DS 520 Homework 1

Show your work.

Name: Solutions

1. Let $\underline{u} = [2, 7, 1]^T$, $\underline{v} = [3\ 0\ 4]$, $\underline{w} = [1, 5, 8]$

Find: (a.) $3\underline{u} - 4\underline{v}$

(b.) $2\underline{u} + 4\underline{v} - 5\underline{w}$

(a) $3\underline{u} - 4\underline{v} = [6, 21, 3]^T + [-12\ 0\ -16] = [-6, 21\ -13]^T$

(b) $2\underline{u} + 4\underline{v} - 5\underline{w} = [4, 14, 2]^T + [12\ 0\ 16]^T + [-5, -25, -40]^T = [11, -11, -22]^T$

2. Find the coefficients $c_1, c_2, c_3$ such that $\sum_{i=1}^{3} c_i \underline{u}_i = [0, 3, 5]^T$. Here, $\underline{u}_1 = [1, -1, 1]^T$, $\underline{u}_2 = [1, 3, 2]^T$, and $\underline{u}_3 = [2, 0, 1]^T$.

Let $\underline{c} = [c_1, c_2, c_3]^T$ and $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}$. Then $A\underline{c} = [0, 3, 5]^T$,

or $\underline{c} = A^{-1}[0, 3, 5]^T$. Here $A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$, and $\underline{c} = [3\frac{1}{2}, 2\frac{1}{6}\ -2\frac{5}{6}]^T$

3. Find $\underline{u}\cdot\underline{v} = \underline{u}^T\underline{v}$, where $\underline{u} = [2, 3, 5, 7]^T$ and $\underline{v} = [1, -1, 1, -1]^T$.

$\underline{u}^T\underline{v} = 2 - 3 + 5 - 7 = -3$

4. Let $\underline{u} = [5, 4, 1]^T$, $\underline{v} = [3, -4, 1]^T$, $\underline{w} = [1, -2, 3]^T$.

(a.) Compute the generalized cosine between all possible pairs.

(b.) Which pair of distinct vectors is most colinear?

(c.) Which pair are most orthogonal?

(d) Are $\underline{u}$, $\underline{v}$, and $\underline{w}$ linearly independent?

(a) Form $W = \left[\frac{\underline{u}}{\|\underline{u}\|}, \frac{\underline{v}}{\|\underline{v}\|}, \frac{\underline{w}}{\|\underline{w}\|}\right]$. Then all gen. cosines are in matrix $W^TW$.

Here $W^TW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.734 \\ 0 & 0.734 & 1 \end{bmatrix}$, so

(b.) $\underline{v}$ and $\underline{w}$ are most colinear.

(c.) $\underline{u}$ and $\underline{v}$ are orthogonal.

(d.) Since $\det(W) = -0.68 \neq 0$, $\{\underline{u}, \underline{v}, \underline{w}\}$ is an independent set.

5. Let $\underline{u} = [1, 2, 1]^T$ and $\underline{v} = [3, 5, 8]^T$. Find

Find:

(a) the generalized cosine between $\underline{u}$ and $\underline{v}$,

(b) the projection of $\underline{u}$ onto $\underline{v}$,

(c.) the $L_2$ distance between $\underline{u}$ & $\underline{v}$,

(d.) the $L_1$ distance between $\underline{u}$ & $\underline{v}$.

(a) $\cos\theta = \dfrac{\underline{u}^T\underline{v}}{\|\underline{u}\| \cdot \|\underline{v}\|} = 0.8660$ (here $\|\underline{u}\|$ is $L_2$-norm)

(b.) $\|\underline{u}\| \dfrac{\cos\theta}{\|\underline{v}\|} \cdot \underline{v} = (0.2143)\underline{v} = [0.6428, 1.071, 1.714]^T$

(c.) $\|\underline{u} - \underline{v}\| = \|\underline{u} - \underline{v}\|_2 = 7.874$

(d.) $\|\underline{u} - \underline{v}\|_1 = 12$

6. A hyperplane can be described by two vectors $\underline{p}$ and $\underline{w}$, and is the set of vectors $\underline{x}$ such that $(\underline{x} - \underline{p})^T \underline{w} = \underline{0}$.

In $\mathbb{R}^2$, a hyperplane consists of all vectors $[x_1\ x_2]^T$ such that $x_2 = m x_1 + b$, for fixed $m$ and $b$. Find $\underline{p}$ and $\underline{w}$ in terms of $m$ and $b$. Let $\underline{p} = [0, ?]^T$ and $\underline{w} = [? \ 1]$ for unique answers.

