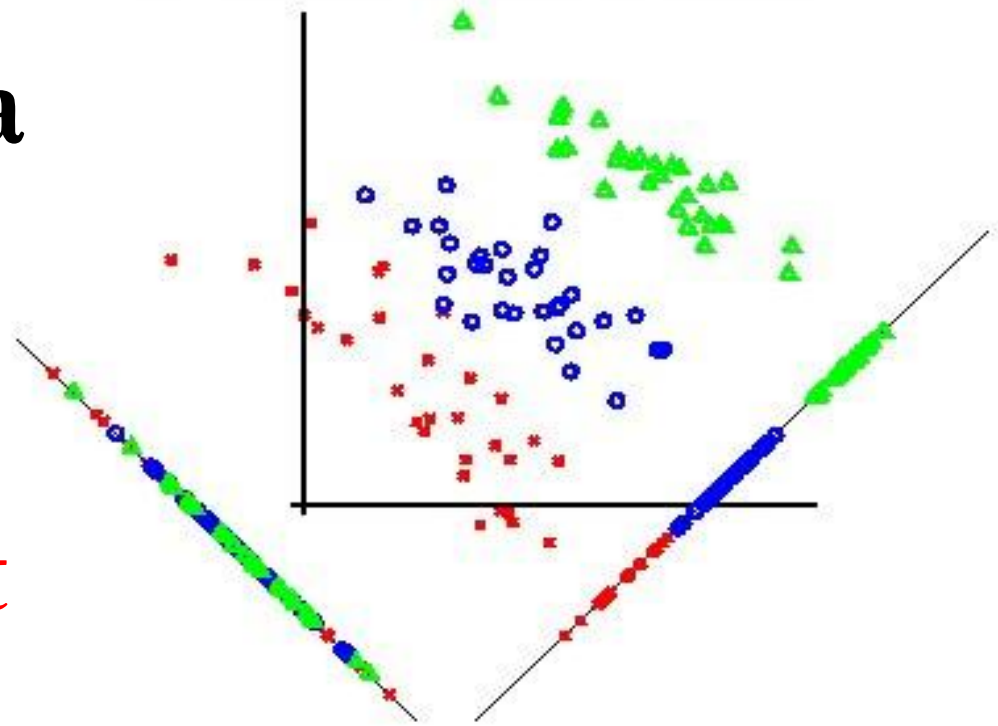


A Tutorial on Data Reduction

Linear Discriminant Analysis (LDA)



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Outline

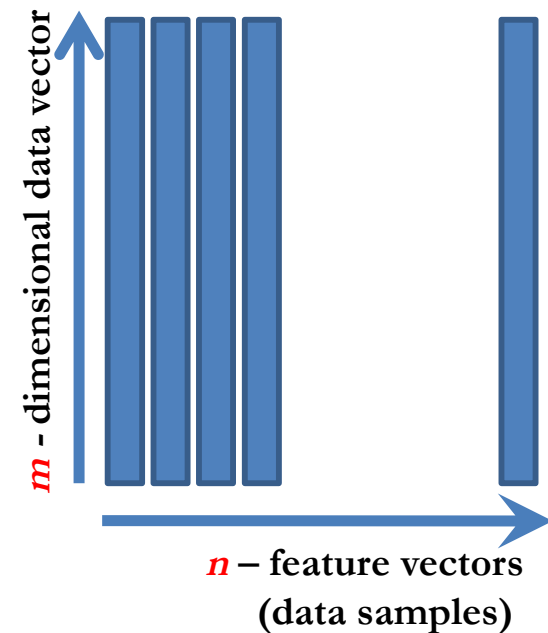
- LDA objective
- Recall ... PCA
- Now ... LDA
- LDA ... Two Classes
 - Counter example
- LDA ... C Classes
 - Illustrative Example
- LDA vs PCA Example
- Limitations of LDA

LDA Objective

- The objective of LDA is to perform dimensionality reduction ...
 - So what, PCA does this ☹...
- However, we want to preserve as much of the class discriminatory information as possible.
 - OK, that's new, let dwell deeper ☺ ...

Recall ... PCA

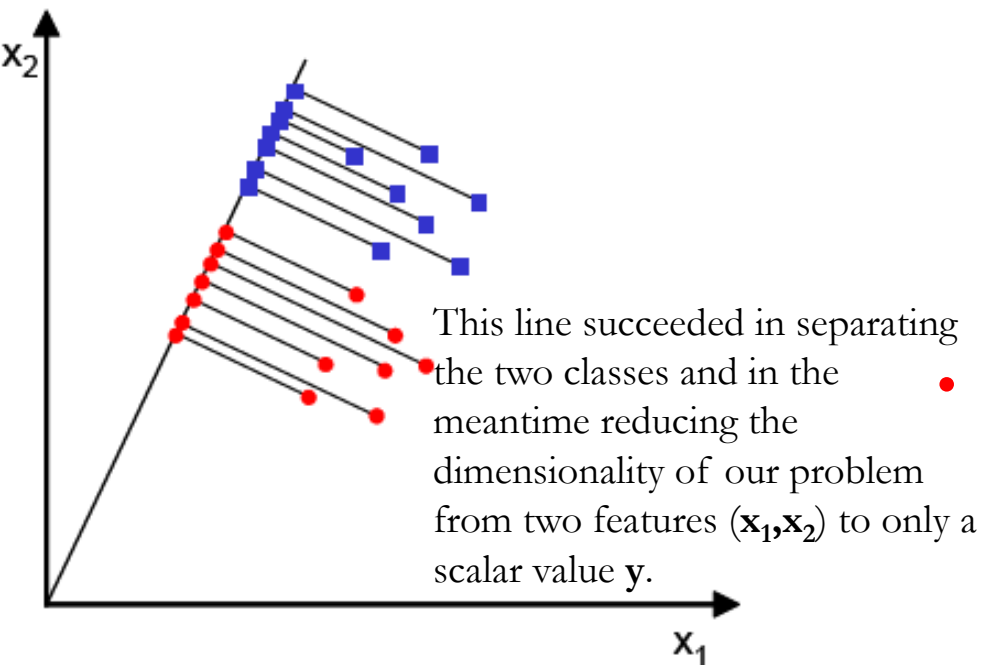
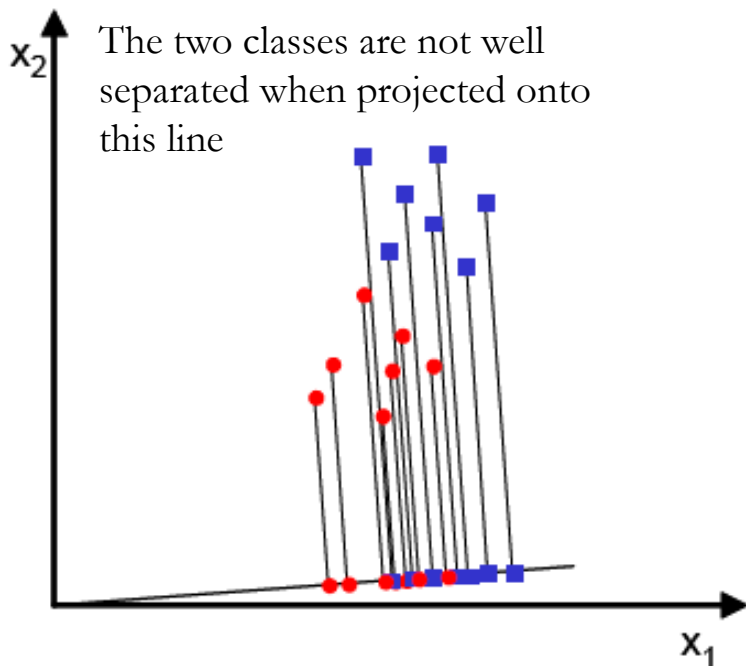
- In PCA, the main idea to re-express the available dataset to extract the relevant information by reducing the redundancy and minimize the noise.
- We didn't care about whether this dataset represent features from one or more classes, i.e. the discrimination power was not taken into consideration while we were talking about PCA.
- In PCA, we had a dataset matrix \mathbf{X} with dimensions $m \times n$, where columns represent different data samples.
- We first started by subtracting the mean to have a zero mean dataset, then we computed the covariance matrix $\mathbf{S}_x = \mathbf{X}\mathbf{X}^T$.
- Eigen values and eigen vectors were then computed for \mathbf{S}_x . Hence the new basis vectors are those eigen vectors with highest eigen values, where the number of those vectors was our choice.
- Thus, using the new basis, we can project the dataset onto a less dimensional space with more powerful data representation.



Now ... LDA

- Consider a pattern classification problem, where we have C -classes, e.g. seabass, tuna, salmon ...
- Each class has N_i m -dimensional samples, where $i = 1, 2, \dots, C$.
- Hence we have a set of m -dimensional samples $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^{N_i}\}$ belong to class ω_i .
- Stacking these samples from different classes into one big fat matrix \mathbf{X} such that each column represents one sample.
- We seek to obtain a transformation of \mathbf{X} to \mathbf{Y} through projecting the samples in \mathbf{X} onto a hyperplane with dimension $C-1$.
- Let's see what does this mean?

LDA ... Two Classes



- Assume we have m -dimensional samples $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$, N_1 of which belong to ω_1 and N_2 belong to ω_2 .
- We seek to obtain a scalar y by projecting the samples \mathbf{x} onto a line ($C-1$ space, $C = 2$).

$$y = \mathbf{w}^T \mathbf{x} \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_m \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ w_m \end{bmatrix}$$

- where \mathbf{w} is the projection vectors used to project \mathbf{x} to y .

- **Of all the possible lines we would like to select the one that maximizes the separability of the scalars.**

LDA ... Two Classes

- In order to find a good projection vector, we need to define a measure of separation between the projections.

- The mean vector of each class in \mathbf{x} and \mathbf{y} feature space is:

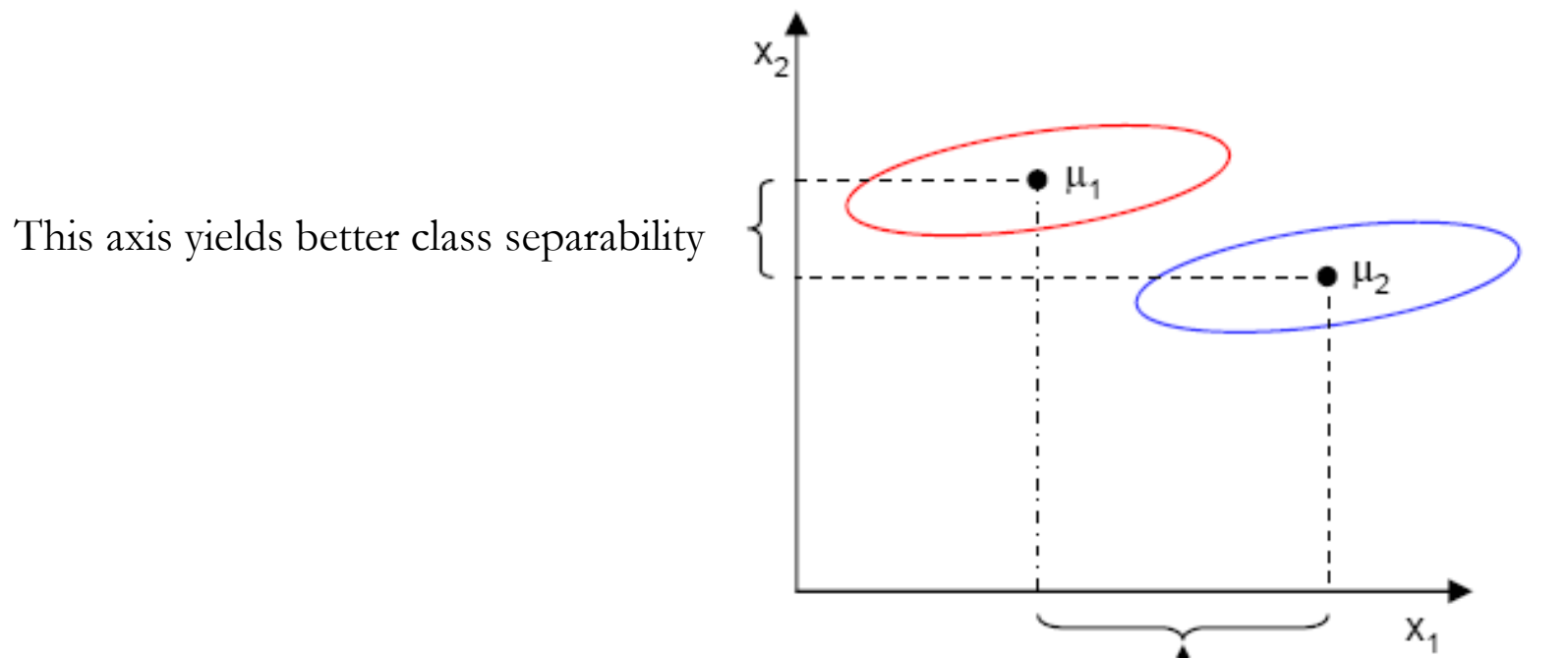
$$\mu_i = \frac{1}{N_i} \sum_{x \in \omega_i} x \quad \text{and} \quad \tilde{\mu}_i = \frac{1}{N_i} \sum_{y \in \omega_i} y = \frac{1}{N_i} \sum_{x \in \omega_i} w^T x$$
$$= w^T \frac{1}{N_i} \sum_{x \in \omega_i} x = w^T \mu_i$$

- i.e. projecting \mathbf{x} to \mathbf{y} will lead to projecting the mean of \mathbf{x} to the mean of \mathbf{y} .
- We could then choose the distance between the projected means as our objective function

$$J(w) = |\tilde{\mu}_1 - \tilde{\mu}_2| = |w^T \mu_1 - w^T \mu_2| = |w^T (\mu_1 - \mu_2)|$$

LDA ... Two Classes

- However, the distance between the projected means is not a very good measure since it does not take into account the standard deviation within the classes.



