:) Find the expected number of trails in a coin toss to get three consecutive Heads.

⇒ State 0 — No heads
State 1 — 1 head
State 2 — 2 consecutive heads
State 3 — 3 consecutive heads

We have to calculate the expected number of tosses to reach state 3 from state 0

E_0 = Expected no. of tosses to reach state 3 from state 0

E_1 = Expected no. of tosses to reach state 3 from State 1

E_2 = Expected no. of tosses to reach state 3 from state 2

following eq^n based on the transitions betn states

1. From state 0 : $E_0 = 1 + \frac{1}{2} E_1 + \frac{1}{2} E_0$
this eq^n means that from state 0, we need 1 toss plus the exp. no. of tosses from either state 1 (with prob. 1/2) or state 0 (with prob. 1/2).

2. from state 1 : $E_1 = 1 + \frac{1}{2} E_2 + \frac{1}{2} E_0$.
this eq^n means that from state 1, we need 1 toss plus the exp. no. of tosses from either state 2 (with prob. 1/2) or state 0 (with prob. 1/2).

3. From state 2 : $E_2 = 1 + \frac{1}{2} \times 0 + \frac{1}{2} E_0$

this eqⁿ means that from state 2, we need 1 toss plus the exp. no. of tosses from either state 3 (with prob 1/2) which is 0 since we reached our goal) or state 0 (with prob. 1/2).

$$E_0 = 1 + \frac{1}{2} E_1 + \frac{1}{2} E_0$$

$$E_0 = 2 + E_1 \quad \text{——} \quad (1)$$

$$E_1 = 1 + \frac{1}{2} E_2 + \frac{1}{2} E_0$$

$$E_1 = 4 + E_2 \quad \text{——} \quad (2)$$

$$E_2 = 1 + \frac{1}{2} E_0$$

$$E_2 = 2 + \frac{1}{2} E_1 \quad \text{——} \quad (3)$$

Sub E_1 in (3)

$$E_2 = 2 + \frac{1}{2}(4 + E_2)$$

$$= 2 + 2 + \frac{1}{2} E_2$$

$$\frac{1}{2} E_2 = 4$$

$$\boxed{E_2 = 8}$$

$$E_1 = 4 + E_2$$

$$E_1 = 4 + 8$$

$$\boxed{E_1 = 12}$$

$$E_0 = 2 + E_1$$

$$= 2 + 12$$

$$\boxed{E_0 = 14}$$

∴ The expected no. of tosses to get 3 cons. heads is 14.

Q.2. What is Minimum Variance Portfolio?

→ The Minimum Variance Portfolio (MVP) is the investment mix that has the lowest risk for a given level of return. It's all about finding the best combination of assets to keep overall volatility as low as possible. This helps you achieve stable returns with minimal risk.

Q.3. Suppose you have 2 assets in your portfolio, what are the parametes you need to find the variance of the portfolio?

→ Variance of Each Asset:
Variance of Asset 1 (σ_1^2)
Variance of Asset 2 (σ_2^2)

Covariance betn the Assets:
Covariance betn Asset 1 & 2 (σ_{12}).

Weight of Each Asset in the portfolio:
Asset 1 ω_1 , Asset 2 ω_2

You can calculate variance of portfolio using the formulae:

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_{12}$$

Q.4. You have a portfolio consisting of three Asset A, Asset B, Asset C, the portfolio weights & covariance matrix of the returns are provided blow:

Portfolio Weights:

$\omega_A = 0.4$ (40% in Asset A)
$\omega_B = 0.35$ (35% in Asset B)
$\omega_C = 0.25$ (25% in Asset C)

Covariance Matrix :

$$\Sigma = \begin{pmatrix} \sigma^2_{A,A} & \sigma_{A,B} & \sigma_{A,C} \\ \sigma_{A,B} & \sigma^2_{B,B} & \sigma_{B,C} \\ \sigma_{A,C} & \sigma_{B,C} & \sigma^2_{C,C} \end{pmatrix} = \begin{pmatrix} 0.0225 & 0.0180 & 0.0150 \\ 0.0180 & 0.0400 & 0.0250 \\ 0.0150 & 0.0250 & 0.0300 \end{pmatrix}$$

Cal. the variance of the portfolio return using the given portfolio weights & the covariance matrix. Explain the role of the covariance terms in determinig the overall portfolio risk.

⇒ Portfolio Variance (σ_p^2)

$$\sigma_p^2 = W_A^2\sigma_A^2 + W_B^2\sigma_B^2 + W_C^2\sigma_C^2 + 2W_A \cdot W_B\,\sigma_{A,B} + 2W_A \cdot W_C\,\sigma_{A,C} + 2W_B W_C\,\sigma_{B,C}$$

$$= (0.4)^2(0.0225) + (0.35)^2(0.0400) + (0.25)^2(0.0300) + 2(0.4)(0.35)(0.0180) + 2(0.4)(0.25)(0.0150) + 2(0.35)(0.25)(0.0250)$$

$$\boxed{\sigma^2_p \approx 0.0217}$$

Role of Covariance Terms

The covariance terms ($\sigma_{A,B}$, $\sigma_{A,C}$, $\sigma_{B,C}$) in the portfolio variance formulae capture the relationship betn the returns of diff assets.

