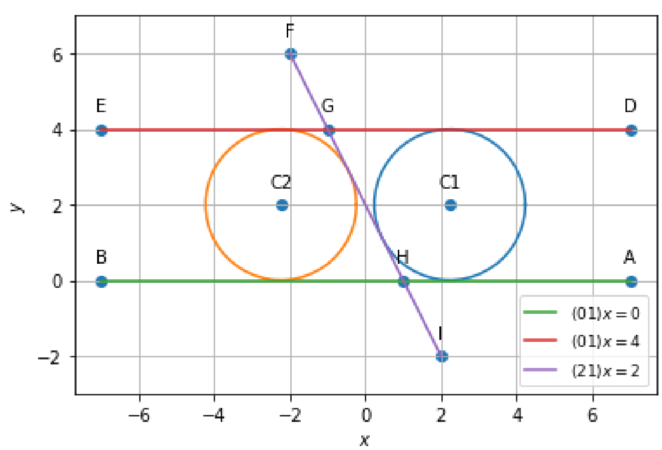

Fig. 0: Circles touching given lines with centres C1,C2