This axis has a larger distance between means

LDA ... Two Classes

- The solution proposed by Fisher is to maximize a function that represents the difference between the means, normalized by a measure of the within-class variability, or the so-called *scatter*.
- For each class we define the **scatter**, an equivalent of the **variance**, as; (sum of square differences between the projected samples and their class mean).

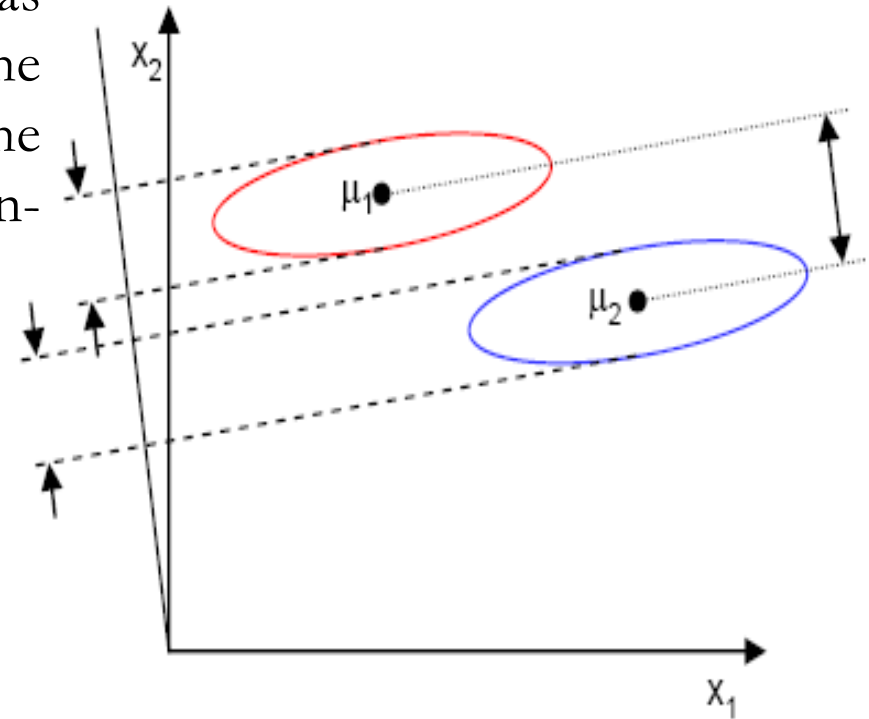
$$\tilde{s}_i^2 = \sum_{y \in \omega_i} (y - \tilde{\mu}_i)^2$$

- \tilde{s}_i^2 measures the variability within class ω_i after projecting it on the y-space.
- Thus $\tilde{s}_1^2 + \tilde{s}_2^2$ measures the variability within the two classes at hand after projection, hence it is called *within-class scatter* of the projected samples.

LDA ... Two Classes

- The Fisher linear discriminant is defined as the linear function $\mathbf{w}^T \mathbf{x}$ that maximizes the criterion function: (the distance between the projected means normalized by the within-class scatter of the projected samples).

$$J(w) = \frac{|\tilde{\mu}_1 - \tilde{\mu}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$$



- Therefore, we will be looking for a projection where examples from the same class are projected very close to each other and, at the same time, the projected means are as farther apart as possible

LDA ... Two Classes

- In order to find the optimum projection w^* , we need to express $J(w)$ as an explicit function of w .
- We will define a measure of the scatter in multivariate feature space \mathbf{x} which are denoted as scatter matrices;

$$S_i = \sum_{x \in \omega_i} (x - \mu_i)(x - \mu_i)^T$$

$$S_w = S_1 + S_2$$

$$J(w) = \frac{|\tilde{\mu}_1 - \tilde{\mu}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$$

- Where \mathbf{S}_i is the covariance matrix of class ω_i , and \mathbf{S}_w is called the within-class scatter matrix.

LDA ... Two Classes

- Now, the scatter of the projection \mathbf{y} can then be expressed as a function of the scatter matrix in feature space \mathbf{x} .

$$\begin{aligned}\tilde{s}_i^2 &= \sum_{y \in \omega_i} (y - \tilde{\mu}_i)^2 = \sum_{x \in \omega_i} (w^T x - w^T \mu_i)^2 \\ &= \sum_{x \in \omega_i} w^T (x - \mu_i)(x - \mu_i)^T w \\ &= w^T \left(\sum_{x \in \omega_i} (x - \mu_i)(x - \mu_i)^T \right) w = w^T S_i w\end{aligned}$$

$$J(w) = \frac{|\tilde{\mu}_1 - \tilde{\mu}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$$

$$\tilde{s}_1^2 + \tilde{s}_2^2 = w^T S_1 w + w^T S_2 w = w^T (S_1 + S_2) w = w^T S_W w = \tilde{S}_W$$

Where \tilde{S}_W is the within-class scatter matrix of the projected samples \mathbf{y} .

LDA ... Two Classes

- Similarly, the difference between the projected means (in y-space) can be expressed in terms of the means in the original feature space (x-space).

$$\begin{aligned}(\tilde{\mu}_1 - \tilde{\mu}_2)^2 &= (w^T \mu_1 - w^T \mu_2)^2 \\&= w^T \underbrace{(\mu_1 - \mu_2)(\mu_1 - \mu_2)^T}_{S_B} w \\&= w^T S_B w = \tilde{S}_B\end{aligned}$$

$$J(w) = \frac{|\tilde{\mu}_1 - \tilde{\mu}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$$

- The matrix \mathbf{S}_B is called the between-class scatter of the original samples/feature vectors, while \tilde{S}_B is the between-class scatter of the projected samples \mathbf{y} .
- Since \mathbf{S}_B is the outer product of two vectors, its rank is at most one.

LDA ... Two Classes

- We can finally express the Fisher criterion in terms of S_W and S_B as:

$$J(w) = \frac{|\tilde{\mu}_1 - \tilde{\mu}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2} = \frac{w^T S_B w}{w^T S_W w}$$

- Hence $J(w)$ is a measure of the difference between class means (encoded in the between-class scatter matrix) normalized by a measure of the within-class scatter matrix.

LDA ... Two Classes

- To find the maximum of $J(w)$, we differentiate and equate to zero.

$$\frac{d}{dw} J(w) = \frac{d}{dw} \left(\frac{w^T S_B w}{w^T S_W w} \right) = 0$$

$$\Rightarrow (w^T S_W w) \frac{d}{dw} (w^T S_B w) - (w^T S_B w) \frac{d}{dw} (w^T S_W w) = 0$$

$$\Rightarrow (w^T S_W w) 2S_B w - (w^T S_B w) 2S_W w = 0$$

Dividing by $2w^T S_W w$:

$$\Rightarrow \left(\frac{w^T S_W w}{w^T S_W w} \right) S_B w - \left(\frac{w^T S_B w}{w^T S_W w} \right) S_W w = 0$$

$$\Rightarrow S_B w - J(w) S_W w = 0$$

$$\Rightarrow S_W^{-1} S_B w - J(w) w = 0$$

LDA ... Two Classes

- Solving the generalized eigen value problem

$$S_W^{-1} S_B w = \lambda w \quad \text{where} \quad \lambda = J(w) = \text{scalar}$$

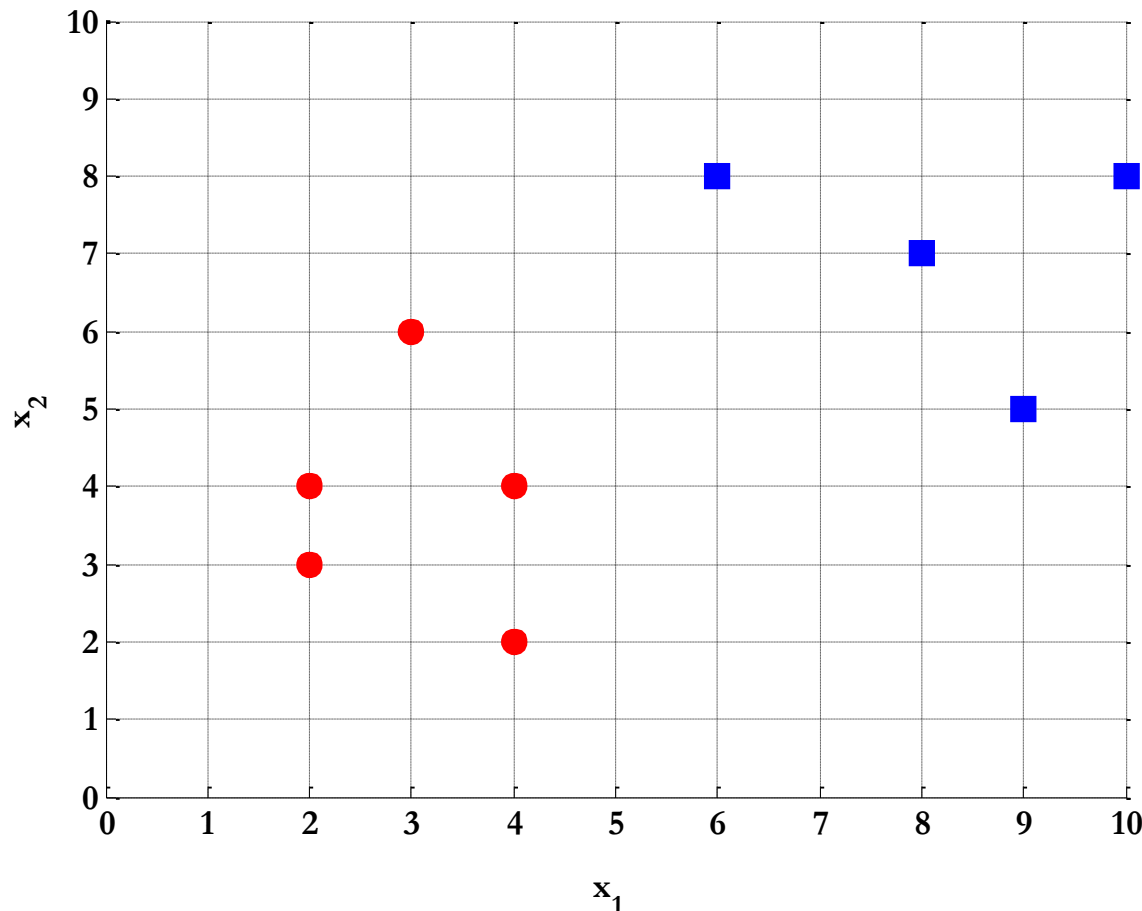
yields

$$w^* = \arg \max_w J(w) = \arg \max_w \left(\frac{w^T S_B w}{w^T S_W w} \right) = S_W^{-1} (\mu_1 - \mu_2)$$

- This is known as Fisher's Linear Discriminant, although it is not a discriminant but rather a specific choice of direction for the projection of the data down to one dimension.
- Using the same notation as PCA, **the solution will be the eigen vector(s) of** $S_X = S_W^{-1} S_B$

