CS510 Fall 2021 Assignment #1 solutions

1) a) Exercise 0.3 solution:
    a. no
    b. yes
    c. {x, y, z}
    d. {x, y}
    e. {(x,x), (x,y), (y,x), (y,y), (z,x), (z,y)}
    f. {{}, {x}, {y}, {x,y}}

b) Exercise 0.4 solution:

For each element x of A, the Cartesian product A × B contains exactly b pairs of the form (x, y), where y is an element of B. Since there are a elements in A, the number of pairs in the Cartesian product is ab.

c) Exercise 0.5 solution:

Power set of C is a set of subsets of C. Therefore, the size of the power set of C is equal to the number of subsets of C. If there are c elements in C, then the number of subsets of C is 2^c. This is because for each element, you have two choices: either you put it in the subset, or you don't; and these choices are independent.

2) Exercise 0.11(a)(b) You have to use mathematical induction.

a. S(n) = 1 + 2 + … + n. Prove that S(n) = $\frac{1}{2}n(n+1)$.

Proof: Let the property *P(n)* be the equation S(n) = $\frac{1}{2}n(n+1)$. ← *P(n)*

Show that P(1) is true: To establish *P(1)*, we must show that when 1 is substituted in place of *n*, the left-hand side equals the right-hand side. But when *n* = 1, the left-hand side is the sum of all the integers from 1 to 1, which is just 1. The right-hand side is $\frac{1}{2}\cdot 1(1+1)$, which also equals 1. So *P(1)* is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is true: Let *k* be any integer with *k* ≥ 1.

Suppose P(k) is true. That is: S(k) = $\frac{1}{2}k(k+1)$. *[This is the inductive hypothesis.]*

We must show that P(k + 1) is true. That is: S(k+1) = $\frac{1}{2}(k+1)(k+1+1)$. ← *P(k + 1)*

The left-hand side of *P(k + 1)* is
S(k+1)
= 1 + 2 + 3 + … + (*k* + 1)
= [1 + 3 + 5 + … + *k*] + (*k* + 1)   - by separating the last term
= $\frac{1}{2}k(k+1)$ + (*k* + 1) by the inductive hypothesis
= $\frac{1}{2}k^2 + \frac{1}{2}k + k + 1 = \frac{1}{2}k^2 + \frac{3}{2}k + 1$

The right-hand side of *P(k + 1)* is

$$\frac{1}{2}(k+1)(k+1+1) = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}(k^2+3k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1$$

which is equal to the left-hand side of *P(k + 1) [as was to be shown.]*

b. C(n) = $1^3 + 2^3 + \ldots + n^3$. Prove that C(n) = $\frac{1}{4}n^2(n+1)^2$.

Proof: Let the property *P(n)* be the equation C(n) = $\frac{1}{4}n^2(n+1)^2$. ← *P(n)*

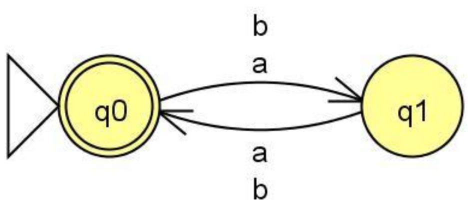
Show that P(1) is true: When *n* = 1, the left-hand side is the sum of cubes of integers from 1 to 1, which is just 1. The right-hand side is $\frac{1}{4}\cdot 1^2(1+1)^2$, which also equals 1. So *P(1)* is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is true: Let k be any integer with $k \geq 1$.
Suppose P(k) is true. That is: C(k) = $\frac{1}{4}k^2(k+1)^2$. *[This is the inductive hypothesis.]*
We must show that P(k + 1) is true. That is: C(k+1) = $\frac{1}{4}(k+1)^2(k+1+1)^2$. ← *P(k* + 1*)*
The left-hand side of *P(k* + 1*)* is
C(k+1)
= $1^3 + 2^3 + \ldots + (k+1)^3$
= $[1^3 + 2^3 + \ldots + k^3] + (k+1)^3$ - by separating the last term
= $\frac{1}{4}k^2(k+1)^2 + (k+1)^3$ by the inductive hypothesis
= $\frac{1}{4}k^2(k+1)^2 + \frac{1}{4} \cdot 4(k+1)^3$
= $\frac{1}{4}(k+1)^2(k^2+4(k+1)) = \frac{1}{4}(k+1)^2(k^2+4k+4) = \frac{1}{4}(k+1)^2(k+2)^2 = \frac{1}{4}(k+1)^2(k+1+1)^2$
which is the right-hand side of *P(k* + 1*) [as was to be shown.]*

