# Diffie-Hellman

# Diffie-Hellman

- Invented by Williamson (GCHQ) and, independently, by D and H (Stanford)
- A "key exchange" algorithm
  - Used to establish a shared symmetric key
- Not for encrypting or signing
- Security rests on difficulty of **discrete log** problem: given $g$, $p$, and $g^k \bmod p$ find $k$

# Diffie-Hellman

- Let $p$ be prime, let $g$ be a **generator**
  - For any $x \in \{1,2,\ldots,p-1\}$ there is $n$ s.t. $x = g^n \bmod p$
- Alice selects secret value $a$
- Bob selects secret value $b$
- Alice sends $g^a \bmod p$ to Bob
- Bob sends $g^b \bmod p$ to Alice
- Both compute shared secret $g^{ab} \bmod p$
- Shared secret can be used as symmetric key

# Diffie-Hellman

- Suppose that Bob and Alice use $g^{ab} \bmod p$ as a symmetric key
- Trudy can see $g^a \bmod p$ and $g^b \bmod p$
- Note $g^a g^b \bmod p = g^{a+b} \bmod p \neq g^{ab} \bmod p$
- If Trudy can find $a$ or $b$, system is broken
- If Trudy can solve **discrete log** problem, then she can find $a$ or $b$

# Diffie-Hellman

- **Public:** g and p
- **Secret:** Alice's exponent a, Bob's exponent b

$g^a \bmod p$ →

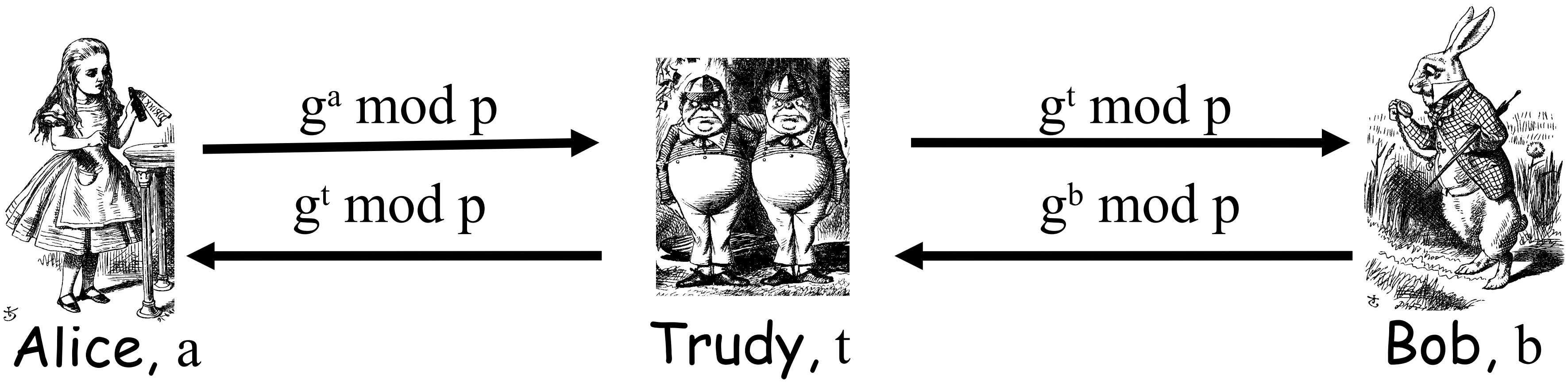
← $g^b \bmod p$

Alice, a — Bob, b

- Alice computes $(g^b)^a = g^{ba} = g^{ab} \bmod p$
- Bob computes $(g^a)^b = g^{ab} \bmod p$
- Could use $K = g^{ab} \bmod p$ as symmetric key

# Diffie-Hellman

- Subject to man-in-the-middle (MiM) attack




- Trudy shares secret $g^{at} \bmod p$ with Alice
- Trudy shares secret $g^{bt} \bmod p$ with Bob
- Alice and Bob don't know Trudy exists!

