

GATE-EC 2016 Q.33

Nelakuditi Rahul Naga - AI20BTECH11029

September 14, 2021

Question

GATE-EC 2016 Q.33

The Discrete Fourier Transform (DFT) of the 4 point sequence $x[n] = \{3, 2, 3, 4\}$ is given by $X[k] = \{12, 2j, 0, -2j\}$. If $X_1[k]$ is the DFT of the 12 point sequence $x_1[n] = \{3, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0\}$, find the value of $\left| \frac{X_1[8]}{X_1[11]} \right|$.

Solution

4-point DFT matrix

The 4-point DFT matrix is given by:

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \quad (2)$$

$$\text{where } \omega = e^{\frac{-2\pi j}{4}} = -j.$$

Now from the given information we can write:

$$X = W_4 x \quad (3)$$

where

$$x = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad (4)$$

and

$$X = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \quad (5)$$

Now we need to find X_1 satisfying the relation :

$$X_1 = W_{12} x_1 \quad (6)$$

where

$$x_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \quad (7)$$

and W_{12} is the 12-point DFT matrix.

12-point DFT matrix

$$W_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \Omega & \Omega^2 & -j & -j\Omega & -j\Omega^2 & -1 & -\Omega & -\Omega^2 & j & j\Omega & j\Omega^2 \\ 1 & \Omega^2 & -j\Omega & -1 & -\Omega^2 & j\Omega & 1 & \Omega^2 & -j\Omega & -1 & -\Omega^2 & j\Omega \\ 1 & -j & -1 & j & 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 & 1 & -j\Omega & -\Omega^2 \\ 1 & -j\Omega^2 & j\Omega & -j & -\Omega^2 & \Omega & -1 & j\Omega^2 & -j\Omega & j & \Omega^2 & -\Omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\Omega & \Omega^2 & j & -j\Omega & j\Omega^2 & -1 & \Omega & -\Omega^2 & -j & j\Omega & -j\Omega^2 \\ 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega & 1 & -\Omega^2 & -j\Omega \\ 1 & j & -1 & -j & 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & j\Omega & -\Omega^2 & -1 & -j\Omega & \Omega^2 & 1 & j\Omega & -\Omega^2 & -1 & -j\Omega & \Omega^2 \\ 1 & j\Omega^2 & j\Omega & j & -\Omega^2 & -\Omega & -1 & -j\Omega^2 & -j\Omega & -j & \Omega^2 & \Omega \end{pmatrix}$$

where

$$\Omega = e^{\frac{-2\pi j}{12}} = \frac{\sqrt{3}-j}{2} \quad (8)$$

We can express x_1 in terms of x as follows :

$$x_1 = Ax \quad (9)$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

Now from (6) have :

$$X_1 = (W_{12}A)x \quad (11)$$

Only the red coloured columns in W_{12} give non-zero output when multiplied with A . We can express the matrix W_{12} as a block matrix in the following way:

$$W_{12} = (\text{c}_1 \text{ c}_2 \text{ c}_3 \text{ c}_4 \text{ c}_5 \text{ c}_6 \text{ c}_7 \text{ c}_8 \text{ c}_9 \text{ c}_{10} \text{ c}_{11} \text{ c}_{12}) \quad (12)$$

where c_i is the i^{th} column matrix of W_{12} . Now we have:

$$\begin{aligned} X_1 &= (\text{c}_1 \text{ c}_2 \text{ c}_3 \text{ c}_4 \text{ c}_5 \text{ c}_6 \text{ c}_7 \text{ c}_8 \text{ c}_9 \text{ c}_{10} \text{ c}_{11} \text{ c}_{12}) A x \\ \Rightarrow X_1 &= (\text{c}_1 \text{ c}_4 \text{ c}_7 \text{ c}_{10}) x \end{aligned} \quad (13)$$

We can also express W_4 as block matrix as follows:

$$W_4 = (w_1 \ w_2 \ w_3 \ w_4) \quad (14)$$

where w_i is the i^{th} column matrix of W_4 .

