Global Well-Posedness and Regularity Issues Associated with Singular Hyperbolic Cauchy Problems

THESIS

submitted in partial fulfillment for the award of the degree of

Doctor of Philosophy

in

Mathematics

Rahul Raju Pattar

(Regd. No. 17010102)

Department of Mathematics and Computer Science

Sri Sathya Sai Institute of Higher Learning

(Deemed to be University)

Prasanthi Nilayam - 515134

Andhra Pradesh, India

February - 2022

Global Well-Posedness and Regularity Issues Associated with Singular Hyperbolic Cauchy Problems

THESIS

submitted in partial fulfillment for the award of the degree of

Doctor of Philosophy

in

Mathematics

Rahul Raju Pattar

(Regd. No. 17010102)

Doctoral Research Supervisor:

Dr. N Uday Kiran

Associate Professor

Department of Mathematics and Computer Science

Sri Sathya Sai Institute of Higher Learning

(Deemed to be University)

Prasanthi Nilayam - 515134

Andhra Pradesh, India

February - 2022

Dedicated to my divine masters
Bhagawan Sri Sathya Sai Baba and
Bhagawan Sri Shirdi Sai Baba

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

SRI SATHYA SAI INSTITUTE OF HIGHER LEARNING

(Deemed to be University)

Prasanthi Nilayam Campus, Prasanthi Nilayam – 515134, Andhra Pradesh, India.

CERTIFICATE

This is to certify that the thesis entitled **“Global Well-Posedness and Regularity Issues Associated with Singular Hyperbolic Cauchy Problems”** being submitted by **Sri Rahul Raju Pattar** in partial fulfillment of the requirements for award of the degree of **Doctor of Philosophy in Mathematics**, is a *bona fide* record of the research work carried out by him in the Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Prasanthi Nilayam Campus, under my supervision. This work is original and has not been submitted or published in part or full for any other degree or diploma of this or any other University or Institution.

Dr. N Uday Kiran

Place: Prasanthi Nilayam
(Doctoral Research Supervisor)

Date:
Associate Professor

DECLARATION

I hereby declare that the research work embodied in this thesis entitled "**Global Well-Posedness and Regularity Issues Associated with Singular Hyperbolic Cauchy Problems**" has been carried out by me in the Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Prasanthi Nilayam Campus, under the supervision of **Dr. N Uday Kiran**, Associate Professor, Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Prasanthi Nilayam Campus. This work is original and has not been submitted or published in part or full for any other degree or diploma of this or any other University or Institution.

Rahul Raju Pattar
(Doctoral Research Scholar)
Department of Mathematics and Computer Science
Sri Sathya Sai Institute of Higher Learning
Prasanthi Nilayam Campus
Prasanthi Nilayam, Andhra Pradesh, INDIA

Place: Prasanthi Nilayam
Date:

Abstract

Loss of regularity of solutions to hyperbolic partial differential equations is one of the central research issues in classical as well as modern analysis of partial differential equations. Suprisingly, this phenomenon of loss which is well-known in the study of degenerate hyperbolic equations with regular coefficients appears even in the case of strictly hyperbolic equations when the coefficients are irregular.

In this thesis, we study Cauchy problems for singular hyperbolic equations of the form

$$\big(\partial_t^2 + A(t,x,D_x)\partial_t + B(t,x,D_x)\big)u(t,x) = f(t,x), \quad (t,x) \in (0,T] \times \mathbb{R}^n,\ T < \infty,$$

where A and B are linear partial differential operators some of whose coefficients or their t-derivatives tend to infinity in some sense as $t \to 0$. In particular, our interest is either blow-up or infinitely many oscillations near $t = 0$ and polynomial growth in x. A study of such problems is motivated by applications in cosmology, transonic gas dynamics and nonlinear Kirchhoff type equations modeling transversal vibrations of elastic string.

We use energy estimates to establish global well-posedness, cone condition and loss of regularity for a class of singular hyperbolic equations with coefficients displaying polynomial growth in x and Cauchy data in an appropriate Sobolev space. In order to study the interplay of the singularity in t and unboundedness in x, we consider a special class of metrics on the phase space. Our methodology relies upon two important techniques: the subdivision of the extended phase space using the Planck function associated to the metric and conjugation of a first order system corresponding to the singular equation. It is well-known in the theory of hyperbolic operators that the irregularity in t needs to be compensated by a higher regularity in the x variable. This "balancing" operation is the key to understand the loss of regularity. The conjugation brings this balance and even encodes the quantity of the loss. In order to overcome the difficulty of tracking a precise loss in our context we introduce a class of parameter dependent pseudodifferential operators of the form $e^{\nu(t)\Theta(x,D_x)}$ for the purpose of conjugation. This operator compensates, microlocally, the loss of regularity of the solutions. The operator $\Theta(x, D_x)$ explains the quantity of the loss by linking it to the metric on the phase space and the singular behavior, while $\nu(t)$ gives a scale for the loss. We call the conjugating operator as *loss operator*. The operators with loss of regularity are transformed to "good" operators by conjugation. This helps us to derive "good" a priori estimates for solutions in the Sobolev space associated with the loss operator. We establish that the metric governing the conjugated operator is conformally equivalent to the initial metric where the conformal factor is given by the symbol of the operator $\Theta(x, D_x)$.

Depending on the order of loss operator we report that the solution experiences zero, arbitrarily small, finite or infinite loss of regularity in relation to the initial datum. Incidentally, our analysis completely settles the well-posedness issue for the oscillatory behavior. Further, we derive anisotropic cone conditions for the singular hyperbolic Cauchy problems. The L^1 integrability of the singularity guarantees that the propagation speed is finite. We report that even the weight function governing the growth of coefficients in x variable influences the geometry of the slope of the cone.

Keywords: Singular hyperbolic Cauchy problems, Blow-up, Oscillations, Global well-posedness, Loss of regularity, Metric on the phase space, Pseudodifferential operators, Energy estimate
MSC 2010: 35L81, 35L15, 35B65, 35B30, 35S05

List of Publications Arising from the Thesis

Journal Publications

1. Pattar, R. R. and Uday Kiran, N. (2021), **Global well-posedness of a class of strictly hyperbolic Cauchy problems with coefficients non-absolutely continuous in time**. *Bulletin des Sciences Mathématiques*, 171, 103037. https://doi.org/10.1016/j.bulsci.2021.103037

2. Pattar, R. R. and Uday Kiran, N. (2021), **Strictly hyperbolic Cauchy problems on $\mathbb{R}^n$ with unbounded and singular coefficients**. *Annali dell Universitá di Ferrara*. https://doi.org/10.1007/s11565-021-00378-2.

3. Pattar, R. R. and Uday Kiran, N. (2022), **Energy Estimates and global well-posedness for a broad class of strictly hyperbolic Cauchy problems with coefficients singular in time**. *Journal of Pseudo-Differential Operators and Applications*, **13**, 9. https://doi.org/10.1007/s11868-021-00439-2.

4. Pattar, R. R. and Uday Kiran, N. (2022), **Global well-posedness of a class of singular hyperbolic Cauchy problems**, *Monatshefte für Mathematik*. https://doi.org/10.1007/s00605-022-01723-z.

Preprints

1. Pattar, R. R. and Uday Kiran, N. (2021), **Strictly hyperbolic equations with coefficients sublogarithmic in time**, arXiv:2111.11701.

List of Conference and Seminar Talks

1. 31^{st} Aug. - 4^{th} Sept. 2020. Delivered a talk on **Global Well-posedness of a Class of Strictly Hyperbolic Cauchy Problems with Coefficients Non-Absolutely Continuous in Time** in International Conference on Generalized Functions. Organized by: International Association for Generalized Functions and Department of Mathematics: Analysis, Logic and Discrete Mathematics at Ghent University, Belgium. (Held via Zoom)

2. 16^{th} and 23^{rd} of Aug. and 6^{th} of Sept. 2021. Delivered a series of 3 talks on **Loss Operator Associated to a Class of Strictly Hyperbolic Cauchy Problems** in Inverse Problems Learning Seminar, Tata Institute of Fundamental Research (TIFR), Centre for Applicable Mathematics. (Held via Zoom)

Acknowledgements

I would like to whole heartedly thank the following wonderful people whose love and support has made the last five years of my life beautiful and reliving.

First and foremost is the revered founder chancellor of our University, Bhagawan Sri Sathya Sai Baba, for providing me the secular and the spiritual education, and instilling in me the Self-confidence.

My PhD supervisor Dr. N Uday Kiran for introducing me to one of the beautiful areas of mathematics- microlocal analysis of hyperbolic operators. He has been an ever-green fount of enthusiasm and new ideas. I am grateful to sir for not only inspiring me mathematically but also instilling in me much needed values for life. He has been my mentor and also a wonderful friend.

Vice Chancellors of our university in the last five years- Prof. K B R Verma and Prof. Carani Sanjeevi for their constant encouragement. I am grateful to Registrar, Prof. B Sai Giridhar and Controller of Examinations, Sri Sanjay Sahni. I thank the Directors of the campus in last five years- Sri Radhakrishn Nair, Prof. Siva Sankar Sai and Dr. Raghavendra Raju for their support.

University Grants Commission, government of India for funding my research and providing me both the junior and senior research fellowships.

Heads of Department of Mathematics and Computer Science (DMACS)- both Dr. Rita Gupta and Dr. Pallav Kumar Baruah for all the support and encouragement. I thank all the teachers of DMACS family for their love and well wishes. I thank DMACS system administrators- Sri C Uday Kiran and Sri Varun V for being resourceful.

Special thanks to the Librarian, Dr. M Pandu Ranga Swamy, and all the library staff. I thank the entire university fraternity for all the support.

Sri Sathya Senior Boys Hostel staff for making my stay comfortable. The hostel has been my second home for the last five years.

All the editors and anonymous referees of the various journals where I have communicated the results of this thesis for their valuable feedback.

My friends- Ram Murthy, Srinivas, Chandrahaas, Hirak, Ananth, Sai Raam, Bhanu and Poddar for filling my PhD time with all the fun and joy.

I thank Kiranmayi akka and K Bala aunty for their motherly affection and their food delicacies which have made my PhD time reliving.

My parents for their unconditional love and constant prayers. I thank my brother for his love and support.