# Diffie-Hellman

- How to prevent MiM attack?
  - Encrypt DH exchange with symmetric key
  - Encrypt DH exchange with public key
  - Sign DH values with private key
  - Other?
- You **MUST** be aware of MiM attack on Diffie-Hellman

# Elliptic Curve Cryptography

# Elliptic Curve Crypto (ECC)

- "Elliptic curve" is **not** a cryptosystem
- Elliptic curves are a different way to do the math in public key system
- Elliptic curve versions of DH, RSA, etc.
- Elliptic curves may be more efficient
  - Fewer bits needed for same security
  - But the operations are more complex

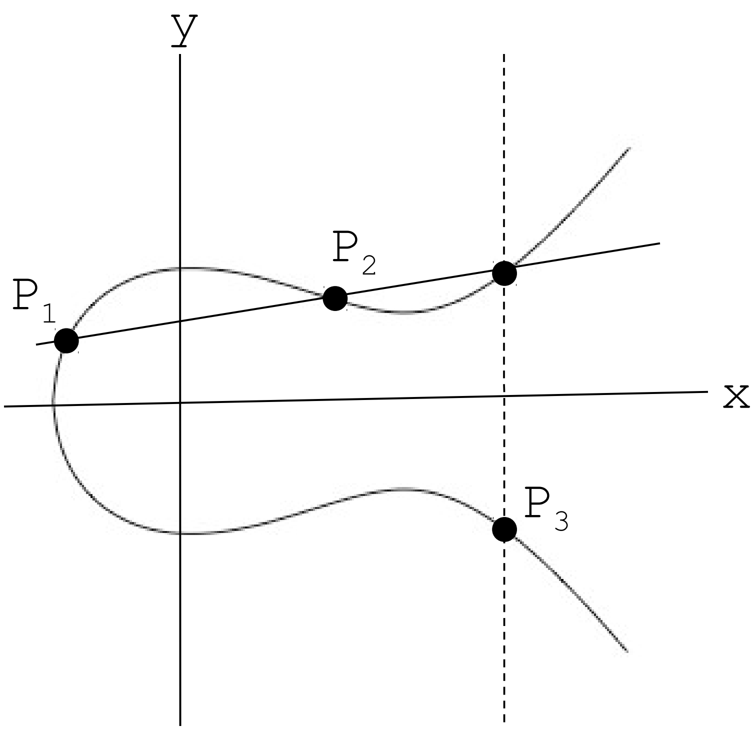
# What is an Elliptic Curve?

- An elliptic curve E is the graph of an equation of the form

$$y^2 = x^3 + ax + b$$

- Also includes a “point at infinity“
- What do elliptic curves look like?
- See the next slide!

# Elliptic Curve Picture




- Consider elliptic curve

  $$E:\ y^2 = x^3 - x + 1$$

- If $P_1$ and $P_2$ are on $E$, we can define

  $$P_3 = P_1 + P_2$$

  as shown in picture
- Addition is all we need

# Points on Elliptic Curve

- Consider $y^2 = x^3 + 2x + 3 \pmod 5$

  $x = 0 \Rightarrow y^2 = 3 \Rightarrow$ no solution (mod 5)

  $x = 1 \Rightarrow y^2 = 6 = 1 \Rightarrow y = 1,4 \pmod 5$

  $x = 2 \Rightarrow y^2 = 15 = 0 \Rightarrow y = 0 \pmod 5$

  $x = 3 \Rightarrow y^2 = 36 = 1 \Rightarrow y = 1,4 \pmod 5$

  $x = 4 \Rightarrow y^2 = 75 = 0 \Rightarrow y = 0 \pmod 5$

- Then points on the elliptic curve are

  (1,1) (1,4) (2,0) (3,1) (3,4) (4,0)

  and the point at infinity: $\infty$

# Elliptic Curve Math

- Addition on: $y^2 = x^3 + ax + b \pmod{p}$

  $P_1=(x_1,y_1), P_2=(x_2,y_2)$

  $P_1 + P_2 = P_3 = (x_3,y_3)$ where

  $x_3 = m^2 - x_1 - x_2 \pmod{p}$

  $y_3 = m(x_1 - x_3) - y_1 \pmod{p}$

  And $m = (y_2-y_1)*(x_2-x_1)^{-1} \bmod p$, if $P_1 \neq P_2$

  $m = (3x_1^2+a)*(2y_1)^{-1} \bmod p$, if $P_1 = P_2$

  Special cases: If $m$ is infinite, $P_3 = \infty$, and

  $\infty + P = P$ for all $P$

# Elliptic Curve Addition

- Consider $y^2 = x^3 + 2x + 3 \pmod{5}$. Points on the curve are (1,1) (1,4) (2,0) (3,1) (3,4) (4,0) and ∞
- What is $(1,4) + (3,1) = P_3 = (x_3, y_3)$?

