SDS 383C: Statistical Modeling I
Fall 2022, Module VI

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"All models are wrong, but some are useful."- George E. P. Box

Normal Mixture Models

$$y_1, \ldots, y_n \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{Normal}(\mu_k, \sigma_k^2)$$

- **Likelihood:** $p(\mathbf{y}_{1:n} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\sum_{k=1}^{K} \pi_k \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{ -\frac{1}{2\sigma_k^2} (y_i - \mu_k)^2 \right\} \right]$

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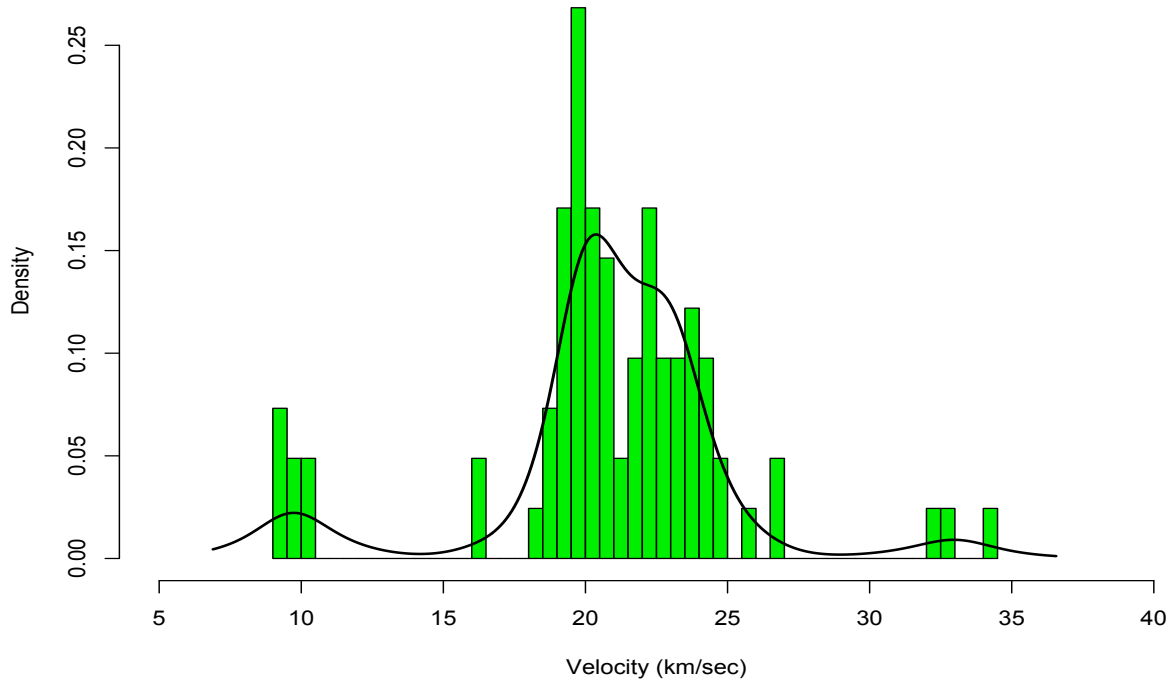

- Theorem: Location mixtures of normals can approximate any pdf on the real line.
- Location-scale mixtures are practically quite versatile.

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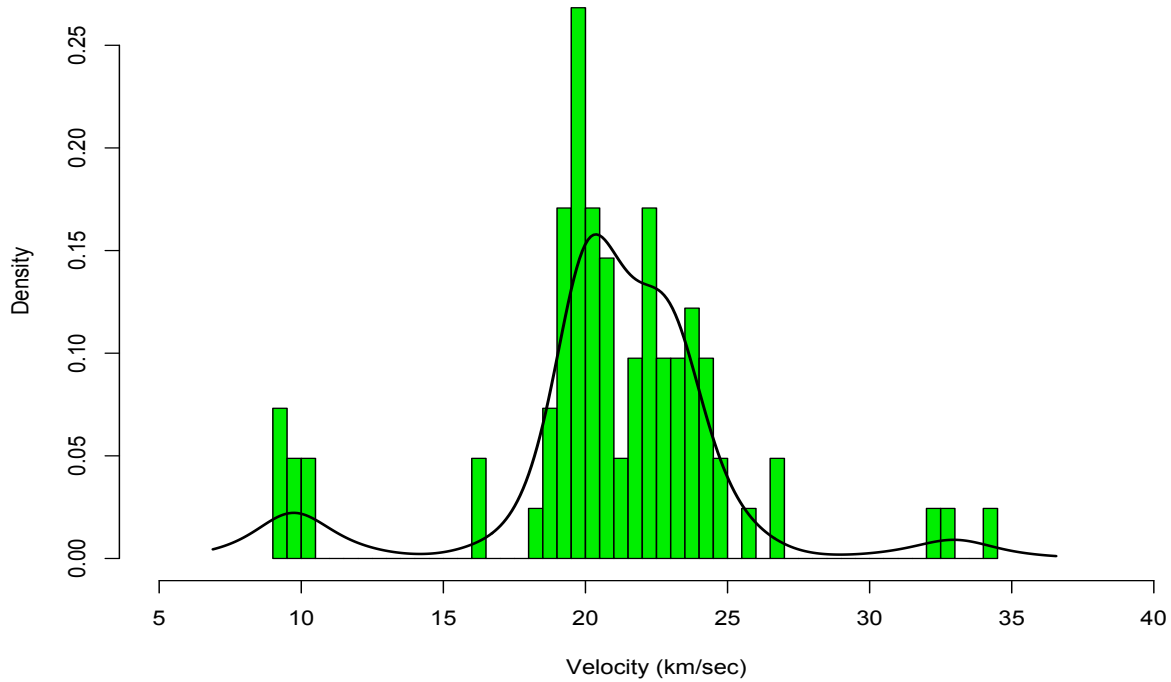

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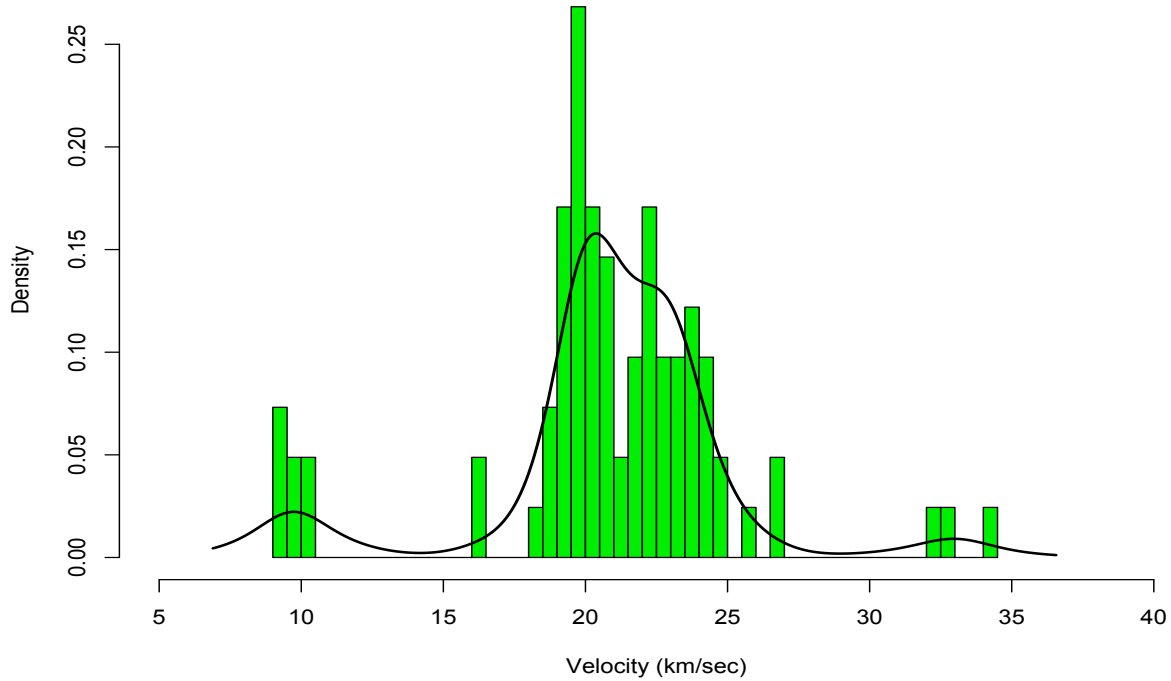

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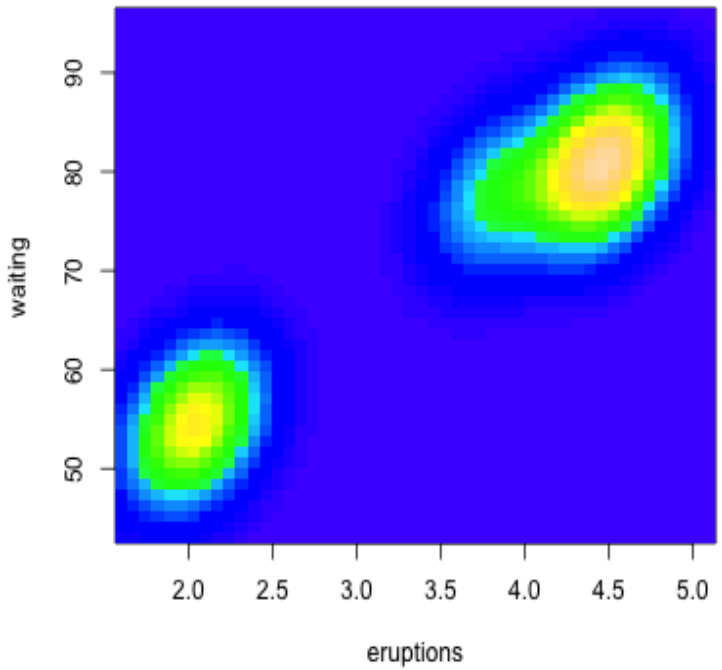