Since $\underline{p}_1 = [0, b]^T$ is in the hyperplane

$\underline{x} - \underline{p}_1 = x_1 [1 \;\; m]^T$. $[1 \;\; m] [w_1, w_2]^T = 0$ requires $w_1 = -m w_2$, or $\underline{w} = w_2 [-m \;\; 1]^T$.

Consider $\underline{p} = \underline{p}_1$. Then $\underline{w} = [-m \;\; 1]^T$.

7. Write the vector $\underline{z} = [1\ 0\ 5]^T$ as a linear combination of $\underline{p}_1 = [1\ 0\ 1]^T$, $\underline{p}_2 = [1\ 1\ 3]^T$, and $\underline{p}_3 = [2\ 0\ -1]^T$.

Solution: $x\underline{p}_1 + y\underline{p}_2 + s\underline{p}_3 = z$, or

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

$\Rightarrow$ the augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & -1 & 5 \end{array}\right] \Leftarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 4 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 4 \end{array}\right], \text{ or}$$

$s = -4/3$, $y = 0$, and

$x - 8/3 = 1$

$x = 11/3$

check: $\begin{bmatrix} 11/3 - 8/3 \\ 0 \\ 11/3 + 4/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

8. Find all $t, K$ so that $\underline{u} = [t \ K]^T$ and $\underline{v} = [3 \ 5]^T$ are orthogonal and $\underline{u}$ has unit length.

Solution:

orthogonal $\underline{u}^T \underline{v} = 0 \Rightarrow 3t + 5K = 0$

$t = -\frac{5}{3} K$

normal: $\underline{u}^T \underline{u} = 1 \Rightarrow t^2 + K^2 = 1$

$K^2 \left( \frac{25}{9} + 1 \right) = 1$

$K^2 = 9/34$

$K = \pm 3/\sqrt{34}$

9. Consider a hyperplane which is normal to $\underline{u}_1 = [1 \ -1 \ 1]^T$ and passes through $\underline{w} = [2 \ 2 \ 2]^T$. Prove or disprove that $\underline{x} = [2 \ 3 \ 3]$ is in the hyperplane.

Solution: since $\underline{u}^T \underline{w} = 2$, any point $\underline{p}$ in the hyperplane satisfies $\underline{u}^T \underline{p} = 2$. As $\underline{u}^T \underline{x} = 2$, $\underline{x}$ is in the hyperplane.

10 Let $L$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, with $L([0\ 1]^T) = [-1\ 0]^T$, and $L([1\ 0]^T) = [3\ 5]^T$.

(a) Find a matrix representation for $L$ in the standard ordered basis.

Solution:

$$M = \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix}$$

(b) Find $L([2\ 2]^T)$ and $L([1\ -1]^T)$

Solution: if $\underline{x} = x_1 \cdot \underline{e}_1 + x_2 \underline{e}_2$, then $L(\underline{x}) = x_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

So,

$$L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 6-2 \\ 10-0 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$L([1\ -1]^T) = [4\ 5]^T$$

(c) Prove or disprove: $[2\ 3]^T$ is an eigenvector of $L$.

Solution: An eigenvector satisfies $x_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\begin{bmatrix} 3-\lambda & -1 \\ 5 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow -3\lambda + \lambda^2 + 5 = 0$

$\lambda_0 = \dfrac{3 \pm \sqrt{9-20}}{2} \rightarrow$ no real roots

So, there are no eigenvectors.

11. Prove or disprove: $\underline{v} = [2\ 3]^T$ is an eigenvector of $M = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

Solution: the eigenvalues are found by solving $\det \begin{bmatrix} \lambda-3 & 2 \\ 2 & \lambda-3 \end{bmatrix} = 0$

$(\lambda-3)^2 - 4 = 0 \Rightarrow \lambda - 3 = \pm 2$, so $\lambda = 1, 5$ are eigenvalues.

The e-vectors for $\lambda = 1$ are found by $\begin{bmatrix} 3-1 & -2 \\ -2 & 3-1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Downarrow$

$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The e-vectors for $\lambda = 5$ are found by $\begin{bmatrix} 3-5 & -2 \\ -2 & 3-5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{v}$ is neither an e-vector of $\lambda = 1$ nor $5$.