$$m = (1-4)*(3-1)^{-1} = -3*2^{-1}$$

$$= 2(3) = 6 = 1 \pmod{5}$$

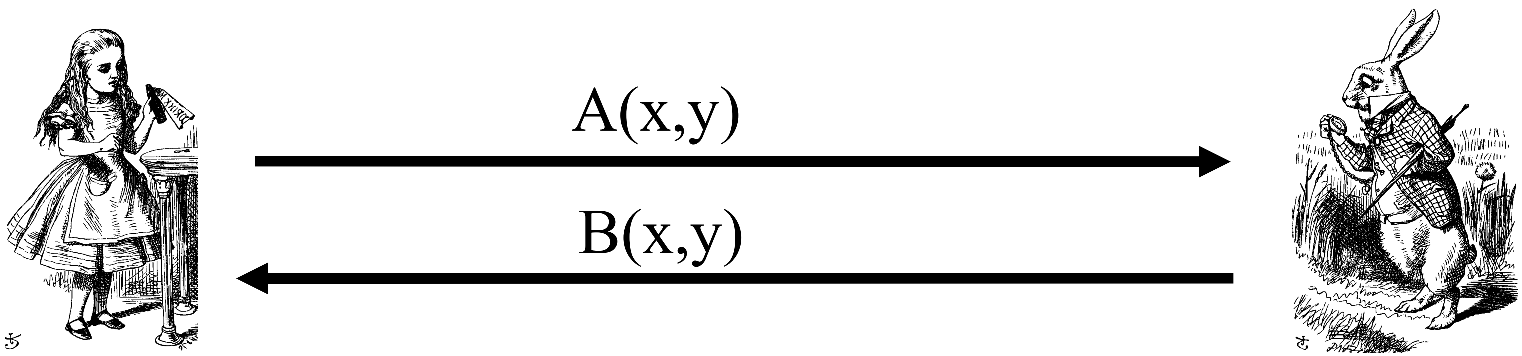
$$x_3 = 1 - 1 - 3 = 2 \pmod{5}$$

$$y_3 = 1(1-2) - 4 = 0 \pmod{5}$$

- On this curve, (1,4) + (3,1) = (2,0)

# ECC Diffie-Hellman

- **Public:** Elliptic curve and point (x,y) on curve
- **Secret:** Alice's A and Bob's B




Alice, A

Bob, B

- Alice computes A(B(x,y))
- Bob computes B(A(x,y))
- These are the same since AB = BA

# ECC Diffie-Hellman

- **Public:** Curve $y^2 = x^3 + 7x + b \pmod{37}$ and point $(2,5) \Rightarrow b = 3$
- **Alice's secret:** $A = 4$
- **Bob's secret:** $B = 7$
- Alice sends Bob: $4(2,5) = (7,32)$
- Bob sends Alice: $7(2,5) = (18,35)$
- Alice computes: $4(18,35) = (22,1)$
- Bob computes: $7(7,32) = (22,1)$

# Uses for Public Key Crypto

# Uses for Public Key Crypto

- Confidentiality
  - Transmitting data over insecure channel
  - Secure storage on insecure media
- Authentication (later)
- Digital signature provides integrity and **non-repudiation**
  - No non-repudiation with symmetric keys

# Non-non-repudiation

- Alice orders 100 shares of stock from Bob
- Alice computes **MAC** using symmetric key
- Stock drops, Alice claims she did not order
- Can Bob prove that Alice placed the order?
- **No!** Since Bob also knows symmetric key, he could have forged message
- **Problem:** Bob knows Alice placed the order, but he can't prove it

# Non-repudiation

- Alice orders 100 shares of stock from Bob
- Alice **signs** order with her private key
- Stock drops, Alice claims she did not order
- Can Bob prove that Alice placed the order?
- **Yes!** Only someone with Alice's private key could have signed the order
- This assumes Alice's private key is not stolen (revocation problem)

# Sign and Encrypt
# vs
# Encrypt and Sign

# Public Key Notation

- **Sign** message M with Alice's **private key:** $[M]_{Alice}$
- **Encrypt** message M with Alice's **public key:** $\{M\}_{Alice}$
- Then