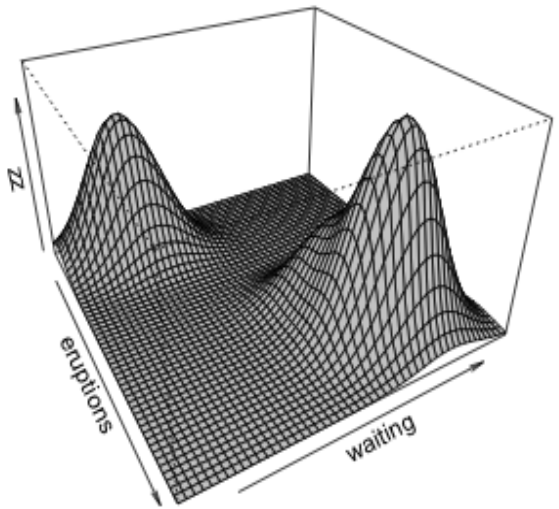

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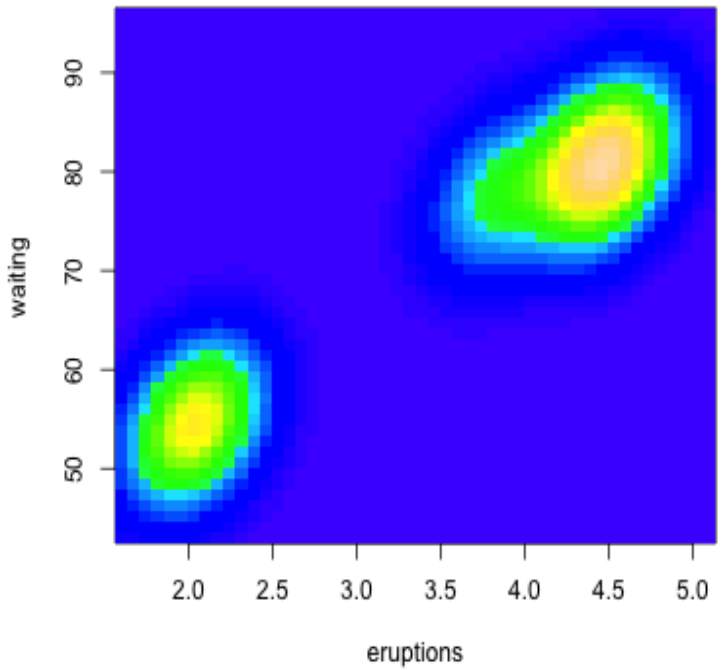

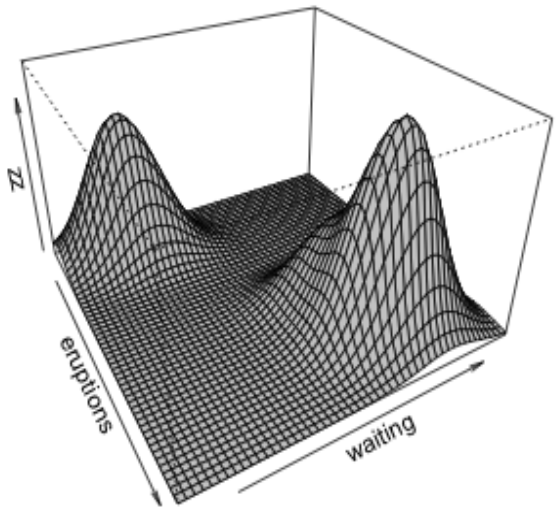

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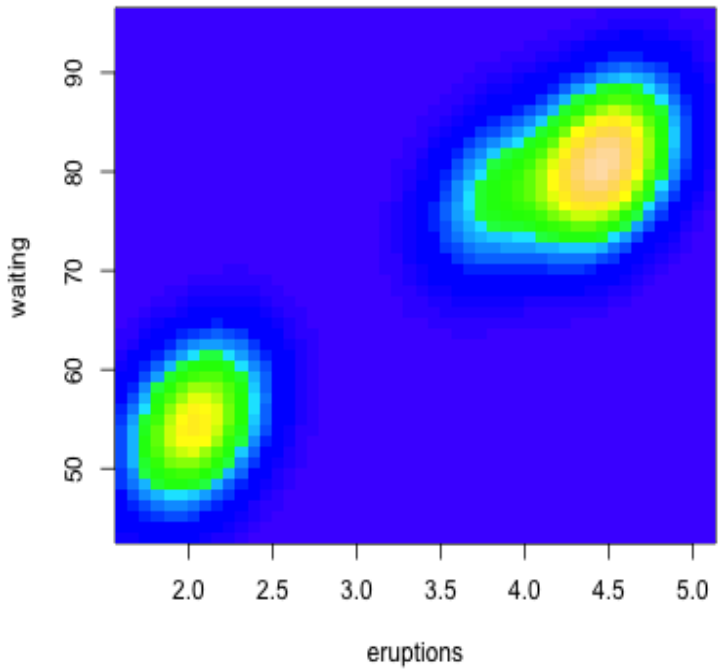

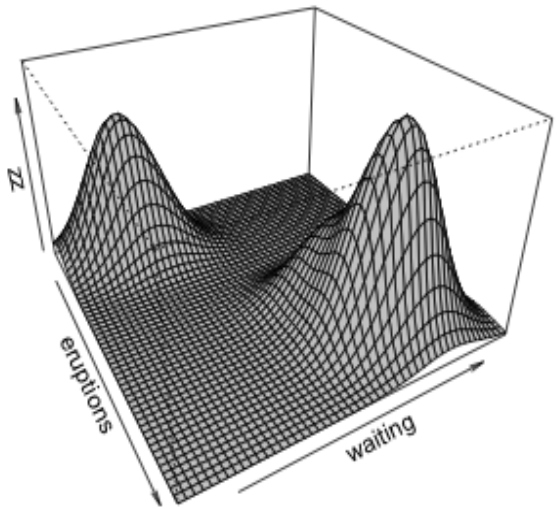

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Mixtures of Random Variables vs Mixtures of Distributions

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$$y = \sum_{k=1}^{K} a_k z_k, \quad a_k \in \mathbb{R}\ \forall k, \quad z_k \overset{ind}{\sim} \text{Normal}(z \mid \mu_k, \sigma_k^2)$$

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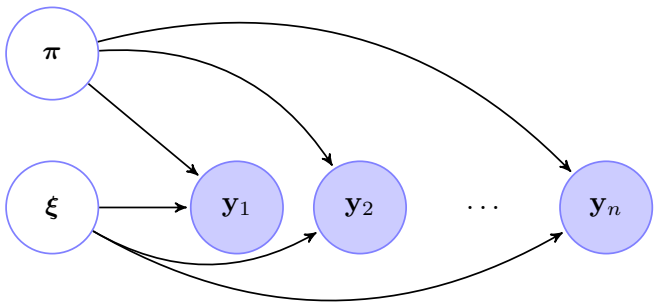

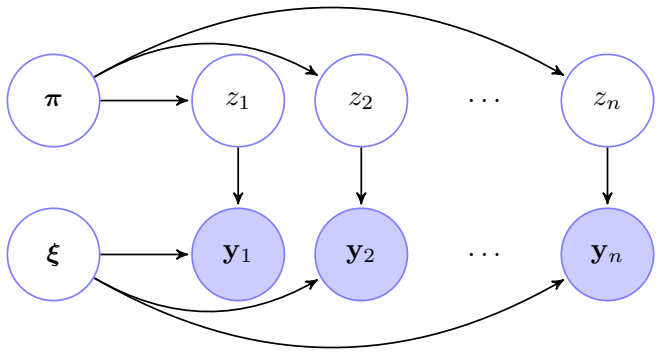

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Expectation-Maximization Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
  $= \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\boldsymbol{\theta}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$

Expectation-Maximization Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{L}(\theta) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \theta) - \log p(\mathbf{z} \mid \mathbf{y}, \theta)\}$
  $= \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \theta) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \theta)$
  $= Q(\theta, \boldsymbol{\theta}^{(m)}) + H(\theta, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\theta) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\theta, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$

Expectation-Maximization Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
  $= \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\boldsymbol{\theta}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$

Expectation-Maximization Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
  $= \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\boldsymbol{\theta}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) = Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - H(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)}) + \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log \frac{p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})}{p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})}$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)}) + D_{KL}\left\{p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}), p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\right\}$
  $\geq Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$

Expectation-Maximization Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
  $= \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\boldsymbol{\theta}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$
- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- $\mathcal{L}(\boldsymbol{\theta}^{(m+1)}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\boldsymbol{\theta}^{(m+1)}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)}) \geq 0$

Expectation-Maximization Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{L}(\boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
  $= \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
  $= Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\boldsymbol{\theta}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$
- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- $\mathcal{L}(\boldsymbol{\theta}^{(m+1)}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) \geq Q(\boldsymbol{\theta}^{(m+1)}, \boldsymbol{\theta}^{(m)}) - Q(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)}) \geq 0$

EM Algorithm - General Mixture Models

$$\mathbf{y}_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)$$

- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.