LDA ... Two Classes - Example

- Compute the Linear Discriminant projection for the following two-dimensional dataset.
 - Samples for class ω_1 : $\mathbf{X}_1=(x_1,x_2)=\{(4,2),(2,4),(2,3),(3,6),(4,4)\}$
 - Sample for class ω_2 : $\mathbf{X}_2=(x_1,x_2)=\{(9,10),(6,8),(9,5),(8,7),(10,8)\}$



```
% samples for class 1
X1 = [4,2;
      2,4;
      2,3;
      3,6;
      4,4];

% samples for class 2
X2 = [9,10;
      6,8;
      9,5;
      8,7;
      10,8];
```

LDA ... Two Classes - Example

- The classes mean are :

$$\mu_1 = \frac{1}{N_1} \sum_{x \in \omega_1} x = \frac{1}{5} \left[\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right] = \begin{pmatrix} 3 \\ 3.8 \end{pmatrix}$$

$$\mu_2 = \frac{1}{N_2} \sum_{x \in \omega_2} x = \frac{1}{5} \left[\begin{pmatrix} 9 \\ 10 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \end{pmatrix} + \begin{pmatrix} 9 \\ 5 \end{pmatrix} + \begin{pmatrix} 8 \\ 7 \end{pmatrix} + \begin{pmatrix} 10 \\ 8 \end{pmatrix} \right] = \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}$$

```
% class means  
Mu1 = mean(X1) ' ;  
Mu2 = mean(X2) ' ;
```

LDA ... Two Classes - Example

- Covariance matrix of the first class:

$$\begin{aligned} S_1 &= \sum_{x \in \omega_1} (x - \mu_1)(x - \mu_1)^T = \left[\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3.8 \end{pmatrix} \right]^2 + \left[\begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3.8 \end{pmatrix} \right]^2 \\ &\quad + \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 3.8 \end{pmatrix} \right]^2 + \left[\begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 3.8 \end{pmatrix} \right]^2 + \left[\begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3.8 \end{pmatrix} \right]^2 \\ &= \begin{pmatrix} 1 & -0.25 \\ -0.25 & 2.2 \end{pmatrix} \end{aligned}$$

```
% covariance matrix of the first class  
S1 = cov(X1) ;
```

LDA ... Two Classes - Example

- Covariance matrix of the second class:

$$\begin{aligned} S_2 &= \sum_{x \in \omega_2} (x - \mu_2)(x - \mu_2)^T = \left[\begin{pmatrix} 9 \\ 10 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right]^2 + \left[\begin{pmatrix} 6 \\ 8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right]^2 \\ &\quad + \left[\begin{pmatrix} 9 \\ 5 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right]^2 + \left[\begin{pmatrix} 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right]^2 + \left[\begin{pmatrix} 10 \\ 8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right]^2 \\ &= \begin{pmatrix} 2.3 & -0.05 \\ -0.05 & 3.3 \end{pmatrix} \end{aligned}$$

```
% covariance matrix of the first class  
S2 = cov(X2);
```

LDA ... Two Classes - Example

- Within-class scatter matrix:

$$\begin{aligned} S_w = S_1 + S_2 &= \begin{pmatrix} 1 & -0.25 \\ -0.25 & 2.2 \end{pmatrix} + \begin{pmatrix} 2.3 & -0.05 \\ -0.05 & 3.3 \end{pmatrix} \\ &= \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix} \end{aligned}$$

```
% within-class scatter matrix  
Sw = S1 + S2 ;
```

LDA ... Two Classes - Example

- Between-class scatter matrix:

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right] \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right]^T$$

$$= \begin{pmatrix} -5.4 \\ -3.8 \end{pmatrix} \begin{pmatrix} -5.4 & -3.8 \end{pmatrix}$$

$$= \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix}$$

```
% between-class scatter matrix  
SB = (Mu1-Mu2) * (Mu1-Mu2) ' ;
```

LDA ... Two Classes - Example

- The LDA projection is then obtained as the solution of the generalized eigen value problem

$$S_W^{-1}S_B w = \lambda w$$

$$\Rightarrow |S_W^{-1}S_B - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}^{-1} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{pmatrix} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 9.2213 - \lambda & 6.489 \\ 4.2339 & 2.9794 - \lambda \end{pmatrix} \right|$$

$$= (9.2213 - \lambda)(2.9794 - \lambda) - 6.489 \times 4.2339 = 0$$

$$\Rightarrow \lambda^2 - 12.2007\lambda = 0 \Rightarrow \lambda(\lambda - 12.2007) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 12.2007$$

LDA ... Two Classes - Example

- Hence

$$\begin{pmatrix} 9.2213 & 6.489 \\ 4.2339 & 2.9794 \end{pmatrix} w_1 = \underbrace{0}_{\lambda_1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and

$$\begin{pmatrix} 9.2213 & 6.489 \\ 4.2339 & 2.9794 \end{pmatrix} w_2 = \underbrace{12.2007}_{\lambda_2} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Thus;

$$w_1 = \begin{pmatrix} -0.5755 \\ 0.8178 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = w^*$$

```
% computing the LDA projection
invSw = inv(Sw);

invSw_by_SB = invSw * SB;

% getting the projection vector
[V,D] = eig(invSw_by_SB)

% the projection vector
W = V(:,1);
```

- The optimal projection is the one that given maximum $\lambda = J(w)$

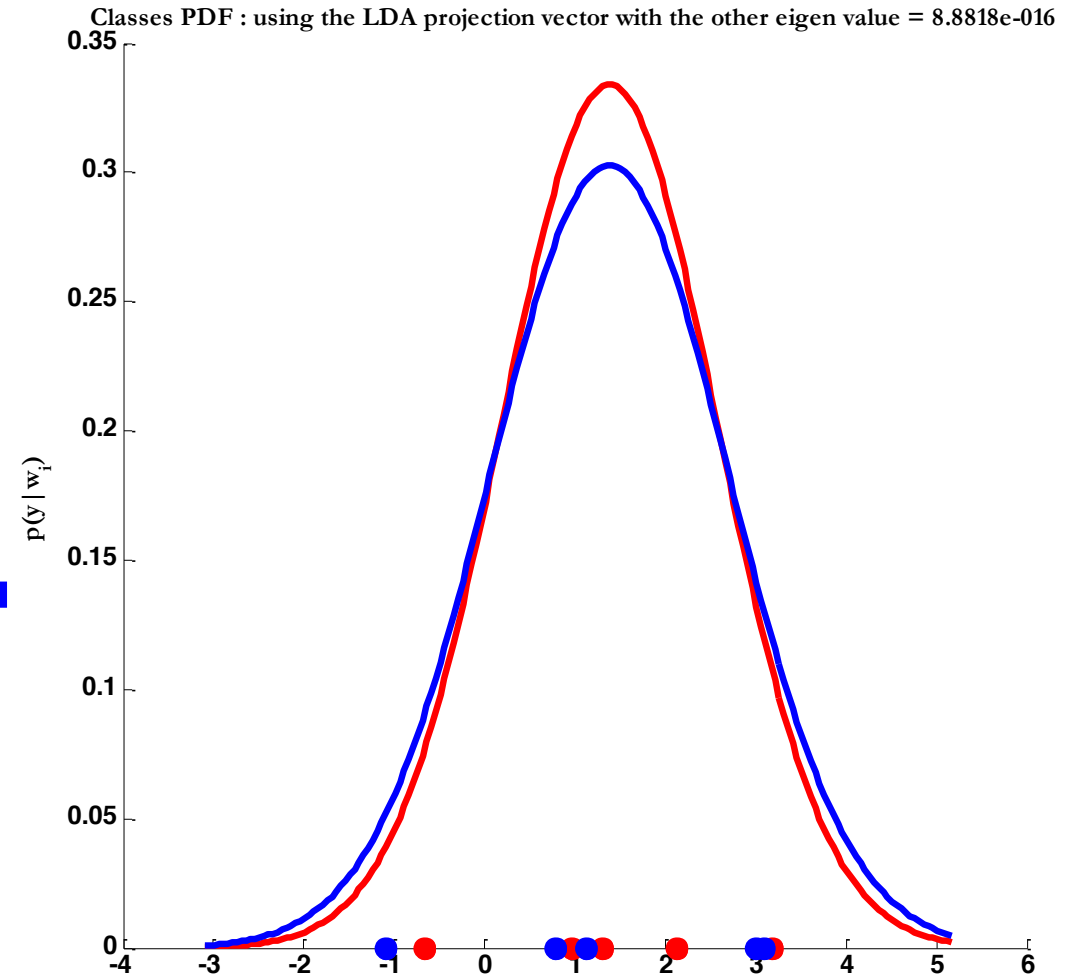
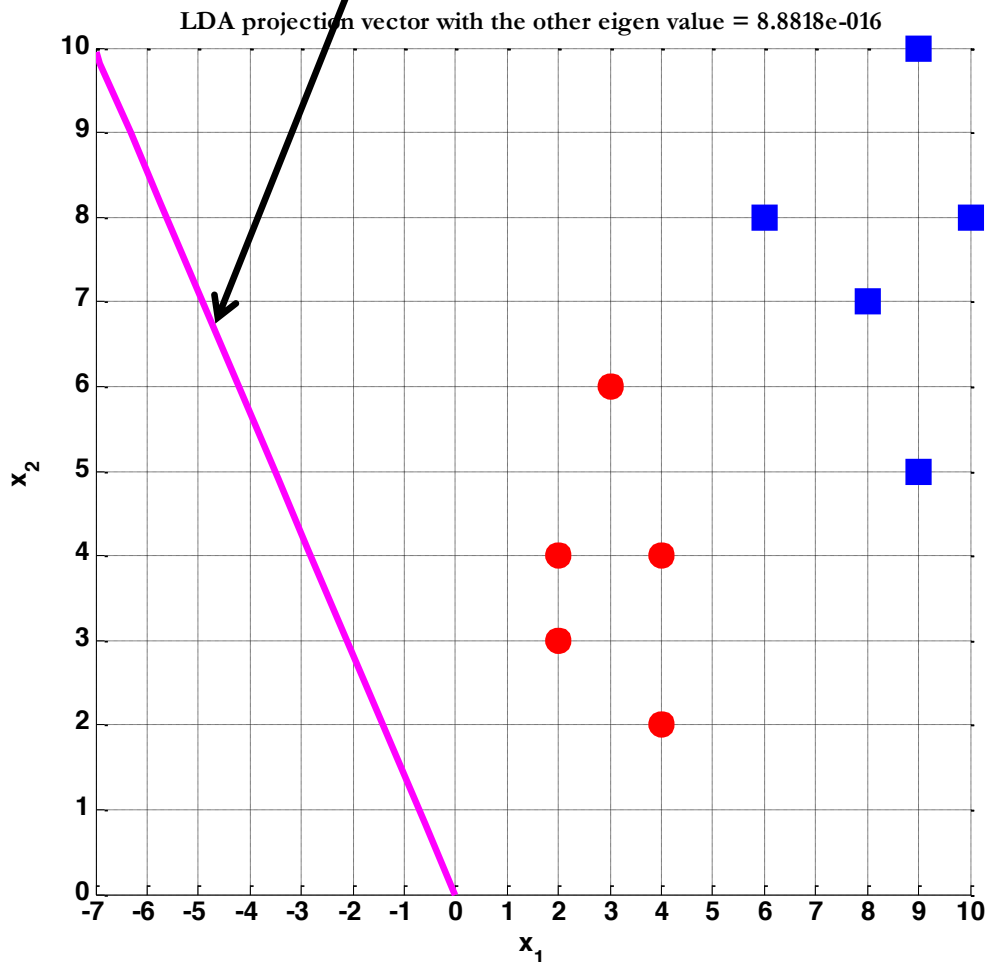
LDA ... Two Classes - Example

Or directly;

$$\begin{aligned}w^* &= S_W^{-1}(\mu_1 - \mu_2) = \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}^{-1} \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix} \right] \\&= \begin{pmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{pmatrix} \begin{pmatrix} -5.4 \\ -3.8 \end{pmatrix} \\&= \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix}\end{aligned}$$

