GATE-EC 2016 Q.33

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Question

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The Discrete Fourier Transform (DFT) of the 4 point sequence $x[n] = \{3, 2, 3, 4\}$ is given by $X[k] = \{12, 2j, 0, -2j\}$. If $X_1[k]$ is the DFT of the 12 point sequence $x_1[n] = \{3, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0\}$, find the value of $\left|\frac{X_1[8]}{X_1[11]}\right|$.

Solution

4-point DFT matrix

The 4-point DFT matrix is given by:

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \tag{2}$$

where $\omega = e^{\dfrac{-2\pi j}{4}} = -j$.

Now from the given information we can write:

$$X = W_4 x \tag{3}$$

where

$$x = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{4}$$

and

$$X = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \tag{5}$$

Now we need to find X_1 satisfying the relation :

$$X_1 = W_{12} x_1 \tag{6}$$

where

$$x_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

and W_{12} is the 12-point DFT matrix.

12-point DFT matrix

$$
W_{12} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \Omega & \Omega^2 & -j & -j\Omega & -j\Omega^2 & -1 & -\Omega & -\Omega^2 & j & j\Omega & j\Omega^2 \\
1 & \Omega^2 & -j\Omega & -1 & -\Omega^2 & j\Omega & 1 & \Omega^2 & -j\Omega & -1 & -\Omega^2 & j\Omega \\
1 & -j & -1 & j & 1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 \\
1 & -j\Omega^2 & j\Omega & -j & -\Omega^2 & \Omega & -1 & j\Omega^2 & -j\Omega & j & \Omega^2 & -\Omega \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\Omega & \Omega^2 & j & -j\Omega & j\Omega^2 & -1 & \Omega & -\Omega^2 & -j & j\Omega & -j\Omega^2 \\
1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega \\
1 & j & -1 & -j & 1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & j\Omega & -\Omega^2 & -1 & -j\Omega & \Omega^2 & 1 & j\Omega & -\Omega^2 & -1 & -j\Omega & \Omega^2 \\
1 & j\Omega^2 & j\Omega & j & -\Omega^2 & -\Omega & -1 & -j\Omega^2 & -j\Omega & -j & \Omega^2 & \Omega
\end{pmatrix}
$$

where

$$
\Omega = e^{\frac{-2\pi j}{12}} = \frac{\sqrt{3}-j}{2} \tag{8}
$$

Only the red coloured columns in W_{12} give non-zero output when multiplied with x_1.We can express the matrix W_{12} as a block matrix in the following way:

$$W_{12} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} \end{pmatrix} \tag{9}$$

where c_i is the i^{th} column matrix of W_{12}. Now we have :

$$X_1 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

The above equation on simplification gives :

$$\mathrm{X}_1 = 3\mathrm{c}_1 + 2\mathrm{c}_4 + 3\mathrm{c}_7 + 4\mathrm{c}_{10} \tag{11}$$

We can also express W_4 as block matrix as follows:

$$\mathrm{W}_4 = \begin{pmatrix} \mathrm{w}_1 & \mathrm{w}_2 & \mathrm{w}_3 & \mathrm{w}_4 \end{pmatrix} \tag{12}$$

where w_i is the i^{th} column matrix of W_4. Now from (3) we can write:

$$\begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} = \begin{pmatrix} \mathrm{w}_1 & \mathrm{w}_2 & \mathrm{w}_3 & \mathrm{w}_4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{13}$$

$$\implies 3\mathrm{w}_1 + 2\mathrm{w}_2 + 3\mathrm{w}_3 + 4\mathrm{w}_4 = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \tag{14}$$