- $p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} p(\mathbf{y}_i, z_i \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} \{p(\mathbf{y}_i \mid z_i, \boldsymbol{\theta}) p(z_i \mid \boldsymbol{\theta})\}$
  $= \prod_{i=1}^{n} \prod_{k=1}^{K} \{p(\mathbf{y}_i \mid z_i = k, \boldsymbol{\theta}) p(z_i = k \mid \boldsymbol{\theta})\}^{1(z_i=k)} = \prod_{i=1}^{n} \prod_{k=1}^{K} \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}^{1(z_i=k)}$

- $\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{k=1}^{K} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$

- $\pi_{i,k}^{(m)} = p(z_i = k \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)}) = \frac{\pi_k^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_k^{(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_j^{(m)})}$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \{\log p(\mathbf{y}_i \mid \boldsymbol{\xi}_k) + \log \pi_k\}$

EM Algorithm - General Mixture Models

$$\mathbf{y}_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)$$

- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- $p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} \prod_{k=1}^{K} \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}^{1(z_i=k)}$
- $\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{k=1}^{K} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
- $\pi_{i,k}^{(m)} = p(z_i = k \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)}) = \frac{\pi_k^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_k^{(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_j^{(m)})}$
- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \{\log p(\mathbf{y}_i \mid \boldsymbol{\xi}_k) + \log \pi_k\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = (\boldsymbol{\pi}^{(m+1)}, \boldsymbol{\xi}^{(m+1)}) = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

EM Algorithm - General Mixture Models

$$\mathbf{y}_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)$$

- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- $p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} \prod_{k=1}^{K} \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}^{1(z_i=k)}$
- $\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{k=1}^{K} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
- $\pi_{i,k}^{(m)} = p(z_i = k \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)}) = \frac{\pi_k^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_k^{(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_j^{(m)})}$
- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \{\log p(\mathbf{y}_i \mid \boldsymbol{\xi}_k) + \log \pi_k\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = (\boldsymbol{\pi}^{(m+1)}, \boldsymbol{\xi}^{(m+1)}) = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

EM Algorithm - General Mixture Models

$$\mathbf{y}_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)$$

- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- $p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} \prod_{k=1}^{K} \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}^{1(z_i=k)}$
- $\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{k=1}^{K} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
- $\pi_{i,k}^{(m)} = p(z_i = k \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)}) = \frac{\pi_k^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_k^{(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_j^{(m)})}$
- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \{\log p(\mathbf{y}_i \mid \boldsymbol{\xi}_k) + \log \pi_k\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = (\boldsymbol{\pi}^{(m+1)}, \boldsymbol{\xi}^{(m+1)}) = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

EM Algorithm - General Mixture Models

$$\mathbf{y}_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)$$

- **Iterative algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- $p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} \prod_{k=1}^{K} \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}^{1(z_i=k)}$
- $\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{k=1}^{K} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
- $\pi_{i,k}^{(m)} = p(z_i = k \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)}) = \frac{\pi_k^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_k^{(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} p(\mathbf{y}_i \mid \boldsymbol{\xi}_j^{(m)})}$
- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \mathbb{E}_{\mathbf{z}\sim p(\mathbf{z}|\mathbf{y},\boldsymbol{\theta}^{(m)})} 1(z_i = k) \log \{p(\mathbf{y}_i \mid \boldsymbol{\xi}_k)\pi_k\}$
  $= \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \{\log p(\mathbf{y}_i \mid \boldsymbol{\xi}_k) + \log \pi_k\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = (\boldsymbol{\pi}^{(m+1)}, \boldsymbol{\xi}^{(m+1)}) = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma_k^2} = 0$$

$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m)}}{\sum_{j=1}^{K} \sum_{i=1}^{n} \pi_{i,j}^{(m)}} \text{ with } \pi_{i,k}^{(m)} = \frac{\pi_k^{(m)} \text{Normal}(y_i \mid \mu_k^{(m)}, \sigma_k^{2(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} \text{Normal}(y_i \mid \mu_j^{(m)}, \sigma_j^{2(m)})},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} y_i}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} y_i,$$

$$\Rightarrow \sigma_k^{2(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} (y_i - \mu_k^{(m+1)})^2.$$

EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{\log p(y_i \mid \mu_k, \sigma_k^2) + \log \pi_k\right\}$

$$= \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{-\tfrac{1}{2}\log \sigma_k^2 - \tfrac{(y_i-\mu_k)^2}{2\sigma_k^2} + \log \pi_k\right\}$$

- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1)\right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma_k^2} = 0$$

$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m)}}{\sum_{j=1}^{K} \sum_{i=1}^{n} \pi_{i,j}^{(m)}} \text{ with } \pi_{i,k}^{(m)} = \frac{\pi_k^{(m)} \text{Normal}(y_i \mid \mu_k^{(m)}, \sigma_k^{2(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} \text{Normal}(y_i \mid \mu_j^{(m)}, \sigma_j^{2(m)})},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} y_i}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} y_i,$$

$$\Rightarrow \sigma_k^{2(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} (y_i - \mu_k^{(m+1)})^2.$$

EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \, \text{Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma_k^2} = 0$$

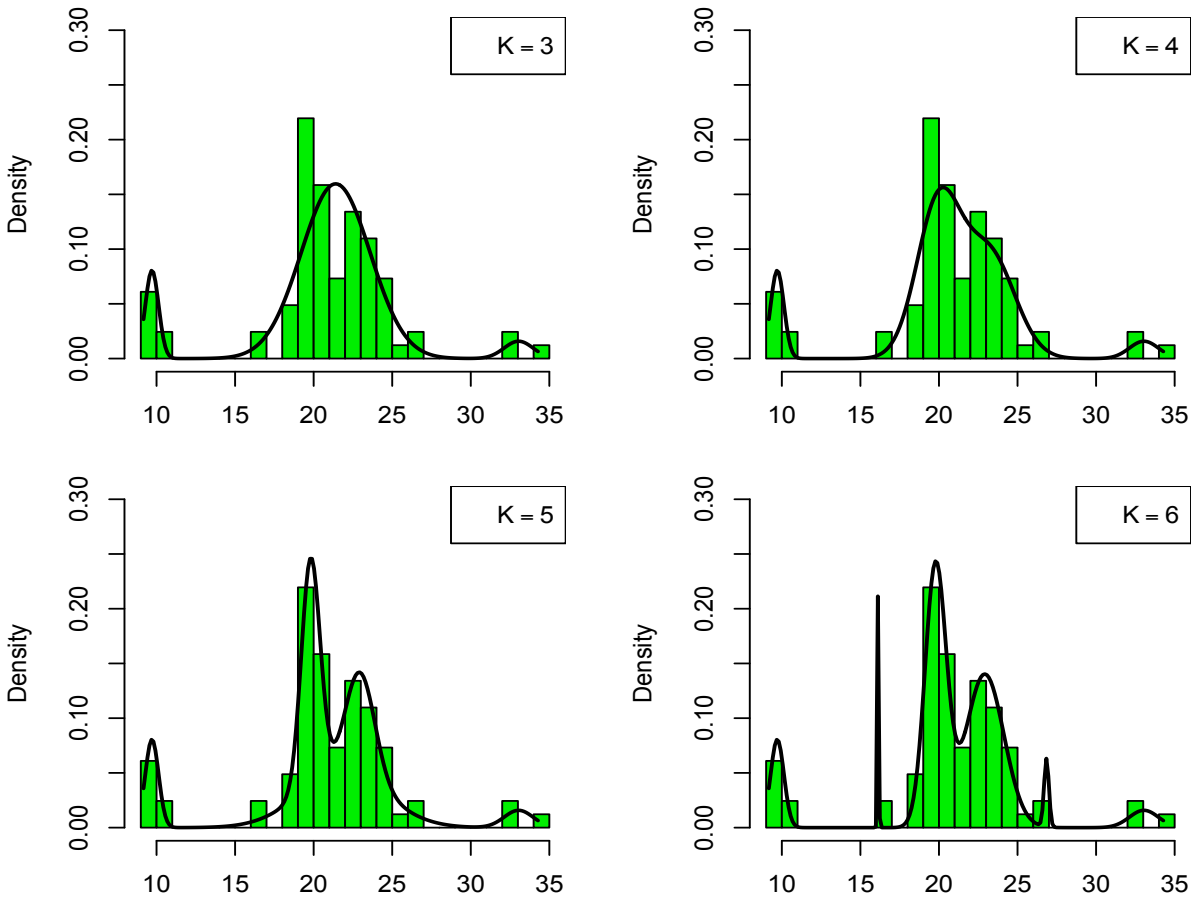
$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m)}}{\sum_{j=1}^{K} \sum_{i=1}^{n} \pi_{i,j}^{(m)}} \text{ with } \pi_{i,k}^{(m)} = \frac{\pi_k^{(m)} \text{Normal}(y_i \mid \mu_k^{(m)}, \sigma_k^{2(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} \text{Normal}(y_i \mid \mu_j^{(m)}, \sigma_j^{2(m)})},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} y_i}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} y_i,$$

$$\Rightarrow \sigma_k^{2(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} (y_i - \mu_k^{(m+1)})^2.$$

