Chapter 21

Linear Difference Equations and Z-Transforms

INTRODUCTION

Difference Equations

Difference equations model processes in which we know relationships between changes or differences rather than rates of changes which lead to differential equations. Thus a difference equation is an equation relating various terms of a sequence a_0, a_1, a_2, ... A string loaded with a finite number of beads at equally spaced points leads to a difference equation. Recurrence relations obtained in the solution of DE by power series or Frobenius method are also difference equations. Numerical solution of DE also leads to difference equations.

Z-transforms

Solution of a discrete system, expressed as a difference equation is obtained using z-transform. Discrete analysis played important role in the development of communication engineering.

21.1 LINEAR DIFFERENCE EQUATIONS

Difference equations are functional equations that define sequences just as differential equations define functions. They arise in several situations as follows.

Finance: Compound Interest

Let P_0 be the amount of money deposited (invested) in a bank earning interest periodically say monthly or quarterly or annually. The conversion period r is the time period between interest payment. Let P_n denote the value of the deposit at the end of the nth conversion period, after n interest payments have accrued.

Then in case of simple interest,

$$P_{n+1} = P_n + rP_0$$

which is a first order difference equation

$$P_{n+1} - P_n = rP_0$$

with solution $P_n = P_0(1 + nr)$

In case of compound interest

$$P_{n+1} = P_n + rP_n$$

which is a homogeneous difference equation

$$P_{n+1} - (1 + r)P_n = 0$$

with solution $P_n = P_0(1 + r)^n$. In this discrete process P is a function of an integer n rather than a continuous variable.

Fibonacci Relation:

Suppose there is only one pair of rabbits male and female just born. Further suppose that every month each pair of rabbits that are one month old produce a new pair of offspring of opposite sexes. Then F_n the number of rabbits after n months is given by the recurrence relation.

$$F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2$$

with the initial conditions $F_0 = F_1 = 1$. This gives rise to a second order homogeneous difference equation

$$F_n - F_{n-1} + F_{n-2} = 0$$

By repeated application of the recurrence relation the equation can be solved recursively. Then we get $F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, F_7 = 21, F_8 = 34$ etc. The disadvantage here is that F_n is calculated upto certain value of n and these values are also dependent on the initial conditions. Instead the general solution

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is function of n and is independent of the initial conditions.

Differential Equations

In the numerical solution of ordinary differential equations, the derivatives are discretized by replacing them by the finite (forward) differences. This gives rise to difference equations of the higher order. Thus a continuous process described by a differential equation is approximated by a discrete process described by its counterpart a difference equation. (see Chapter on Numerical Analysis 33). For example, in a third order ordinary differential equation

$$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = F(x)$$

the derivatives can be replaced by $y' = \dfrac{y_{n+1} - y_n}{h}$,

$y'' = \dfrac{y_{n+2} - 2y_{n+1} + y_n}{h^2}$

$y''' = \dfrac{y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n}{h^3}$, which results in a third order differences equation of the form

$$b_3 y_{n+3} + b_2 y_{n+2} + b_1 y_{n+1} + b_0 y_n = F(x)$$

Recall that a sequence is a numerical valued function whose domain of definition is the set of integers. It is denoted by $\{a_n\}$ or a_n or $a(n)$.

A kth order linear difference equation in the sequence y_n is of the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_1 y_{n+1} + a_0 y_n = f(n) \qquad (1)$$

where $n = 0, 1, 2, 3, \ldots$. Thus (1) represents not just a single equation but an infinite system of equations one equation for every n. Here the coefficients $a_0, a_1, \ldots, a_k$ are all constants and do not depend on n. Here $f(n)$ depends only on n. When a_k is chosen as one, (1) is said to be in the standard form. If $f_n \neq 0$ for all n, then (1) is said to be non-homogeneous. Otherwise it is said to be homogeneous if $f_n = 0$ for all n. The *order* of the difference equation (1) is the positive integer k which is the greatest difference in the index of non-zero values of y. Equation (1) is linear because each term in (1) is of first degree (linear) in y_n. Thus (1) is a non-homogeneous kth order linear difference equation with constant coefficients.

