Research Statement

Take K to be a local field of any characteristic[1], D a division ring that is finite-dimensional over K, and W a finite-dimensional D-module. Varadarajan defined [Var97] a stochastic process generalizing classical Brownian motion with paths in W. This is the starting point for my research. I study these processes, their components, and their symmetries. A general principle motivates the study of Brownian motion in these local field settings: All completions of the rationals ought to be treated on even footing.

Every direct physical measurement results in a rational number. The real numbers are the traditional setting for analysis of these rational measurements. Following the general principle, any question about the reals should also be asked of all the other completions, namely $\mathbb{Q}_p$ for each prime p. Doing so helps determine which properties of our processes are intrinsic, and which are artifacts of the real number setting. Additionally, since p-adic spaces have a natural tree structure, these processes are a specific case of random processes on (infinite) graphs.

For a concrete example, $\mathbb{R}^d$-valued Brownian motion valued can be defined (see, e.g. [SP12; KS98]) through its finite-dimensional distributions, which come from the fundamental solution to the heat equation

$$\partial_t u = \frac{\sigma}{2}\Delta u.$$

It is common (see, for example [Dur19; Law23]) to instead first define $W_t^{(i)}$ to be independent, identically-distributed, one-dimensional Brownian motions, and then define d-dimensional Brownian motion to be the process

$$\vec{W}_t = \left(W_t^{(1)}, \dots, W_t^{(d)}\right).$$

These constructions are not equivalent [RW23] in the local field setting:

Theorem 1 (Rajkumar, Weisbart) *The component processes of $\mathbb{Q}_p^d$-valued Brownian motion are stochastically dependent for all time.*

Despite this, the component processes are still themselves Brownian motions [RW23]:

Theorem 2 (Rajkumar, Weisbart) *The component processes of $\mathbb{Q}_p^d$-valued Brownian motion are identically-distributed $\mathbb{Q}_p$-valued Brownian motions.*

The component dependence strongly influences the first exit times from balls in $\mathbb{Q}_p^d$ ([RW23], Theorem 4.2). Brownian motion having iid component processes should therefore be viewed as a theorem due to the process being $\mathbb{R}^d$-valued.

We are currently extending our investigation to the setting of finite-dimensional local fields over $\mathbb{Q}_p$. We would like to better understand how Brownian motion captures the properties of the underlying spaces, in the spirit of Varadhan's formula. The symmetries induced by the multiplicative structure are analogous to a choice of complex structure, and appear to nontrivially influence the component processes. We suspect that in this setting the component processes are no longer identically distributed, though still Brownian motions. To this end, we would like to achieve the following goal:

[1] K is a topological field which is isomorphic to either the real numbers, the complex numbers, a finite extension of the p-adic numbers for some prime p, or the field of Laurent polynomials over the field $\mathbb{F}_q$ for some prime power q (see [Mil20], Remark 7.49)

Goal: Determine the relationship between component processes of Brownian motion in Varadarajan's full generality.

This goal is further motivated by a problem that Varadarajan suggested. As he noted, quantum mechanics defined over general abelian groups goes back to Weyl [WW09] and Schwinger [Sch70]. His paper came from studies of quantum systems over spaces analogous to the reals [DVV94; DHV99; BD15], specifically understanding the spectra of and semigroups generated by p-adic operators of the form

$$H = \Delta_b + V.$$

Here, Δ_b is the analogue of the Laplacian used to define the Brownian motion, and the potential V is a multiplication operator. Varadarajan suggests taking V to be multiplication by $1/|x|$, the Coulomb potential, and solving the corresponding Schrödinger equation. This is an analogue of the Coulomb problem, used to model the hydrogen atom [Tha05]. Although Varadarajan promised to consider the problem in a future paper, none was published prior to his passing.

Goal: Understand solutions to the local field Coulomb problem as posed by Varadarajan.

The minimal setting in which to study this problem is a module over a p-adic quaternion algebra. In order to accomplish this goal, we will need to understand the properties of Brownian motion in significantly greater generality. Our current project is a step in this direction, considering certain local fields which are algebras over $\mathbb{Q}_p$. However, these algebras are only two-dimensional, and moreover abelian. We will need to better understand how the more general, non-commutative algebra setting affects the Brownian motion before we can consider the Coulomb problem.

In addition to the component properties of p-adic Brownian motion, we study the scaling limit properties. It is well-known that classical Brownian motion is a scaling limit of simple random walk. However, it was not until [BW19; Wei24] that similar results were shown for p-adic Brownian motion. In [Pie+24] we extended this framework to more general vector spaces over local fields. In the classical setting, we can use quantities such as the mean and the variance to determine classes of random walk that all converge to the same process under scaling limit. In the p-adic setting, such quantities do not exist, and our results do not provide a framework for determining the limiting process for a given sequence of random walks. Instead, we witness p-adic Brownian motion as a scaling limit of a *particular* random walk.

Goal: Determine governing features of random walks that give rise to a p-adic Brownian motion under scaling limits.

Here we benefit from our ongoing project to understand the influence of multiplicative structure on Brownian motion, since we have multiple distinct Brownian motions living on the same underlying space. Our framework provides examples of random walks that converge under scaling to different Brownian motions on the underlying $\mathbb{Q}_p$-vector spaces, depending on the multiplicative structure. We can compare these to determined which features are shared and which differ, allowing us to constrain which aspects can influence the limiting process.

Another project that I am currently working on relates to determining the Onsager-Machlup functional for p-adic Brownian motion.

Goal: Determine the Onsager-Machlup functional for p-adic Brownian motion.

The Onsager-Machlup functional of a stochastic process is a probabilistic analogue of the Lagrangian of a dynamical system [CW23], introduced in [MO53; OM53]. The Onsager-Machlup functional greatly strengthens the content of Levy's Forgery Theorem (see e.g. [Wen18] for a discussion by that name) by providing a quantitative determination of how well Brownian motion approximates a given curve. Determining the Onsager-Machlup function for various stochastic processes is a well-known and widely studied problem in probability theory (see e.g. the discussion in [CG23]).

Our scaling limit framework should allow us to prove an analogue of Levy's Forgery Theorem for p-adic Brownian motion. The basic framework of stochastic analysis in $\mathbb{Q}_p$ was defined in [Koc97], but to identify the Onsager-Machlup functional will likely require further utilizing and adapting tools of classical stochastic analysis to the p-adic setting. We believe that the choice of multiplicative structure imposed on the underlying space will also be evident in the resulting Onsager-Machlup functional.

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