EM Algorithm - Normal Location-Scale Mixtures

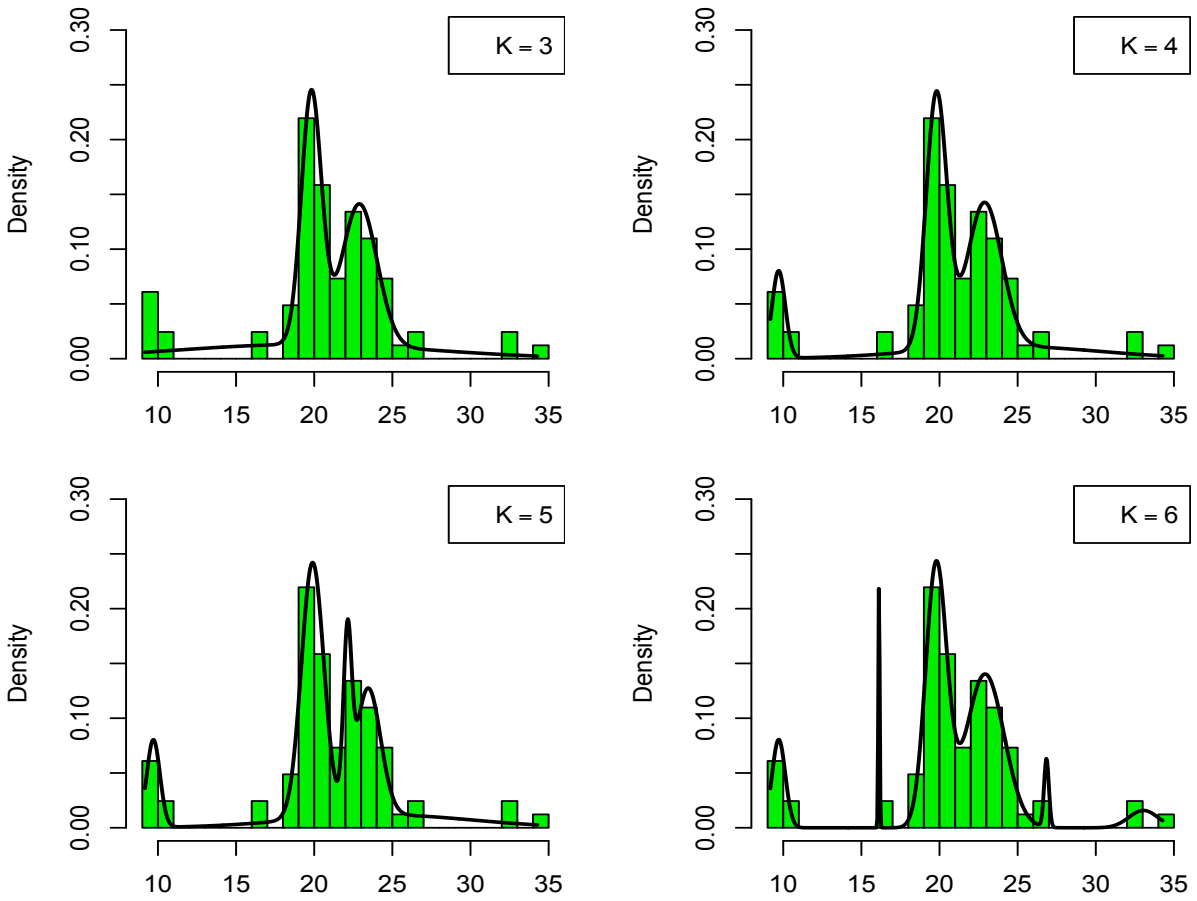
$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$



- Performance if $\pi_{i,k}$'s are NOT updated with $\pi_k^{(m+1)}$ and $\mu_k^{(m+1)}$ before updating $\sigma_k^{2(m+1)}$.

EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$



- Performance when $\pi_{i,k}$'s are updated with $\pi_k^{(m+1)}$ and $\mu_k^{(m+1)}$ before updating $\sigma_k^{2(m+1)}$.

EM Algorithm - Normal Location Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma^2 - \frac{(y_i - \mu_k)^2}{2\sigma^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma^2} = 0$$

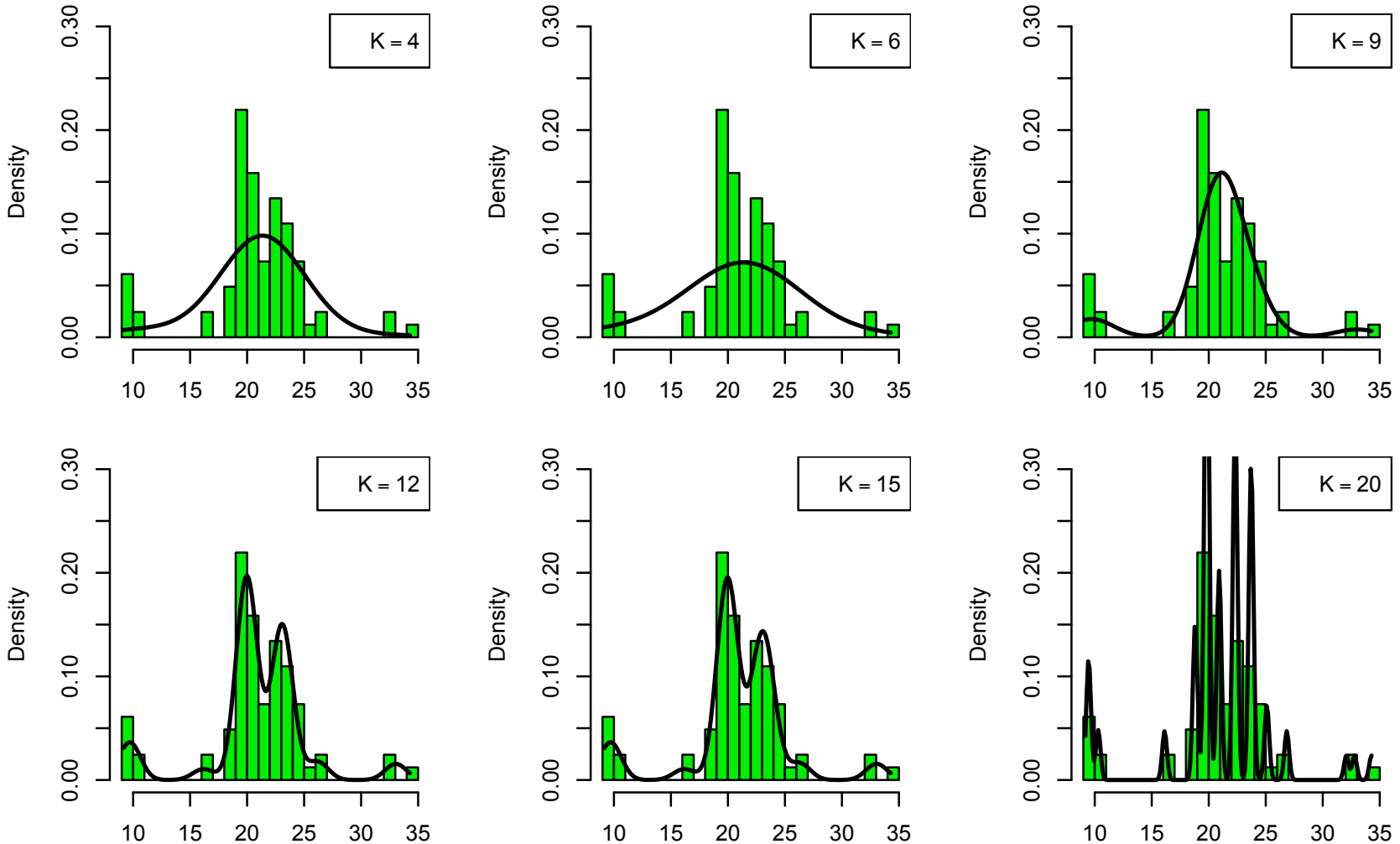
$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m)}}{\sum_{j=1}^{K} \sum_{i=1}^{n} \pi_{i,j}^{(m)}} \text{ with } \pi_{i,k}^{(m)} = \frac{\pi_k^{(m)} \text{Normal}(y_i \mid \mu_k^{(m)}, \sigma^{2(m)})}{\sum_{j=1}^{K} \pi_j^{(m)} \text{Normal}(y_i \mid \mu_j^{(m)}, \sigma^{2(m)})},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} y_i}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \frac{\pi_{i,k}^{(m+1)}}{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}} y_i,$$

$$\Rightarrow \sigma^{2(m+1)} = \frac{\sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m+1)} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m+1)}} = \sum_{i=1}^{n} \sum_{k=1}^{K} \frac{\pi_{i,k}^{(m+1)}}{n} (y_i - \mu_k^{(m+1)})^2.$$