Most importantly, my spiritual master and my very own- my beloved Swami for making my life beautiful by filling it with His unconditional love and answering every prayer during the challenging times. Thank you and love you, Swami!

Contents

Notations

x	Space variable in $\mathbb{R}^n$
t	Time variable in $[0,T], T>0$
$\langle x\rangle$	$(1+\lvert x\rvert^2)^{\frac{1}{2}},\ x\in\mathbb{R}^n$
$\langle \xi\rangle_k$	$(k^2+\lvert\xi\rvert^2)^{\frac{1}{2}},\ \xi\in\mathbb{R}^n$
$đ\xi$	$\dfrac{1}{(2\pi)^n}d\xi$
D_x	$-i\nabla_x$
$\mathbb{N}_0$	$\mathbb{N}\cup\{0\}$
$C([0,T])$	Set of continuous functions on $[0,T]$
$C^k([0,T]), k\geq 1$	Set of k times continuously differentiable functions on $[0,T]$
$C^\infty(\mathbb{R}^n)$	Set of smooth functions on $\mathbb{R}^n$
$B^\infty(\mathbb{R}^n)$	Set of smooth functions on $\mathbb{R}^n$ that are bounded together with all their derivatives
$G^\sigma(\mathbb{R}^n)$	Set of Gevrey functions of index $\sigma\geq 1$ on $\mathbb{R}^n$
$H^s(\mathbb{R}^n), s\in\mathbb{R}$	The Sobolev Space $W^{s,2}(\mathbb{R}^n)$
$\mathcal{S}(\mathbb{R}^n)$	Space of Schwartz class test functions on $\mathbb{R}^n$
$\mathcal{S}'(\mathbb{R}^n)$	Space of tempered distributions on $\mathbb{R}^n$
$\mathcal{S}^\mu_\nu(\mathbb{R}^n)$	Gelfand-Shilov space of indices $\mu,\nu>0$ on $\mathbb{R}^n$

List of Figures

List of Tables

Chapter 1

Introduction

> Analysis takes back with one hand what it gives with the other. I recoil with fear and loathing from that deplorable evil: continuous functions with no derivatives.
>
> — Hermite to Stieltjes, 1893

Let us consider a second order partial differential operator of the form

$$P(t, x, D_t, D_x) = D_t^2 + \sum_{\substack{j+|\alpha|\le 2 \\ j<2}} a_{j,\alpha}(t,x) D_x^\alpha D_t^j, \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where $D_t = -i\partial_t$, $D_x = -i\nabla_x$, $\alpha \in \mathbb{N}_0^n$ is a multi-index and the variables t and x represent time and space, respectively. The operator P is said to be *hyperbolic in the direction* t if the roots $\tau_j(t,x,\xi), j = 1,2,$ of the characteristic equation

$$\tau^2 + \sum_{\substack{j+|\alpha|=2 \\ j<2}} a_{j,\alpha}(t,x)\xi^\alpha \tau^j = 0, \quad (t,x,\xi) \in [0,T] \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n,$$

are real. If the real roots τ_j are also distinct for $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, then we say that the operator P is *strictly hyperbolic in the direction* t. Further, P is *weakly hyperbolic* if it is hyperbolic but not strictly hyperbolic.

Given a Cauchy problem for the operator P,

$$\left.\begin{aligned} &Pu(t,x) = f(t,x) \\ &u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{aligned}\right\} \tag{1.0.1}$$

it is natural to ask the question of well-posedness in the sense of Hadamard.

Definition 1.0.1. *(Well-Posedness [38, 29]) Wellposedness of a Cauchy problem with respect to chosen topological spaces for - the initial data, the non-homogeneous term and the solution, means the existence, uniqueness and continuous dependence of the solution in the topologies of the given spaces.*

In the case of smooth coefficients, way back in 1932 Hadamard [38] (see also [52, 57]) proved that the hyperbolicity is a necessary condition for C^∞ well-posedness of the Cauchy problem. But hyperbolicity alone does not guarantee well-posedness in C^∞. A sufficient condition for well-posedness in C^∞ is strict hyperbolicity, see [67, 53, 36]. In this thesis, we are interested in strictly hyperbolic Cauchy problems with non-smooth coefficients.

Second order hyperbolic equations with Lipschitz coefficients that depend only on time were the subject of research starting in the 1970s and 1980s. When the coefficients of the equation are in $L^\infty([0,T])$ and Lipschitz continuous then the Cauchy problem is C^∞ well-posed. More precisely, the Cauchy problem is well-posed in Sobolev spaces and one can prove that for all $(u_0,u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, $s \in \mathbb{R}$, there is a unique solution in $C\big([0,T];H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T];H^{s-1}(\mathbb{R}^n)\big)$, see [40, Chapter 9].

Non-Lipschitz coefficients arise naturally in wave propagation in non-smooth media and are of practical importance, for example in seismology [26, 45]. This warrants the study of strictly hyperbolic Cauchy problems with non-Lipschitz coefficients. This study presents an interesting phenomenon called *loss of regularity index in Sobolev spaces*, that is, the Sobolev regularity index of the solution is lesser than that of the Cauchy data.

Broadly speaking, in the literature, the non-Lipschitzness is characterized by

(T1) Singular behavior: blow-up rate and oscillations characterized by the coefficients and their derivatives

(T2) Modulus of continuity based irregularity: ranging till Hölder continuity

(T3) Second variation based irregularity: Zygmund and log-Zygmund continuity

Works on (**T1**) - (**T3**) irregularities have a long standing history in the study of strictly hyperbolic equations (a brief literature survey on well-posedness results related to these irregularities is presented in the following sections). In this work, we primarily focus on the irregularity of type (**T1**) which has found fruitful applications in cosmology, transonic gas dynamics and the study of nonlinear Kirchhoff type equations (see Appendix D for the details).

1.1 Overview

We study the Cauchy problem (1.0.1) when the operator coefficients or their t-derivatives tend to infinity in some sense as $t \to 0$. Such problems are called singular hyperbolic Cauchy problems, see [7]. Our main interest is the optimality of loss when the coefficients are singular in time and unbounded in space. In particular, our interest is either blow-up or infinitely many oscillations near $t=0$ and polynomial growth in x. The polynomial growth in x is characterized by generic weights $\omega(x)$ and $\Phi(x)$. We consider the coefficients $a_{j,\alpha}(t,x)$ such that $a_{j,\alpha}(\cdot,x) \in C^\infty(\mathbb{R}^n)$ satisfies the estimate

$$|\partial_x^\beta a_{j,\alpha}(\cdot,x)| \leq C_\beta\, \omega(x)^{2-j}\Phi(x)^{-|\beta|}, \tag{1.1.1}$$

for some positive constant C_β and multi-index $\beta \in \mathbb{N}_0^n$. The functions $\omega(x)$ and $\Phi(x)$ are positive monotone increasing in $|x|$ such that $1 \leq \omega(x) \lesssim \Phi(x) \lesssim \langle x \rangle = (1+|x|^2)^{1/2}$.

These functions specify the structure of the differential equation in x variable. We discuss the properties of these functions in Section 2.1. In order to study the interplay of the singularity in t and unboundedness in x, we consider a class of metrics on the phase space of the form

$$g_{\Phi,k} = \frac{|dx|^2}{\Phi(x)^2} + \frac{|d\xi|^2}{\langle\xi\rangle_k^2},$$

where $\Phi(x)$ is as in (1.1.1) and $\langle\xi\rangle_k = (k^2 + |\xi|^2)^{1/2}$ for an appropriately chosen $k \geq 1$. Note that the functions $\omega(x)$ and $\Phi(x)$ are associated with the weight and the metric respectively. We discuss the properties of these metrics in Chapter 2.

Example 1.1.1. *The function* $f(x) = \langle x\rangle^{\frac{2}{3}}\left(2 + \sin\left(\langle x\rangle^{\frac{1}{3}}\right)\right)$ *satisfies the estimate*

$$|\partial_x^\beta f(x)| \lesssim \omega(x)\Phi(x)^{-|\beta|},$$

for $\omega(x) = \langle x\rangle^{\frac{2}{3}}$ *and* $\Phi(x) = \langle x\rangle^{\frac{1}{3}}$.

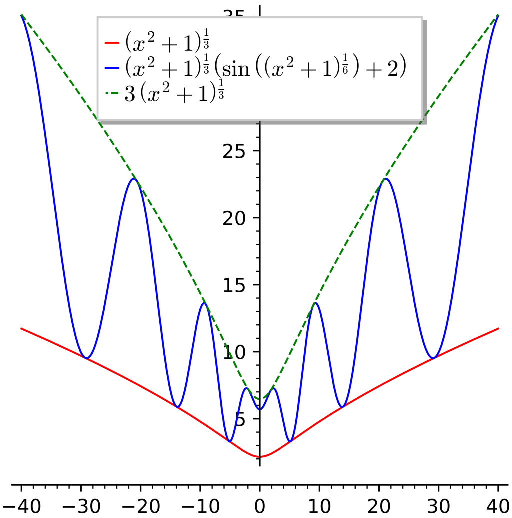

Figure 1.1: Plot of Example 1.1.1

We use energy estimates to establish global well-posedness, cone condition and loss of regularity for a class of singular hyperbolic equations with coefficients displaying polynomial growth in x and Cauchy data in an appropriate Sobolev space tailored to a metric on the phase space and singular behavior. Our methodology relies upon two important techniques: the subdivision of the extended phase space using the Planck function associated to the metric and conjugation of a first order system corresponding to the singular equation. The irregularity in t needs to be compensated by a higher regularity in the x variable. This balancing operation is the key to understand the loss of regularity. The conjugation operation brings this balance and even encodes the quantity of the loss. In order to overcome the difficulty of tracking a precise loss in our context, we introduce a class of parameter dependent pseudodifferential operators of the form

$$e^{\nu(t)\Theta(x,D_x)},$$

for the purpose of conjugation. This operator compensates, microlocally, the loss of regularity of the solutions. The operator $\Theta(x, D_x)$ explains the quantity of the loss by linking it to the metric on the phase space and the singular behavior while $\nu(t)$ gives a scale for the loss. We call the conjugating operator as *loss operator*. The operators with loss of regularity are transformed to good operators by the conjugation. This helps us to derive good a priori estimates of solutions in the Sobolev space associated with the loss operator. We establish that the metric governing the conjugated operator is conformally equivalent to the metric $g_{\Phi,k}$ where the conformal factor is given by the symbol of the operator $\Theta(x, D_x)$.