Difference equation is also referred to as recurrence relation since it expresses y_{n+k} in terms of one or more of the previous terms (of the sequence) namely $y_{n+k-1}, \ldots, y_{n+1}, y_n$. In this case (1) can be written as $y_{n+k} = -a_{k-1} y_{n+k-1} \ldots - a_1 y_{n+1} - a_0 y_n + f(n)$. The difference equation (1) models a physical system. So f_n is known as system input (system excitation or forcing sequence or driving sequence) while y_n is referred to as system output (system response). The structure of the system is defined by the values of the coefficients and order of the equation. Thus any system output depends on the system input and the structure of the system. The general solution of (1) determines the output y_n which depends only on n (but no longer on the prior terms of the sequence) and describes the complete sequence y_n in the closed form. Thus any sequence y_n that satisfies the difference equation (1) is a solution of (1). The solution of (1) can be obtained by (a) classical approach similar to those used for solving linear non-homogeneous differential equations with constant coefficients (b) Laplace transform method (c) z-transform method (d) recursive method (which is a numerical solution yielding a finite number of terms of the sequence and is disadvantageous because it is influenced by the change of initial conditions). Here we consider only the classical approach since theory of difference equations is analogous to that of differential equations (which is considered in chapter 9).

Homogeneous Equations

First order homogeneous difference equation

Consider a first order linear homogeneous difference equation $y_{n+1} - b y_n = 0$ for $n \geq 0$ and b is a constant. Assume the solution of the form $y_n = cr^n$ where $c \neq 0, r \neq 0$. Then $y_{n+1} = cr^{n+1}$. Substituting in the given difference equation, we have

$$cr^{n+1} - bcr^n = cr^n(r - b) = 0 \Rightarrow r - b = 0$$

or $b = r$. Thus the general solution of the difference equation $y_{n+1} - by_n = 0$ is given by $y_n = cb^n$. In addition, if a boundary condition $y_0 = d$ then $d = y_0 = cb^0 \quad \therefore c = d$. Then the particular solution is $y_n = db^n$. The solution y_n defines a discrete function whose domain is the set N of all non-negative integers.

Second-order linear homogeneous difference equation with constant coefficients

Consider $a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0$ (2)

Assume $y_n = cr^n$ (with $c \neq 0, r \neq 0$) (3)

as a solution of (2). Then substituting (3) in (2), we get

$$a_2 cr^{n+2} + a_1 cr^{n+1} + a_0 cr^n = 0$$

or $$cr^n(a_2 r^2 + a_1 r + a_0) = 0$$

Thus (3) is solution of (2) if

$$a_2 r^2 + a_1 r + a_0 = 0 \tag{4}$$

since $c \neq 0$ and $r \neq 0$. The equation (4) which is a quadratic in r is known as the *characteristic or auxiliary equation* of (2). Three cases arise.

Case 1: When the roots of the characteristic equation are *real and distinct* given by r_1 and r_2 then r_1^n and r_2^n are two linearly independent solutions. Thus the general solution of (2) is

$$\boxed{y_n = c_1 r_1^n + c_2 r_2^n} \tag{5}$$

where c_1, c_2 are two arbitrary constants.

Case 2: If the roots are *real and equal*, say r, then the general solution of (2) is

$$\boxed{y_n = c_1 r^n + c_2 \cdot n \cdot r^n = (c_1 + n \cdot c_2) r^n} \tag{6}$$

Case 3: Suppose the roots of (4) are *complex conjugate* given by $a \pm bi$. Then the general solution of (2) is

$$\boxed{y_n = r^n(c_1 \cos n\theta + c_2 \sin n\theta)} \tag{7}$$

where $r = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$.

This analysis can easily be extended to kth order difference equation by considering the nature of the k roots of the auxilary equation which is a kth degree polynomial. (see Chapter 9).

Note 1: The forward-difference or advancing-difference operator Δ is defined by $\Delta f_k = f_{k+1} - f_k$

Note 2: The shift operator E is defined as the operator that increases the argument of a function by one tabular interval. Thus

$$Ef_k = Ef(x_k) = f(x_k + h) = f(x_{k+1}) = f_{k+1}$$

Note 3: Δ and E are related by $E = 1 + \Delta$.