We can importantly note that :

$$c_1 = \begin{pmatrix} w_1 \\ w_1 \\ w_1 \end{pmatrix} \tag{15}$$

$$c_4 = \begin{pmatrix} w_2 \\ w_2 \\ w_2 \end{pmatrix} \tag{16}$$

$$c_7 = \begin{pmatrix} w_3 \\ w_3 \\ w_3 \end{pmatrix} \tag{17}$$

$$c_{10} = \begin{pmatrix} w_4 \\ w_4 \\ w_4 \end{pmatrix} \tag{18}$$

$$\therefore X_1 = 3c_1 + 2c_4 + 3c_7 + 4c_{10} \quad = \begin{pmatrix} 3w_1 + 2w_2 + 3w_3 + 4w_4 \\ 3w_1 + 2w_2 + 3w_3 + 4w_4 \\ 3w_1 + 2w_2 + 3w_3 + 4w_4 \end{pmatrix} \tag{19}$$

From (14) and (19) we have :

$$X_1 = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \\ 12 \\ 2j \\ 0 \\ -2j \\ 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \tag{20}$$

Therefore we have:

$$\left|\frac{X_1[8]}{X_1[11]}\right| = \left|\frac{12}{-2j}\right| = |6j| = 6 \tag{21}$$

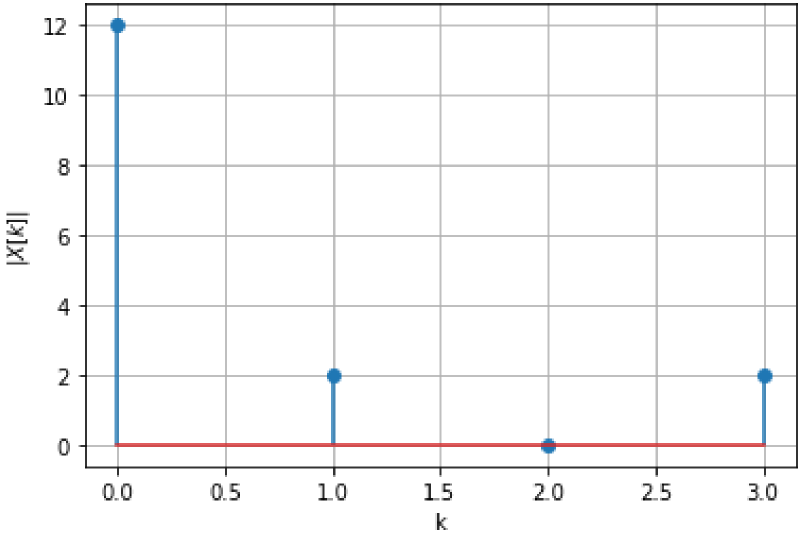

Figure: Magnitude of X[k] vs k

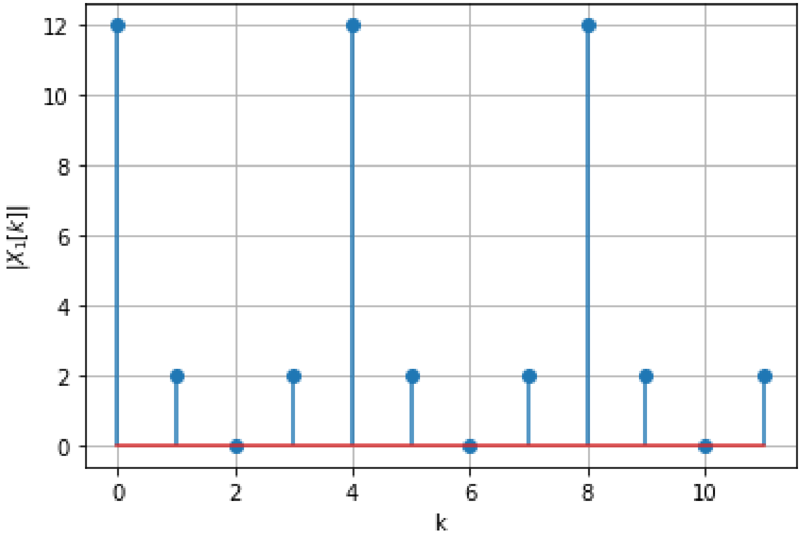

Figure: Magnitude of $X_1[k]$ vs k