Depending on the order of the loss operator, we report that the solution experiences zero, arbitrarily small, finite or infinite loss of regularity in relation to the initial datum. Incidentally, our analysis completely settles the well-posedness issue for the oscillatory behavior.

Further, we derive optimal cone conditions for the solutions of the singular hyperbolic Cauchy problems. The L^1 integrability of the singularity guarantees that the propagation speed is finite. We report that even the weight function governing the growth of coefficients in x variable influences the geometry of the slope of the cone in such a manner that the slope grows as $|x|$ grows.

1.2 Literature Survey

In this section, we present various well-posedness and regularity results from the literature when the coefficients are non-Lipschitz in time and smooth in space, and outline the results from this thesis in the context of singular behavior.

1.2.1 Singular Behavior

The main interest of this thesis is the singular behavior characterized by either blow-up or infinitely many oscillations near the hyperplane $t = 0$. Let us look at these two cases in detail.

Oscillatory Behavior

Following is a classification of oscillations given by Reissig [69, 50].

Definition 1.2.1 (Oscillatory Behavior [69, 50]). *Let $c = c(t) \in L^\infty([0,T]) \cap C^2((0,T])$ satisfy the estimate*

$$\left| \frac{d^j}{dt^j} c(t) \right| \lesssim \left(\frac{|\ln t|^{\tilde{\gamma}}}{t} \right)^j, \quad j = 1, 2. \tag{1.2.1}$$

We say that the oscillating behavior of the function $c(t)$ is

- *very slow if $\tilde{\gamma} = 0$*
- *slow if $\tilde{\gamma} \in (0, 1)$*
- *fast if $\tilde{\gamma} = 1$*

- *very fast if (1.2.1) is not satisfied for $\tilde{\gamma} = 1$.*

We modify the above classification in order to refine the definition for very fast oscillation.

Definition 1.2.2 (Oscillatory Behavior)**.** *Let $c = c(t) \in L^\infty([0,T]) \cap C^2((0,T])$ satisfy the estimate*

$$\left| \frac{d^j}{dt^j} c(t) \right| \lesssim \left(\frac{|\ln t|^{\tilde{\gamma}\mathbf{I}_q}}{t^q} \right)^j, \tag{1.2.2}$$

for $j = 1, 2$ and $q \geq 1$. The function $\mathbf{I}_q$ is such that $\mathbf{I}_q \equiv 1$ if $q = 1$ else $\mathbf{I}_q \equiv 0$. We say that the oscillating behavior of the function $c(t)$ is

- *very slow if $q = 1$ and $\tilde{\gamma} = 0$*
- *slow if $q = 1$ and $\tilde{\gamma} \in (0, 1)$*
- *fast if $q = \tilde{\gamma} = 1$*
- *very fast if $q > 1$ or else $\tilde{\gamma} > 1$ when $q = 1$.*

When $q = 1$, note that both the above definitions match. Based on our work, we have redefined very fast oscillation by introducing the case of $q > 1$. Observe that when $q > 1$ the contribution from the logarithmic factor is negligible as $|\ln|^{\tilde{\gamma}} \lesssim t^{-\varepsilon}$ for any $\varepsilon > 0$ and one can replace $\frac{|\ln t|^{\tilde{\gamma}}}{t^q}$ by $\frac{1}{t^{\tilde{q}}}$ where $\tilde{q} = q + \varepsilon$.

The borderline case $q = 1$ is more challenging. A pioneering work in this direction was done by Yamazaki [77] in 1990 who considered very slowly oscillating ($\tilde{\gamma} = 0, q = 1$) coefficients that depend only on time. The author reports Sobolev well-posedness without any loss in regularity index.

Order of Oscillations q	Order of Oscillations $\tilde{\gamma}$	Regularity in t of coefficients	Growth in x of coefficients ω	Growth in x of coefficients Φ	Loss of regularity index for solution	Ref.
1	$(0,1)$	$L^\infty([0,T]) \cap C^2((0,T])$	1	1	Zero to arbitrarily small	[69]
1	$\boldsymbol{(0,1)}$	$\boldsymbol{C^2((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Zero to arbitrarily small**	**[65]**
1	1	$L^\infty([0,T]) \cap C^\infty((0,T])$	1	1	Finite	[50]
1	1	$L^\infty([0,T]) \cap C^\infty((0,T])$	$\langle x \rangle$	$\langle x \rangle$	Finite	[75]
1	$\boldsymbol{[1,\infty)}$	$\boldsymbol{C^2((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Finite to Infinite**	**[65]**
$(1,\infty)$	-	$C^1((0,T])$	1	1	Infinite	[9]
$\boldsymbol{(1,\frac{3}{2})}$	-	$\boldsymbol{C^1((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Infinite**	**[62]**

Table 1.1: Loss of regularity in case of oscillatory coefficients. Rows in **bold** correspond to the results of this thesis.

Reissig [69, Theorem 8] considered the coefficients independent of x and oscillating in t. The author reports no loss, arbitrary small loss, finite loss and infinite loss of derivatives for the cases very slow, slow, fast and very fast oscillations, respectively. These results were partially extended to the case of coefficients depending on both t and x in [69, Theorem 13] and [50, Theorem 1.2] where C^∞ well-posedness is established through the construction of a parametrix. But this extension requires the condition

$$|\partial_t^j \partial_x^\beta a_{j,\alpha}(t,x)| \leq C_{j,\beta} \left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^{\tilde{\gamma}}\right)^j, \qquad \text{for all } j \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n, \tag{1.2.3}$$

where $\tilde{\gamma} \in [0,1]$. Uday Kiran et al. [75] extended these results to the SG setting, i.e., $\Phi(x) = \omega(x) = \langle x \rangle$ in (1.1.1) and an additional factor $\langle x \rangle^{2-j}$ multiplied to the right hand side of the inequality in (1.2.3). The authors report well-posedness in $\mathcal{S}'(\mathbb{R}^n)$ with finite loss in regularity index. It should be noted that these extensions partially settle the well-posedness issue for oscillatory behavior case as they require the coefficients to possess all the t-derivatives in $(0,T]$.

In this thesis we consider the Cauchy problem (1.0.1) with coefficients displaying varied rates of oscillations in t (specified by just the first and second t-derivatives of the coefficients) and a polynomial growth in x governed by generic weights $\Phi(x)$ and $\omega(x)$. In this thesis we completely settle the well-posedness issue for the oscillatory behavior. In Chapters 4 and 5, we show that the solution experiences zero, arbitrarily small, finite and infinite loss in the regularity index in relation to the Cauchy data defined in a Sobolev space tailored to the metric and the order of oscillations for the cases very slow, slow, fast and very fast oscillations, respectively. Table 1.1 summarizes the results of this thesis in the context of oscillatory behavior.

Blow-up

Inspired from the classification for oscillations, we have come up with the following scale for the blow-up rate based on the amount of loss in regularity of solution for the associated Cauchy problem.

Definition 1.2.3 (Blow-up Rate). *Let $c = c(t) \in L^1((0,T]) \cap C^1((0,T])$ satisfy the estimates*

$$\left.\begin{aligned} |c(t)| &\lesssim \frac{1}{t^p}|\ln t|^{\tilde{\gamma}\mathbf{I}_q}, \\ |\partial_t c(t)| &\lesssim \frac{1}{t^q}|\ln t|^{(\tilde{\gamma}-1)\mathbf{I}_q}, \end{aligned}\right\} \tag{1.2.4}$$

with $q \in [1,\infty), p \in [0,1), p \leq q-1$ and $\tilde{\gamma} > 0$. The function $\mathbf{I}_q$ is such that $\mathbf{I}_q \equiv 1$ if $q = 1$ else $\mathbf{I}_q \equiv 0$. We say that the blow-up rate of the function $c(t)$ is

- *mild if $q = 1, p = 0, \tilde{\gamma} \in (0,1)$*
- *logarithmic if $q = 1, p = 0, \tilde{\gamma} = 1$*
- *strong if $q = 1, p = 0, \tilde{\gamma} \in (1,\infty)$*
- *very strong if $q > 1, p \in [0,1)$.*

Rate of Blow-up			Regularity in t of coefficients	Growth in x of coefficients		Loss of regularity index for solution	Ref.
p	q	$\tilde{\gamma}$		ω	Φ		
0	**1**	$\boldsymbol{(0,1)}$	$\boldsymbol{C^1((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Arbitrarily small**	**[64]**
0	1	1	$C^1((0,T])$	-	-	Finite	[15]
0	1	1	$C^1((0,T])$	1	1	Finite	[9]
0	**1**	1	$\boldsymbol{C^1((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Finite**	**[63]**
0	**1**	$\boldsymbol{(1,\infty)}$	$\boldsymbol{C^2((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Infinite**	**[65]**
$[0,1)$	$(1,\infty)$	-	$C^1((0,T])$	-	-	Infinite	[15]
$\boldsymbol{(0,\frac{1}{2})}$	$\boldsymbol{(1,\frac{3}{2})}$	-	$\boldsymbol{C^1((0,T])}$	$\boldsymbol{\omega(x)}$	$\boldsymbol{\Phi(x)}$	**Infinite**	**[66]**

Table 1.2: Loss of regularity in case of coefficients blowing-up at $t = 0$. Rows in **bold** correspond to the results of this thesis.

Cicognani [9] studied well-posedness of (1.0.1) for the case $\omega(x) = \Phi(x) = 1$ and atmost logarithmic blow-up in t. The author reports Sobolev well-posedness for the Cauchy problem (1.0.1) with a finite loss of derivatives. Colombini et al. [15] considered the Cauchy problem (1.0.1) with operator coefficients independent of x with very strong blow-up in t. They report well-posedness in Gevrey space $G^\sigma, 1 \leq \sigma < \frac{q-p}{q-1}$, with infinite loss of derivatives.