Note 4: The difference equation

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_1 y_{n+1} + a_0 y_n = f(n) \tag{1}$$

can be written in terms of E as follows

$$(a_k E^k + a_{k-1} E^{k-1} + \ldots + a_1 E + a_0) y_n = f(n) \tag{8}$$

Non-homogeneous Equations

The general solution of a non-homogeneous linear difference equation with constant coefficients (1) is the sum of the complementary function and any particular solution. Here the complementary function (C.F.) of (1) is the general solution of the corresponding homogeneous equation (2). Particular solution, more often known as particular integral of (1) can be obtained by (a) method of undetermined coefficients (b) short cut inverse operator methods.

(a) In the method of undetermined coefficients

The particular integral is assumed in a particular form depending on the form of the RHS function f_n.

(b) Inverse operator methods

The non-homogeneous equation (8) can be written as

$$F(E) y_n = f(n) \tag{9}$$

where $F(E) = (a_k E^k + a_{k-1} E^{k-1} + \ldots +$
$a_1 E + a_0)$ (10)
is a function of the operator E.

Then the particular integral is

$$\text{P.I.} = \frac{1}{F(E)} f(n)$$

Case 1: If $f(n) = a^n$ then

$$\text{P.I.} = \frac{1}{F(E)} a^n = \frac{1}{F(a)} a^n, \text{ provided } F(a) \neq 0.$$

Case 2: Failure case: If $F(a) = 0$, then

$$\text{P.I.} = \frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)}{3!} a^{n-3}$$

Case 3: If $f(n) = \sin \alpha n$ then

$$\text{P.I.} = \frac{1}{F(E)} \sin \alpha n = \frac{1}{F(E)} \left(\frac{e^{i\alpha n} - e^{-i\alpha n}}{2i} \right)$$

$$= \frac{1}{2i} \left[\frac{1}{F(E)} a^n - \frac{1}{F(E)} b^n \right]$$

where $a = e^{i\alpha n}$ and $b = e^{-i\alpha n}$.
Similarly if $f(n) = \cos \alpha n$, replace

$$\cos \alpha_n = \frac{1}{2}(e^{i\alpha n} + e^{-i\alpha n}) = \frac{1}{2}(a^n + b^n)$$

Case 4: If $f(n) = n^m$ or polynomial in n. Replace E by $1 + \Delta$ and expand $1/F(1+\Delta)$ in binomial series in ascending powers of Δ upto Δ^m. Express $f(n)$ in factorials and use $\Delta[x]^n = n[x]^{n-1}$

Case 5: If $f(n) = a^n V(n)$ where $V(n)$ is a polynomial in n. Then

$$\text{P.I.} = \frac{1}{F(E)} \{a^n V(n)\} = a^n \frac{1}{F(aE)} V(n)$$

Example 2: Solve $a_{n+2} - 6a_{n+1} + 5a_n = 2^n$ with $a_0 = 0, a_1 = 0$.

Solution: The auxiliary equation is $m^2 - 6m + 5 = 0$ with real distinct roots $m_1 = 1, m_2 = 5$. So the complementary function is $c_1 \cdot 1^n + c_2 5^n$. The particular integral is obtained by inverse operator method. The given difference equation is rewritten in terms of the shift operator E as $(E^2-6E+5)a_n = 2^n$. Then the particular integral is

$$\text{P.I.} = \frac{1}{E^2-6E+5}\cdot 2^n = \frac{1}{2^2-6\cdot 2+5}\cdot 2^n$$
$$= -\frac{1}{3}2^n$$

Here E is replaces by 2. Thus the general solution is

$$a_n = \text{C.F.} + \text{P.I.} = c_1 + c_2 5^n - \frac{1}{3}2^n$$

Since $0 = a_0 = c_1 + c_2 - \frac{1}{3}$ $\therefore$ $c_1 + c_2 = \frac{1}{3}$

Since $0 = a_1 = c_1 + 5c_2 - \frac{2}{3}$ $\therefore$ $c_1 + 5c_2 = \frac{2}{3}$

Solving $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{12}$. Then

$$a_n = \frac{1}{4} + \frac{1}{12}5^n - \frac{1}{3}2^n$$

Failure case

Example 3: Solve $y_{n+3} - 12y_{n+2} + 48y_{n+1} - 64y_n = 5\cdot 4^n$.

