The Pumping Lemma for Context Free Languages

Introduction and Motivation

In this lecture we present the ***Pumping Lemma*** for ***Context Free Languages***.

This lemma enables us to prove that some languages are not CFL and hence are not recognizable by any PDA.

The Pumping Lemma

Let A be a context free language. There exists a number p such that for every $w \in A$, if $|w| \geq p$ then w may be divided into **five** parts, $w = uvxyz$ satisfying:

1. for each $i \geq 0$, it holds that $uv^i xy^i z \in A$.
2. $|vy| > 0$.
3. $|vxy| \leq p$.

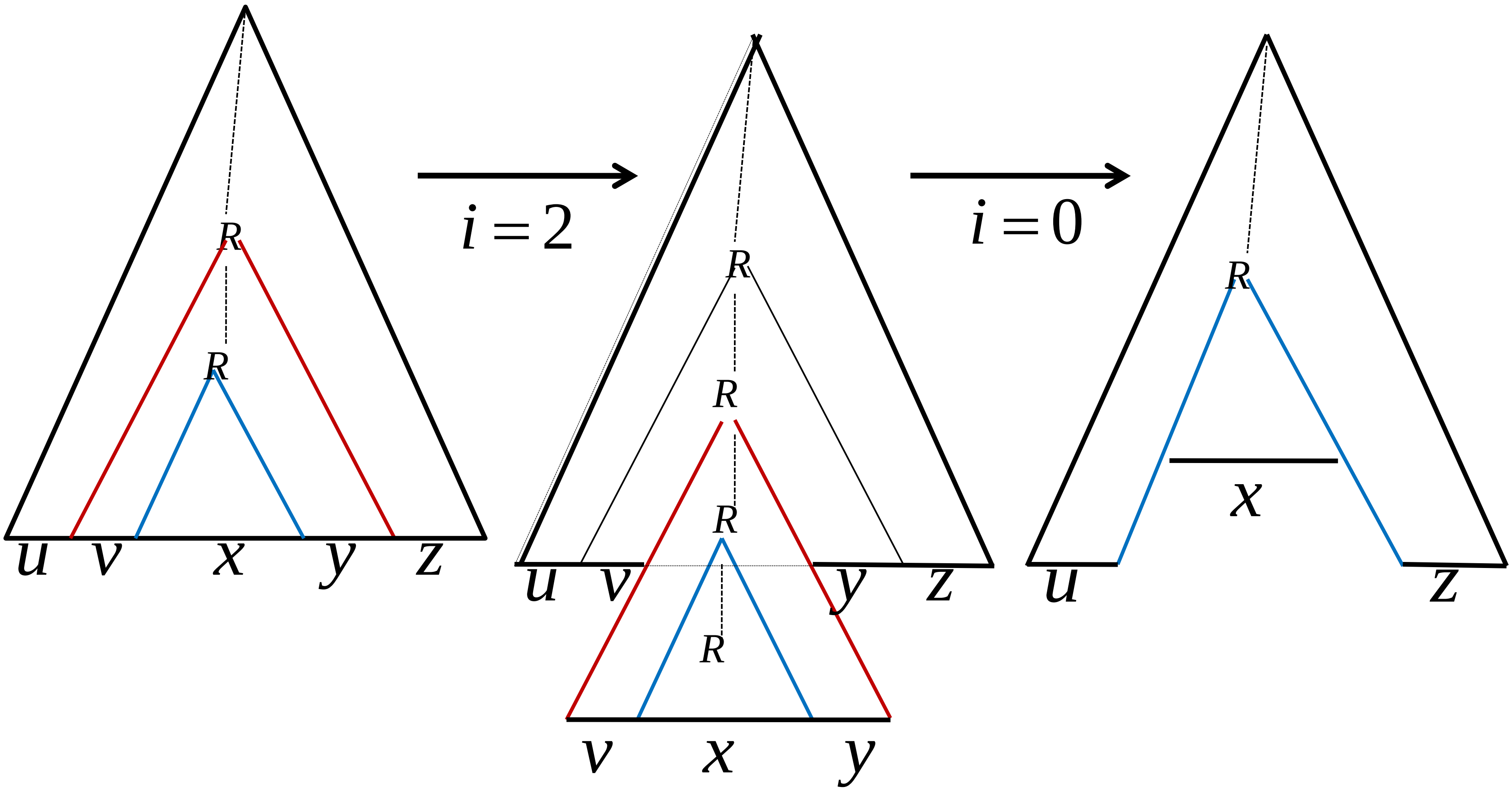
Note: Without req. 2 the Theorem is **trivial**.