In this thesis we consider the Cauchy problem (1.0.1) with coefficients displaying varied rates of blow-up in t and a polynomial growth in x governed by generic weights $\Phi(x)$ and $\omega(x)$. In Chapters 3, 4 and 6, we show that the solution experiences arbitrarily small, finite and infinite loss in the regularity index in relation to the Cauchy data defined in Sobolev space tailored to the metric and the order of blow-up for the cases mild, logarithmic, strong and very strong blow-up, respectively. Table 1.2 summarizes the results of this thesis in the context of blow-up near $t = 0$.

1.2.2 Other Types of Non-Lipschitz Behavior

In this section, we look at some of well-posedness and regularity results from the literature when the modulus of continuity or second variation is used to describe the regularity of the coefficients with respect to time.

Modulus of Continuity Based Irregularity

Let us first recall what we mean by the term modulus of continuity.

Definition 1.2.4 (Modulus of continuity and μ-continuity)**.** *We call $\mu : [0,1] \to [0,1]$ a modulus of continuity, if μ is continuous, concave and increasing and satisfies $\mu(0) = 0$. A function $f \in C(\mathbb{R}^n)$ is μ continuous if and only if*

$$|f(x) - f(y)| \leq C\mu(|x - y|),$$

for all $x, y \in \mathbb{R}^n$, $|x - y| \leq 1$ and some constant C.

Modulus of continuity	Commonly called
$\mu(s) = s$	Lipschitz-continuity
$\mu(s) = s\left(\ln\left(\frac{1}{s}\right)+1\right)$	Log-Lipschitz-continuity
$\mu(s) = s^{\alpha},\ \alpha \in (0,1)$	Hölder-continuity

Table 1.3: Typical examples of moduli of continuity

Colombini, De Giorgi and Spagnolo pioneered in 1979 the study of loss of regularity of solution to a strictly hyperbolic Cauchy problem with non-Lipschitz coefficients depending only on time. They proved in [13] that log-Lipschitz regularity is the optimal one for Sobolev and hence, C^∞ well-posedness with loss of derivatives. In this case, more precisely, they showed that there exists $\delta > 0$ (depending on the log-Lipschitz norm of the coefficients) such that for all $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, $s \in \mathbb{R}$, there is a unique solution in $C\big([0,T]; H^{s-\delta}(\mathbb{R}^n)\big) \cap C^1\big([0,T]; H^{s-1-\delta}(\mathbb{R}^n)\big)$. Further, if the coefficients are Hölder continuous of exponent $\alpha \in (0,1)$, then (1.0.1) is G^σ well-posed for all $\sigma < \frac{1}{1-\alpha}$.

Colombini and Lerner [17] considered second-order operators with L^∞ coefficients which are log-Lipschitz in both time and space. They proved well-posedness in Sobolev-spaces with finite loss of derivatives and established the log-Lipschitz regularity as the natural threshold beyond which no Sobolev well-posedness could be expected. Considering operators whose coefficients are Hölder continuous in time and Gevrey in the spatial variables, Nishitani [60] and Jannelli [46] were able to extend the results of [13]. Cicognani [8] extended the results of [17, 60, 46] to equations of order m and considered coefficients that are log-Lipschitz and Hölder continuous in time and $B^\infty(\mathbb{R}^n)$ regular in space.

For second order equations, Cicognani and Colombini [11] provided a classification, linking the loss of derivatives to the modulus of continuity of the coefficients with respect to time. In 2017, Cicognani and Lorenz [12] extended these results to m^{th} order equations with coefficients diplaying $B^\infty(\mathbb{R}^n)$ regularity in x and linked the loss of derivatives not only to the modulus of continuity but also to the weight sequence (see Definition 2.2 in [12]) in spacial variable of the coefficients. The results in [12] demonstrate a well-known fact in the context of the study that one has to compensate for the irregularity in time by assuming higher regularity in space.

Ascanelli and Cappiello [3, 2] initiated the study of loss of regularity when the coefficients are polynomially growing in x and log-Lipschitz or Hölder continuous in t. The authors report that the solution experiences finite loss of not only derivatives but also decay in weighted Sobolev spaces for the log-Lipschitz case [3] while for the Hölder continuous case [2] they prove well-posedness in Gelfand-Shilov spaces with infinite loss of regularity.

Second Variation Based Irregularity

Let us now look at some of the well-posedness and regularity results when the irregularity in time variable of the coefficients is dictated by second variation based regularities: Zygmund and log-Zygmund.

Definition 1.2.5. *A function $f \in L^\infty(\mathbb{R}^n)$ is said to be Zygmund continuous if*

$$\sup_{x \in \mathbb{R}^n} |f(x+y) + f(x-y) - 2f(x)| \leq C|y|, \quad |y| < 1,$$

and log-Zygmund if right-hand-side of the above inequality is replaced by $C|y| \ln\left(1 + \left(\frac{1}{|y|}\right)\right)$.

Though we have the embedding Lipschitz $\hookrightarrow$ Zygmund $\hookrightarrow$ log-Lipschitz, Zygmund regularity is a condition on second variation, hence it is not related to the modulus of continuity based irregularity considered by Cicognani and Colombini [11].

Hyperbolic equations with Zygmund coefficients appear in various geophysical applications [45] where wave propagation in a media described by multifractal behavior is studied. As for the well-posedness results, Colombini e.al [14, 19] studied the Cauchy problem (1.0.1) with Zygmund and log-Zygmund type assumptions on time, and proved Sobolev well-posedness without loss and with finite loss of derivatives, respectively. The authors also investigated the case of coefficients log-Zygmund in time and log-Lipschitz in space. They report in [18] that solution experiences time depedent loss of derivatives in the Sobolev spaces $H^s(\mathbb{R}^n), s \in \mathbb{R}$.

Remark 1.2.1. *It should be noted that singular behavior is different from the modulus of continuity based irregularity. It is possible to construct a function $f_1 \in C^1(0,T]$ and log-Lipschitz continuous in $[0,T]$, such that*

$$\limsup_{t \to 0^+} \; t^q \, |f_1'(t)| = +\infty$$

for all $q \geq 1$. Conversely, it is easy to find a function $f_2 \in C([0,T]) \cap C^1(0,T]$ but Hölder-continuous on $[0,T]$ for no $\alpha < 1$, such that

$$\limsup_{t \to 0^+} \; t \, |f_2'(t)| < +\infty.$$

We refer to Appendix C for the details.

1.3 Motivating Examples and Counterexamples

One may ask, in the context of singular hyperbolic Cauchy problems, the following pertinent questions:

1. Does loss of regularity really appear? If so, what is the amount of loss?
2. Can we have problems where uniqueness is compromised?

In this section we provide certain examples answering the above questions. We cover various cases such as finite loss, no loss, nonuniqueness and infinite loss. The case of infinite loss is challenging as it involves construction of a singular coefficient using techniques from spectral theory of pseudodifferential operators on $\mathbb{R}^n$ and Baire categorical arguments.

1.3.1 Finite Loss

The following example shows that one can encounter finite loss when the coefficients of lower order terms are singular. Let $(t,x) \in [0,T] \times \mathbb{R}$.

Example 1.3.1. *(Pattar-Kiran[66])*

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 + \frac{1}{2t} \left(\partial_t - (4m+1)\partial_x \right) \right) u(t,x) = 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = (4m+1)\partial_x u_0(x), \end{cases} \tag{1.3.1}$$

for some $m \in \mathbb{N}_0$. The solution to the above Cauchy problem is given by

$$u(t,x) = \sum_{j=0}^{m} C_j^{(m)} t^j \partial_x^j u_0(x+t), \tag{1.3.2}$$

for $C_j^{(m)}$ of the form

$$C_0 = 1, \quad C_j^{(m)} = \frac{(-2)^j}{j!} \frac{(m)_j}{\left(-\frac{1}{2}\right)_j}, \quad j \geq 1,$$

where $(y)_j, y \in \mathbb{R}$, is the j^{th} falling factorial of y [20, page 6] given by

$$(y)_j = y(y-1)\cdots(y-j+1).$$

Notice that the solution $u(t,x)$ in (1.3.2) is expressed using first m derivatives of the initial datum $u_0(x)$. This leads to a loss in regularity index by m amount. For example, let $m=1$, and

$$u_0(x) = H(x) + \sin(x^2) \in H^{s_1,-1}_{\langle x \rangle,1}, \; s_1 < \frac{1}{2},$$

where $H(x)$ is Heaviside function and $H^{s_1,s_2}_{\Phi,k}, (s_1,s_2) \in \mathbb{R}^2$ is a weighted Sobolev space as in Definition 3.1.2. Then, $u(t,x)$ is given by

$$u(t,x) = H(x+t) + \sin(x+t)^2 + 4t\left(\delta(x+t) + 2(x+t)\cos(x+t^2)\right) \in H^{s_1-1,-2}_{\langle x \rangle,1},$$

where $\delta(x)$ is Dirac delta distribution.

1.3.2 No Loss

It is not always that we have loss of regularity in the case of singular hyperbolic Cauchy problems. Following examples demonstrate this point.

Example 1.3.2. *(Pattar-Kiran[66])*

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 - \frac{2}{t}\partial_x \right) u(t,x) = 0 \\ u(0,x) = 0, \; \partial_t u(0,x) = u_0(x). \end{cases} \tag{1.3.3}$$

The solution to the above Cauchy problem is given by

$$u(t,x) = t u_0(x+t).$$

The following example demonstrates that when the coefficient of the top order term is oscillatory but in $C^1((0,T]) \cap W^{1,1}((0,T])$ and that of the lower order term is in $C^1((0,T]) \cap L^1((0,T])$, one may have no loss.

Example 1.3.3. *(Pattar-Kiran[66])*

$$\begin{cases} \left(\partial_t^2 - (2+\sin\sqrt{t})^2\partial_x^2 - \frac{\cos\sqrt{t}}{2\sqrt{t}}\partial_x\right)u(t,x) = 0 \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = 2\partial_x u_0(x). \end{cases} \tag{1.3.4}$$

The solution to the above Cauchy problem is given by

$$u(t,x) = u_0\left(x + \int_0^t (2+\sin\sqrt{s})ds\right).$$

1.3.3 Nonuniqueness

The following example demonstrates that one may even encounter nonuniqueness when the coefficients are singular.