Solution: A.E. is $m^3 - 12m^2 + 48m - 64 = 0$ or $(m-4)^3 = 0$. The roots are real, equal repeated thrice $m = 4, 4, 4$. So the

$$\text{C.F.} = c_1 4^n + c_2\cdot n\cdot 4^n + c_3 n^2 4^n$$

$$\text{Now P.I.} = \frac{1}{E^3-12E^2+48E-64}5\cdot 4^n$$
$$= \frac{5}{(E-4)^3}\cdot 4^n$$

Here E can *not* be replaced by 4. Then by result (case 2 on page 21.4) we have

$$\text{P.I.} = 5\cdot\frac{n(n-1)(n-2)}{3!}4^{n-3}$$

Thus the general solution is

$$y_n = \text{C.F.} + \text{P.I.} = (c_1 + c_2\cdot n + c_3\cdot n^2)4^n + \frac{5n(n-1)(n-2)}{3!}4^{n-3}$$

Trigonometric function

Example 4: Find the general solution of $(E^2 + 4)y_n = \cos\alpha n$.

Solution: A.E. is $m^2+4=0$ with conjugate complex roots $\pm 2i$. So $a=0$, $b=2$, then $r = \sqrt{a^2+b^2} = 2$, and $\tan\theta = \frac{2}{0} = \infty$. $\therefore$ $\theta = \frac{\pi}{2}$. Thus

$$\text{C.F.} = r^n(c_1\cos n\theta + c_2\sin n\theta)$$
$$\text{C.F.} = 2^n\left(c_1\cos\frac{n\pi}{2} + c_2\sin\frac{n\pi}{2}\right)$$

Now P.I. $= \frac{1}{E^2+4}\cos\alpha n = \frac{1}{(E+2i)(E-2i)}\cos\alpha n$.

But $\cos\alpha n = \frac{1}{2}(e^{in\alpha} + e^{-in\alpha})$. So

$$\text{P.I.} = \frac{1}{(E+2i)(E-2i)}\left\{\frac{1}{2}(e^{in\alpha}+e^{-in\alpha})\right\}$$
$$= \frac{1}{(E+2i)(E-2i)}\frac{1}{2}(a^n+b^n)$$

where $a = e^{i\alpha}$, $b = e^{-i\alpha}$

$$= \frac{1}{2}\frac{1}{(a+2i)(a-2i)}a^n + \frac{1}{2}\frac{1}{(b+2i)(b-2i)}b^n$$
$$= \frac{1}{2}\cdot\frac{1}{a^2+4}a^n + \frac{1}{2}\frac{1}{(b^2+4)}b^n$$
$$= \frac{1}{2}\left[\frac{1}{e^{2i\alpha}+4}e^{i\alpha n} + \frac{1}{e^{-2i\alpha}+4}e^{-i\alpha n}\right]$$
$$= \frac{1}{2}\left[\frac{(e^{-2i\alpha}+4)e^{i\alpha n} + (e^{2i\alpha}+4)e^{-i\alpha n}}{(e^{2i\alpha}+4)(e^{-2i\alpha}+4)}\right]$$
$$= \frac{4\cos\alpha n + \frac{1}{2}\{e^{i\alpha(n-2)} + e^{i\alpha(n-2)}\}}{1+4(e^{2i\alpha}+e^{-2i\alpha})+16}$$
$$= \frac{4\cos\alpha n + \cos\alpha(n-2)}{17+8\cos 2\alpha}$$

Thus the general solution is

$$y_n = 2^n\left(c_1\cos\frac{n\pi}{2} + c_2\sin\frac{\pi}{2}\right) + \left(\frac{4\cos\alpha n + \cos\alpha(n-2)}{17+8\cos 2\alpha}\right)$$

Polynomial

Example 5: Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2n^2 - 6n - 1$.