$$\{[M]_{Alice}\}_{Alice} = M$$

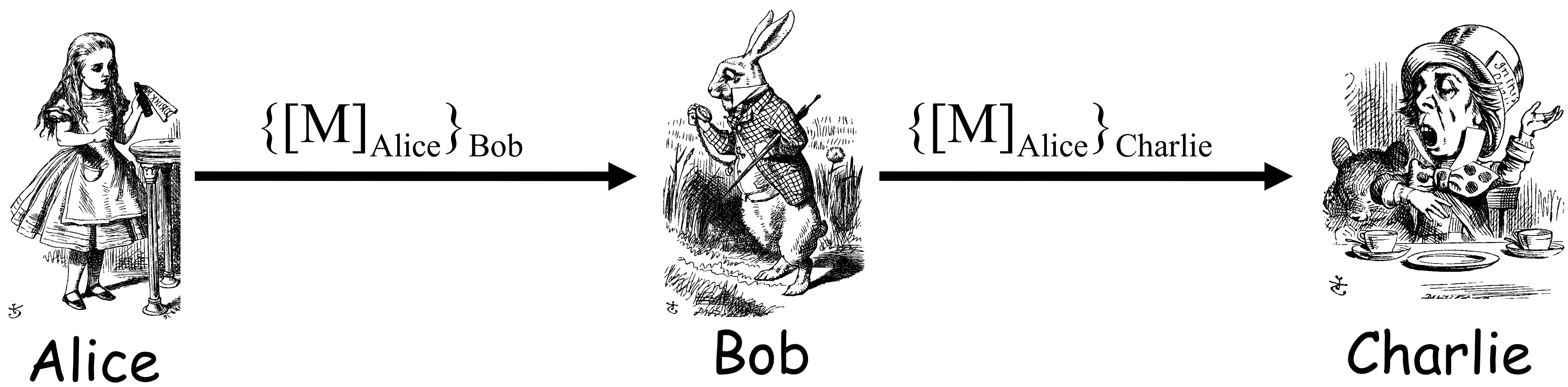
$$[\{M\}_{Alice}]_{Alice} = M$$

# Confidentiality and Non-repudiation

- Suppose that we want confidentiality and non-repudiation
- Can public key crypto achieve both?
- Alice sends message to Bob
  - **Sign and encrypt** $\{[M]_{Alice}\}_{Bob}$
  - **Encrypt and sign** $[\{M\}_{Bob}]_{Alice}$
- Can the order possibly matter?

# Sign and Encrypt

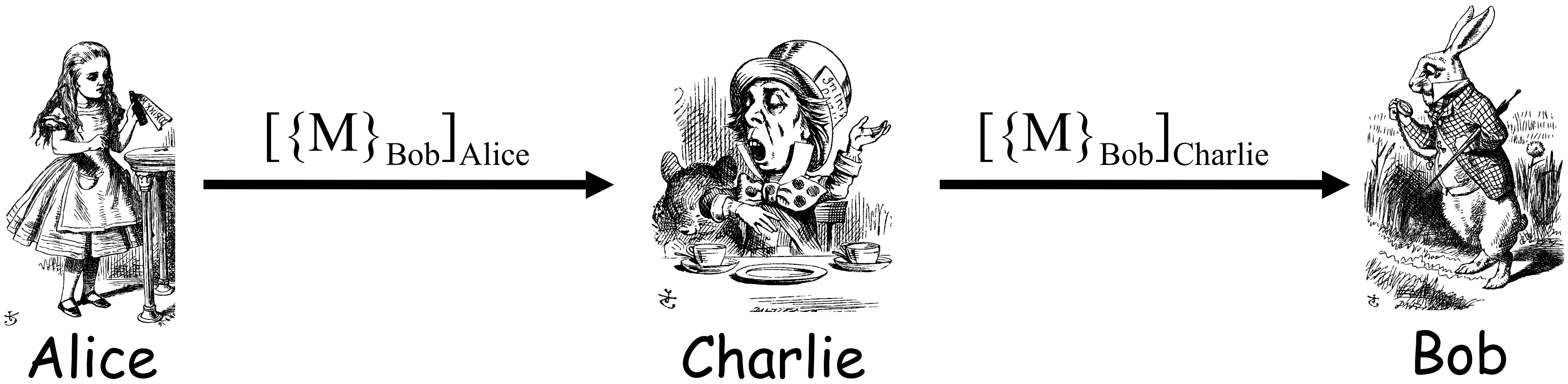
- M = "I love you"




- **Q:** What is the problem?
- **A:** Charlie misunderstands crypto!

# Encrypt and Sign

- M = "My theory, which is mine...."




- **Note** that Charlie cannot decrypt M
- **Q:** What is the problem?
- **A:** Bob misunderstands crypto!