Proof Idea

If w is "long enough" (to be precisely defined later) it has a large parse tree which has a "long enough" path α from its root to one of its leaves.

Under these conditions, some variable on α should appear **twice**. This enables pumping of w as demonstrated in the next slide:

Proof Idea



Pumping up

Pumping down

Reminder

Let T be a binary tree.

The 0-*th* level of T has $1 = 2^0$ nodes.

The 1-*th* level of T has at most $2 = 2^1$ nodes.

…

The i-*th* level of T has at most 2^i nodes.

If T' is a b-*ari* tree then its i-*th* level has at most b^i nodes.

The Proof

Let G be a grammar for the language L. Let b be the maximum number of symbols (variables and constants) in the right hand side of a rule of G (assume $b \geq 2$). In any parse tree T, for generating w from G, a node of T may have no more than b children.

If the height of T is h then $|w| \leq b^h$.

The Proof (cont.)

If the height of T is h then $|w| \leq b^h$. Conversely,

If $|w| > b^h$ then the height of T is at least $h+1$.

Assume that G has $|V|$ variables. Then we set

$$p = b^{|V|+1}.$$

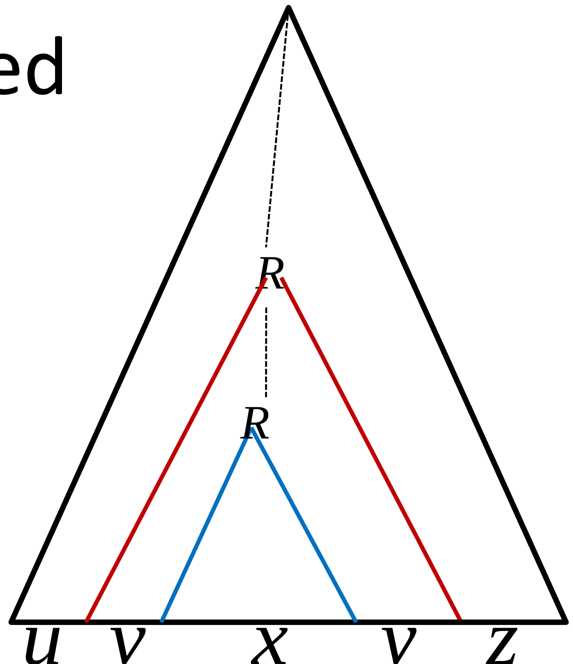
Conclusion: For any $w \in L$, if $|w| \geq p$, then the height of any parse tree of w is **at least** $|V|+1$ (because $|w| \geq p = b^{|V|+1} \geq b^{|V|} + 1 > b^{|V|}$).

The Proof (cont.)

To see how pumping works let τ be the parse tree of w with a **minimal number of nodes**. The height of the tree τ, is at least $|V|+1$, so it has a path, α with at least $|V|+2$ nodes, from its root until some leaf. The path α has at least $|V|+1$ variables and a single terminal.

The Proof (cont.)