LDA - Projection

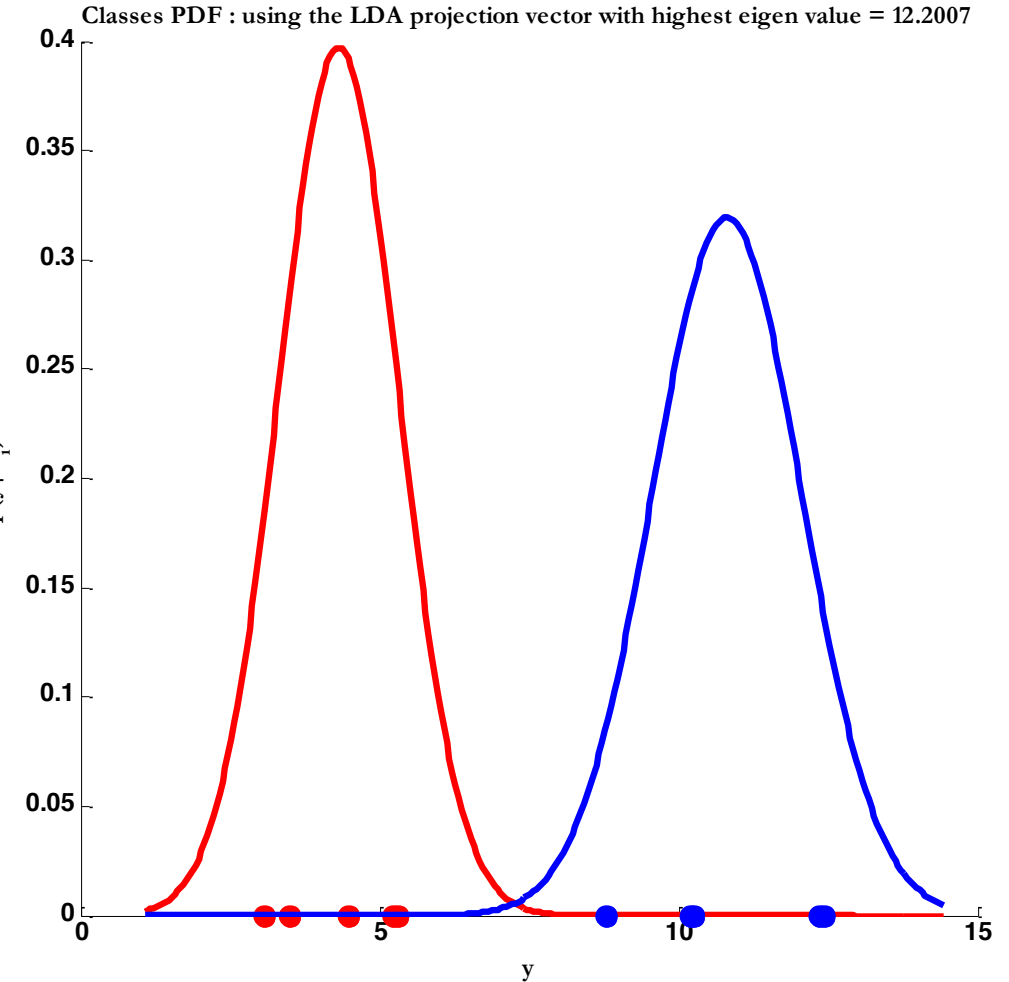
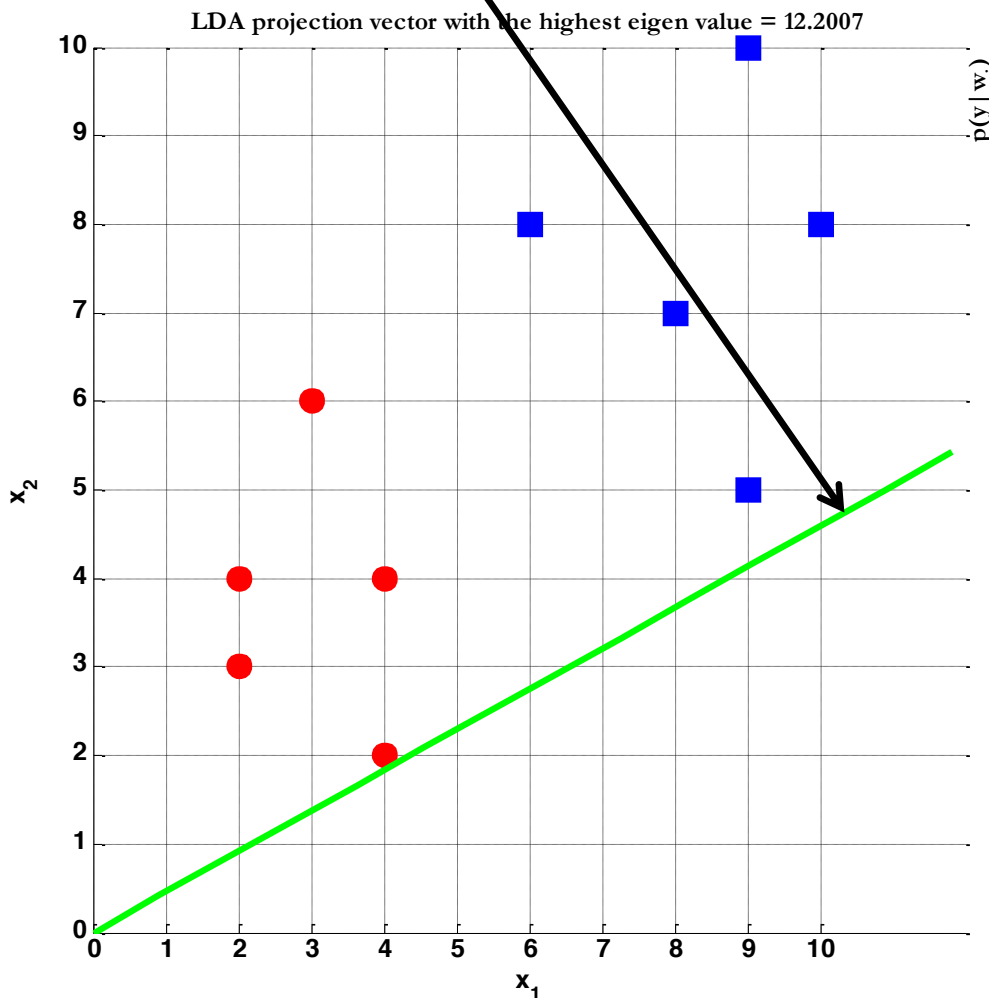
The projection vector corresponding to the **smallest** eigen value



Using this vector leads to **bad separability** between the two classes

LDA - Projection

The projection vector
corresponding to the
highest eigen value



Using this vector leads to
good separability
between the two classes

LDA ... C-Classes

- Now, we have C -classes instead of just two.
- We are now seeking $(C-1)$ projections $[y_1, y_2, \dots, y_{C-1}]$ by means of $(C-1)$ projection vectors \mathbf{w}_i .
- \mathbf{w}_i can be arranged by *columns* into a projection matrix $\mathbf{W} = [\mathbf{w}_1 | \mathbf{w}_2 | \dots | \mathbf{w}_{C-1}]$ such that:

$$y_i = w_i^T x \quad \Rightarrow \quad y = W^T x$$

where $x_{m \times 1} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_m \end{bmatrix}$, $y_{C-1 \times 1} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_{C-1} \end{bmatrix}$

and $W_{m \times C-1} = \begin{bmatrix} w_1 & w_2 & \dots & w_{C-1} \end{bmatrix}$

LDA ... C-Classes

- If we have n -feature vectors, we can stack them into one matrix as follows;

$$Y = W^T X$$

$$\text{where } X_{m \times n} = \begin{bmatrix} x_1^1 & x_1^2 & \cdot & x_1^n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_m^1 & x_m^2 & \cdot & x_m^n \end{bmatrix}, \quad Y_{C-1 \times n} = \begin{bmatrix} y_1^1 & y_1^2 & \cdot & y_1^n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_{C-1}^1 & y_{C-1}^2 & \cdot & y_{C-1}^n \end{bmatrix}$$

$$\text{and } W_{m \times C-1} = [w_1 \mid w_2 \mid \dots \mid w_{C-1}]$$

LDA – C-Classes

- Recall the two classes case, the *within-class scatter* was computed as:

$$S_w = S_1 + S_2$$

- This can be generalized in the C -classes case as:

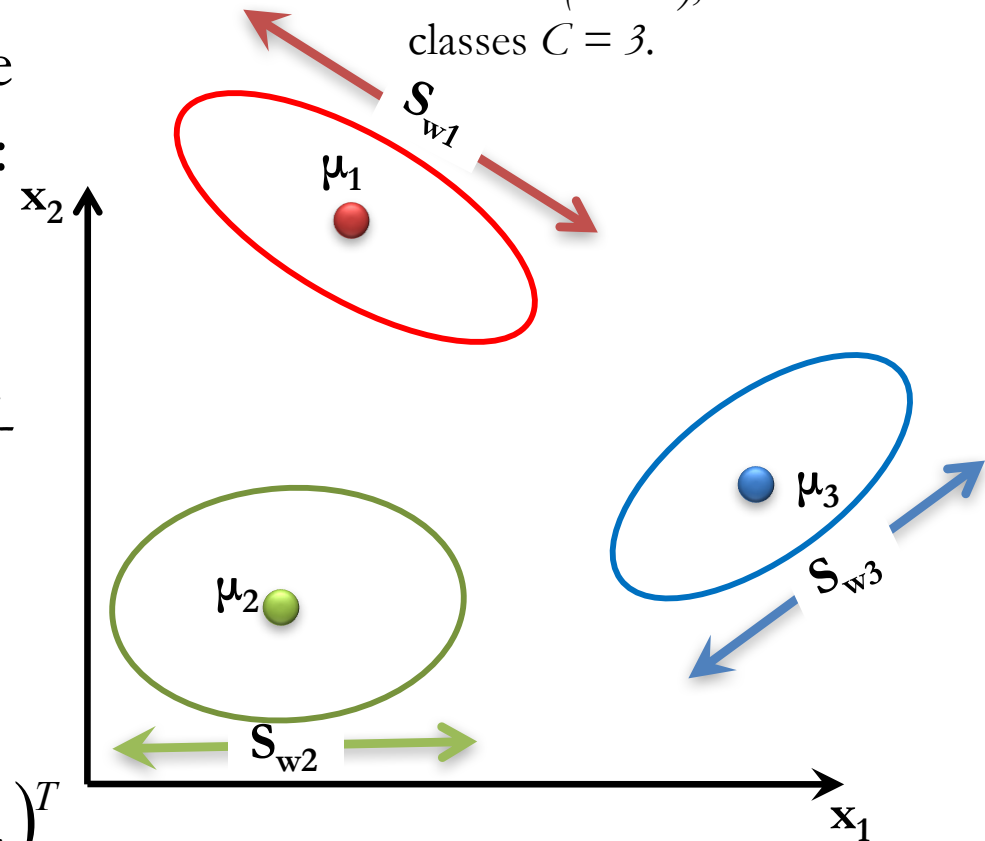
$$S_W = \sum_{i=1}^C S_i$$

where
$$S_i = \sum_{x \in \omega_i} (x - \mu_i)(x - \mu_i)^T$$

and
$$\mu_i = \frac{1}{N_i} \sum_{x \in \omega_i} x$$

N_i : number of data samples in class ω_i .

Example of two-dimensional features ($m = 2$), with three classes $C = 3$.