Example 1.3.4. *(Pattar-Kiran[66])*

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 - \frac{1}{t}(\partial_t + 3\partial_x)\right)u(t,x) = 0 \\ u(0,x) = 0,\ \partial_t u(0,x) = 0. \end{cases} \tag{1.3.5}$$

The solution to the above Cauchy problem is given by

$$u(t,x) = t^2\varphi(x+t),$$

for any function $\varphi(x)$.

1.3.4 Infinite Loss

In Sections 3.6 and 4.7 we generate a residual set of singular coefficients that lead to infinite loss of regularity. It should be noted that when the assumptions that guarantee a finite loss in regularity are not satisfied, the infinite loss is the common behavior for solutions.

1.4 Content of the Thesis

In this section, we briefly outline the contents of the thesis. For detailed discussions, see the respective chapters.

In Chapter 2, we outline the methods and tools necessary for the analysis. Chapters 3 to 6 are devoted to addressing global well-posedness issues for varied rates of oscillations and blow-up. Following is the chapterwise break up of the singular behavior.

Chapter 3: Mild and Logarithmic Blow-up

Chapter 4: Oscillations and Strong Blow-up: $q = 1$ Case

Chapter 5: Very fast Oscillations: $q > 1$ Case

Chapter 6: Very Strong Blow-up

We conclude in Chapter 7 by summarising the results obtained in this thesis and identify some interesting questions for future research.

Chapter 2

Our Methods and Tools

> The method of energy estimates is much more robust and general method than the explicit construction of the parametrix.
>
> — Victor Ivrii, Microlocal Analysis, Sharp Spectral Asymptotics and Applications I (page XLV).

In this chapter we outline our methodology in dealing with singular hyperbolic Cauchy problems by introducing the tools and techniques that we employ.

Observe that the operator coefficients of the Cauchy problem (1.0.1) under our consideration are dependent on x as well. This means that we can not use Fourier transform directly as in [15, 16]. One can bypass this problem by using the following three methods:

- Pseudodifferential calculus (see for example [9, 3, 2, 62, 63, 65])
- Littlewood-Paley decomposition and paradifferential calculus (see for example [17, 14, 18])
- Construction of a parametrix (see for example [50, 16, 75])

In this thesis we use pseudofifferential calculus to obtain energy estimates. It is worth noting that paradifferential calculus is available only for the case of coefficients bounded with respect to the space variable. Developing the paradifferential calculus first for the SG setting ($\Phi(x) = \omega(x) = \langle x \rangle$) would be an interesting issue to consider. The construction of a parametrix depends on the Fourier integral operator calculus which is available only for the cases $\Phi(x) = \omega(x) = 1$ (see [44]) or $\Phi(x) = \omega(x) = \langle x \rangle$ (see [23, 22]). Weighing these issues of the last two methods against the availability of the sophisticated pseudodifferential operator calculus for a generic metric on the phase space (see for example, [43, 21, 58, 54]), we have chosen the pseudodifferential operator calculus in dealing with the singular hyperbolic Cauchy problems.

In the following, we introduce certain tools from the pseudodifferential operator theory and phase space analysis. We employ these tools in the forthcoming chapters to deal with the singular behavior in our context.

2.1 Metric on the Phase Space

Although the pseudodifferential operator theory has its roots in the theory of singular integrals and Fourier analysis, it was reinitiated and popularized in 1960s by Kohn and Nirenberg [59]. It was subsequently refined and extended using a generic Riemannian metric on the phase space by many authors, notably Hörmander [43], Rodino [58] and Lerner [54]. The theory has become one of the powerful tools in the modern theory of partial differential equations as it offers a meaningful and flexible way of applying Fourier techniques to the study of variable coefficient operators.

Metrics on the phase space are now widely used in the pseudodifferential operator theory to address the solvability and regularity issues [54]. Last decade has seen the complete resolution of the Nirenberg-Treves conjecture [28] and more recently in obtaining the loss of derivatives in the Ivrii-Petkov conjecture [6, 61] - thanks to the metric on the phase space for both these achievements. Furthermore, global issues in the pseudodifferential theory require a direct application of the metrics on phase space.

The notion of a metric on phase space $T^*\mathbb{R}^n$ ($\cong \mathbb{R}^{2n}$) was first introduced by Hörmander [43, Chapter 18] who studied smooth functions $p(x,\xi)$ called symbols using the metric

$$g_{x,\xi} = \langle\xi\rangle^{2\delta}|dx|^2 + \frac{|d\xi|^2}{\langle\xi\rangle^{2\rho}}, \quad 0 \le \delta < \rho \le 1.$$

That is, the symbol $p(x,\xi)$ satisfies for some $m \in \mathbb{R}$ and $C_{\alpha\beta} > 0$,

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $\alpha, \beta \in \mathbb{N}_0^n$ are multi-indices. In a more general framework created by Beals and Fefferman [4, 5, 58], we can consider the symbol $p(x,\xi)$ that satisfies the estimate for some $C_{\alpha\beta} > 0$,

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha\beta} M(x,\xi)\Psi(x,\xi)^{-|\alpha|}\Phi(x,\xi)^{-|\beta|},$$

where the positive functions $M(x,\xi), \Psi(x,\xi)$ and $\Phi(x,\xi)$ are specially chosen. As given in [54, Chapter 2], the above symbol estimate can be expressed using the following Riemannian structure on the phase space

$$g_{x,\xi} = \frac{|dx|^2}{\Phi(x,\xi)^2} + \frac{|d\xi|^2}{\Psi(x,\xi)^2}. \tag{2.1.1}$$

More generally, one can define symbol classes using a metric g on the phase space satisfying certain geometric restrictions of both Riemannian type and symplectic type. In order to understand these restrictions more clearly, let us first review some notation and terminology used in the study of metrics on the phase space, see [54, Chapter 2] and [58] for further details. Let us denote by $\Omega(X,Y)$ the standard symplectic form [25] on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$: if $X = (x,\xi)$ and $Y = (y,\eta)$, then Ω is given by

$$\Omega(X,Y) = \xi \cdot y - \eta \cdot x.$$

We can identify Ω with the isomorphism of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ such that $\Omega^* = -\Omega$, with the formula $\Omega(X,Y) = \langle \Omega X, Y\rangle$.

Consider a Riemannian metric g which is a (measurable) mapping from $\mathbb{R}^{2n}$ into $\mathcal{C}$, the cone of positive definite quadratic forms on $\mathbb{R}^{2n}$: for each $X \in \mathbb{R}^{2n}, g_X$ is a positive definite quadratic form on $\mathbb{R}^{2n}$. To g_X we associate the dual metric g_X^Ω by

$$g_X^\Omega(Y) = \sup_{0 \neq Y' \in \mathbb{R}^{2n}} \frac{\langle \Omega Y, Y' \rangle^2}{g_X(Y')}, \qquad \text{for all } Y \in \mathbb{R}^{2n}.$$

Considering g_X as a matrix associated to positive definite quadratic form on $\mathbb{R}^{2n}$, $g_X^\Omega = \Omega^* g_X^{-1} \Omega$. We define the Planck function [58] which plays a crucial role in the development of pseudodifferential calculus as

$$h_g(x,\xi) := \sup_{0 \neq Y \in \mathbb{R}^{2n}} \left(\frac{g_X(Y)}{g_X^\Omega(Y)} \right)^{1/2}.$$

Basically, a pseudodifferential calculus is the datum of the metric satisfying some local and global conditions. In our case, it amounts to the following conditions on g. We say that

(M1) g is a slowly varying metric on $\mathbb{R}^{2n}$ if

$$\exists C_0 > 0, \exists r_0 > 0, \forall X, Y, T \in \mathbb{R}^{2n}$$
$$g_X(Y - X) \leq r_0^2 \Longrightarrow C_0^{-1} g_Y(T) \leq g_X(T) \leq C_0 g_Y(T).$$

This property is used to introduce a partition of unity related to the metric, see [54, Theorem 2.2.7] to take care of the local analysis of the pseudodifferential calculus.

(M2) g satisfies Heisenberg's uncertainty principle if

$$g_X \leq g_X^\Omega, \quad X \in \mathbb{R}^{2n}.$$

Note that $g_X \leq h_g(X)^2 g_X^\Omega, \quad h_g(X) \leq 1$. We often make use of the strong uncertainty principle, that is, for some $\kappa > 0$, we have

$$h_g(x,\xi) \leq (1 + |x| + |\xi|)^{-\kappa}, \quad (x,\xi) \in \mathbb{R}^{2n}.$$

Heisenberg uncertainty principle prevents the localization from being too sharp on conjugate axes.

(M3) g is temperate if

$$\exists C > 0, \exists N \geq 0, \forall X, Y, T \in \mathbb{R}^{2n}, \quad \frac{g_X(T)}{g_Y(T)} \leq C \left(1 + \left(g_X^\Omega \wedge g_Y^\Omega\right)(X - Y)\right)^N,$$

where $g_X^\Omega \wedge g_Y^\Omega$ is harmonic mean of the quadratic forms. This property is used to take care of the nonlocal nature of the composition fomula in the pseudodifferential calculus.

A couple of well-known examples of symbol classes of the form $S(M,g)$ are given below:

1. $S^m_{\rho,\delta} = S\left(\langle\xi\rangle^m, \dfrac{|dx|^2}{\langle\xi\rangle^{-2\delta}} + \dfrac{|d\xi|^2}{\langle\xi\rangle^{2\rho}}\right)$: used to determine parametrices for hypoelliptic operators.
2. $S^m_{scl} = S\left(h^{-m}, |dx|^2 + h^2|d\xi|^2\right)$: used in semi-classical analysis, where the operators depending on a small parameter h are studied.
3. $\Sigma^m = S\left(\left(1+|x|^2+|\xi|^2\right)^m, \dfrac{|dx|^2+|d\xi|^2}{1+|x|^2+|\xi|^2}\right)$: used for studying spectral properties of pseudodifferential operators on $\mathbb{R}^n$.
4. $S^{m_1,m_2}_{\Psi,\Phi} = S\left(\Psi^{m_1}\Phi^{m_2}, \dfrac{|dx|^2}{\Phi^2} + \dfrac{|d\xi|^2}{\Psi^2}\right)$: used for studying solvability of principal-type differential operators satisfying Nirenberg-Treves' condition.