1) Positive Covariance :- If the returns of two assets move in the same direction (i.e. both increase or decrease together), their covariance is positive. This means that when one asset's return is above its average, the other asset's return is also likely to be above avg. in this case pos. covariance contributes to the overall portfolio risk.

2) Negative covariance :- If the returns of two assets move in opposite directions (i.e. both Increases while other decreasy), their covariance is negative. This means that when one asset's return above its avg. the other assets return is likely to be below avg. in this case, the negative covariance can help to reduce the overall portfolio risk.

So, the covariance terms means the degree to which the returns of different assets are correlated. by carefully selecting assets with low or negative corr. investor can potentially redue the overall risk of their portfolio.

5) What is the difference betn geometric dist & negative binomial dist.
Frame a question based on both of them & solve the question.

=> Key Differences

1. Geometric Distribution:

Purpose → Models the no. of trials needed to get the first success.

Parameter → One parameter, p, which is the prob. of success on each trial.

Probability mass fun:- (PMF) => $P(X=k) = (1-p)^{k-1}$
for $k = 1, 2, 3, \ldots$

Mean : $\frac{1}{p}$

Variance :- $\frac{1-p}{p^2}$

2. Negative Binomial Distribution:-

Purpose → Models the number of trials needed to get a fixed no. of successes.

Parameter → Two parameter, r (number of successes) & p (prob. of success on each trial).

· Prob. Mass function (PMF):

$$P(X=K) = \binom{K+r-1}{r-1} \cdot p^r \cdot (1-p)^K$$

for $K = 0, 1, 2 \ldots\ldots$

· Mean : $\frac{r(1-p)}{p}$

· Variance: $\frac{r(1-p)}{p^2}$

Sample Question

Suppose you are rolling a fair die. Let X be the number of rolls needed to get the first 6 & let Y be the no. of rolls needed to get 6s. What is the prob. that $X=4$ & $Y=10$.

$\Rightarrow$ Prob. for X (Geometric Dist.):

the prob. of getting 6 on any roll is $p = \frac{1}{6}$.

for $X=4$. (i.e getting the 6 on the 4th roll):

$$P(X=4) = \left(1-\frac{1}{6}\right)^{4-1} \cdot \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6}$$

$$= 0.032$$

2. Prob. for Y (Negative Bionomial dist.)
We need prob. of getting three 6s with 10th roll being the last successfull roll.

for $r = 3$, $p = 1/6$, & $k = 10 - 3 = 7$:

$$P(Y=10) = \binom{10-1}{3-1} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(5/6\right)^7$$

$$= \binom{9}{2} \left(\frac{1}{6}\right)^3 \cdot \left(5/6\right)^7$$

$$= 36 \cdot \frac{1}{216} \cdot \frac{78125}{279936}$$

$$= 0.085$$

So the prob. that $X = 4$ is about 0.032 &
prb. that $Y = 10$ is about 0.085

Q.6. You are participating in a game where you start with an initial amount of \$100. In each round of the game. you have a 50% chance of winnig & 50% chance of losing. If you win a round, your money doubles. If you lose a round, your money is halved.
Lets X_n reprent the amount of money you have after n rounds.

• Question:

1) What is the expected amount of money $E(X_1)$ you will have after the first round?

→ Expected Amount After the First Round

• Start with \$100. In each round:

1) if you win (50% chance), your money doubles to \$200.

• 2) If you loose (50% chance), your money is halved to \$50.

To find the expected amount after the first round, you calculate:

$$E(x_1) = (0.5 \times 200) + (0.5 \times 50)$$
$$= 100 + 25$$
$$= 125$$

∴ after the first round you can expect to have \$125.

2) If you play two rounds, what is the expected amount of money $E(x_2)$ you will have after second round?

→ Expected Amount After Two Rounds.

If you play a second round:

- win again : \$400
- Lose : \$100

The expected amount after the second round if you started with \$200 is :

$= (0.5 \times 400) + (0.5 \times 100)$
$= 200 + 50$
$= 250$

if you had \$ 50 after the first round :

- win again : \$100
- Lose : \$25

The expected amount after the second round if you started with \$50 is

$= (0.5 \times 100) + (0.5 \times 25)$
$= 50 + 12.5$
$= 62.5$

Combining these :

$$E(x_2) = (0.5 \times 250) + (0.5 \times 62.5)$$
$$= 125 + 31.25$$
$$\boxed{E(x_2) = 156.25}$$

Q3. Generalize the result to find the expected amount of money $E(x_n)$ after n rounds. What does this say about the long-term expectation of your money if you keep playing the game?

→ If you keep playing, you notice that the expected amount grows by a factor of 1.25 each round. So, if you play n rounds:

$$E(x_n) = 100 \times (1.25)^n$$

This means the longer you play, the more your expected amount of money grows. Over time, the amount increases exponentially, so in the long run, you would expect to have more & more money as you continue playing.