LDA – C-Classes

- Recall the two classes case, the *between-class scatter* was computed as:

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

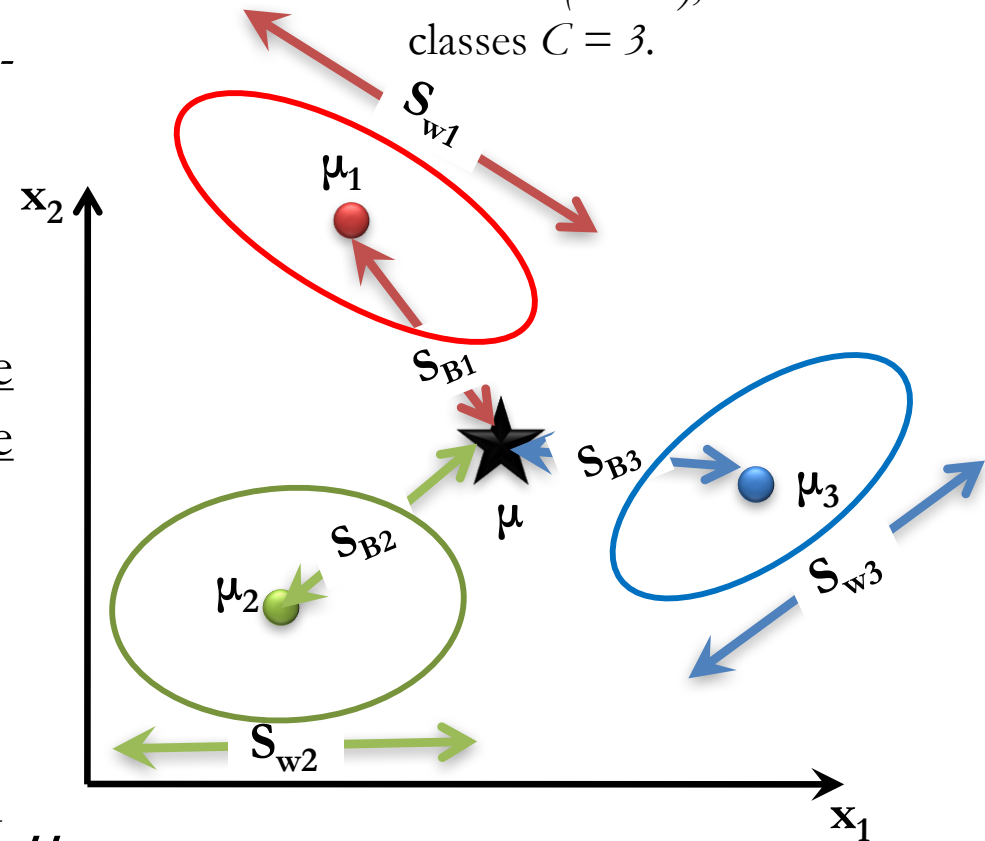
- For C -classes case, we will measure the between-class scatter with respect to the mean of all classes as follows:

$$S_B = \sum_{i=1}^C N_i (\mu_i - \mu)(\mu_i - \mu)^T$$

where $\mu = \frac{1}{N} \sum_{\forall x} x = \frac{1}{N} \sum_{\forall x} N_i \mu_i$

and $\mu_i = \frac{1}{N_i} \sum_{x \in \omega_i} x$

Example of two-dimensional features ($m = 2$), with three classes $C = 3$.



N : number of all data .

N_i : number of data samples in class ω_i .

LDA – C-Classes

- Similarly,
 - We can define the mean vectors for the projected samples \mathbf{y} as:

$$\tilde{\mu}_i = \frac{1}{N_i} \sum_{y \in \omega_i} y \quad \text{and} \quad \tilde{\mu} = \frac{1}{N} \sum_{\forall y} y$$

- While the scatter matrices for the projected samples \mathbf{y} will be:

$$\tilde{S}_W = \sum_{i=1}^C \tilde{S}_i = \sum_{i=1}^C \sum_{y \in \omega_i} (y - \tilde{\mu}_i)(y - \tilde{\mu}_i)^T$$

$$\tilde{S}_B = \sum_{i=1}^C N_i (\tilde{\mu}_i - \tilde{\mu})(\tilde{\mu}_i - \tilde{\mu})^T$$

LDA – C-Classes

- Recall in two-classes case, we have expressed the scatter matrices of the projected samples in terms of those of the original samples as:

$$\tilde{S}_W = W^T S_W W$$

$$\tilde{S}_B = W^T S_B W$$

This still hold in C -classes case.

- Recall that we are looking for a projection that maximizes the ratio of between-class to within-class scatter.
- Since the projection is no longer a scalar (it has $C-1$ dimensions), we then use the determinant of the scatter matrices to obtain a scalar objective function:

$$J(W) = \frac{|\tilde{S}_B|}{|\tilde{S}_W|} = \frac{|W^T S_B W|}{|W^T S_W W|}$$

- And we will seek the projection \mathbf{W}^* that maximizes this ratio.

LDA – C-Classes

- To find the maximum of $J(W)$, we differentiate with respect to W and equate to zero.
- Recall in two-classes case, we solved the eigen value problem.

$$S_W^{-1} S_B w = \lambda w \quad \text{where} \quad \lambda = J(w) = \text{scalar}$$

- For C -classes case, we have $C-1$ projection vectors, hence the eigen value problem can be generalized to the C -classes case as:

$$S_W^{-1} S_B w_i = \lambda_i w_i \quad \text{where} \quad \lambda_i = J(w_i) = \text{scalar} \quad \text{and} \quad i = 1, 2, \dots, C-1$$

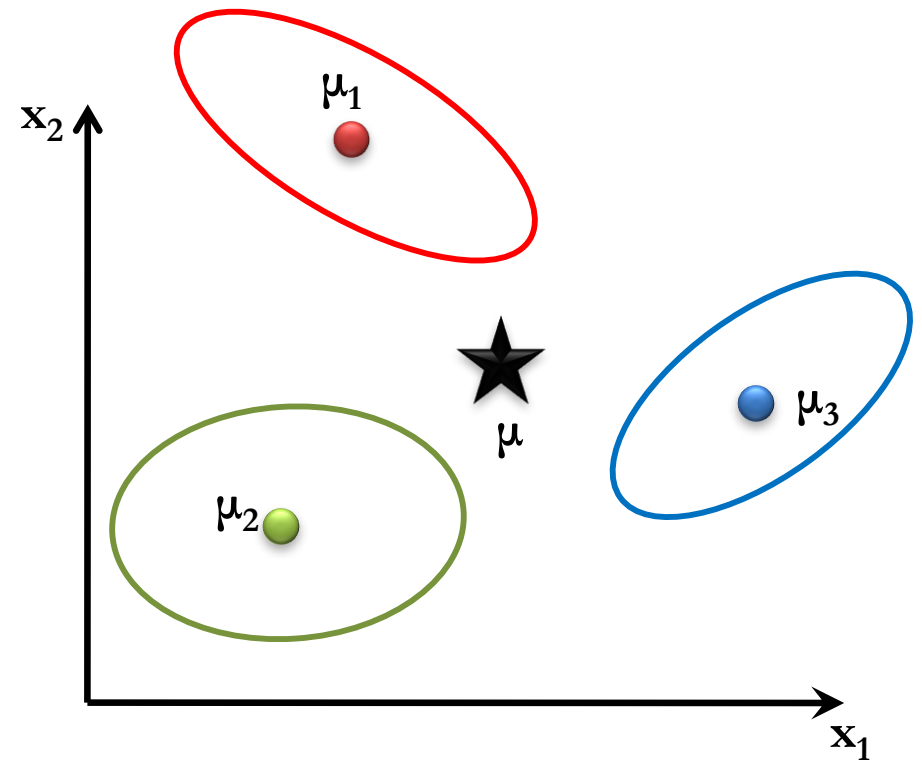
- Thus, It can be shown that the optimal projection matrix W^* is the one whose columns are the eigenvectors corresponding to the largest eigen values of the following generalized eigen value problem:

$$S_W^{-1} S_B W^* = \lambda W^*$$

$$\text{where} \quad \lambda = J(W^*) = \text{scalar} \quad \text{and} \quad W^* = \begin{bmatrix} w_1^* & w_2^* & \dots & w_{C-1}^* \end{bmatrix}$$

Illustration – 3 Classes

- Let's generate a dataset for each class to simulate the three classes shown
- For each class do the following,
 - Use the random number generator to generate a uniform stream of 500 samples that follows $U(0,1)$.
 - Using the Box-Muller approach, convert the generated uniform stream to $N(0,1)$.



- Then use the method of eigen values and eigen vectors to manipulate the standard normal to have the required mean vector and covariance matrix .
- Estimate the mean and covariance matrix of the resulted dataset.

Dataset Generation

- By visual inspection of the figure, classes parameters (means and covariance matrices) can be given as follows:

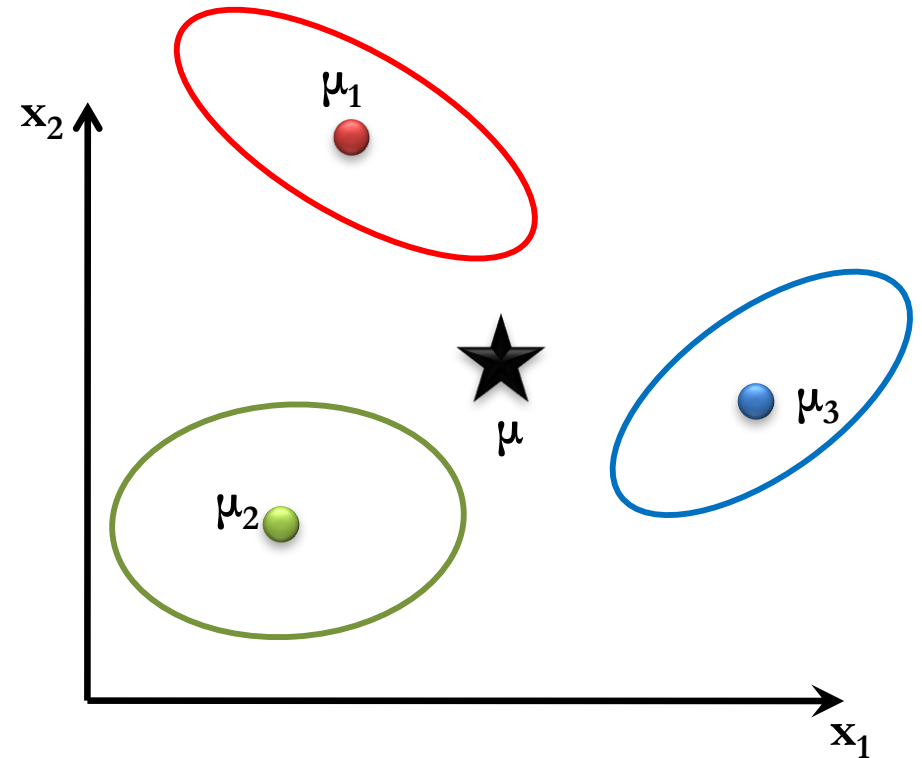
Overall mean $\mu = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$$\mu_1 = \mu + \begin{bmatrix} -3 \\ 7 \end{bmatrix}, \quad \mu_2 = \mu + \begin{bmatrix} -2.5 \\ -3.5 \end{bmatrix}, \quad \mu_3 = \mu + \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$S_1 = \begin{pmatrix} 5 & -1 \\ -3 & 3 \end{pmatrix} \longrightarrow \text{Negative covariance to lead to data samples distributed along the } y = -x \text{ line.}$$

$$S_2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \longrightarrow \text{Zero covariance to lead to data samples distributed } \textit{horizontally}.$$

$$S_3 = \begin{pmatrix} 3.5 & 1 \\ 3 & 2.5 \end{pmatrix} \longrightarrow \text{Positive covariance to lead to data samples distributed along the } y = x \text{ line.}$$