12. Consider the linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $V = \{[1\ 1]^T, [1\ -1]^T\}$, both in the standard ordered basis.

(a) Show that $V$ is a basis for $\mathbb{R}^2$

Solution: We need to show linear independence.

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow c_1 = -c_2$ & $c_1 = c_2$

$\Rightarrow c_1 = c_2 = 0$

(b) Find matrix $K$ to express the linear operator in the basis $V$.

Solution:

$B\underline{v_1} = B \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v_1}} = 3\underline{e_1} + 7\underline{e_2} = \frac{3+7}{2}\underline{v_1} + \frac{3-7}{2}\underline{v_2}$

$B\underline{v_2} = B \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{v_2}} = -1\underline{e_1} - 1\cdot\underline{e_2} = \frac{-1-1}{2}\underline{v_1} + \frac{-1+1}{2}\underline{v_2} = -\underline{v_1}$

$K = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}$

13. Let $\underline{X}$ denote a Gaussian column vector with mean vector $\underline{m}_X = [2\ 3]^T$, and covariance matrix $C_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The random vector $\underline{Y} = A\underline{X}$, where $A = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$.

(a.) Determine $E[\underline{Y}]$.

$$
\begin{aligned}
E[\underline{Y}] &= E[A\underline{X}] \\
&= A\,E[\underline{X}] \\
&= A\,\underline{m}_X \\
&= \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= [-8 \quad 13]^T
\end{aligned}
$$

(b.) Find and simplify the covariance matrix of $\underline{Y}$.

$$
\begin{aligned}
\mathrm{cov}(\underline{Y}) &= \mathrm{cov}(A\underline{X}) \\
&= A\,\mathrm{cov}(\underline{X})A^T \\
&= A I A^T \\
&= AA^T \\
&= \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 5 & -8 \\ -8 & 13 \end{bmatrix}
\end{aligned}
$$

(c.) Completely specify the probability density function of vector $\underline{Y}$.

$$\underline{Y} \sim \mathcal{N}\left( \underbrace{\begin{bmatrix} -8 \\ 13 \end{bmatrix}}_{\underline{m}_Y}, \underbrace{\begin{bmatrix} 5 & -8 \\ -8 & 13 \end{bmatrix}}_{C_Y} \right)$$

$$f_{\underline{Y}}(\underline{y}) = \frac{1}{2\pi} \frac{1}{\sqrt{\det C_Y}} \exp\left[ -\frac{1}{2} (\underline{y} - \underline{m}_y)^T C_Y^{-1} (\underline{y} - \underline{m}_y) \right]$$

14. Let $X$ and $Y$ be random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} \frac{K}{16}, & \text{if } -4 < x < 4 \text{ and } 2 < y < 4, \\ 0, & \text{otherwise} \end{cases}$$

(a) Find constant $K$.

$$\iint f_{XY}(x,y)\,dx\,dy = 1 \quad \text{, or}$$

$$\frac{K}{16} \cdot 8 \cdot 2 = 1$$

$$K = 1$$

(b.) Prove or disprove: $X$ and $Y$ are orthogonal.

$$E[XY] = \int_{-4}^{4} \int_{2}^{4} \frac{xy}{16}\,dx\,dy$$

$$= \frac{1}{64} x^2 \Big|_{-4}^{4} \cdot y^2 \Big|_{2}^{4}$$

$$= 0 \longrightarrow \text{orthogonal}$$

(c) Prove or disprove: X and Y are independent.

$$f_X(x) = \int f_{XY}(x,y)\,dy$$

$$= \begin{cases} \int_2^4 \frac{dy}{16}, & -4 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{8}, & -4 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int f_{XY}(x,y)\,dx$$

$$= \begin{cases} \int_{-4}^4 \frac{dx}{16}, & 2 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & 2 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) \cdot f_Y(y) = \begin{cases} 1/16, & -4 < x < 4 \text{ and } 2 < y < 4 \\ 0, & \text{otherwise} \end{cases}$$

$f_X(x)\, f_Y(y) = f_{XY}(x,y)$, so X and Y are independent

(d.) Find $P[Y \le 3 \mid X \ge 0]$.

$$P[Y \le 3, X \ge 0] = \int_2^3 \int_0^4 \frac{dx\,dy}{16} = \frac{1 \cdot 4}{16} = \frac{1}{4}$$

$$P[X \ge 0] = \int_0^4 \frac{1}{8}\,dx = \frac{1}{2}$$ , so

$$P[Y \le 3 \mid X \ge 0] = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(or, $= P[Y \le 3]$ directly)

15. Coin $C_1$ is selected with probability $1/3$; otherwise coin $C_2$ is chosen for a single flip. $C_1$ produces heads with probability $1/7$ and $C_2$ produces heads with probability $7/8$.