When $\Psi = \langle\xi\rangle$ and $\Phi = \langle x\rangle$ in $S^{m_1,m_2}_{\Psi,\Phi}$, the symbol classes correspond to the SG setting used in [3, 2].

The notion ellipticity for the general symbol class $S(M,g)$ is defined as below.

Definition 2.1.1. *A symbol a is called globally elliptic (or G-elliptic) in the class $S(M,g)$ if $a \in S(M,g)$ and for some $R > 0$,*

$$|a(x,\xi)| \gtrsim M(x,\xi), \qquad \textit{for } |x| + |\xi| \geq R.$$

We refer to [58, Chapter 1], [54, Chapter 2] and [43, Chpater 18] for the pseudodifferential operator calculus related to the symbol class $S(M,g)$.

In this thesis, we consider a metric of the form

$$g_{\Phi,k} = \frac{|dx|^2}{\Phi(x)^2} + \frac{|d\xi|^2}{\langle\xi\rangle_k^2}, \tag{2.1.2}$$

which corresponds to metric in (2.1.1) with $\Phi(x,\xi) = \Phi(x)$ and $\Psi(x,\xi) = \langle\xi\rangle_k = (k^2 + |\xi|^2)^{1/2}$, for a large positive parameter k. The weight function is of the form $M(x,\xi) = \omega(x)^{m_2}\langle\xi\rangle_k^{m_1}$ for $m_1, m_2 \in \mathbb{R}$. Here the functions $\omega(x)$ and $\Phi(x)$ are positive monotone increasing in $|x|$ such that $1 \leq \omega(x) \lesssim \Phi(x) \lesssim \langle x\rangle$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. Further, note that $\omega(x)$ and $\Phi(x)$ are associated with the weight and metric respectively, and they specify the structure of the differential equation in the space variable.

The metric $g_{\Phi,k}$ satisfies the conditions (**M1**)-(**M3**) if $\Phi(x)$ satisfies the following properties:

$$\begin{array}{rclcl} 1 & \leq & \Phi(x) & \lesssim 1+|x| & \text{(sub-linear)} \\ |x-y| & \leq & r\Phi(y) & \implies C^{-1}\Phi(y) \leq \Phi(x) \leq C\Phi(y) & \text{(slowly varying)} \\ & & \Phi(x+y) \lesssim \Phi(x)(1+|y|)^s & & \text{(temperate)} \end{array}$$

for all $x, y \in \mathbb{R}^n$ and for some $r, s, C > 0$. For the sake of calculations arising in the development of symbol calculus related to the metrics $g_{\Phi,k}$, we need to impose following additional conditions:

$$\begin{aligned} |\Phi(x)-\Phi(y)| \leq &\Phi(x+y) \leq \Phi(x)+\Phi(y), && \text{(Subadditive)}\\ |\partial_x^\beta \Phi(x)| &\lesssim \Phi(x)\langle x\rangle^{-|\beta|}, &&\\ \Phi(ax) &\leq a\Phi(x), \text{ if } a>1, &&\\ a\Phi(x) &\leq \Phi(ax), \text{ if } a\in[0,1], && \end{aligned}$$

where $\beta \in \mathbb{Z}_+^n$. It can be observed that the above conditions are quite natural in the context of symbol classes.

The Planck function associated with the metric $g_{\Phi,k}$ is $(\Phi(x)\langle\xi\rangle_k)^{-1}$. In general, we need the metrics of the form

$$g_{\Phi,k}^{\rho,r} = \left(\frac{\langle\xi\rangle_k^{\rho_2}}{\Phi(x)^{r_1}}\right)^2 |dx|^2 + \left(\frac{\Phi(x)^{r_2}}{\langle\xi\rangle_k^{\rho_1}}\right)^2 |d\xi|^2. \tag{2.1.3}$$

where $\rho = (\rho_1,\rho_2)$, $r = (r_1,r_2)$ for $\rho_j, r_j \in [0,1]$, $j = 1,2$ are such that $0 \leq \rho_2 < \rho_1 \leq 1$ and $0 \leq r_2 < r_1 \leq 1$. The Planck function associated to the metric in (2.1.3) is $\Phi(x)^{r_2-r_1}\langle\xi\rangle_k^{\rho_2-\rho_1}$.

In our work, we need the weight function ω to satisfy the above stated properties of Φ as well. In arriving at the energy estimate using the sharp Gårding inequality, we also need

$$\omega(x) \lesssim \Phi(x), \quad x\in\mathbb{R}^n.$$

2.2 Our Philosophy of Conjugation

It is well-known in the theory of hyperbolic operators (see [48, 12]) that the low-regularity in t needs to be compensated by a higher regularity in the x variable. For example, when the coefficients are Hölder continuous ($C^\alpha([0,T]), \alpha\in(0,1)$) in t, one needs to compensate this irregularity in t with Gevrey regularity ($G^\sigma(\mathbb{R}^n), \sigma < 1/(1-\alpha)$) in x, as seen in [13, 12]. This balancing operation is the key to understand the loss of regularity. As seen in the literature, conjugation brings this balance. Infact, conjugation links irregularity in t, growth with respect to x variable of the coefficients to the weight functions defining the solution spaces in a precise manner, see [12, Proposition 2.8] and [2, Proposition 3.1]. For example, when the coefficients are in $C^\alpha([0,T];B^\infty(\mathbb{R}^n))$, the conjugating operator is of the form $e^{c_0\langle D_x\rangle^{1/\sigma}}, c_0>0$ while for the SG setting with coefficients in $C^\alpha([0,T];C^\infty(\mathbb{R}^n))$, the conjugating operator used in the literature is of the form $e^{\nu(t)(\langle x\rangle^{1/\sigma}+\langle D_x\rangle^{1/\sigma})}$ for some function $\nu\in C([0,T])\cap C^1((0,T])$.

In the global setting, solutions experience an infinite loss of both derivatives and decay when the coefficients display strong blow-up, very strong blow-up or very fast oscillations in t. In order to overcome the difficulty of tracking a precise loss in our context we introduce a class of parameter dependent infinite order pseudodifferential operators of the form $e^{\nu(t)\Theta(x,D_x)}$ for the purpose of conjugation. These operators compensate, microlocally, the loss of regularity of the solutions. The operator $\Theta(x,D_x)$ is in general nonselfadjoint and it explains the compensation for the singularity in t and decides the quantity of the loss while the monotone continuous function $\nu(t)$ gives a scale for the loss. Hence, we call the conjugating operator as the “loss operator”.

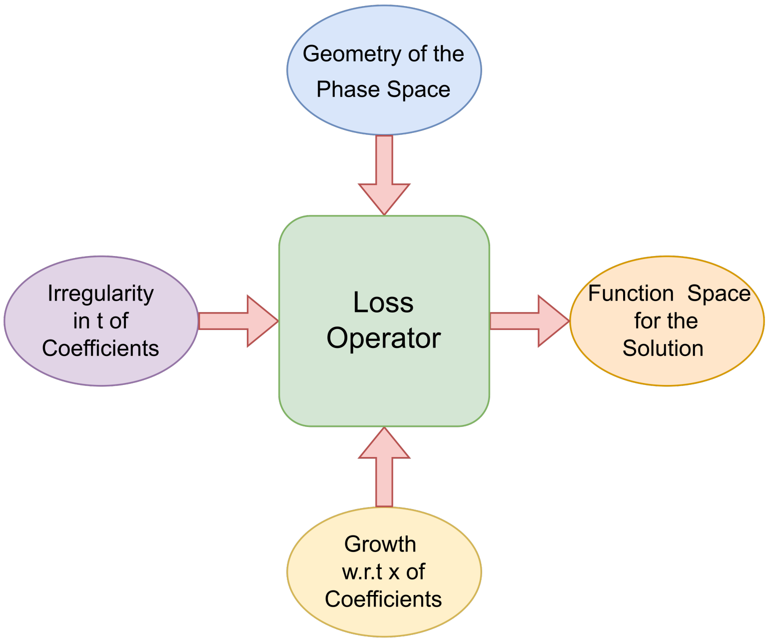

Figure 2.1: Our Philosophy of Conjugation

Our philosophy of conjugation is to design the loss operator that links the singular behavior, growth rate in x and the geometry of the phase space to the loss of regularity. In fact, the operator $\Theta(x, D_x)$ is such that $\Theta(x, \xi)$ is a function of $h(x, \xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$ which is the Planck function associated to the metric $g_{\Phi,k}$ in (2.1.2) and the quantity $\nu'(t)\Theta(x, \xi)$ majorizes the symbols corresponding to the lower order terms obtained after the application of a suitable diagonalization technique to the first order system corresponding to the Cauchy problem in (1.0.1). See Theorems 4.4.1 and 5.3.1 for more details. The appearance of Planck function $h(x, \xi)$ in the definition of $\Theta(x, \xi)$ is justified as it controls the extent of localization on the phase space.

One of our key observations is that the symbol of the operator arising after the conjugation is governed by a metric $\tilde{g}_{\Phi,k}$ that is conformally equivalent to the initial metric $g_{\Phi,k}$. The metric $\tilde{g}_{\Phi,k}$ is of the form

$$\tilde{g}_{\Phi,k} = \Theta(x, \xi)^2 g_{\Phi,k}.$$

The operators with loss of regularity index are transformed to good operators by conjugation with loss operator. This helps us to derive good a priori estimates of solutions in the Sobolev space associated with the loss operator by an application of sharp Gårding inequality followed by Gronwall inequality.

Apart from the well-posedness and regularity results that we obtain for the various cases of singular behavior, the conjugation results (Theorems 4.4.1 and 5.3.1) of this thesis are our key contributions to the literature.

2.3 Subdivision of the Extended Phase Space

Before we perform a conjugation, we need to "preprocess" the operator with singular coefficients. This preprocessing step uses a careful amalgam of a localization technique on the extended phase space ($[0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$) and a diagonalization procedure already available in the literature [50] to handle the singularity. It helps in diagonalizing the top order terms in a first order system obtained from a singular hyperbolic partial differential equation while the lower order terms are handled through conjugation.