In Matlab ☺

```
% let the center of all classes be
Mu = [ 5;5];

%% for the first class
Mu1 = [Mu(1)-3; Mu(2)+7];
CovM1 = [5 -1; -3 3];

% Generating feature vectors using Box-Muller approach
% Generate a random variable following uniform(0,1) having two features and
% 1000 feature vectors
U = rand(2,1000);

% Extracting from the generated uniform random variable two independent
% uniform random variables
u1 = U(:,1:2:end);
u2 = U(:,2:2:end);

% Using u1 and u2, we will use Box-Muller method to generate the feature
% vectors to follow standard normal
X = sqrt((-2).*log(u1)) .* (cos(2*pi.*u2));
clear u1 u2 U;

% Now ... Manipulating the generated Features N(0,1) to following certain
% mean and covariance other than the standard normal

% First we will change its variance then we will change its mean

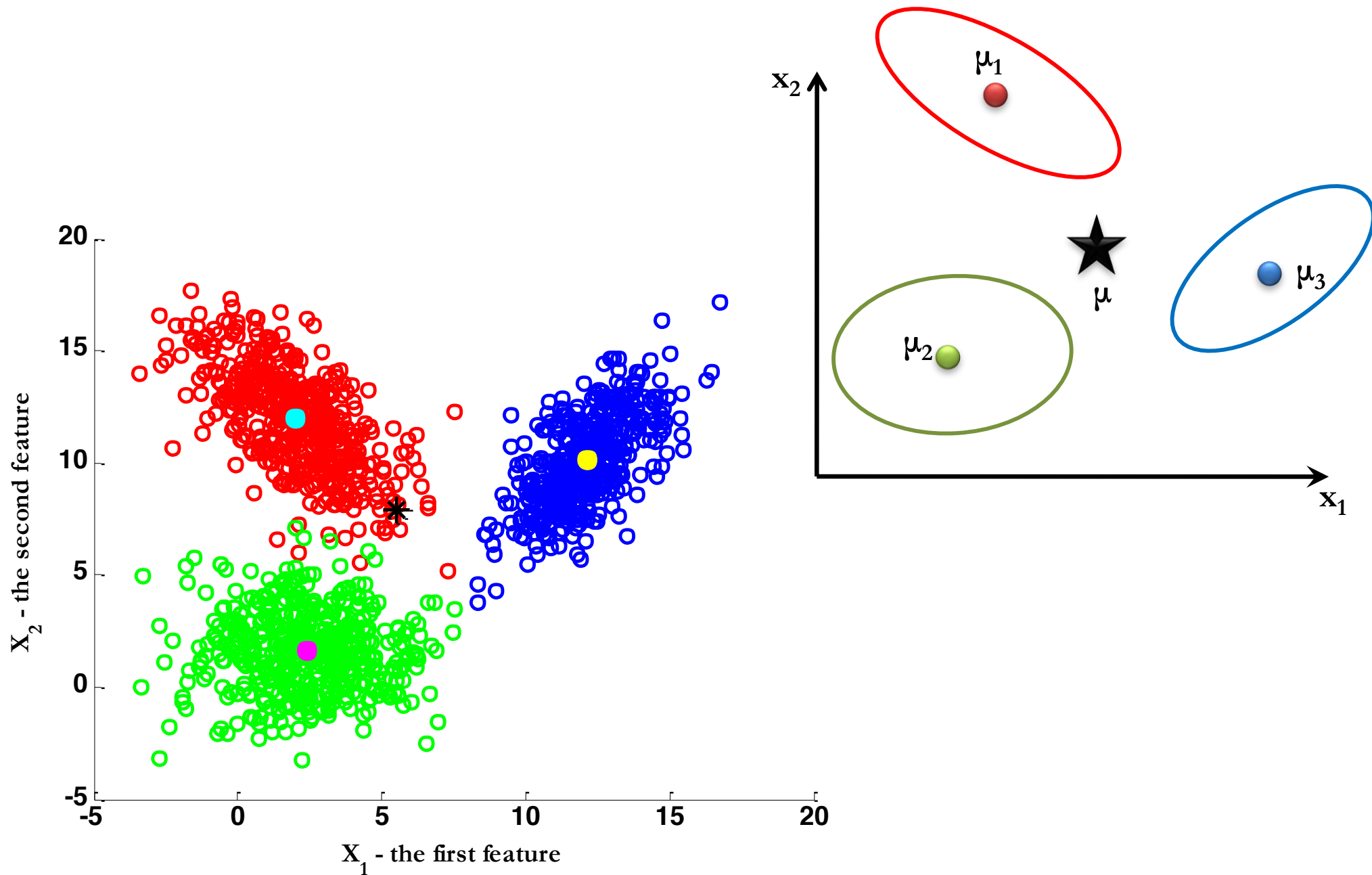
% Getting the eigen vectors and values of the covariance matrix
[V,D] = eig(CovM1); % D is the eigen values matrix and V is the eigen vectors
matrix
newX = X;
for j = 1 : size(X,2)
    newX(:,j) = V * sqrt(D) * X(:,j);
end

% changing its mean
newX = newX + repmat(Mu1,1,size(newX,2));

% now our dataset for the first class matrix will be
X1 = newX ; % each column is a feature vector, each row is a single feature

% ... do the same for the other two classes with difference means and
covariance matrices
```

It's Working ... ☺



Computing LDA Projection Vectors

```

%% computing the LDA
% class means
Mu1 = mean(X1')';
Mu2 = mean(X2')';
Mu3 = mean(X3')';

% overall mean
Mu = (Mu1 + Mu2 + Mu3) ./ 3;

% class covariance matrices
S1 = cov(X1');
S2 = cov(X2');
S3 = cov(X3');

% within-class scatter matrix
Sw = S1 + S2 + S3;

% number of samples of each class
N1 = size(X1,2);
N2 = size(X2,2);
N3 = size(X3,2);

% between-class scatter matrix
SB1 = N1 .* (Mu1-Mu)*(Mu1-Mu)';
SB2 = N2 .* (Mu2-Mu)*(Mu2-Mu)';
SB3 = N3 .* (Mu3-Mu)*(Mu3-Mu)';

SB = SB1 + SB2 + SB3;

% computing the LDA projection
invSw = inv(Sw);
invSw_by_SB = invSw * SB;

% getting the projection vectors
%[V,D] = EIG(X) produces a diagonal matrix D of eigenvalues and a
%full matrix V whose columns are the corresponding eigenvectors
[V,D] = eig(invSw_by_SB);

% the projection vectors - we will have at most C-1 projection vectors,
% from which we can choose the most important ones ranked by their
% corresponding eigen values ... lets investigate the two projection
% vectors
W1 = V(:,1);
W2 = V(:,2);
    
```

Recall ...

$$S_W = \sum_{i=1}^C S_i$$

where $S_i = \sum_{x \in \omega_i} (x - \mu_i)(x - \mu_i)^T$

and $\mu_i = \frac{1}{N_i} \sum_{x \in \omega_i} x$

$$S_B = \sum_{i=1}^C N_i (\mu_i - \mu)(\mu_i - \mu)^T$$

where $\mu = \frac{1}{N} \sum_{\forall x} x = \frac{1}{N} \sum_{\forall x} N_i \mu_i$

and $\mu_i = \frac{1}{N_i} \sum_{x \in \omega_i} x$

$$S_W^{-1} S_B$$

Let's visualize the projection vectors W

```
% lets visualize them ...
% we will plot the scatter plot to better visualize the features
hfig = figure;
axes1 = axes('Parent',hfig,'FontWeight','bold','FontSize',12);
hold('all');

% Create xlabel
xlabel('X_1 - the first feature','FontWeight','bold','FontSize',12,...
      'FontName','Garamond');

% Create ylabel
ylabel('X_2 - the second feature','FontWeight','bold','FontSize',12,...
      'FontName','Garamond');

% the first class
scatter(X1(1,:),X1(2,:), 'r','LineWidth',2,'Parent',axes1);
hold on

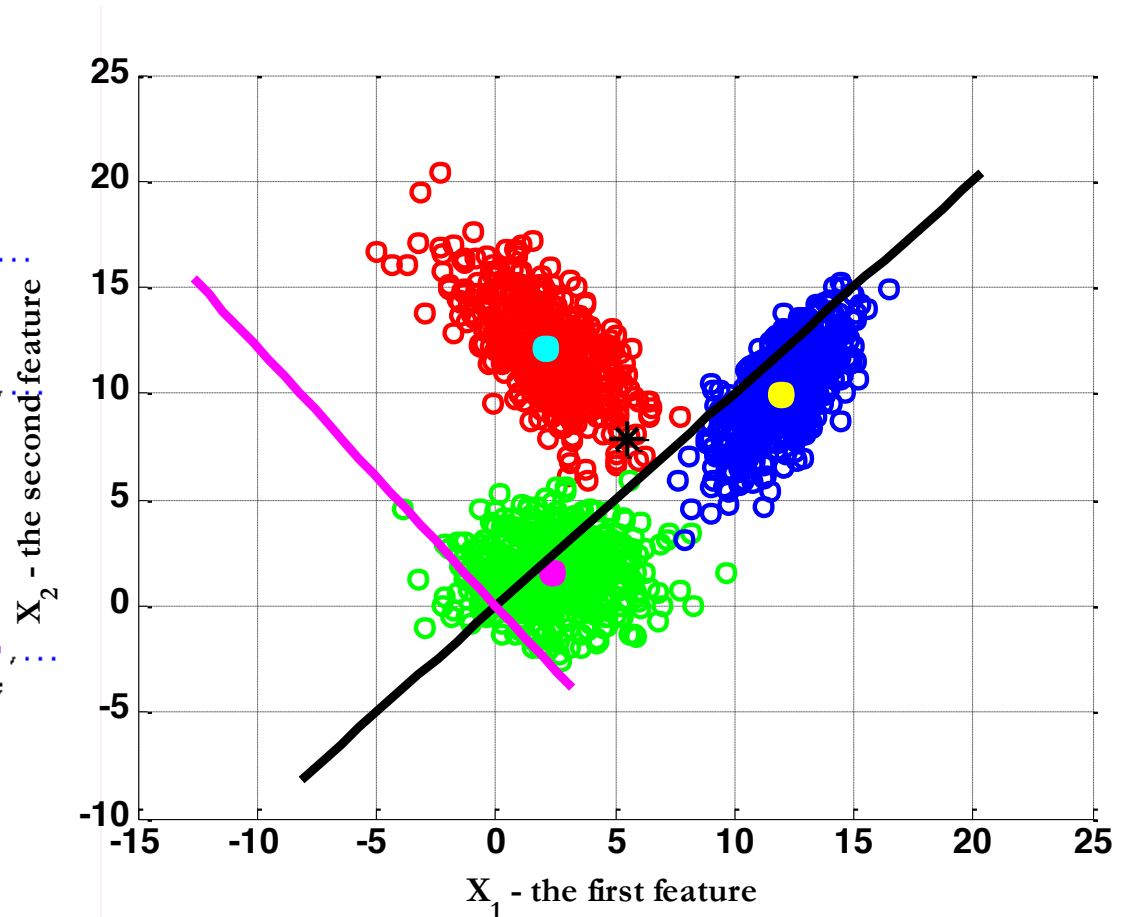
% class's mean
plot(Mu1_est(1),Mu1_est(2),'co','MarkerSize',8,'MarkerEdgeColor','c',...
     'Color','c','LineWidth',2,'MarkerFaceColor','c','Parent',axes1);
hold on

% the second class
scatter(X2(1,:),X2(2,:), 'g','LineWidth',2,'Parent',axes1);
hold on

% class's mean
plot(Mu2_est(1),Mu2_est(2),'mo','MarkerSize',8,'MarkerEdgeColor','m',...
     'Color','m','LineWidth',2,'MarkerFaceColor','m','Parent',axes1);
hold on

% the third class
scatter(X3(1,:),X3(2,:), 'b','LineWidth',2,'Parent',axes1);
hold on

% class's mean
plot(Mu3_est(1),Mu3_est(2),'yo','LineWidth',2,'MarkerSize',8,'MarkerEdgeColor',...
     'y','Color','y','MarkerFaceColor','y','Parent',axes1);
hold on
```



```
% drawing the projection vectors
% the first vector
t = -10:25;
line_x1 = t .* W1(1);
line_y1 = t .* W1(2);

% the second vector
t = -5:20;
line_x2 = t .* W2(1);
line_y2 = t .* W2(2);

plot(line_x1,line_y1,'k-', 'LineWidth',3);
hold on
plot(line_x2,line_y2,'m-', 'LineWidth',3);
grid on
```


Projection ... $y = W^T x$

Along first projection vector

```
% project data samples along the projections axes
% the first projection vector
y1_w1 = W1'*X1;
y2_w1 = W1'*X2;
y3_w1 = W1'*X3;
```