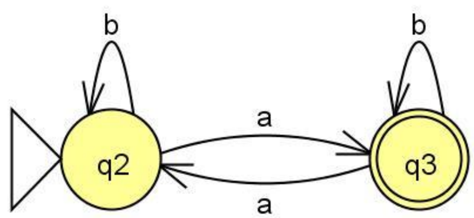
3) Exercise 1.4 (g) (Note: you will have 3 automata here)

Solution:

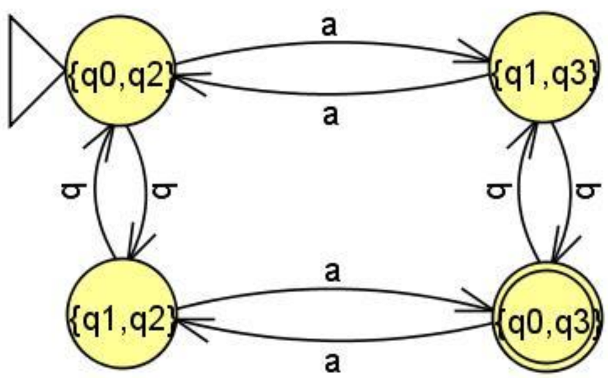
{w| w has even length}



{w| w has an odd number of *a*'s }



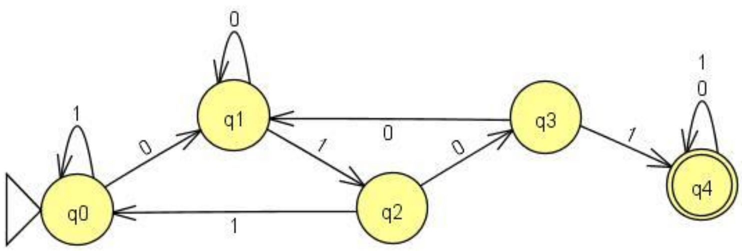
{w| w has even length and an odd number of *a*'s}



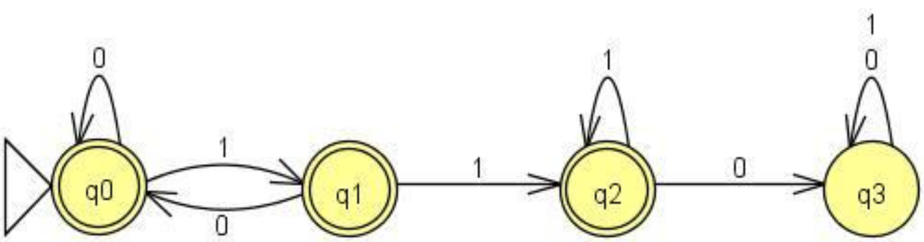
4) Exercise 1.6 (c), (f), (g), (i) (Note: one automaton for each part (c, f, g, i)) Note: in Exercise 1.6(i) positions start with position 1 (not 0). That is, string 101 is accepted

Solution:

