Ramsey 4.2 Tangent and Normal Q.16

Nelakuditi Rahul Naga - AI20BTECH11029

August 25, 2021

Question

Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{2}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{3}$$

Solution

General Equation of a Circle

The general equation of a circle can be expressed as:

$$\mathrm{x}^{\top}\mathrm{x} + 2\mathrm{u}^{\top}\mathrm{x} + f = 0 \tag{4}$$

If r is radius and c is the centre of the circle we have:

$$f = \mathrm{u}^{\top}\mathrm{u} - r^2 \tag{5}$$

$$\mathrm{c} = -\mathrm{u} \tag{6}$$

The standard basis vectors in 2D plane are given by:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

We note that the lines (1) and (2) are **parallel** with a common normal along $\mathbf{e}_2$.
Thus the common normal passing through the centre of the circle is of the form :

$$\mathbf{e}_1{}^T \mathbf{x} = k \tag{9}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{10}$$

where k is a constant.

Let the circles of the form (4) touch the lines (1) and (2) at M and N. M is the point of intersection of the following lines:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathrm{x} = k \tag{11}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{x} = 0 \tag{12}$$

The above equations can be expressed as the matrix equation :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{x} = \begin{pmatrix} k \\ 0 \end{pmatrix} \tag{13}$$

The augmented matrix for the above equation is given by :

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \end{pmatrix} \tag{14}$$

As the left part is already a identity matrix ,the intersection point M is given by :

$$\mathrm{M} = \begin{pmatrix} k \\ 0 \end{pmatrix} \tag{15}$$

N is the point of intersection of the following lines:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathrm{x} = k \tag{16}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathrm{x} = 4 \tag{17}$$

The above equations can be expressed as the matrix equation :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{x} = \begin{pmatrix} k \\ 4 \end{pmatrix} \tag{18}$$

The augmented matrix for the above equation is given by :

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 4 \end{pmatrix} \tag{19}$$

As the left part is already a identity matrix ,the intersection point N is given by :

$$\mathrm{N} = \begin{pmatrix} k \\ 4 \end{pmatrix} \tag{20}$$

Also the radius r of the circles is given by:

$$r = \frac{\|\mathbf{M} - \mathbf{N}\|}{2} \tag{21}$$

$$r = \frac{\sqrt{(k-k)^2 + (0-4)^2}}{2} \tag{22}$$

$$r = 2 \tag{23}$$

From (5) we have:

$$f = \begin{pmatrix} -k & -2 \end{pmatrix} \begin{pmatrix} -k \\ -2 \end{pmatrix} - r^2 \tag{24}$$

$$f = k^2 + 2^2 - 2^2 \tag{25}$$

$$f = k^2 \tag{26}$$

The equation of the remaining line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{27}$$

General equation of a 2nd degree conic and point of contact of the tangent

The general equation of a second degree can be expressed as :

$$\mathbf{x}^{\top}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\top}\mathbf{x} + f = 0 \tag{28}$$

The points of contact q, of a line with a normal vector n to the conics in (28) are given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^{\top}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n}}} \tag{30}$$

We know that, for a circle,

$$\mathsf{V} = \mathsf{I} \tag{31}$$

and from the properties of an Identity matrix,

$$\mathsf{I}^{-1} = \mathsf{I} \tag{32}$$

$$\mathsf{IX} = \mathsf{X} \tag{33}$$

From (5), (27) and (31) we have:

$$\kappa = \pm\sqrt{\frac{r^2}{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}} \tag{34}$$

$$= \pm\sqrt{\frac{4}{5}} \tag{35}$$

$$= \pm\frac{2}{\sqrt{5}} \tag{36}$$

Therefore from (29) we have:

$$q = \pm \begin{pmatrix} \frac{4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} - \begin{pmatrix} -k \\ -2 \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \frac{\pm 2}{\sqrt{5}} + 2 \end{pmatrix} \tag{38}$$

Now q lies on the line (27) therefore,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \frac{\pm 2}{\sqrt{5}} + 2 \end{pmatrix} = 2 \tag{39}$$

$$\implies k = \pm\sqrt{5} \tag{40}$$

$$\implies f = k^2 = 5 \tag{41}$$

Hence the tangent circles are given by the equations:

$$\mathrm{x}^{\mathrm{T}}\mathrm{x} + \begin{pmatrix} 2\sqrt{5} & -4 \end{pmatrix} \mathrm{x} + 5 = 0 \tag{42}$$

$$\mathrm{x}^{\mathrm{T}}\mathrm{x} + \begin{pmatrix} -2\sqrt{5} & -4 \end{pmatrix} \mathrm{x} + 5 = 0 \tag{43}$$

The illustration of the circles and the lines is shown below :

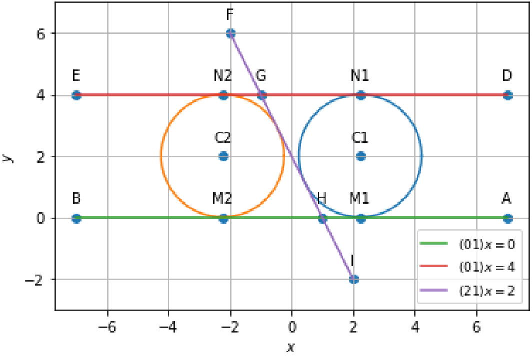

Figure: Circles touching given lines with centres C1,C2