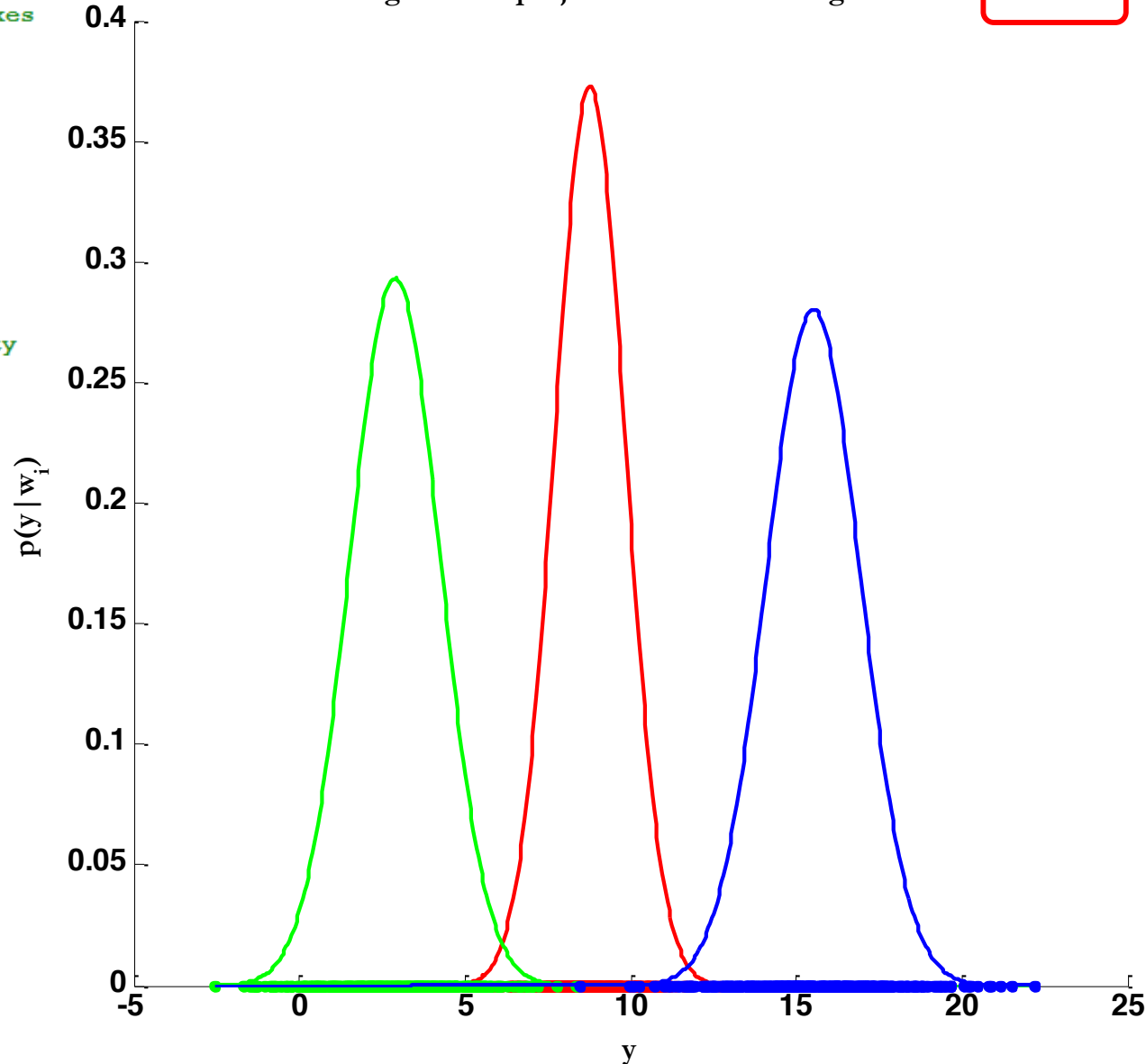
```
% projection limits
minY = min([min(y1_w1),min(y2_w1),min(y3_w1)]);
maxY = max([max(y1_w1),max(y2_w1),max(y3_w1)]);
y_w1 = minY:0.05:maxY;
```

```
% for visualization lets compute the probability
% density function of the
% classes after projection
% the first class
y1_w1_Mu = mean(y1_w1);
y1_w1_sigma = std(y1_w1);
y1_w1_pdf = mvnpdf(y_w1',y1_w1_Mu,y1_w1_sigma);
```

```
% the second class
y2_w1_Mu = mean(y2_w1);
y2_w1_sigma = std(y2_w1);
y2_w1_pdf = mvnpdf(y_w1',y2_w1_Mu,y2_w1_sigma);
```

```
% the third class
y3_w1_Mu = mean(y3_w1);
y3_w1_sigma = std(y3_w1);
y3_w1_pdf = mvnpdf(y_w1',y3_w1_Mu,y3_w1_sigma);
```

Classes PDF : using the first projection vector with eigen value = 4508.2089



Projection ... $y = W^T x$

Along second projection vector

```
% project data samples along the projections axes
% the second projection vector
y1_w2 = W2'*X1;
y2_w2 = W2'*X2;
y3_w2 = W2'*X3;
```

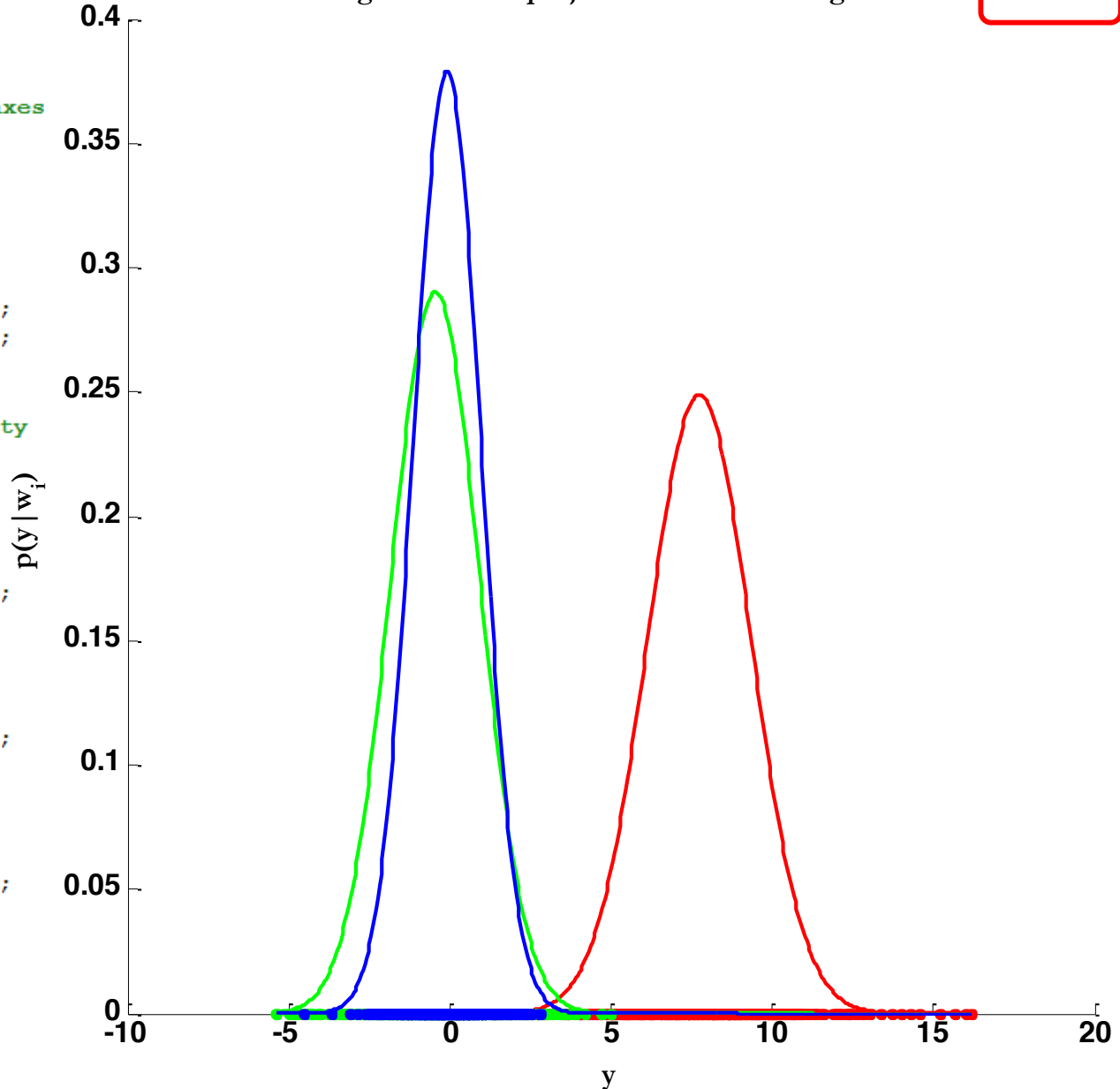
```
% projection limits
minY = min([min(y1_w2),min(y2_w2),min(y3_w2)]);
maxY = max([max(y1_w2),max(y2_w2),max(y3_w2)]);
y_w2 = minY:0.05:maxY;
```

```
% for visualization lets compute the probability
% density function of the
% classes after projection
% the first class
y1_w2_Mu = mean(y1_w2);
y1_w2_sigma = std(y1_w2);
y1_w2_pdf = mvnpdf(y_w2',y1_w2_Mu,y1_w2_sigma);
```

```
% the second class
y2_w2_Mu = mean(y2_w2);
y2_w2_sigma = std(y2_w2);
y2_w2_pdf = mvnpdf(y_w2',y2_w2_Mu,y2_w2_sigma);
```

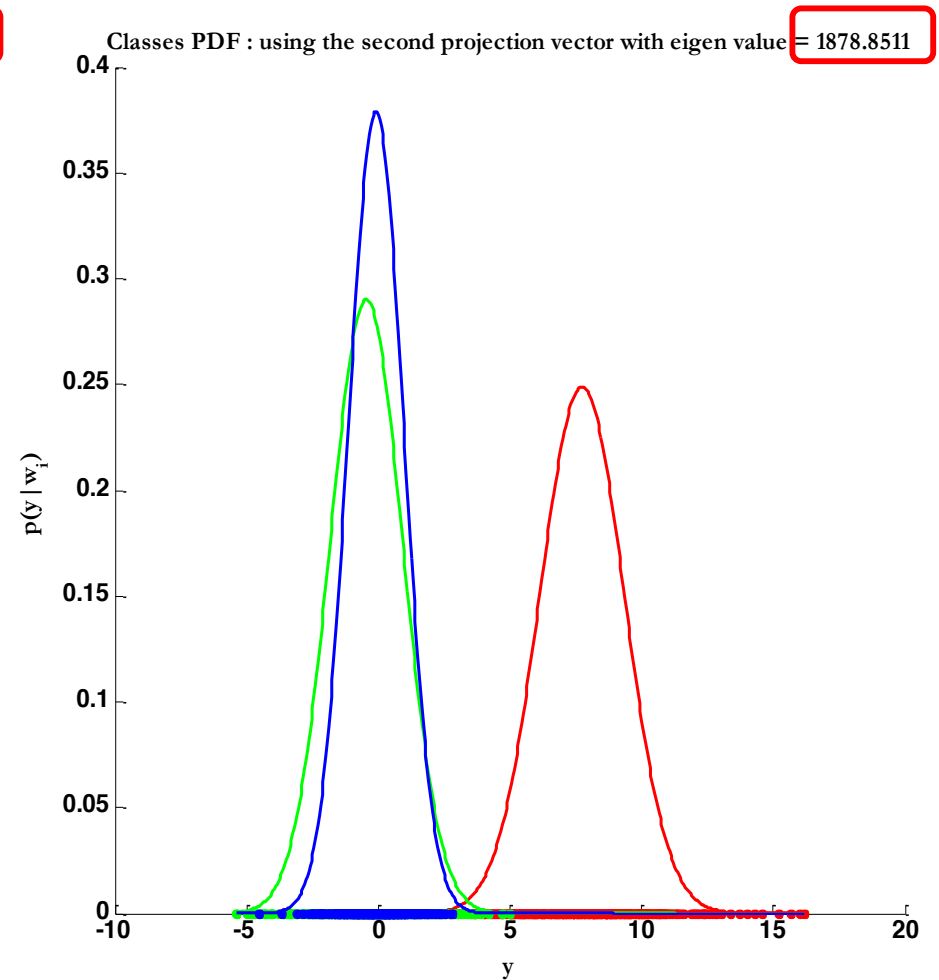
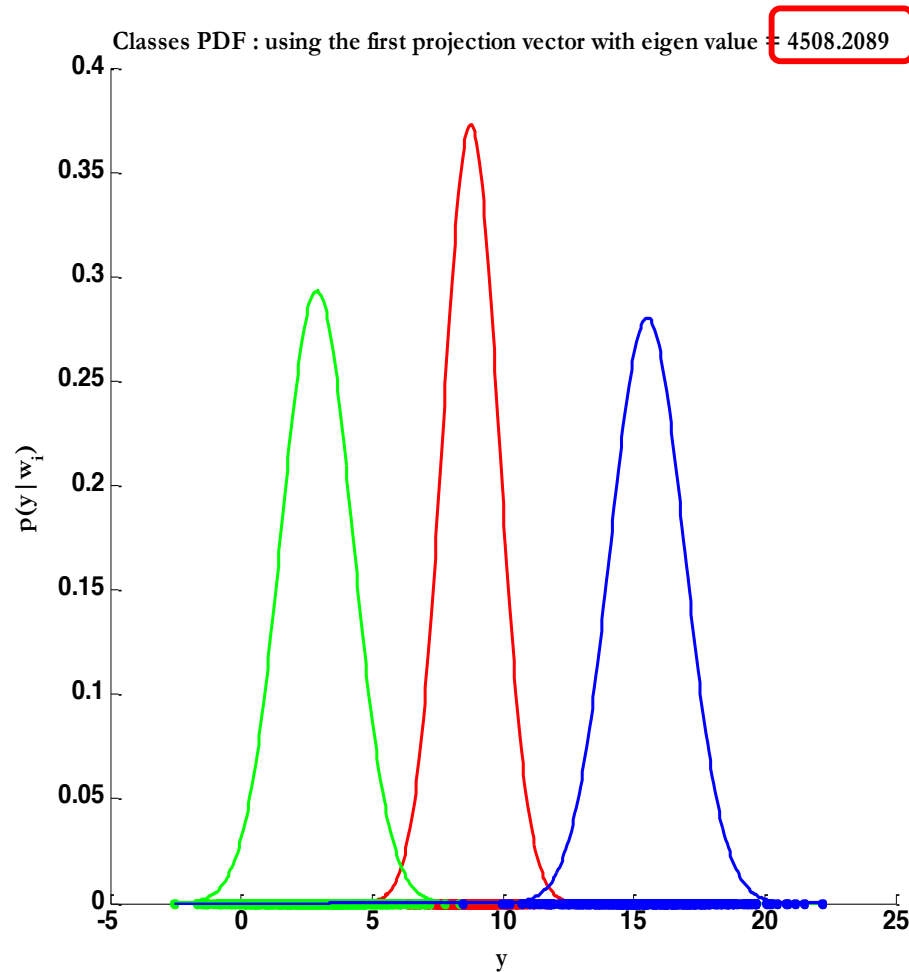
```
% the third class
y3_w2_Mu = mean(y3_w2);
y3_w2_sigma = std(y3_w2);
y3_w2_pdf = mvnpdf(y_w2',y3_w2_Mu,y3_w2_sigma);
```

Classes PDF : using the second projection vector with eigen value = 1878.8511

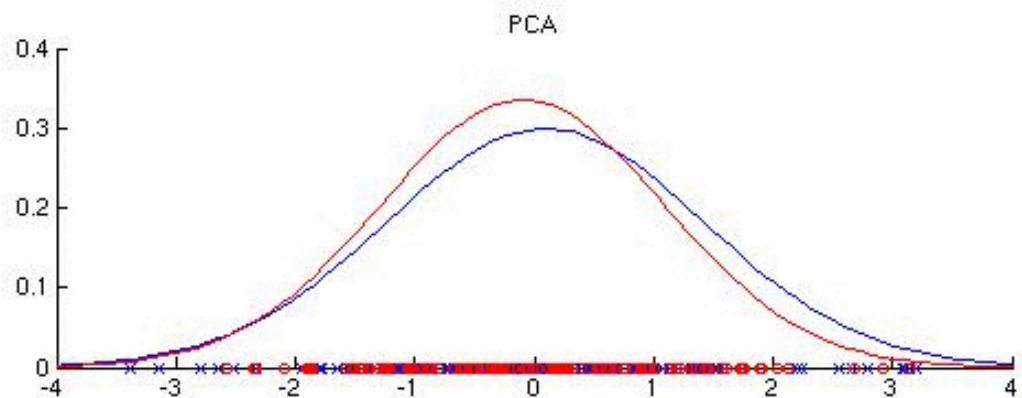
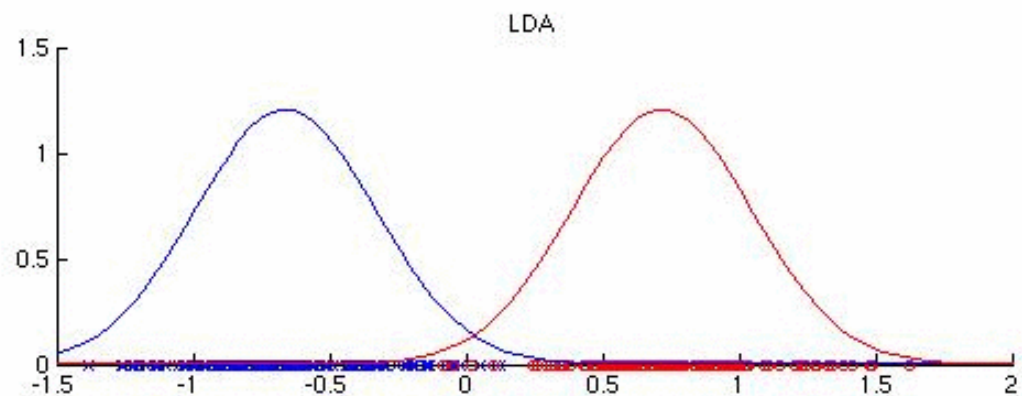
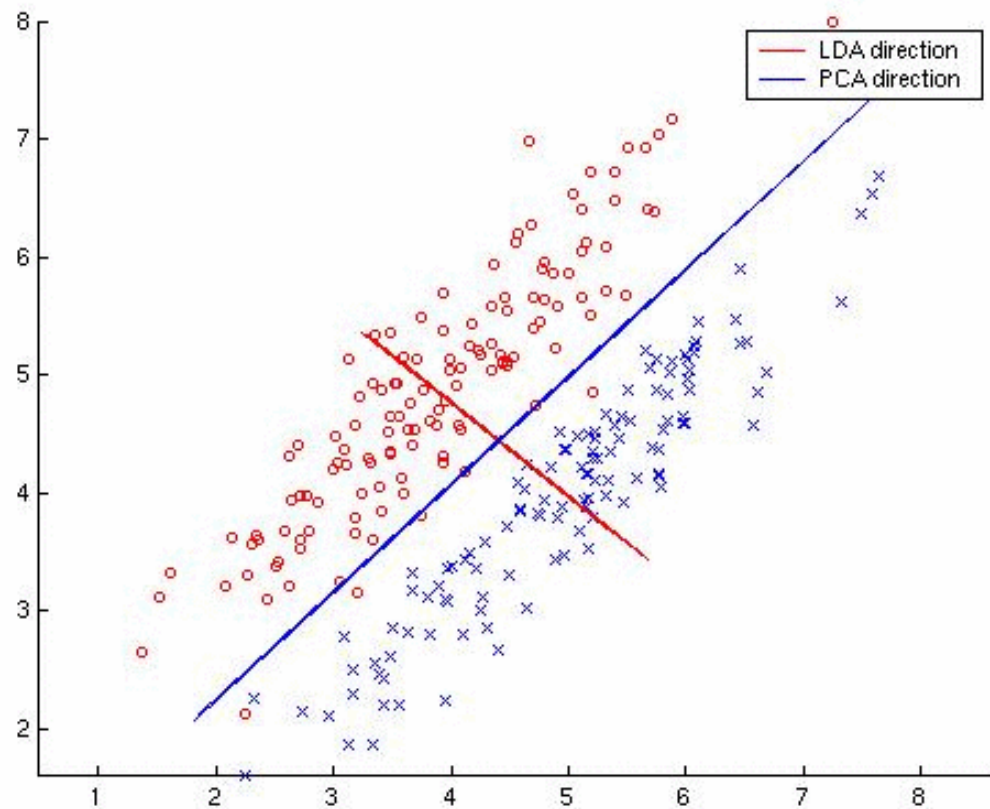


Which is Better?!!!

- Apparently, the projection vector that has the **highest eigen value** provides higher discrimination power between classes

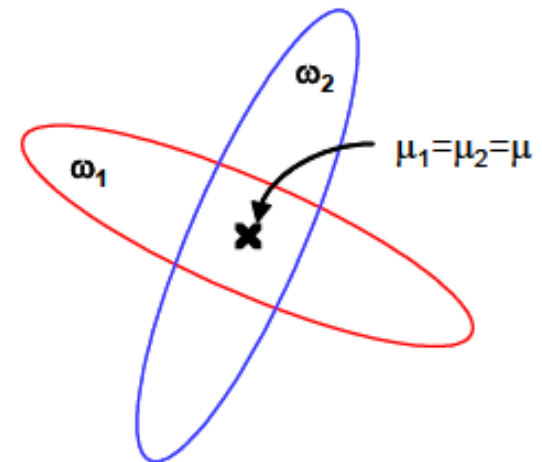
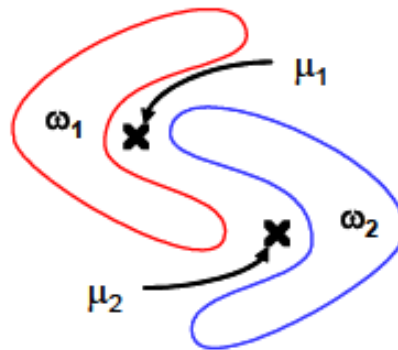
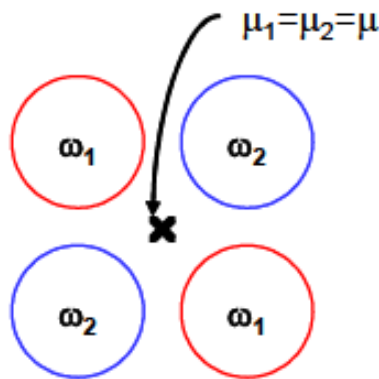


PCA vs LDA



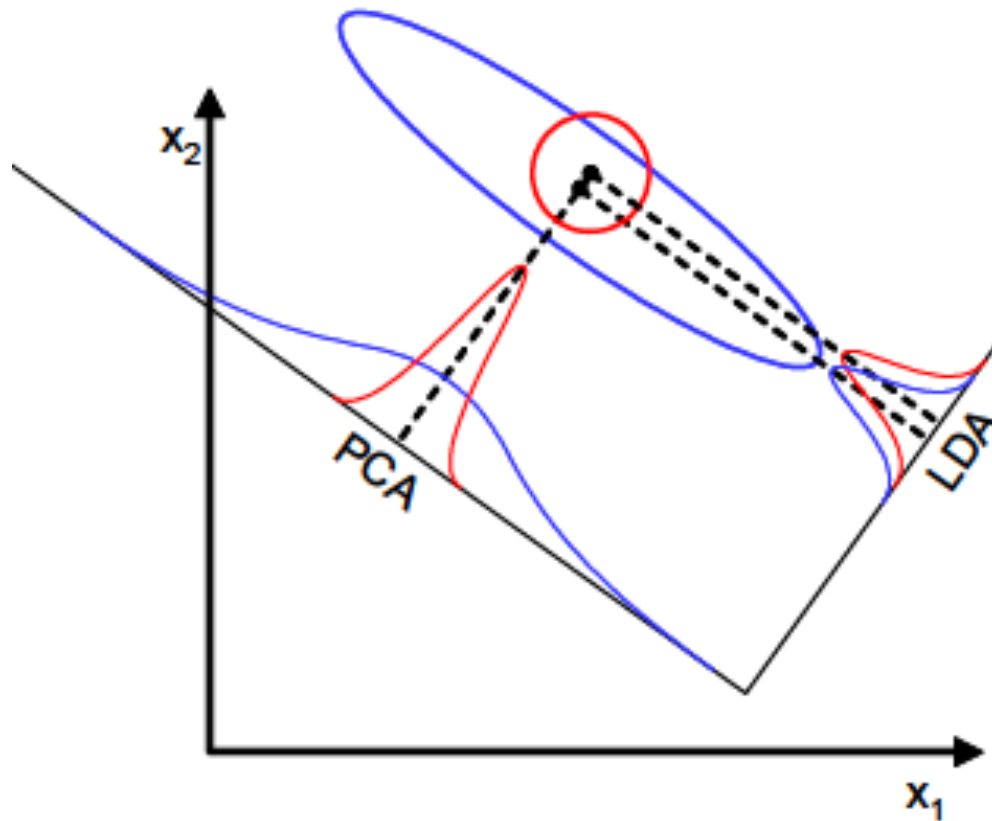
Limitations of LDA ☹️

- **LDA produces at most $C-1$ feature projections**
 - If the classification error estimates establish that more features are needed, some other method must be employed to provide those additional features
- **LDA is a parametric method since it assumes unimodal Gaussian likelihoods**
 - If the distributions are significantly non-Gaussian, the LDA projections will not be able to preserve any complex structure of the data, which may be needed for classification.



Limitations of LDA ☹️

- LDA will fail when the discriminatory information is not in the mean but rather in the variance of the data



Thank You