As the Planck function holds information about the extent of localization on the phase space, our localization technique that dictates the subdivision of the extended phase space is dependent on this critical information. For a fixed (x,ξ), we define the time splitting points $t^{(j)}_{x,\xi}, 1 \leq j \leq m_0$, for some $m_0 \in \mathbb{N}$ as

$$t^{(j)}_{x,\xi} = NG_j(h(x,\xi))$$

where N is the positive integer, $G_j(r), r > 0$ is a function that depends on the order of singularity (see for example Section 4.2) and $h(x,\xi)$ is the Planck function associated with the metric $g_{\Phi,k}$. For a fixed (x,ξ) we split the time interval as

$$[0,T] = [0, t^{(1)}_{x,\xi}) \cup [t^{(1)}_{x,\xi}, t^{(2)}_{x,\xi}) \cup \cdots \cup [t^{(m_0)}_{x,\xi}, T].$$

We define the regions as below:

$$\begin{aligned} Z_j(N) &= \{((t,x,\xi)) : t^{(j-1)}_{x,\xi} \leq t < t^{(j)}_{x,\xi}\}, \quad j = 1, \dots, m_0 - 1 \\ Z_{m_0}(N) &= \{((t,x,\xi)) : t^{(m_0)}_{x,\xi} \leq t \leq T\} \end{aligned}$$

with $t^{(0)}_{x,\xi} = 0$. In all our cases $m_0 \leq 2$.

The way we deploy these regions is that we first perform an excision of the irregular symbol (for example, see Sections 3.4.1, 4.5.1 and 6.4.1) so that the resulting symbol is smooth near $t = 0$ that is in $Z_1(N)$. The difference of these symbols is localized in $Z_1(N)$. Further, we deploy the diagonalization procedure (for example, see Sections 3.4.2, 4.5.1, 4.5.2, 5.5.2 and 6.4.2) to restrict the singularities arising from t-derivatives to the appropriate regions. This kind of localization of the singularities allows one to come with a function $\nu(t)\Theta(x,\xi)$ that is used to define the loss operator (for example, see Sections 3.4.1 and 4.5.3). More details and interpretation will follow in the respective chapters.

Chapter 3

Mild and Logarithmic Blow-up

> Knowing what is big and what is small is more important than being able to solve partial differential equations.
>
> — Stanislaw Ulam

Let us start with simplest case of blow-up rate - at most logarithmic. In this chapter, we investigate the asymptotic behavior of the solutions to Cauchy problems as $|x| \to \infty$ when the singularity of the coefficients with respect to x-derivatives and t-derivative is of order $\mathrm{O}(t^{-\delta}), \delta \in [0,1)$, and $\mathrm{O}(t^{-1}|\ln t|^{\tilde{\gamma}-1}), \tilde{\gamma} \in (0,1]$, respectively. We report that the solutions experience an arbitrarily small (when $\tilde{\gamma} \in (0,1)$) or finite (when $\tilde{\gamma} = 1$) loss in the Sobolev space index in relation to the initial datum defined in the Sobolev space tailored to the metric and the order of singularity.

3.1 Introduction and Statement of Main Result

Let us consider the following protypical Cauchy problem:

$$\left.\begin{aligned} &\partial_t^2 u - a(t,x)\Delta_x u = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^n, \\ &u(0,x) = u_1(x), \quad \partial_t u(0,x) = u_2(x), \end{aligned}\right\} \tag{3.1.1}$$

where the coefficient $a(t,x)$ is in $C^1((0,T]; C^\infty(\mathbb{R}^n))$ and satisfies the following estimates

$$a(t,x) \geq C_0 \omega(x)^2, \tag{3.1.2}$$

$$|\partial_x^\beta \partial_t a(t,x)| \leq C_\beta \omega(x)^2 \Phi(x)^{-|\beta|} \, \frac{1}{t} \left(\ln \left(1 + \frac{1}{t} \right) \right)^{\tilde{\gamma}(1+|\beta|)-1} \tag{3.1.3}$$

where $\beta \in \mathbb{N}_0^n$, $C, C_0, C_\beta > 0$. From (3.1.3) with $|\beta| = 0$, we have

$$|a(t,x)| - |a(T,x)| \leq |a(T,x) - a(t,x)| \leq \int_t^T |\partial_s a(s,x) ds| \leq C \left(\ln \left(1 + \frac{1}{t} \right) \right)^{\tilde{\gamma}} \omega(x)^2.$$

Since $a(t,\cdot) \in C^1([T_0,T])$ for any $T_0 > 0$, we have $|a(T,x)| \le C\omega(x)^2$. Implying

$$|a(t,x)| \le C\left(\ln\left(1+\frac{1}{t}\right)\right)^{\tilde{\gamma}} \omega(x)^2, \tag{3.1.4}$$

i.e., the coefficient $a(t,x)$ has at most logarithmic blow-up at $t=0$.

Remark 3.1.1. *The singular behavior quantified by (3.1.3) is a generalization of Cicognani [9] to a global setting. Our conditions (3.1.3) confirm with the conditions on the coefficients given in [9] for $\omega(x) = \Phi(x) = 1$ and $\tilde{\gamma} = 1$. Moreover, we can also replace (3.1.3) with the following general estimates as given in [9],*

$$\left.\begin{aligned} |\partial_x^\beta a(t,x)| &\le C_\beta \frac{1}{t^{\delta_1}}\omega(x)^2\Phi(x)^{-|\beta|}, \quad |\beta|>0,\\ |\partial_x^\beta \partial_t a(t,x)| &\le C_\beta \frac{(\ln(1+1/t))^{\tilde{\gamma}-1}}{t^{1+\delta_2|\beta|}}\omega(x)^2\Phi(x)^{-|\beta|}, \quad |\beta|\ge 0, \end{aligned}\right\} \tag{3.1.3*}$$

for $\delta_1, \delta_2 \in [0,1)$.

An example of a coefficient $a(t,x)$ satisfying (3.1.2) and (3.1.3*) is given below.

Example 3.1.1. *The function $a(t) = \left(\ln\left(1+\left(\frac{1}{t}\right)\right)\right)^{\frac{1}{4}}$ satisfies (3.1.3*) for $\omega(x) = \Phi(x) = 1$, for $\delta_2 = 0$, $\tilde{\gamma} = 1/4$, and for any $\delta_1 \in (0,1)$ and .*

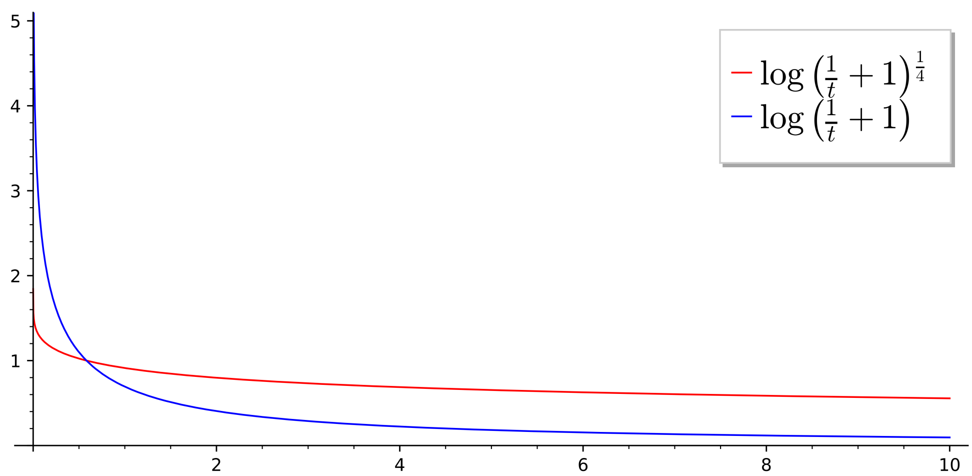

Figure 3.1: Sublogarithmic function $\left(\ln\left(1+\left(\frac{1}{t}\right)\right)\right)^{\frac{1}{4}}$

Example 3.1.2. *Let $n=1$, $T=1$, $\kappa_1 \in [0,1]$ and $\kappa_2 \in (0,1]$ such that $\kappa_1 \le \kappa_2$. Then,*

$$a(t,x) = \langle x\rangle^{2\kappa_1}\left(2+\sin\left(\langle x\rangle^{1-\kappa_2} + \cos x\,(\ln t)^{\tilde{\gamma}}\right) + \left(2+\cos\langle x\rangle^{1-\kappa_2}\right)(\ln(1+1/t))^{\tilde{\gamma}}\right)$$

satisfies the estimates (3.1.3) for $\omega(x) = \langle x\rangle^{\kappa_1}$, $\Phi(x) = \langle x\rangle^{\kappa_2}$ and for any $\delta_1, \delta_2 \in (0,1)$.*

In [9], Cicognani discussed well-posedness of (3.1.1) for the case $\tilde{\gamma} = \omega(x) = \Phi(x) = 1$ in (3.1.2) and (3.1.3*), where the author reports well-posedness in $C^\infty(\mathbb{R}^n)$ for the Cauchy problem (3.1.1) with a finite loss of derivatives. In this chapter, we extend the result of Cicognani to the case of generic weight functions ω and Φ and for any $\tilde{\gamma} \in (0,1]$.

3.1.1 Sobolev Spaces

The Sobolev spaces related to the metric $g_{\Phi,k}$ in (2.1.2) and tailored to the order of singularity are defined below. Let $s = (s_1, s_2) \in \mathbb{R}^2$ and $k \geq 1$.

Definition 3.1.1. *The Sobolev space $\mathcal{H}^{s,\mu,\tilde{\gamma}}_{\Phi,k}(\mathbb{R}^n)$ for $\mu \in \mathbb{R}$ and $\tilde{\gamma} \in (0,1)$ is defined as*

$$\mathcal{H}^{s,\mu,\tilde{\gamma}}_{\Phi,k}(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : \Phi(x)^{s_2} \langle D \rangle_k^{s_1} e^{\mu(\ln(1+\Phi(x)\langle D_x \rangle_k))^{\tilde{\gamma}}} v \in L^2(\mathbb{R}^n)\}, \tag{3.1.5}$$

equipped with the norm $\|v\|_{\Phi,k;s,\mu,\tilde{\gamma}} = \|\Phi(\cdot)^{s_2} \langle D \rangle_k^{s_1} e^{\mu(\ln(1+\Phi(\cdot)\langle D \rangle_k))^{\tilde{\gamma}}} v\|_{L^2}$.