We can importantly note that :

$$c_1 = \begin{pmatrix} w_1 \\ w_1 \\ w_1 \end{pmatrix} \quad (15)$$

$$c_4 = \begin{pmatrix} w_2 \\ w_2 \\ w_2 \end{pmatrix} \quad (16)$$

$$c_7 = \begin{pmatrix} w_3 \\ w_3 \\ w_3 \end{pmatrix} \quad (17)$$

$$c_{10} = \begin{pmatrix} w_4 \\ w_4 \\ w_4 \end{pmatrix} \quad (18)$$

Therefore from (13) we have:

$$X_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} x = \begin{pmatrix} X \\ X \\ X \end{pmatrix} = \begin{pmatrix} 12 \\ 2j \\ 0 \\ -2j \\ 12 \\ 2j \\ 0 \\ -2j \\ 12 \\ 2j \\ 0 \\ -2j \end{pmatrix} \quad (19)$$

Therefore we have:

$$\left| \frac{X_1[8]}{X_1[11]} \right| = \left| \frac{12}{-2j} \right| = |6j| = 6 \quad (20)$$

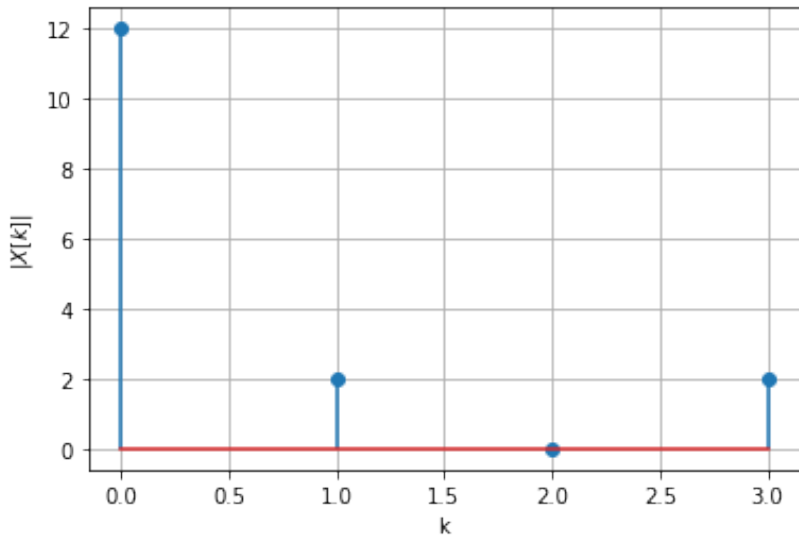


Figure: Magnitude of $X[k]$ vs k

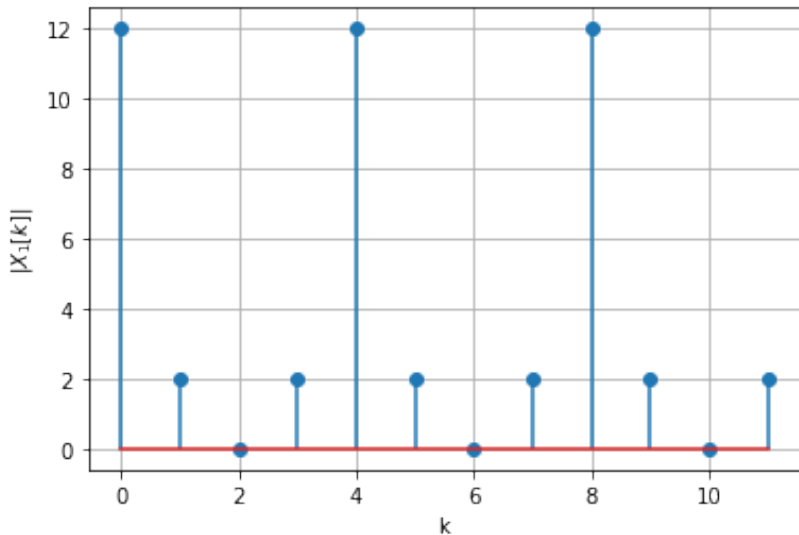


Figure: Magnitude of $X_1[k]$ vs k