EE3900 : Assignment-3

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Download all python codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.py

and latex-tikz codes from

https://github.com/Rahul27n/EE3900/blob/main/Assignment_3/Assignment_3.tex

1 Question: Ramsey 4.2 Tangent and Normal Q.16

Find the equations of the circles that touch the lines:

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.0.1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 4 \tag{1.0.2}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 2 \tag{1.0.3}$$

2 Solution

The general equation of a circle can be expressed as:

$$\mathbf{x^T}\mathbf{x} + 2\mathbf{u^T}\mathbf{x} + f = 0 \tag{2.0.1}$$

If r is radius and $\mathbf{c}$ is the centre of the circle we have:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.2}$$

$$\mathbf{c} = -\mathbf{u} \tag{2.0.3}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.4}$$

The points of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conics in (2.0.4) are given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{2.0.5}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.6}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.7}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{2.0.8}$$

$$\mathbf{IX} = \mathbf{X} \tag{2.0.9}$$

The touch points of the circles of the form (2.0.1) with line (1.0.1) are determined by:

$$\kappa_1 = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{u} - f}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{2.0.10}$$

$$\kappa_1 = \pm\sqrt{\frac{r^2}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{2.0.11}$$

$$= \pm r \tag{2.0.12}$$

Therefore we have:

$$\mathbf{q_1} = \pm r\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u} \tag{2.0.13}$$

Now $\mathbf{q_1}$ lies on the line (1.0.1) therefore,

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\pm r\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u}\right) = 0 \tag{2.0.14}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{u} = \pm r \tag{2.0.15}$$

The touch points of the circles of the form (2.0.1) with line (1.0.2) are determined by:

$$\kappa_2 = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{u} - f}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{2.0.16}$$

$$\kappa_2 = \pm\sqrt{\frac{r^2}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{2.0.17}$$

$$= \pm r \tag{2.0.18}$$

Therefore we have:

$$\mathbf{q_2} = \pm r\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{u} \tag{2.0.19}$$

Now $\mathbf{q_2}$ lies on the line (1.0.2) therefore,

$$\begin{pmatrix}0 & 1\end{pmatrix}\left(\pm r\begin{pmatrix}0\\1\end{pmatrix} - \mathbf{u}\right) = 4 \tag{2.0.20}$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{u} = \pm r - 4 \tag{2.0.21}$$

The touch points of the circles of the form (2.0.1) with line (1.0.3) are determined by:

$$\kappa_3 = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{u} - f}{\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}} \tag{2.0.22}$$

$$\kappa_3 = \pm\sqrt{\frac{r^2}{\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}} \tag{2.0.23}$$

$$= \pm\frac{r}{\sqrt{5}} \tag{2.0.24}$$

Therefore we have:

$$\mathbf{q_3} = \pm\frac{r}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} - \mathbf{u} \tag{2.0.25}$$

Now $\mathbf{q_3}$ lies on the line (1.0.3) therefore,

$$\begin{pmatrix}2 & 1\end{pmatrix}\left(\pm\frac{r}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} - \mathbf{u}\right) = 2 \tag{2.0.26}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{u} = \pm\sqrt{5}r - 2 \tag{2.0.27}$$

Now we need to solve the equations (2.0.15), (2.0.21) and (2.0.27) written below to obtain $\mathbf{u}$ and r :

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{u} = \pm r$$
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{u} = \pm r - 4$$
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{u} = \pm\sqrt{5}r - 2$$

The first two equations are consistent and give a positive solution for r only when they are of the form :

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{u} = -r$$
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{u} = r - 4$$

which upon solving give :

$$r = 2 \tag{2.0.28}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{u} = -2 \tag{2.0.29}$$

Now putting $r = 2$ in the third equation we have:

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{u} = \pm 2\sqrt{5} - 2 \tag{2.0.30}$$

Let us say $\mathbf{u} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$. Substituting $\mathbf{u}$ in (2.0.29) we have:

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = -2 \tag{2.0.31}$$

$$\implies \beta = -2 \tag{2.0.32}$$

Substituting $\mathbf{u}$ in (2.0.30) we have:

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \pm 2\sqrt{5} - 2 \tag{2.0.33}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}\alpha\\-2\end{pmatrix} = \pm 2\sqrt{5} - 2 \tag{2.0.34}$$

$$\implies \alpha = \pm\sqrt{5} \tag{2.0.35}$$

Therefore we have:

$$\mathbf{u} = \begin{pmatrix}\pm\sqrt{5}\\-2\end{pmatrix} \tag{2.0.36}$$

Hence the value of f is given by:

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{2.0.37}$$

$$f = \begin{pmatrix}\pm\sqrt{5} & -2\end{pmatrix}\begin{pmatrix}\pm\sqrt{5}\\-2\end{pmatrix} - 2^2 \tag{2.0.38}$$

$$f = 5 \tag{2.0.39}$$

Hence the tangent circles are given by the equations:

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + \begin{pmatrix}2\sqrt{5} & -4\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.0.40}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + \begin{pmatrix}-2\sqrt{5} & -4\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.0.41}$$

The illustration of the circles and the lines is shown below :

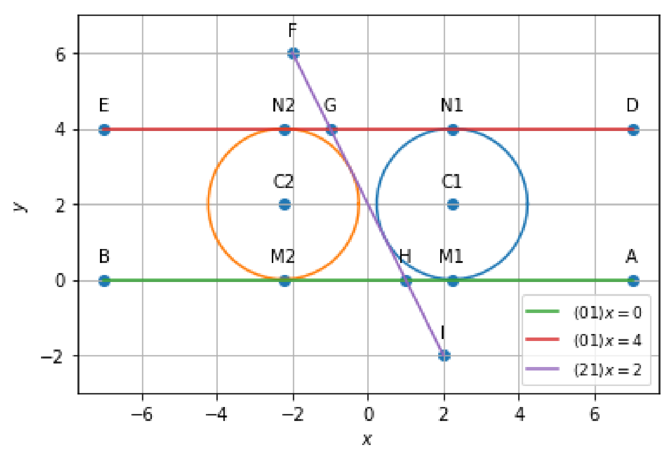

Fig. 0: Circles touching given lines with centres C1,C2