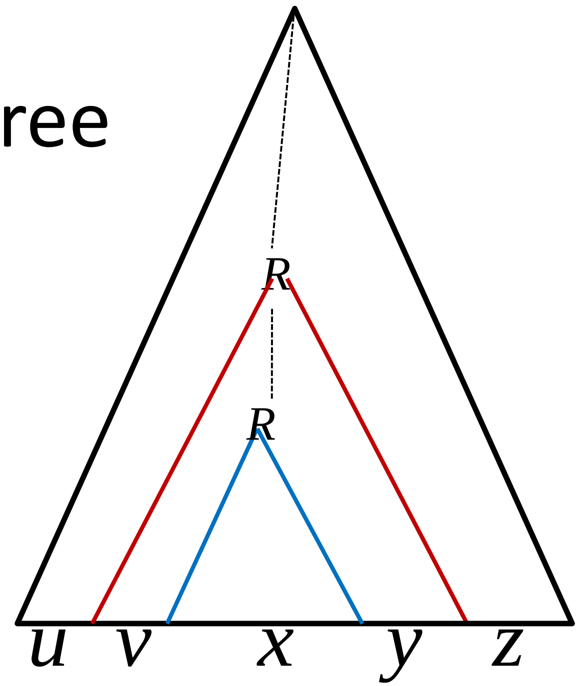
Since G has $|V|$ variables and α has at least $|V|+1$ variables, there exists a variable, R , that repeats itself among the $|V|+1$ lowest variables of α, as depicted in the following picture:



The Proof (cont.)

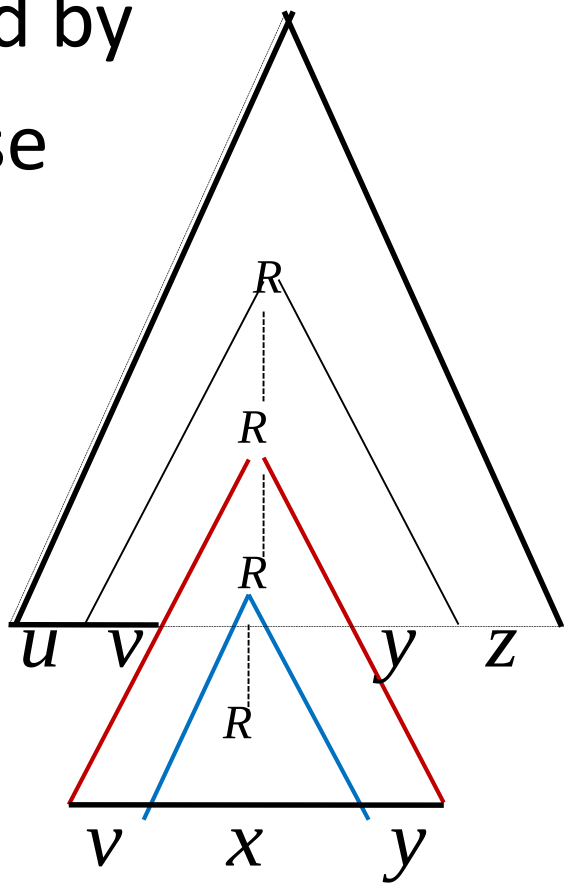
Each occurrence of R has a sub-tree rooted at it:

Let vxy be the word generated by the upper occurrence of R and let x be the word generated by the lower occurrence of R.



The Proof (cont.)