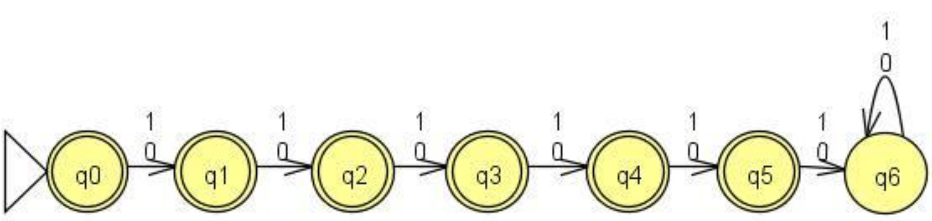
1.6(c) {w| w contains the substring 0101}



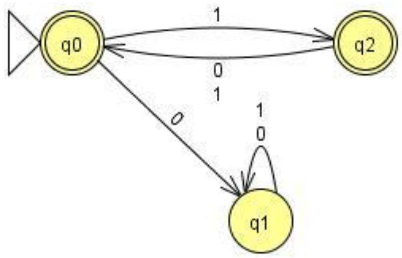
1.6(f) {w| w doesn’t contain the substring 110}



1.6(g) {w| the length of w is at most 5}

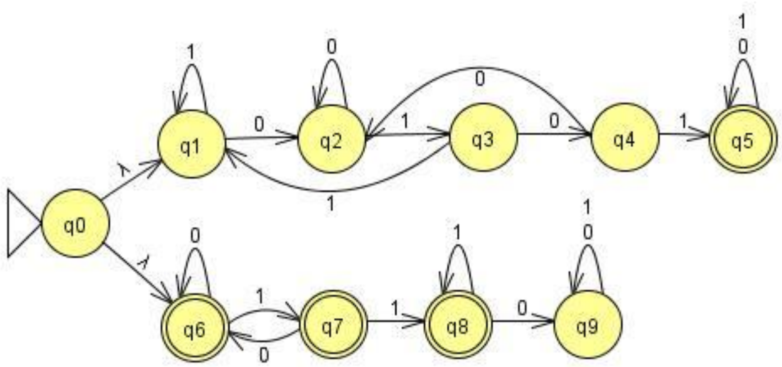


1.6(i) {w| every odd position of w is a 1}



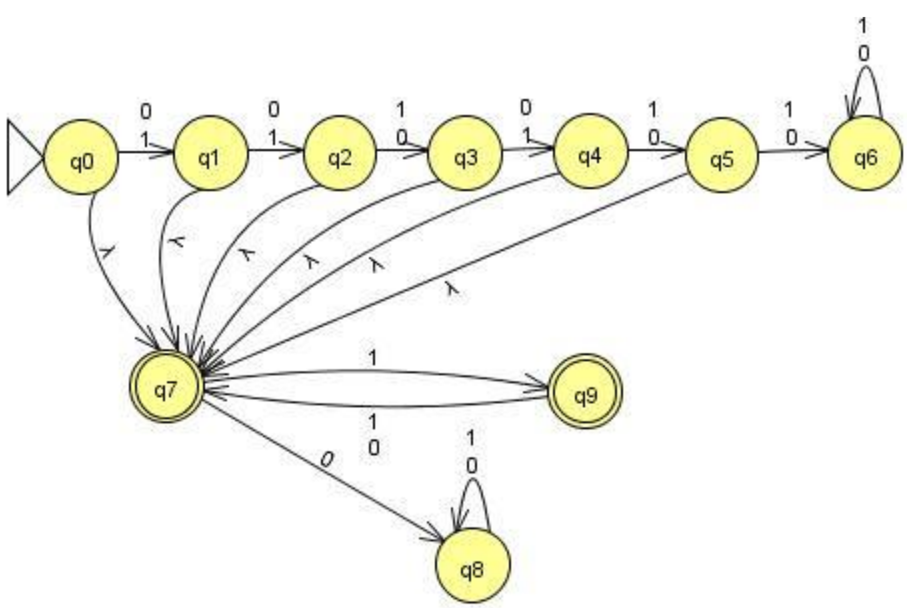
5) Exercise 1.8 (b)

Solution:



6) Exercise 1.9 (a)

Solution:



7) Exercise 1.16 (b)

Solution: