CS510 Assignment #4 solutions

1. **Exercise 4.2.**

   Let $EQ_{\text{DFA, REX}}$ = {<A, R> | A is a DFA, R is a regular expression and L(A) = L(R)}. The following TM E decides $EQ_{\text{DFA, REX}}$.

   E = "On input <A, R>:
   i. Convert regular expression, R to an equivalent DFA B using the procedure given in Theorem 1.54.
   ii. Use the TM F for deciding EQ_{DFA} given in Theorem 4.5, on input <A, B>.
   iii. If F accepts *accept*. If F rejects, *reject*."

2. **Exercise 4.3**

   Let ALL_{DFA} = {<A> | A is a DFA that recognizes $\sum^*$}. The following TM, L decides ALL_{DFA}.

   L = "On input <A> where A is a DFA:
   i. Construct DFA B that recognizes the complement of L(A), $\overline{L(A)}$. It can be constructed by swapping the accept and the nonaccept states in A.
   ii. Run TM T from Theorem 4.4 on input <B>, where T decides E_{DFA}.
   iii. If T accepts *accept*. If T rejects, *reject*."

3. **Exercise 4.4**

   Let $A\varepsilon_{\text{CFG}}$ = {<G> | G is a CFG that generates ε}. The following TM V decides $A\varepsilon_{\text{CFG}}$.

   V = "On input <G> where G is a CFG:
   i. Run TM, S from Theorem 4.7 on input <G, ε>, where S is a decider of A_{CFG} .
   ii. If S accepts *accept*. If S rejects, *reject*."

4. **Exercise 4.6 (b, c, e, f)**

   (b) *f* is not onto because there does not exist x ∈ X such that *f*(x) = 10.
   (c) *f* is not a correspondence since *f* is not one-to-one or onto.
   (e) *g* is onto.
   (f) *g* is a correspondence since *g* is one-to-one or onto.

5. **Let A be the set of all functions from natural numbers (N) to natural numbers:**
   A={f | f is a function, f: N → N }, where N ={1, 2, 3, …}.
   Show that A is uncountable using a proof by diagonalization.

   Solution:

   Proof by contradiction. We assume that A is countable, i.e., we can give an enumeration

$f_1, f_2, f_3, \ldots$ of A. To come to a contradiction, we construct a new function $\bar{f}$ as $\bar{f}(x) = f_x(x) + 1$ for $x \in$ N. The function $\bar{f}$ is constructed from the diagonal of the function values of $f_i \in$ A as represented in the figure below. For each x, $\bar{f}$ differs from f_x on input x. Hence, $\bar{f}$ does not appear in the given enumeration. However $\bar{f}$ is a function and $\bar{f}$: N → N . Such an $\bar{f}$ can be given for any chosen enumeration. This leads to a contradiction. Therefore A cannot be enumerated. Hence A is uncountable.

f_1	$f_1(1)$	$f_1(2)$	$f_1(3)$	…
f_2	$f_2(1)$	$f_2(2)$	$f_2(3)$	…
f_3	$f_3(1)$	$f_3(2)$	$f_3(3)$	…
…	…	…	…	…

6. **Exercise 4.8**

Solution 1: To show that a set is countable it is sufficient to show that there is a one-to-one function from the set to the set of natural numbers N. We define a one-to-one function f: $T \rightarrow N$ as follows. Let $f(i, j, k) = 2^i 3^j 5^k$. Function f is one-to-one because if $a \neq b$, then $f(a) \neq f(b)$. So T is countable.

Solution 2: T can be enumerated as follows. For each triple (i, j, k), we call i + j + k the sum of the triple. For each natural number n, we have only finitely many triples with sum equal to n. So we can first enumerate all triples with sum 1, and then all triples with sum 2, and then all triples with sum 3, and so on. This clearly allows us to enumerate all the triples in T.

7. **Exercise 4.13**

We observe that L(R) ⊆ L(S) if and only if $L(R) \cap \overline{L(S)} = \emptyset$. The following TM X decides A.

X = "On input <R, S> where R and S are regular expressions:

i. Construct DFA E such that L(E) = $L(R) \cap \overline{L(S)}$.
ii. Run TM T on <E>, where T decides E_{DFA} from Theorem 4.4.
iii. If T accepts *accept*. If T rejects, *reject*."