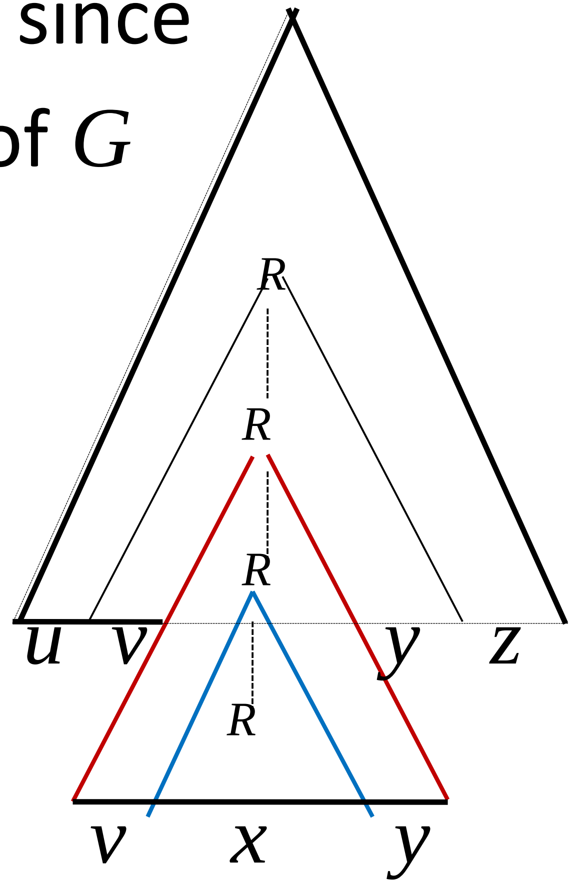
Since both sub-trees are generated by the same variable, each of these sub-trees can be replaced by another . This tree is obtained from τ by substituting the upper sub-tree at the lower occurrence of R.



The Proof (cont.)

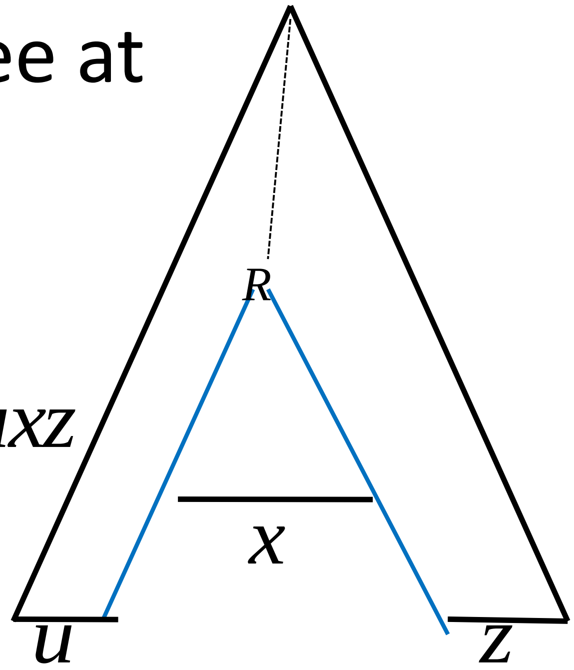
The word generated is uv^2xy^2z, and since

- It is generated by a parse tree of G we get $uv^2xy^2z \in A$. Additional substitutions of the upper sub-tree at the lower occurrence of R, yield the conclusion $uv^ixy^iz \in A$ for each $i > 0$.



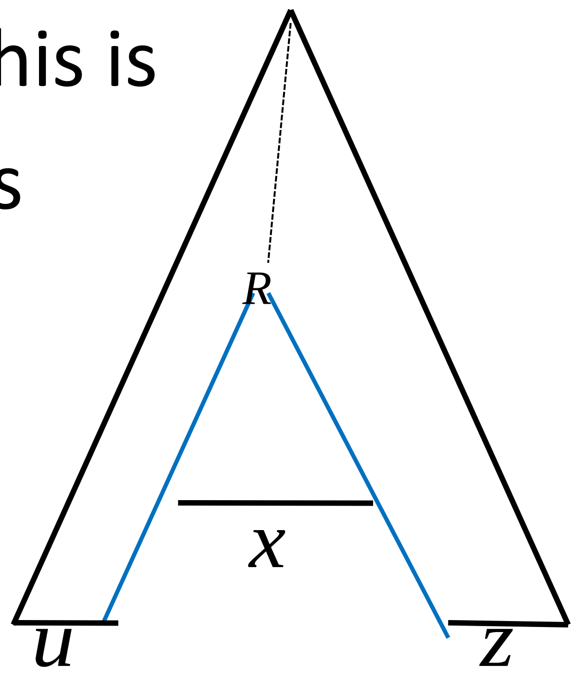
The Proof (cont.)

Substitution of the lower sub-tree at the upper occurrence of R yields this pars tree whose generated word is $uv^0xy^0z = uxz$ Since once again this is a legitimate parse tree we get $uxz \in A$.



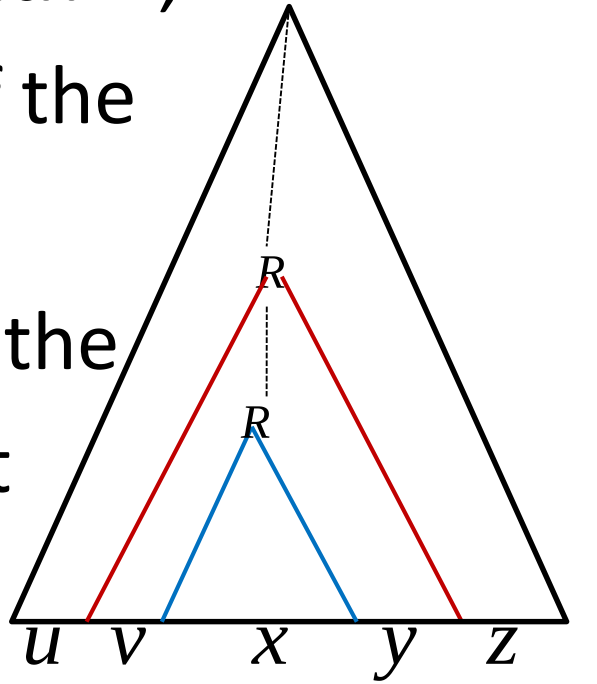
The Proof (cont.)

To see that $|vy|>0$, assume that this is the situation. In this case, this tree is a parse tree for w with less nodes then τ , in contradiction with the choice of τ as a parse tree for w with a minimal number of nodes.



The Proof (cont.)

In order to show that $|vxy| \le p$ recall that we chose R so that both its occurrences fall within the bottom $|V|+1$ nodes of the path α, where α is the longest path of the tree so the height of the red sub-tree is at most $|V|+1$ and the number of its leaves is at most $b^{|V|+1} = p$.



Using the Pumping Lemma

Now we use the pumping lemma for CFL to show that the language $L = \{a^n b^n c^n \mid n \geq 0\}$ is not CFL.

Assume towards a contradiction that L is CFL and let p be the pumping constant. Consider $w = a^p b^p c^p$. Obviously $w \in L$.

Using the Pumping Lemma

By the pumping lemma, there exist a partition $w = uvxyz$ where $|vy| > 0$, $|vxy| \leq p$ and for each i, it holds that $uv^i xy^i z \in L$.

Case 1: Both v and y contain one type of symbol each:

Together they may hold 2 types of symbols, so in uv^2xy^2z, the third symbol appears less often than the other two.

Using the Pumping Lemma

Case 2: Either v or y contains more than one type of symbol:

In this case, the word uv^2xy^2z has more than three blocks of identical letters: In other words: $uv^2xy^2z \notin a^+b^+c^+$. End of proof.

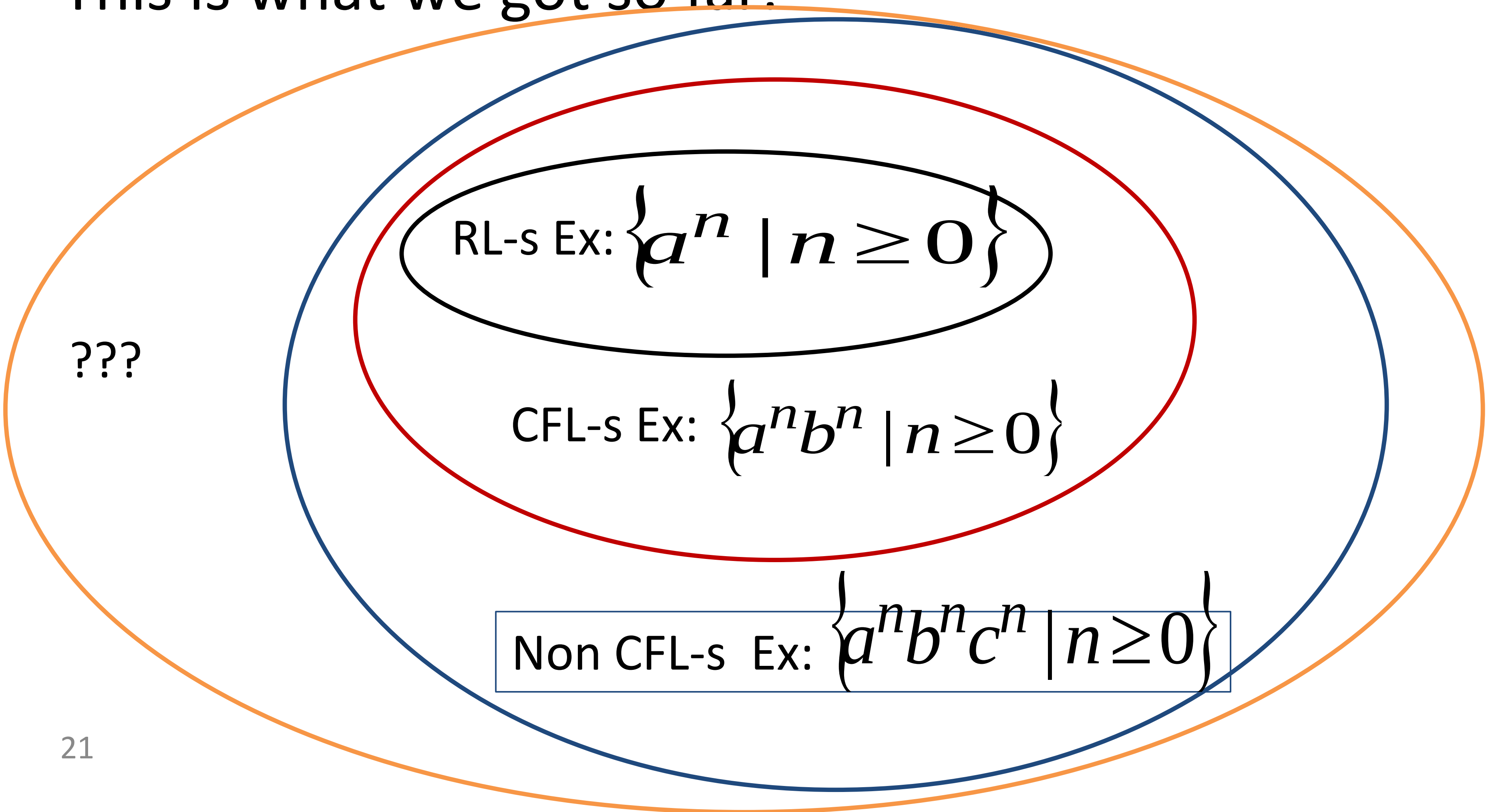
Discussion

Some weeks ago we started our quest to find out **"What can be computed and what cannot?"**

So far we identified two classes: RL-s and CFL-s and found some examples which do not belong in neither class

Discussion

This is what we got so far:



Discussion

Moreover: Our most complex example, namely, the language $L=\{a^n b^n c^n \mid n \geq 0\}$ is easily recognizable by your everyday computer, so we did not get so far yet.

Our next attempt to grasp the essence of "What's Computable?" are **Turing Machines**.

Recap

In this lecture we introduced and proved the **Pumping Lemma for CFL-s**

Using this lemma we managed to prove that the fairly simple language $L = \{a^n b^n c^n \mid n \geq 0\}$, is not CFL.

The next step is to define **Turing Machines**.