(a) A head is produced. Find the chance that $C_2$ was selected.

$$P[C_2 | H] = P[H | C_2] \cdot \frac{P[C_2]}{P[H]}$$

$$= \frac{1}{\frac{P[H/C_1]\,P[C_1]}{P[H/C_2]\,P[C_2]} + 1}$$

$$= \frac{1}{1 + \frac{\frac{1}{7}\cdot\frac{1}{3}}{\frac{7}{8}\,\frac{2}{3}}} = \frac{1}{1 + \frac{8}{49}\cdot\frac{1}{2}}$$

$$= \frac{98}{106} = \frac{49}{53}$$

(b) Prove or disprove: the chance of heads is $1/2$.

$$P[H] = P[C_1]\cdot P[H/C_1] + P[C_2]\,P[H/C_2]$$

$$= 1/3 \cdot 1/7 + 2/3 \cdot 7/8$$

$$= (1/7 + 14/8)\,1/3$$

$$= (8 + 98)\,\frac{1}{3}\cdot\frac{1}{56} = \frac{106}{3\cdot 56} = \frac{2\cdot 53}{3\cdot 56} \neq \frac{1}{2}$$

16. Let $X$ have probability density function $f_X(x) = \begin{cases} 2x, & 0 < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$.
Find the characteristic function of $X$.

$$E\left[e^{j\omega X}\right] = \int_0^{\frac{1}{2}} 2x\, e^{j\omega x}\, dx$$

By parts:

$$u = 2x \qquad du = 2\,dx$$
$$dv = e^{j\omega x}\,dx, \qquad v = \frac{1}{j\omega} e^{j\omega x}$$

so

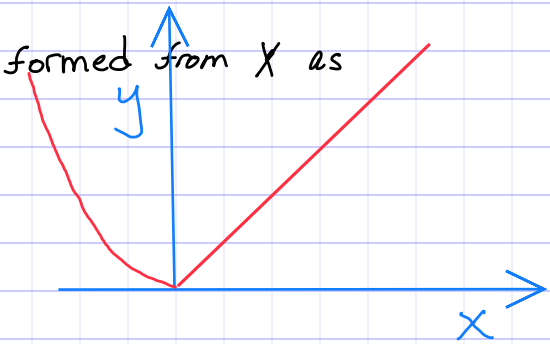
$$E\left[e^{j\omega X}\right] = \frac{2}{j\omega} x e^{j\omega x}\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{2}{j\omega} e^{j\omega x}\,dx$$

$$= \frac{1}{j\omega} e^{j\frac{\omega}{2}} \underbrace{- \frac{2}{(j\omega)^2} e^{j\omega x}\Big|_0^{\frac{1}{2}}}_{1 - \frac{2}{(j\omega)^2} e^{j\frac{\omega}{2}}}$$

$$= 1 + \frac{1}{j\omega} e^{j\frac{\omega}{2}} + \frac{2}{\omega^2} e^{j\frac{\omega}{2}}$$

17. Let $Y$ be a random variable formed from $X$ as

$$Y = \begin{cases} X, & X > 0 \\ 2X^2, & X \leq 0 \end{cases}$$




(a.) Compute the cdf of $Y$, $F_Y(y)$, in terms of the cdf of $X$, $F_X(x)$.

$$F_Y(y) = P[Y \leq y]$$
$$= \begin{cases} P\left[-\sqrt{\frac{y}{2}} \leq X \leq y\right], & y \geq 0 \\ 0, & y < 0 \end{cases}$$
$$= \begin{cases} F_X(y) - F_X\left(-\sqrt{\frac{y}{2}}\right), & y \geq 0 \\ 0, & y < 0 \end{cases}$$

(b) Find the pdf of $Y$, $f_Y(y)$, when $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$y \geq 0$: $f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= -\frac{d}{dy} F_X\left(-\sqrt{\frac{y}{2}}\right) + \frac{d}{dy} F_X(y)$$
$$= +\frac{1}{2}\frac{1}{\sqrt{y}} f_X\left(-\sqrt{\frac{y}{2}}\right) + f_X(y)$$

$y < 0$: $f_Y(y) = 0$