# Public Key Infrastructure

# Public Key Certificate

- Contains name of user and user's public key (and possibly other info)
- Certificate is **signed** by the issuer (such as VeriSign) who vouches for it
- Signature on certificate is verified using signer's public key

# Certificate Authority

- Certificate authority (CA) is a trusted 3rd party (TTP) that issues and signs cert's
  - Verifying signature verifies the identity of the owner of corresponding private key
  - Verifying signature does **not** verify the identity of the source of certificate!
  - Certificates are public!
  - Big problem if CA makes a mistake (a CA once issued Microsoft certificate to someone else!)
  - Common format for certificates is X.509

# PKI

- Public Key Infrastructure (PKI) consists of all pieces needed to securely use public key cryptography
  - Key generation and management
  - Certificate authorities
  - Certificate revocation (CRLs), etc.
- No general standard for PKI
- We consider a few "trust models"

# PKI Trust Models

- Monopoly model
  - One universally trusted organization is the CA for the known universe
  - Favored by VeriSign (for obvious reasons)
  - Big problems if CA is ever compromised
  - Big problem if you don't trust the CA!

# PKI Trust Models

- Oligarchy
  - Multiple trusted CAs
  - This approach used in browsers today
  - Browser may have 80 or more certificates, just to verify signatures!
  - User can decide which CAs to trust

# PKI Trust Models

- Anarchy model
  - Everyone is a CA!
  - Users must decide which "CAs" to trust
  - This approach used in PGP (Web of trust)
  - Why do they call it "anarchy"? Suppose cert. is signed by Frank and I don't know Frank, but I do trust Bob and Bob says Alice is trustworthy and Alice vouches for Frank. Should I trust Frank?
- Many other PKI trust models

# Confidentiality in the Real World

# Symmetric Key vs Public Key

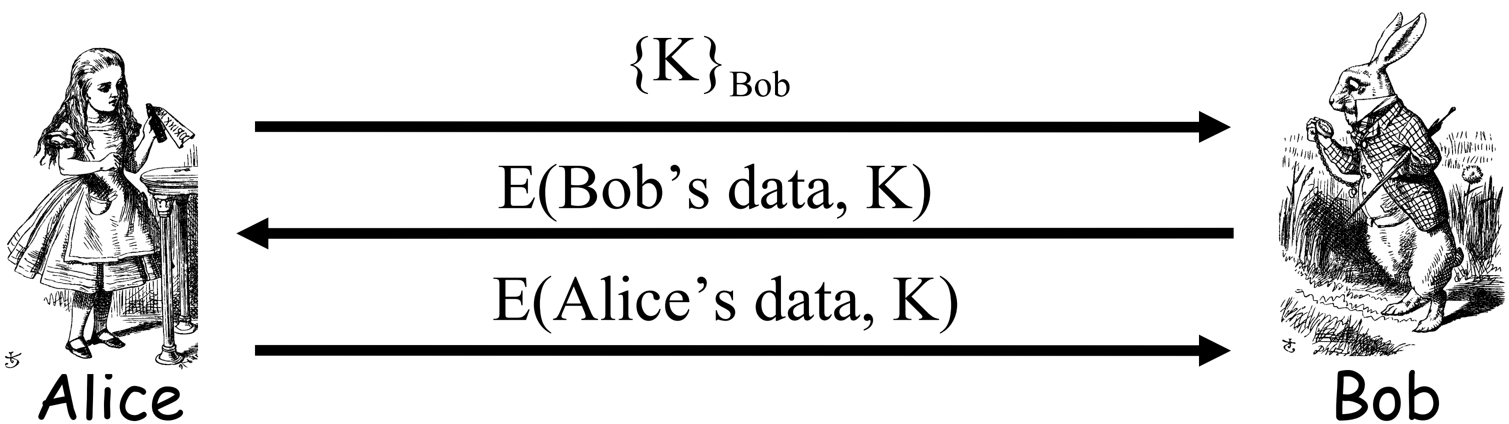
- Symmetric key +'s
  - **Speed**
  - No public key infrastructure (PKI) needed
- Public Key +'s
  - **Signatures** (non-repudiation)
  - No shared secret

# Notation Reminder

- Public key notation
  - Sign message M with Alice's **private key**
    - $[M]_{Alice}$
  - Encrypt message M with Alice's **public key**
    - $\{M\}_{Alice}$
- Symmetric key notation
  - Encrypt plaintext P with symmetric key K
    - $C = E(P,K)$
  - Decrypt ciphertext C with symmetric key K
    - $P = D(C,K)$

# Real World Confidentiality

- Hybrid cryptosystem
  - Public key crypto to establish a key
  - Symmetric key crypto to encrypt data
  - Consider the following




- Can Bob be sure he's talking to Alice?

Next time ...
Hash Functions