GATE EC 1998 Q1.4

Nelakuditi Rahul Naga - AI20BTECH11029

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Question

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The trigonometric Fourier series of a periodic time function can have only:

1. cosine terms
2. sine terms
3. d.c. ,cosine and sine terms
4. d.c. and cosine terms

Solution

Fourier series of a periodic function

The trigonometric Fourier series of a periodic function $x(t)$ with period T is given by:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{2\pi kt}{T} + \sum_{k=1}^{\infty} b_k \sin\frac{2\pi kt}{T} \tag{1}$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt \tag{2}$$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos\frac{2\pi kt}{T}\,dt \tag{3}$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin\frac{2\pi kt}{T}\,dt \tag{4}$$

The Fourier series of some example functions are demonstrated below.

We have a even periodic function having period 2π defined in $[-\pi, \pi]$ as follows :

$$x(t) = \begin{cases} \frac{\pi}{2} + t, & \text{if } -\pi \le t \le 0 \\ \frac{\pi}{2} - t, & \text{if } 0 < t \le \pi \end{cases}$$

The Fourier series of $x(t)$ is determined as follows:

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x(t)\,dt + \int_{0}^{\pi} x(t)\,dt\right) = 0$$

$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} x(t)\cos kt\,dt + \int_{0}^{\pi} x(t)\cos kt\,dt\right) = \frac{2(1-(-1)^k)}{\pi k^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin kt\,dt = 0$$

$$\implies x(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)t}{(2k-1)^2} \tag{5}$$

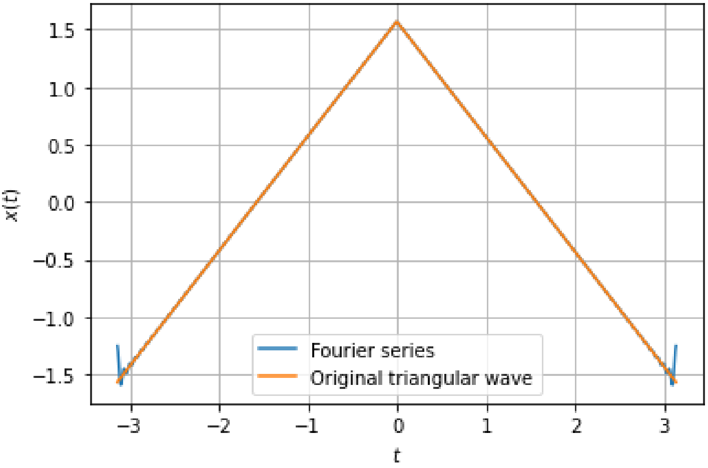

Figure: $x(t)$ vs t

We have a odd periodic function having period 2π defined in $[0, 2\pi]$ as follows :

$$x(t) = \begin{cases} 1, & \text{if } 0 < t < \pi \\ 0, & \text{if } t = 0, \pi, 2\pi \\ -1, & \text{if } \pi < t < 2\pi \end{cases}$$

The Fourier series of $x(t)$ is determined as follows:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\,dt = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos kt\,dt = 0$$

$$b_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} x(t)\sin kt\,dt + \int_{0}^{\pi} x(t)\sin kt\,dt\right) = \frac{2(1-(-1)^k)}{\pi k}$$

$$\implies x(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)t}{2k-1} \tag{6}$$

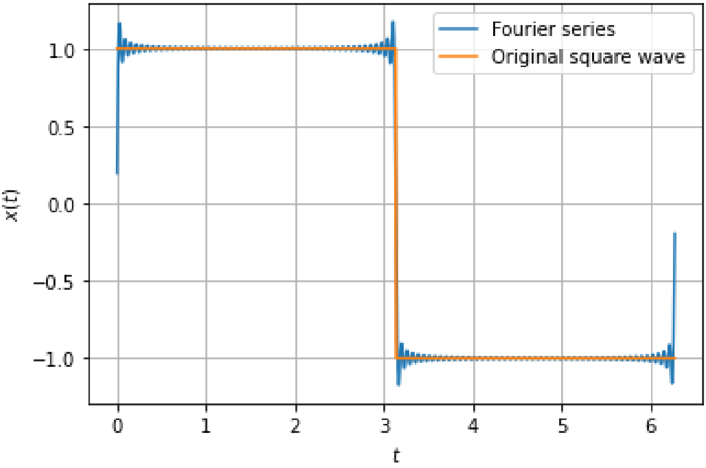

Figure: $x(t)$ vs t

We have a neither even nor odd periodic function having period 2π defined in $(-\pi, \pi)$ as follows :

$$x(t) = \begin{cases} -\pi, & \text{if } -\pi < t < 0 \\ -\frac{\pi}{2}, & \text{if } t = 0 \\ t, & \text{if } 0 < t < \pi \end{cases}$$