EM Algorithm - Normal Location Mixtures

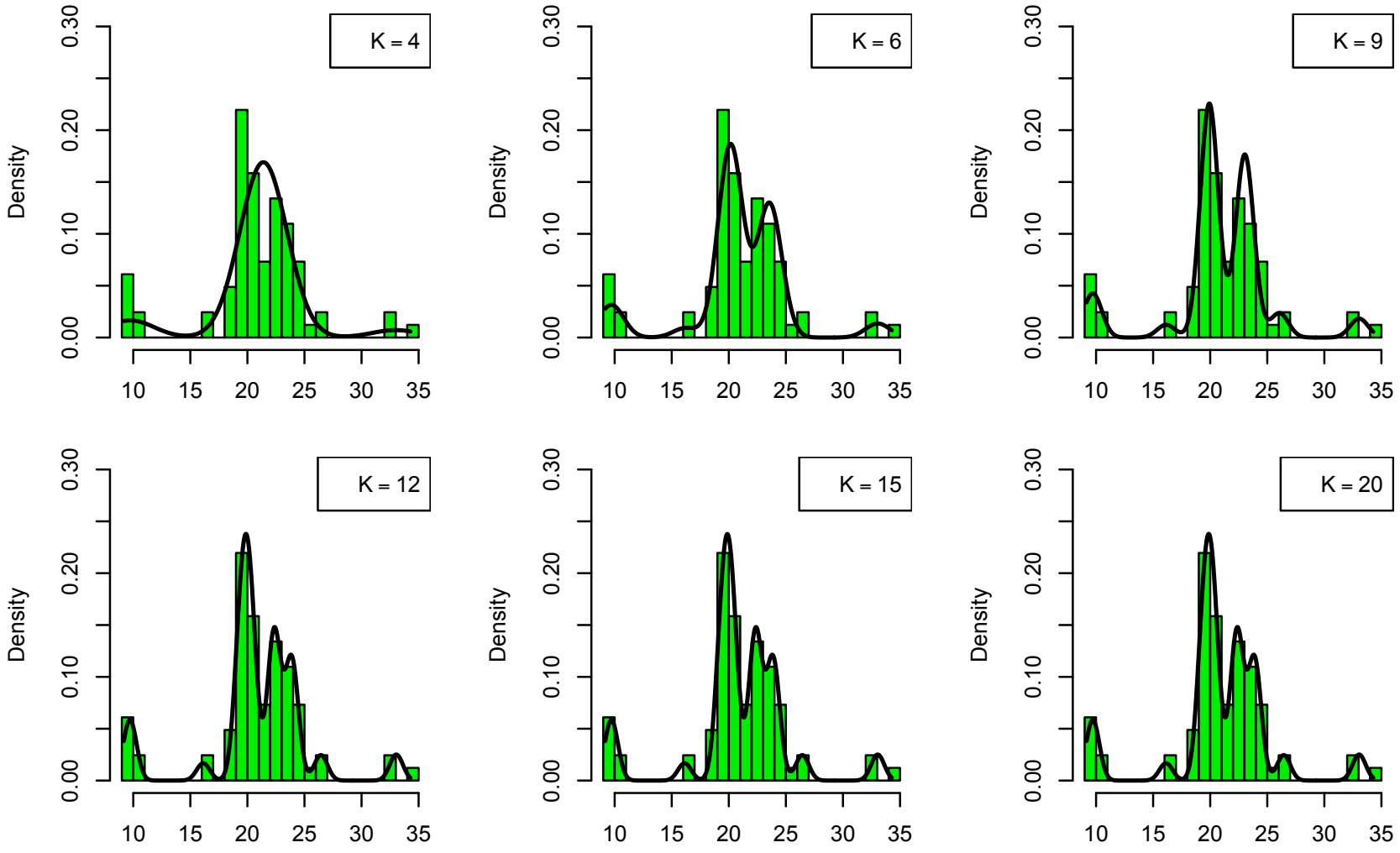
$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma^2)$$



- Performance if $\pi_{i,k}$'s are NOT updated with $\pi_k^{(m+1)}$ and $\mu_k^{(m+1)}$ before updating $\sigma^{2(m+1)}$.

EM Algorithm - Normal Location Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma^2)$$



- Performance when $\pi_{i,k}$'s are updated with $\pi_k^{(m+1)}$ and $\mu_k^{(m+1)}$ before updating $\sigma^{2(m+1)}$.

EM Algorithm - MAP Estimation - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Posterior:** $p(\boldsymbol{\theta} \mid \mathbf{y}) \propto p(\boldsymbol{\theta}) p(\mathbf{y} \mid \boldsymbol{\theta})$
- $\widetilde{\mathcal{L}}(\boldsymbol{\theta} \mid \mathbf{y}) = \log p(\boldsymbol{\theta}) + \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\boldsymbol{\theta}) + \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- $\widetilde{\mathcal{L}}(\boldsymbol{\theta} \mid \mathbf{y}) = \log p(\boldsymbol{\theta}) + \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\widetilde{\mathcal{L}}(\boldsymbol{\theta} \mid \mathbf{y}) = \log p(\boldsymbol{\theta}) + \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$

  $= \log p(\boldsymbol{\theta}) + \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)}) \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$

  $= \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \widetilde{Q}(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + H(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$
- $\widetilde{\mathcal{L}}(\boldsymbol{\theta} \mid \mathbf{y}) - \widetilde{\mathcal{L}}(\boldsymbol{\theta}^{(m)} \mid \mathbf{y}) \geq \widetilde{Q}(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - \widetilde{Q}(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)})$
- **Iterative algorithm:**

  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.

  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$.

  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) \right\}$.
- $\widetilde{\mathcal{L}}(\boldsymbol{\theta}^{(m+1)} \mid \mathbf{y}) - \widetilde{\mathcal{L}}(\boldsymbol{\theta}^{(m)} \mid \mathbf{y}) \geq \widetilde{Q}(\boldsymbol{\theta}^{(m+1)}, \boldsymbol{\theta}^{(m)}) - \widetilde{Q}(\boldsymbol{\theta}^{(m)}, \boldsymbol{\theta}^{(m)}) \geq 0$

EM Algorithm - MAP Estimation - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) \right\}$
- **Non-informative Improper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto 1$
  - $\widehat{\theta}_{MAP} = \widehat{\theta}_{MLE}$
- **Proper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto p(\boldsymbol{\pi}) \prod_{k=1}^{K} \{p(\mu_k) p(\sigma_k^2)\}$

$$\propto \pi_1^{\alpha_1 - 1} \cdots \pi_K^{\alpha_K - 1} \prod_{k=1}^{K} \left[\exp\left\{ -\frac{1}{2\sigma_0^2} (\mu_k - \mu_0)^2 \right\} (\sigma_k^2)^{-a_0 - 1} \exp\left\{ -\frac{b_0}{\sigma_k^2} \right\} \right]$$

  - M-step:

EM Algorithm - MAP Estimation - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) \right\}$
- **Non-informative Improper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto 1$
  - $\widehat{\boldsymbol{\theta}}_{MAP} = \widehat{\boldsymbol{\theta}}_{MLE}$
- **Proper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto p(\boldsymbol{\pi}) \prod_{k=1}^{K} \{p(\mu_k) p(\sigma_k^2)\}$

$$\propto \pi_1^{\alpha_1 - 1} \cdots \pi_K^{\alpha_K - 1} \prod_{k=1}^{K} \left[\exp\left\{ -\frac{1}{2\sigma_0^2} (\mu_k - \mu_0)^2 \right\} (\sigma_k^2)^{-a_0 - 1} \exp\left\{ -\frac{b_0}{\sigma_k^2} \right\} \right]$$

  - M-step:

EM Algorithm - MAP Estimation - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) \right\}$
- **Non-informative Improper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto 1$
  - $\widehat{\boldsymbol{\theta}}_{MAP} = \widehat{\boldsymbol{\theta}}_{MLE}$
- **Proper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto p(\boldsymbol{\pi}) \prod_{k=1}^{K} \{p(\mu_k) p(\sigma_k^2)\}$

  $\propto \pi_1^{\alpha_1 - 1} \cdots \pi_K^{\alpha_K - 1} \prod_{k=1}^{K} \left[\exp\left\{ -\frac{1}{2\sigma_0^2} (\mu_k - \mu_0)^2 \right\} (\sigma_k^2)^{-a_0 - 1} \exp\left\{ -\frac{b_0}{\sigma_k^2} \right\} \right]$
  - M-step:

EM Algorithm - MAP Estimation - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) \right\}$
- **Non-informative Improper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto 1$
  - $\widehat{\boldsymbol{\theta}}_{MAP} = \widehat{\boldsymbol{\theta}}_{MLE}$
- **Proper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto p(\boldsymbol{\pi}) \prod_{k=1}^{K} \{p(\mu_k) p(\sigma_k^2)\}$

$$\propto \pi_1^{\alpha_1 - 1} \cdots \pi_K^{\alpha_K - 1} \prod_{k=1}^{K} \left[\exp\left\{ -\frac{1}{2\sigma_0^2} (\mu_k - \mu_0)^2 \right\} (\sigma_k^2)^{-a_0 - 1} \exp\left\{ -\frac{b_0}{\sigma_k^2} \right\} \right]$$

$$\frac{\partial \left\{ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \sum_{k=1}^{K} (\alpha_k - 1) \log \pi_k + \lambda (\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0,$$

- **M-step:**

$$\frac{\partial \left\{ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - (\mu_k - \mu_0)^2 / (2\sigma_0^2) \right\}}{\partial \mu_k} = 0,$$

$$\frac{\partial \left\{ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) - (a_0 + 1) \log \sigma_k^2 - b_0 / \sigma_k^2 \right\}}{\partial \sigma_k^2} = 0$$

EM Algorithm - MAP Estimation - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \sum_{k=1}^{K} \pi_{i,k}^{(m)} \left\{ -\frac{1}{2} \log \sigma_k^2 - \frac{(y_i - \mu_k)^2}{2\sigma_k^2} + \log \pi_k \right\}$
- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log p(\boldsymbol{\theta}) + Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) \right\}$
- **Non-informative Improper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto 1$
  - $\widehat{\boldsymbol{\theta}}_{MAP} = \widehat{\boldsymbol{\theta}}_{MLE}$
- **Proper Prior:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \propto p(\boldsymbol{\pi}) \prod_{k=1}^{K} \{p(\mu_k) p(\sigma_k^2)\}$

$$\propto \pi_1^{\alpha_1 - 1} \cdots \pi_K^{\alpha_K - 1} \prod_{k=1}^{K} \left[\exp\left\{ -\frac{1}{2\sigma_0^2} (\mu_k - \mu_0)^2 \right\} (\sigma_k^2)^{-a_0 - 1} \exp\left\{ -\frac{b_0}{\sigma_k^2} \right\} \right]$$