Solution: A.E. is $m^2 - 5m + 6 = 0$ with real distinct roots 2 and 3. So C.F. $= c_1 2^n + c_2 3^n$. The given difference equation in shift oeprator E is

$$(E^2 - 5E + 6)a_n = 2n^2 - 6n - 1$$

So P.I. $= \dfrac{1}{E^2 - 5E + 6)}(2n^2 - 6n - 1)$

Replace E by $1 + \Delta$ then

$$E^2 - 5E + 6 = (1+\Delta)^2 - 5(1+\Delta) + 6$$

$$= 1 + \Delta^2 + 2\Delta - 5 - 5\Delta + 6 = \Delta^2 - 3\Delta + 2$$

Then

$$\text{P.I.} = \frac{1}{\Delta^2 - 3\Delta + 2}(2n^2 - 6n - 1)$$

$$= \frac{1}{2}\frac{1}{1 + \left(\frac{\Delta^2 - 3\Delta}{2}\right)}(2n^2 - 6n - 1)$$

Expanding in binomial series

$$= \frac{1}{2}\left[1 - \left(\frac{\Delta^2 - 3\Delta}{2}\right) + \left(\frac{\Delta^2 - 3\Delta}{2}\right)^2 + \ldots\right] \times$$

$$\times (2n^2 - 6n - 1)$$

Neglecting powers of Δ more than 2, we get

$$\text{P.I.} = \frac{1}{2}\left[1 + \frac{3}{2}\Delta + \frac{7}{4}\Delta^2\right][2n^2 - 2n - 4n - 1]$$

$$= \frac{1}{2}\left[1 + \frac{3}{2}\Delta + \frac{7}{4}\Delta^2\right]\{2[n]^2 - 4[n] - 1\}$$

where $[n]^2 = n(n-1)$. Then

$$\text{P.I.} = \frac{1}{2}\left[\{2[n]^2 - 4[n] - 1\} + \frac{3}{2}\{4[n] - 4\} + \frac{7}{4}\{4\}\right]$$

$$= \frac{1}{2}\left[\{2n^2 - 6n - 1\} + 6\{(n-1)\} + 7\right] = n^2$$

Therefore the general solution is

$$a_n = c_1 2^n + c_2 3^n + n^2$$

Exponential shift

Example 6: Solve $(E^2 + E - 56)a_n = 2^n(n^2 - 3)$

Solution: A.E. is $m^2 + m - 56 = (m+8)(m-7) = 0$ with distinct real roots -8, 7. So C.F. $= c_1(-8)^n + c_2(7)^n$.

Since the R.H.S. is of the form $a^n F(n)$, apply shift result for obtaining particular integral. So replace E by $2E$, we get

$$\text{P.I.} = \frac{1}{E^2 + E - 56}\{2^n(n^2 - 3)\}$$

$$= \frac{2^n}{(2E)^2 + 2E - 56}\{(n^2 - 3)\}.$$

$$= \frac{2^n}{2(2E^2 + E - 28)}(n^2 - 3)$$

$$= \frac{2^n}{2[2(1+\Delta)^2 + (1+\Delta) - 28)]}(n^2 - 3)$$

where E is replaced by $1 + \Delta$.

$$\text{P.I.} = \frac{2^n}{2[2\Delta^2 + 5\Delta - 25]}(n^2 - 3)$$

$$= -\frac{2^n}{50}\left[1 - \left(\frac{5\Delta + 2\Delta^2}{25}\right)\right]^{-1}(n^2 - 3)$$

Expanding in binomial series we get

$$\text{P.I.} = -\frac{2^n}{50}\left[1 + \frac{5\Delta + 2\Delta^2}{25} + \left(\frac{5\Delta + 2\Delta^2}{25}\right)^2 + \ldots\right] \times$$

$$\times (n^2 - 3).$$

Neglecting powers of Δ more than 2 we have

$$\text{P.I.} = -\frac{2^n}{50}\left[1 + \frac{5}{25}\Delta + \frac{3}{25}\Delta^2\right]\{[n]^2 + [n] - 3\}$$

where $n^2 - 3 = n^2 - n + n - 3 = n(n-1) + n - 3 = [n]^2 + [n] - 3$.

$$\text{P.I.} = -\frac{2^n}{50}[\{[n]^2 + [n] - 3\} + \frac{5}{25}\{2[n] + 1\} + \frac{3}{25}\{2\}]$$

$$\text{P.I.} = -\frac{2^n}{50}\left[n^2 + \frac{2}{5}n - \frac{64}{25}\right]$$

Therefore the general solution is

$$a_n = c_1(-8)^n + c_2(7)^n - \frac{2^{n-1}}{25}\left(n^2 + \frac{2}{5}n - \frac{64}{25}\right)$$