The Fourier series of $x(t)$ is determined as follows:

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x(t)\,dt + \int_{0}^{\pi} x(t)\,dt\right) = -\frac{\pi}{4}$$

$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} x(t)\cos kt\,dt + \int_{0}^{\pi} x(t)\cos kt\,dt\right) = \frac{(-1)^k - 1}{\pi k^2}$$

$$b_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} x(t)\sin kt\,dt + \int_{0}^{\pi} x(t)\sin kt\,dt\right) = \frac{2(-1)^k + 1}{k}$$

$$\implies x(t) = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)t}{(2k-1)^2} + \sum_{k=1}^{\infty}\frac{2(-1)^k + 1}{k}\sin kt \tag{7}$$

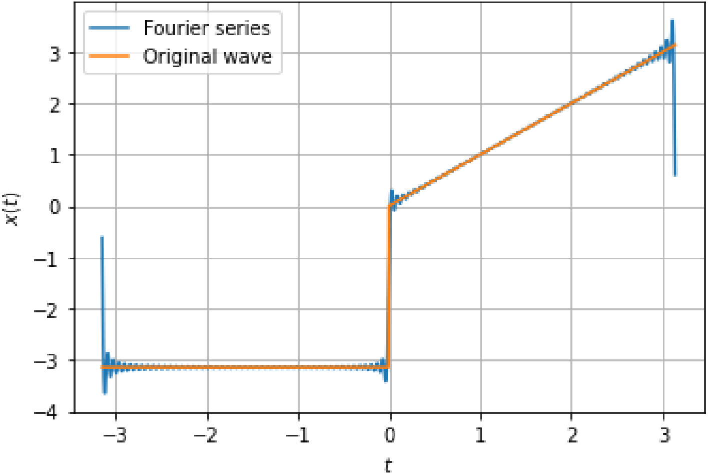

Figure: $x(t)$ vs t

We have a even periodic function having period 2π defined in $(0, 2\pi)$ as follows :

$$x(t) = \begin{cases} t, & \text{if } 0 < t \leq \pi \\ 2\pi - t, & \text{if } \pi \leq t < 2\pi \end{cases}$$

The Fourier series of $x(t)$ is determined as follows:

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x(t)\,dt + \int_{0}^{\pi} x(t)\,dt\right) = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} x(t)\cos kt\,dt + \int_{0}^{\pi} x(t)\cos kt\,dt\right) = \frac{2((-1)^k - 1)}{\pi k^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin kt\,dt = 0$$

$$\implies x(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos{(2k-1)t}}{(2k-1)^2} \tag{8}$$

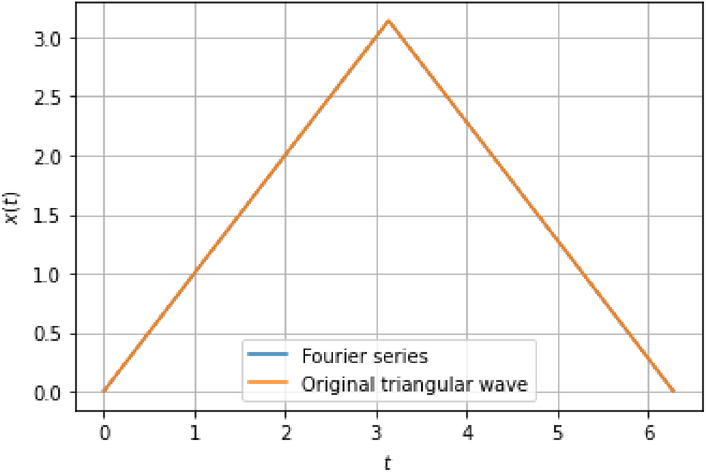

Figure: $x(t)$ vs t

Hence the correct answer is option (3).