  - **M-step:**

$$\pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} \pi_{i,k}^{(m)} + (\alpha_k - 1)}{\sum_{j=1}^{K} \{\sum_{i=1}^{n} \pi_{i,j}^{(m)} + (\alpha_j - 1)\}},$$

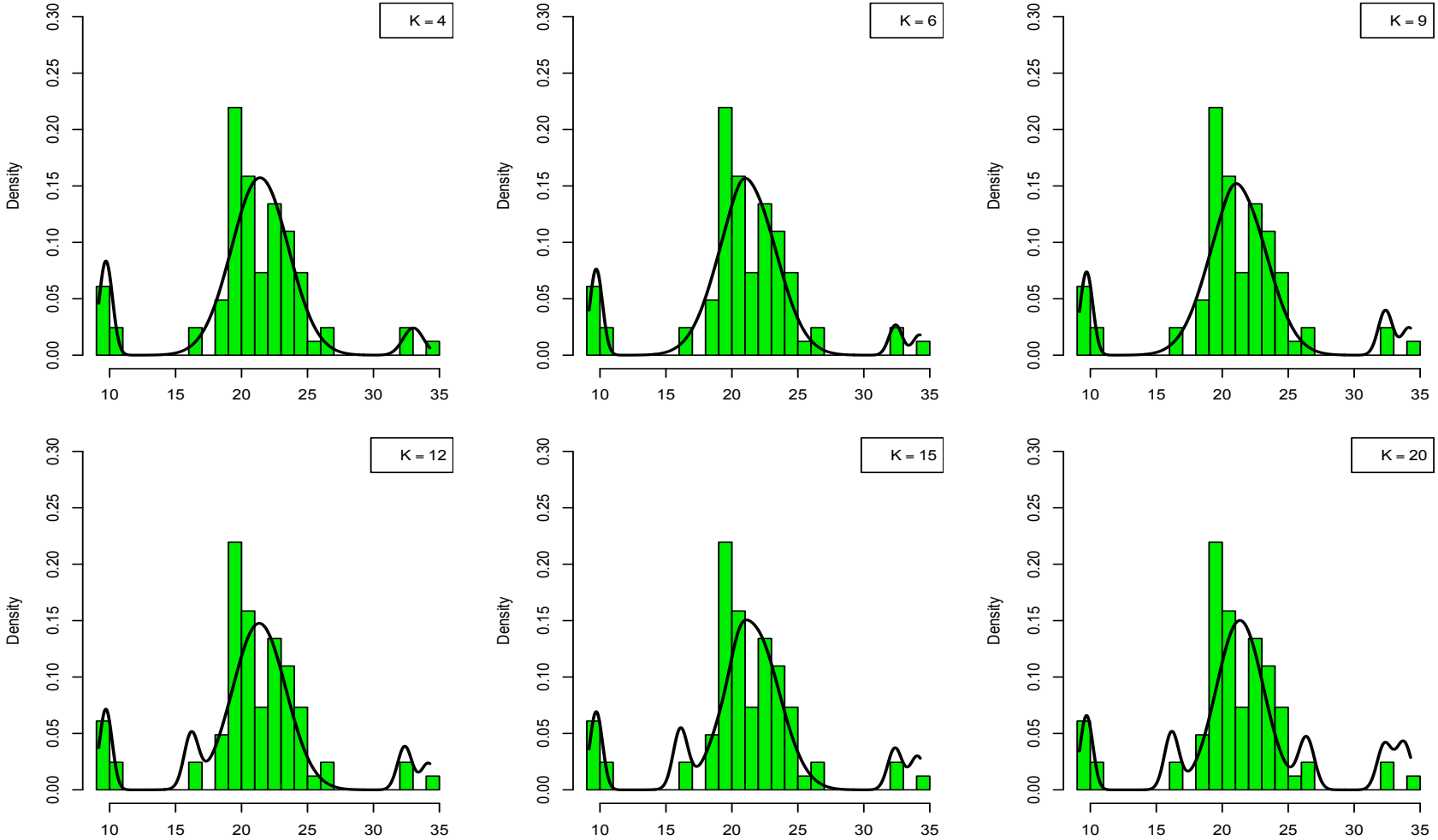
$$\mu_k^{(m+1)} = \left(\frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)}}{\sigma_k^{2(m)}} + \frac{1}{\sigma_0^2} \right)^{-1} \left(\frac{\sum_{i=1}^{n} \pi_{i,k}^{(m+1)} y_i}{\sigma_k^{2(m)}} + \frac{\mu_0}{\sigma_0^2} \right),$$

$$\sigma_k^{2(m+1)} = \left(a_0 + 1 + \frac{1}{2} \sum_{i=1}^{n} \pi_{i,k}^{(m+1)} \right)^{-1} \left\{ b_0 + \frac{1}{2} \sum_{i=1}^{n} \pi_{i,k}^{(m+1)} (y_i - \mu_k^{(m+1)})^2 \right\}.$$

EM Algorithm - MAP Estimation - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2),$$

$$\boldsymbol{\pi} \sim \text{Dir}(2, \ldots, 2), \qquad \mu_k \overset{iid}{\sim} \text{Normal}(\overline{y}, 5s_y^2), \qquad \sigma_k^2 \overset{iid}{\sim} \text{Inv-Ga}(1, 1)$$



- MAP estimation with proper priors usually does NOT lead to singularities!

Normal Location-Scale Mixtures - Gibbs Sampling

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **Equivalent representation:**

$$(z_i \mid \boldsymbol{\pi}) \overset{iid}{\sim} \text{Mult}(1, \boldsymbol{\pi})$$

$$(y_i \mid z_i = k, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \overset{ind}{\sim} \text{Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **Priors:** $p(\boldsymbol{\theta}) = p(\boldsymbol{\pi})p(\boldsymbol{\mu})p(\boldsymbol{\sigma}^2)$

$$= \text{Dir}\left(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K}\right) \cdot \prod_{k=1}^{K} \left\{\text{Normal}(\mu_k \mid \mu_0, \sigma_0^2) \cdot \text{Inv-Ga}\left(\sigma_k^2 \mid a_0, b_0\right)\right\}$$

- **Full Conditionals:**

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  - $p(\mu_k \mid -) = \text{Normal}(\mu_{k,n}, \sigma_{k,n}^2), \quad \mu_{k,n} = \sigma_{k,n}^2 \left(\frac{\mu_0}{\sigma_0^2} + \frac{n_k \overline{y}_k}{\sigma_k^2}\right),$

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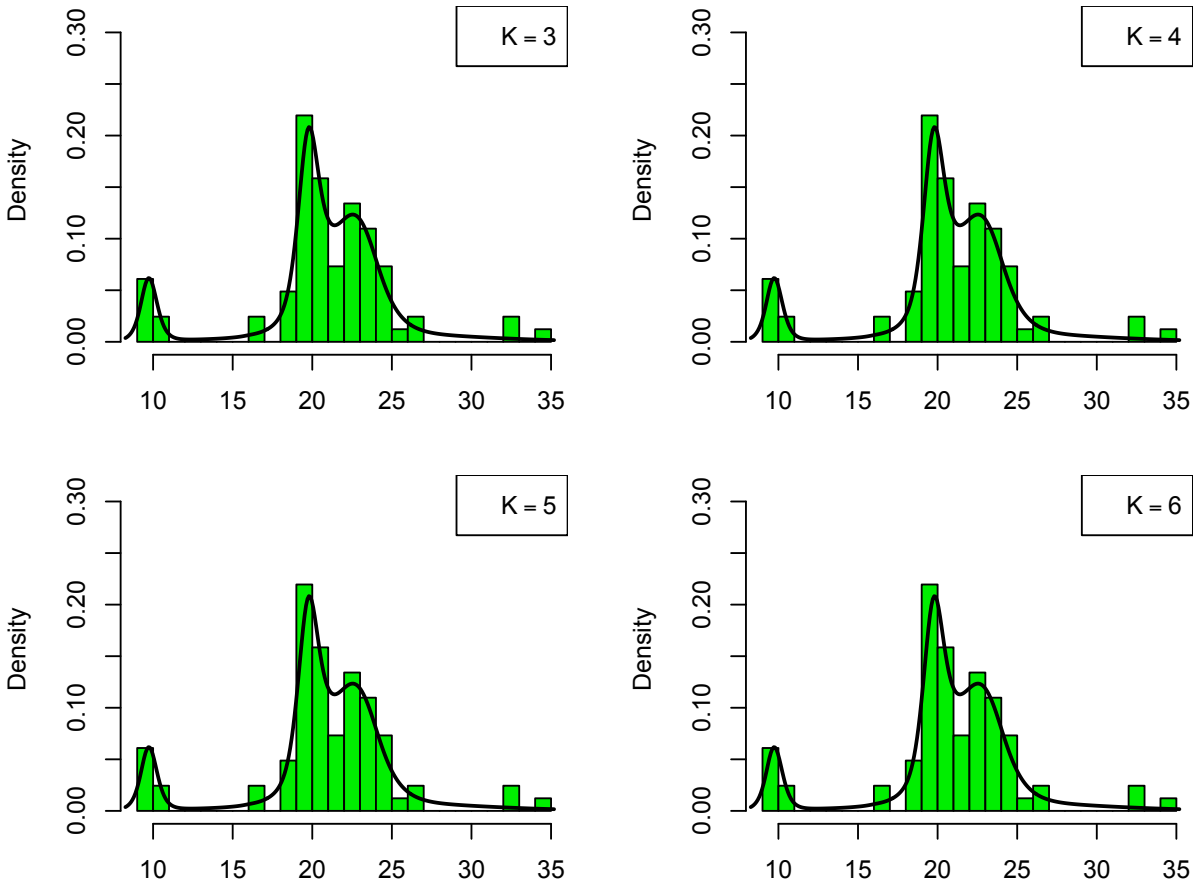
→ **Compare with results from Module 5**

