

Ramsey 4.2 Tangent and Normal Q.16

Nelakuditi Rahul Naga - AI20BTECH11029

August 28, 2021

Question

Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$(0 \ 1) x = 0 \quad (1)$$

$$(0 \ 1) x = 4 \quad (2)$$

$$(2 \ 1) x = 2 \quad (3)$$

Solution

General Equation of a Circle

The general equation of a circle can be expressed as:

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4)$$

If r is radius and \mathbf{c} is the centre of the circle we have:

$$f = \mathbf{u}^T \mathbf{u} - r^2 \quad (5)$$

$$\mathbf{c} = -\mathbf{u} \quad (6)$$

The standard basis vectors in 2D plane are given by:

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (7)$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8)$$

We note that the lines (1) and (2) are **parallel** with a common normal along e_2 .

Thus the common normal passing through the centre of the circle is of the form :

$$e_1^T x = k \quad (9)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} x = k \quad (10)$$

where k is a constant.

Let the circles of the form (4) touch the lines (1) and (2) at M and N.
M is the point of intersection of the following lines:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} x = k \quad (11)$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} x = 0 \quad (12)$$

The above equations can be expressed as the matrix equation :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} k \\ 0 \end{pmatrix} \quad (13)$$

The augmented matrix for the above equation is given by :

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \end{pmatrix} \quad (14)$$

As the left part is already a identity matrix ,the intersection point M is given by :

$$M = \begin{pmatrix} k \\ 0 \end{pmatrix} \quad (15)$$

N is the point of intersection of the following lines:

$$(1 \ 0) x = k \quad (16)$$

$$(0 \ 1) x = 4 \quad (17)$$

The above equations can be expressed as the matrix equation :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} k \\ 4 \end{pmatrix} \quad (18)$$

The augmented matrix for the above equation is given by :

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 4 \end{pmatrix} \quad (19)$$

As the left part is already a identity matrix ,the intersection point N is given by :

$$N = \begin{pmatrix} k \\ 4 \end{pmatrix} \quad (20)$$

The centre c of the circle must be the mid-point of M and N as M and N are the touch points of parallel tangents to a circle. Therefore we have:

$$c = \frac{\binom{k}{0} + \binom{k}{4}}{2} \quad (21)$$

$$c = \binom{k}{2} \quad (22)$$

Also the radius r of the circles is given by:

$$r = \frac{\|M - N\|}{2} \quad (23)$$

$$r = \frac{\sqrt{(k - k)^2 + (0 - 4)^2}}{2} \quad (24)$$

$$r = 2 \quad (25)$$

We have :

$$c = \begin{pmatrix} k \\ 2 \end{pmatrix} \quad (26)$$

$$\implies u = -c = \begin{pmatrix} -k \\ -2 \end{pmatrix} \quad (27)$$

From (5) we have:

$$f = (-k \quad -2) \begin{pmatrix} -k \\ -2 \end{pmatrix} - r^2 \quad (28)$$

$$f = k^2 + 2^2 - 2^2 \quad (29)$$

$$f = k^2 \quad (30)$$

The equation of the remaining line is

$$(2 \quad 1)x = 2 \quad (31)$$

General equation of a 2nd degree conic and point of contact of the tangent

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (32)$$

The points of contact \mathbf{q} , of a line with a normal vector \mathbf{n} to the conics in (32) are given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \quad (33)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (34)$$

We know that, for a circle,

$$V = I \quad (35)$$

and from the properties of an Identity matrix,

$$I^{-1} = I \quad (36)$$

$$IX = X \quad (37)$$

From (5), (31) and (35) we have:

$$\kappa = \pm \sqrt{\frac{r^2}{(2 \ 1) \begin{pmatrix} 2 \\ 1 \end{pmatrix}}} \quad (38)$$

$$= \pm \sqrt{\frac{4}{5}} \quad (39)$$

$$= \pm \frac{2}{\sqrt{5}} \quad (40)$$

Therefore from (33) we have:

$$q = \pm \begin{pmatrix} \frac{4}{\sqrt{5}} \\ 2 \end{pmatrix} - \begin{pmatrix} -k \\ -2 \end{pmatrix} \quad (41)$$

$$= \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \pm 2 \end{pmatrix} \quad (42)$$

Now q lies on the line (31) therefore,

$$(2 \quad 1) \begin{pmatrix} \frac{\pm 4}{\sqrt{5}} + k \\ \pm 2 \end{pmatrix} = 2 \quad (43)$$

$$\implies k = \pm\sqrt{5} \quad (44)$$

$$\implies f = k^2 = 5 \quad (45)$$

Hence the tangent circles are given by the equations:

$$x^T x + (2\sqrt{5} \quad -4) x + 5 = 0 \quad (46)$$

$$x^T x + (-2\sqrt{5} \quad -4) x + 5 = 0 \quad (47)$$

The illustration of the circles and the lines is shown below :

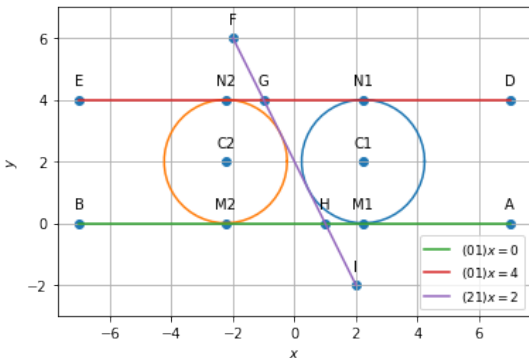


Figure: Circles touching given lines with centres C1,C2