Normal Location-Scale Mixtures - Gibbs Sampling

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$



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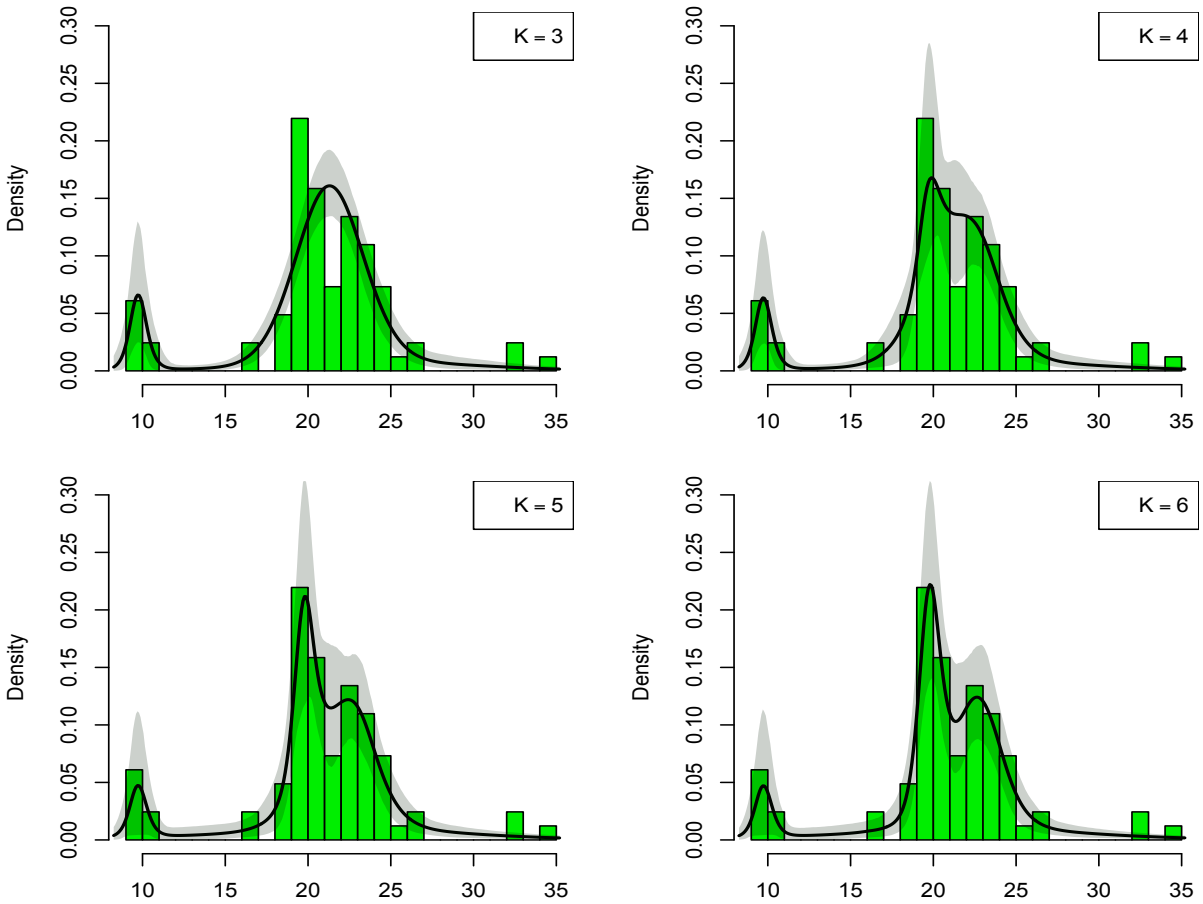

- Posterior mean and point-wise credible intervals are straightforwardly computed!

Normal Location Mixtures - Gibbs Sampling

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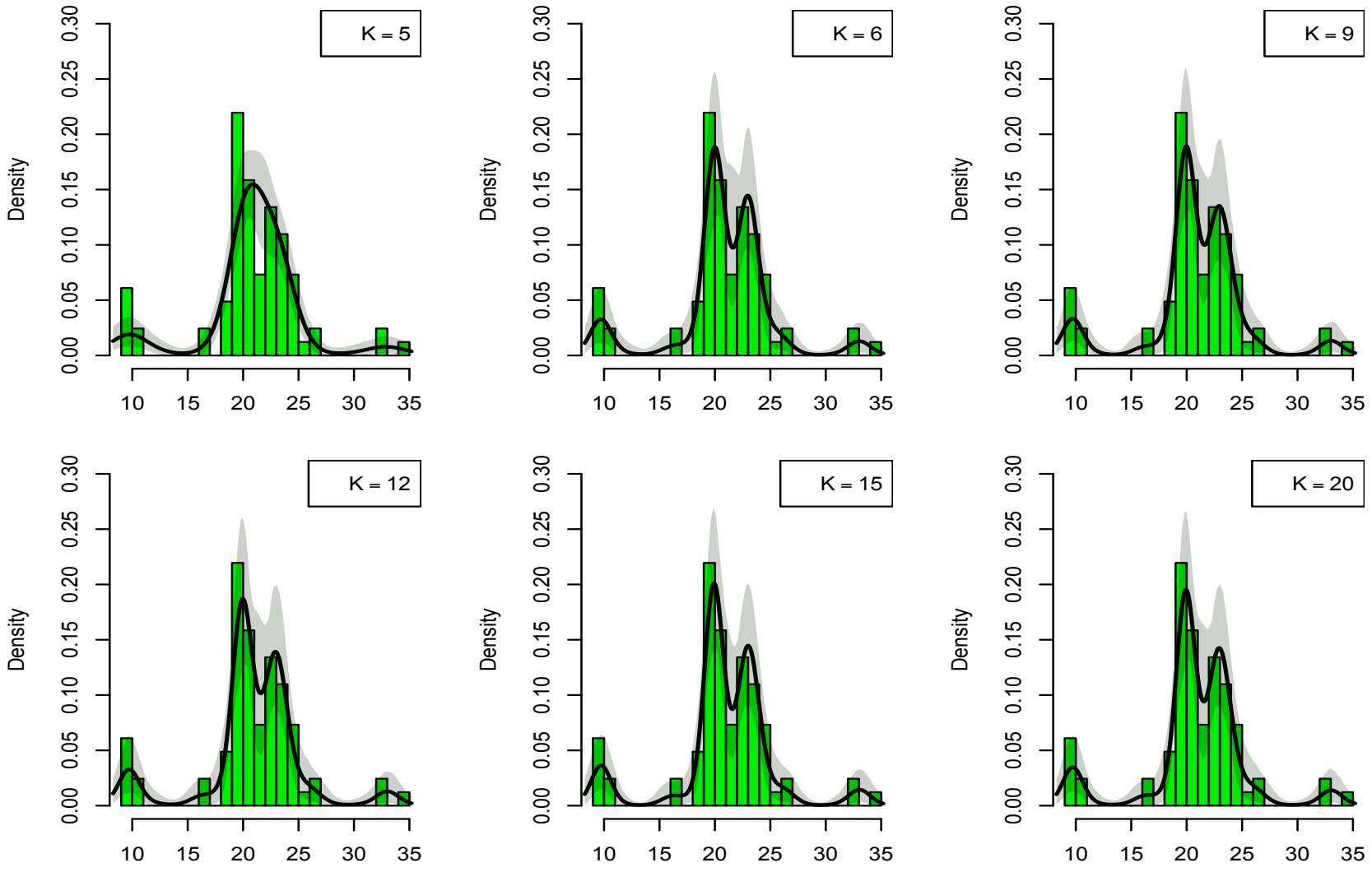

General Mixture Models - Asymptotic Behavior of the Posterior

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \, p(y_i \mid \boldsymbol{\xi}_k)$$

- Let the true model be a finite mixture with K_0 components.
- The fitted model has K components with $K > K_0$.
- **Theorem:** When $\alpha_k < L/2$, where $L = |\xi_k|$ denotes the number of parameters specifying the component kernels, the posterior is stable and concentrates in regions with empty redundant components.
- Markov chains expected to reach steady states with empty redundant components.

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Module VI Summary

- Mixture models
  - Identifiability issues - label switching and overfitting
  - High flexibility - can approximate large classes of distributions
    - Location mixtures of normals can approximate essentially any continuous density
    - Location-scale mixtures of normals are practically more efficient
    - Mixtures of Poissons are somewhat limited for modeling distributions in the tails
  - Likelihood function becomes difficult to directly work with
- EM algorithm
  - Applicable to a broad class of problems where the likelihood conditional on some 'missing' data is easier to handle compared to the marginal likelihood based only on observed data
  - Iteratively updates an optimization function making clever use of 'missing' data
  - Suffers from typical optimization algorithm issues
- EM algorithm for mixture models
  - Updating equations are typically straightforward
  - MLE often leads to singularity problems
  - Singularity problems can be averted with MAP estimation under proper priors
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  - Gibbs sampling straightforward for conjugate/semi-conjugate models
  - Unlike EM, prior elaboration and whole posterior not just point estimates

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  - Gibbs sampling straightforward to implement and tends to mix well
  - Unlike EM, uncertainty about which component each observation belongs to is easily assessed

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    - MAP: iteratively maximizes the posterior
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    - Can be very sensitive to initial conditions and get stuck at local optima
    - Can be very sensitive to small variations in updating rules
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Additional topics

- EM algorithm - an alternative view
- Stochastic EM algorithm

EM Algorithm - an Alternative View of the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \rightarrow \text{observed}, \ \mathbf{z} \rightarrow \text{latent}$$

$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

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  $= \sum_{\mathbf{z}} q(\mathbf{z}) \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \sum_{\mathbf{z}} q(\mathbf{z}) \log q(\mathbf{z}) + \sum_{\mathbf{z}} q(\mathbf{z}) \log \frac{q(\mathbf{z})}{p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})}$
  $= \mathcal{J}(q, \boldsymbol{\theta}) + D_{KL} \{q(\mathbf{z}), p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
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- $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \{\log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\}$
- $\mathcal{J}(q, \boldsymbol{\theta}) = \underbrace{\mathcal{L}(\boldsymbol{\theta})}_{no\ q} - D_{KL}\{q(\mathbf{z}), p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})\} = \mathbb{E}_{\mathbf{z} \sim q(\mathbf{z})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \underbrace{\sum_{\mathbf{z}} q(\mathbf{z}) \log q(\mathbf{z})}_{no\ \boldsymbol{\theta}}$
- **EM algorithm:**

  Starting with some $(q^{(0)}, \boldsymbol{\theta}^{(0)})$, iteratively update $(q^{(m)}, \boldsymbol{\theta}^{(m)})$ until convergence.

  (a) **E-step:** $q^{(m+1)} = \arg\min_{q} D_{KL}\left\{q(\mathbf{z}), p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})\right\} = p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})$

  (b) **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta})$
- (a) **After the E-step:** $\mathcal{J}(q^{(m+1)}, \boldsymbol{\theta}^{(m)}) \geq \mathcal{J}(q^{(m)}, \boldsymbol{\theta}^{(m)})$
  (b) **After the M-step:** $\mathcal{J}(q^{(m+1)}, \boldsymbol{\theta}^{(m+1)}) \geq \mathcal{J}(q^{(m+1)}, \boldsymbol{\theta}^{(m)})$
- $\mathcal{L}(\boldsymbol{\theta}^{(m+1)}) - \mathcal{L}(\boldsymbol{\theta}^{(m)}) = \mathcal{J}(q^{(m+1)}, \boldsymbol{\theta}^{(m+1)}) - \mathcal{J}(q^{(m)}, \boldsymbol{\theta}^{(m)}) + (0 - 0) \geq 0$

Stochastic EM Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \to \text{observed}, \ \mathbf{z} \to \text{latent}$$
$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- If $\mathbf{y}_i \overset{iid}{\sim} p(\mathbf{y}_i \mid \boldsymbol{\theta})$, then $\mathcal{L}(\boldsymbol{\theta}) = \sum_{i=1}^n \mathcal{L}_i(\boldsymbol{\theta}) = \sum_{i=1}^n \log p(\mathbf{y}_i \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^n \{\log p(\mathbf{y}_i, \mathbf{z}_i \mid \boldsymbol{\theta}) - \log p(\mathbf{z}_i \mid \mathbf{y}_i, \boldsymbol{\theta})\}$.
- **EM algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^n \mathbb{E}_{\mathbf{z}_i \sim p(\mathbf{z}_i \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}_i, \mathbf{z}_i \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.

Stochastic EM Algorithm - the General Case

$$p(\mathbf{y} \mid \boldsymbol{\theta}) = \sum_{\mathbf{z}} p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}), \quad \mathbf{y} \to \text{observed}, \ \mathbf{z} \to \text{latent}$$
$$p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{y} \mid \boldsymbol{\theta}) \ \Rightarrow \ p(\mathbf{y} \mid \boldsymbol{\theta}) = p(\mathbf{z}, \mathbf{y} \mid \boldsymbol{\theta}) / p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$$

- **Log-likelihood:** $\mathcal{L}(\boldsymbol{\theta}) = \log p(\mathbf{y} \mid \boldsymbol{\theta}) = \log p(\mathbf{y}, \mathbf{z} \mid \boldsymbol{\theta}) - \log p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$
- If $\mathbf{y}_i \overset{iid}{\sim} p(\mathbf{y}_i \mid \boldsymbol{\theta})$, then $\mathcal{L}(\boldsymbol{\theta}) = \sum_{i=1}^n \mathcal{L}_i(\boldsymbol{\theta}) = \sum_{i=1}^n \log p(\mathbf{y}_i \mid \boldsymbol{\theta})$
  $= \sum_{i=1}^n \{\log p(\mathbf{y}_i, \mathbf{z}_i \mid \boldsymbol{\theta}) - \log p(\mathbf{z}_i \mid \mathbf{y}_i, \boldsymbol{\theta})\}$.
- **EM algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^n \mathbb{E}_{\mathbf{z}_i \sim p(\mathbf{z}_i \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)})} \log p(\mathbf{y}_i, \mathbf{z}_i \mid \boldsymbol{\theta})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.
- **Stochastic EM algorithm:**
  Starting with some $\boldsymbol{\theta}^{(0)}$, iteratively update $\boldsymbol{\theta}^{(m)}$ until convergence.
  (a) **E-step:** Compute $Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^n \log p(\mathbf{y}_i, \mathbf{z}_i^{(m)} \mid \boldsymbol{\theta})$,
  where $\mathbf{z}_i^{(m)} \sim p(\mathbf{z}_i \mid \mathbf{y}_i, \boldsymbol{\theta}^{(m)})$.
  (b) **M-step:** Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$.

Stochastic EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \left\{ -\frac{1}{2} \log \sigma^2_{z_i^{(m)}} - \frac{(y_i - \mu_{z_i^{(m)}})^2}{2\sigma^2_{z_i^{(m)}}} + \log \pi_{z_i^{(m)}} \right\}$,

  $z_i^{(m)} \sim p(z_i \mid y_i, \boldsymbol{\theta}^{(m)}) \equiv \text{Mult}(1, \boldsymbol{\pi}_i^{(m)})$, $\quad \pi_{ik}^{(m)} = \frac{\pi_k^{(m)} \times \text{Normal}_{(y_i \mid \mu_k^{(m)}, \sigma_k^{(m)2})}}{\sum_{j=1}^{K} \pi_j^{(m)} \times \text{Normal}_{(y_i \mid \mu_j^{(m)}, \sigma_j^{(m)2})}}$

- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{ Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma_k^2} = 0$$

$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}}{n},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\} y_i}{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}},$$

$$\Rightarrow \sigma_k^{2(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}}.$$

Stochastic EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \text{ Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \left\{ -\frac{1}{2} \log \sigma^2_{z_i^{(m)}} - \frac{(y_i - \mu_{z_i^{(m)}})^2}{2\sigma^2_{z_i^{(m)}}} + \log \pi_{z_i^{(m)}} \right\}$,

  $z_i^{(m)} \sim p(z_i \mid y_i, \boldsymbol{\theta}^{(m)}) \equiv \text{Mult}(1, \boldsymbol{\pi}_i^{(m)})$, $\quad \pi_{ik}^{(m)} = \frac{\pi_k^{(m)} \times \text{Normal}(y_i \mid \mu_k^{(m)}, \sigma_k^{(m)2})}{\sum_{j=1}^{K} \pi_j^{(m)} \times \text{Normal}(y_i \mid \mu_j^{(m)}, \sigma_j^{(m)2})}$

- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{ Q_S(\theta, \theta^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma_k^2} = 0$$

$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}}{n},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\} y_i}{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}},$$

$$\Rightarrow \sigma_k^{2(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}}.$$

Stochastic EM Algorithm - Normal Location-Scale Mixtures

$$y_i \overset{iid}{\sim} \sum_{k=1}^{K} \pi_k \, \text{Normal}(y_i \mid \mu_k, \sigma_k^2)$$

- **E-step:** $Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) = \sum_{i=1}^{n} \left\{ -\frac{1}{2} \log \sigma^2_{z_i^{(m)}} - \frac{(y_i - \mu_{z_i^{(m)}})^2}{2\sigma^2_{z_i^{(m)}}} + \log \pi_{z_i^{(m)}} \right\}$,

  $z_i^{(m)} \sim p(z_i \mid y_i, \boldsymbol{\theta}^{(m)}) \equiv \text{Mult}(1, \boldsymbol{\pi}_i^{(m)})$, $\quad \pi_{ik}^{(m)} = \frac{\pi_k^{(m)} \times \text{Normal}_{(y_i \mid \mu_k^{(m)}, \sigma_k^{(m)2})}}{\sum_{j=1}^{K} \pi_j^{(m)} \times \text{Normal}_{(y_i \mid \mu_j^{(m)}, \sigma_j^{(m)2})}}$

- **M-step:** $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})$

$$\frac{\partial \left\{ Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)}) + \lambda(\sum_{k=1}^{K} \pi_k - 1) \right\}}{\partial \pi_k / \partial \lambda} = 0, \quad \frac{\partial Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \mu_k} = 0, \quad \frac{\partial Q_S(\boldsymbol{\theta}, \boldsymbol{\theta}^{(m)})}{\partial \sigma_k^2} = 0$$

$$\Rightarrow \pi_k^{(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}}{n},$$

$$\Rightarrow \mu_k^{(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\} y_i}{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}},$$

$$\Rightarrow \sigma_k^{2(m+1)} = \frac{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\} (y_i - \mu_k^{(m+1)})^2}{\sum_{i=1}^{n} 1\{z_i^{(m)} = k\}}.$$