The subscript k in the notation $\mathcal{H}^{s,\mu,\tilde{\gamma}}_{\Phi,k}(\mathbb{R}^n)$ is related to the parameter in the operator $\langle D \rangle_k = (k^2 - \Delta_x)^{1/2}$. Observe that $e^{\mu(\ln(1+\Phi(x)\langle D_x \rangle_k))^{\tilde{\gamma}}}$ is a pseudodifferential operator of an arbitrarily small positive order in both x and D_x for $\tilde{\gamma} \in (0,1)$. When $\mu = 0$, the above spaces correspond to the following spaces.

Definition 3.1.2. *The Sobolev space $H^s_{\Phi,k}(\mathbb{R}^n)$ for $s = (s_1, s_2) \in \mathbb{R}^2$ and $k \geq 1$, is defined as*

$$H^s_{\Phi,k}(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : \Phi(x)^{s_2} \langle D \rangle_k^{s_1} v \in L^2(\mathbb{R}^n)\}, \tag{3.1.6}$$

equipped with the norm $\|v\|_{\Phi,k;s} = \|\Phi(\cdot)^{s_2} \langle D \rangle_k^{s_1} v\|_{L^2}$.

When $\Phi(x)$ is bounded and $k = 1$, $H^s_{\Phi,1}(\mathbb{R}^n)$ correspond to the usual Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Some remarks on the relation between the spaces $\mathcal{H}^{s,\mu,\tilde{\gamma}}_{\Phi,k}(\mathbb{R}^n)$ and $H^s_{\Phi,k}(\mathbb{R}^n)$ are in order.

Remark 3.1.2. *For any $\mu \in \mathbb{R}$, we have*

1. $H^{s+\nu e}_{\Phi,k}(\mathbb{R}^n) \hookrightarrow \mathcal{H}^{s,\mu,\tilde{\gamma}}_{\Phi,k}(\mathbb{R}^n) \hookrightarrow H^{s-\nu e}_{\Phi,k}(\mathbb{R}^n)$, *when* $\tilde{\gamma} \in (0,1)$,
2. $\mathcal{H}^{s,\mu,1}_{\Phi,k}(\mathbb{R}^n) \equiv H^{s+\mu e}_{\Phi,k}(\mathbb{R}^n)$,

where $\nu > 0$ is arbitrarily small and $e = (1,1)$.

3.1.2 Main Result

Let us generalize the problem (3.1.1) and consider

$$\begin{cases} P(t,x,\partial_t,D_x)u(t,x) = f(t,x), \qquad D_x = -i\nabla_x, \ (t,x) \in (0,T] \times \mathbb{R}^n, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x), \end{cases} \tag{3.1.7}$$

with the strictly hyperbolic operator $P(t,x,\partial_t,D_x) = \partial_t^2 + a(t,x,D_x) + b(t,x,D_x)$ where

$$a(t,x,\xi) = \sum_{i,j=1}^n a_{i,j}(t,x)\xi_i\xi_j \quad \text{and} \quad b(t,x,\xi) = i\sum_{j=1}^n b_j(t,x)\xi_j + b_{n+1}(t,x). \tag{3.1.8}$$

Here, the matrix $(a_{i,j}(t,x))$ is real symmetric for all $(t,x) \in (0,T]\times\mathbb{R}^n$, $a_{i,j} \in C^1((0,T];C^\infty(\mathbb{R}^n))$ and $b_j \in C([0,T];C^\infty(\mathbb{R}^n))$. Similar to the estimates in Remark 3.1.1, we have the following assumptions on $a(t,x,\xi)$ and $b(t,x,\xi)$

$$\left.\begin{aligned} &a(t,x,\xi) \ge C_0\omega(x)^2\langle\xi\rangle_k^2, \quad C_0>0,\\ |\partial_\xi^\alpha\partial_x^\beta a(t,x,\xi)| &\le C_{\alpha\beta}\frac{1}{t^{\delta_1}}\omega(x)^2\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{2-|\alpha|}, \quad |\alpha|\ge 0, |\beta|>0,\\ |\partial_\xi^\alpha\partial_x^\beta b(t,x,\xi)| &\le C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}, \end{aligned}\right\} \tag{3.1.9}$$

and either of the following estimates

$$|\partial_\xi^\alpha\partial_x^\beta\partial_t a(t,x,\xi)| \le C_{\alpha\beta}\frac{(\ln(1+1/t))^{\tilde\gamma-1}}{t^{1+\delta_2|\beta|}}\omega(x)^2\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{2-|\alpha|}, \tag{3.1.10}$$

$$|\partial_\xi^\alpha\partial_x^\beta\partial_t a(t,x,\xi)| \le C_{\alpha\beta}\frac{1}{t^{\delta_3+\delta_2|\beta|}}\omega(x)^2\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{2-|\alpha|}, \tag{3.1.10*}$$

where $\delta_j \in \big[0,1\big), j=1,2,3$ and $(t,x,\xi)\in(0,T]\times\mathbb{R}^n\times\mathbb{R}^n$. Note that $C_{\alpha\beta}$ is a generic constant.

The following theorem states that the solution to the Cauchy problem (3.1.7) experiences no loss when (3.1.10*) is satisfied. On the other hand, when (3.1.10) satisfied, the solution experiences arbitrarily small loss and finite loss for $\tilde\gamma \in (0,1)$ and $\tilde\gamma=1$, respectively. Let $e=(1,1)$.

Theorem 3.1.1. *(Zero/ Arbitrary Small/ Finite Loss) Consider the strictly hyperbolic Cauchy problem (3.1.7) satisfying the conditions (3.1.9) and (3.1.10 or 3.1.10*). Let the initial data f_j belong to $H^{s+(2-j)e}_{\Phi,k}$, $j=1,2$ and the right hand side $f\in C([0,T];H^s_{\Phi,k})$. Then, denoting $\delta=\max\{\delta_1,\delta_2\}$, for every $\varepsilon\in(0,1-\delta)$ there are $\kappa_0,\kappa_1>0$ such that for every $s\in\mathbb{R}^2$ there exists a unique global solution*

$$u \in C\left([0,T];\mathcal{H}^{s+e,-\Lambda(t),\tilde\gamma}_{\Phi,k}\right)\bigcap C^1\left([0,T];\mathcal{H}^{s,-\Lambda(t),\tilde\gamma}_{\Phi,k}\right),$$

where

$$\Lambda(t) = \begin{cases}\kappa_0+\kappa_1 t^\varepsilon/\varepsilon, & \text{when (3.1.10) is satisfied,}\\ 0, & \text{when (3.1.10*) is satisfied.}\end{cases}$$

More specifically, the solution satisfies an a priori estimate

$$\sum_{j=0}^{1}\|\partial_t^j u(t,\cdot)\|_{\Phi,k;s+(1-j)e,-\Lambda(t),\tilde\gamma} \le C\left(\sum_{j=1}^{2}\|f_j\|_{\Phi,k;s+(2-j)e}+\int_0^t\|f(\tau,\cdot)\|_{\Phi,k;s,-\Lambda(\tau),\tilde\gamma}\,d\tau\right) \tag{3.1.11}$$

for $0\le t\le T$, $C=C_s>0$.

3.2 Subdivision of the Phase Space

One of the main techniques in proving Theorem 3.1.1 is the division of the extended phase space $J=[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, where $T>0$, into two regions using the Planck function $h(x,\xi)=(\Phi(x)\langle\xi\rangle_k)^{-1}$ of the metric $g_{\Phi,k}$ in (2.1.2). As we will see in Section 3.4.1, the main utility of these regions is to handle the low regularity in t. To this end we define $t_{x,\xi}$, for a fixed (x,ξ), as the solution to the equation

$$t = \frac{N}{\Phi(x)\langle\xi\rangle_k},$$

where N is the positive constant chosen appropriately later. Using $t_{x,\xi}$ we define the interior region

$$Z_{int}(N) = \{(t,x,\xi) \in J : 0 \leq t \leq t_{x,\xi}\} \tag{3.2.1}$$

and the exterior region

$$Z_{ext}(N) = \{(t,x,\xi) \in J : t_{x,\xi} < t \leq T\}. \tag{3.2.2}$$

In the following section, we use these regions to define the parameter dependent global symbol classes.

3.3 Parameter Dependent Global Symbol Classes

We now define certain parameter dependent global symbol classes that are associated with the study of the Cauchy problem (3.1.7). Let $m = (m_1, m_2) \in \mathbb{R}^2$. Consider the metric $g^{\rho,r}_{\Phi,k}$ as in (2.1.3).

Definition 3.3.1. *$G^{m_1,m_2}(\omega, g^{\rho,r}_{\Phi,k})$ is the space of all functions $p = p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying*

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{(x,\xi)\in\mathbb{R}^n} \langle\xi\rangle_k^{-m_1+\rho_1|\alpha|-\rho_2|\beta|}\omega(x)^{-m_2}\Phi(x)^{r_1|\beta|-r_2|\alpha|}|\partial_\xi^\alpha D_x^\beta p(x,\xi)| < +\infty$$

For the sake of simplicity, we denote the symbol class $G^{m_1,m_2}(\omega, g^{(1,0),(1,0)}_{\Phi,k})$ as $G^{m_1,m_2}(\omega, g_{\Phi,k})$.

Observe that the derivatives of $\sqrt{a(t,x,\xi)}$ show stronger singular behavior compared to $a(t,x,\xi)$ defined in (3.1.8) due to the blow-up. Thus, in order to handle the stronger singular behavior of the characteristics of operator P in (3.1.7), we have the following symbol classes. Let us denote

$$\tilde{\theta}(t) = \left(\ln\left(1+\frac{1}{t}\right)\right)^{\tilde{\gamma}}. \tag{3.3.1}$$

Definition 3.3.2. *$G^{m_1,m_2}\{l_1,l_2;\tilde{\gamma},p\}_{int,N}(\omega, g_{\Phi,k})$ for $l_1,l_2 \in \mathbb{R}$ and $p \in [0,1)$is the space of all t-dependent symbols $a = a(t,x,\xi)$ in $C^1((0,T];G^{m_1,m_2}(\omega, g_{\Phi,k}))$ satisfying*

$$|\partial_\xi^\alpha a(t,x,\xi)| \leq C_{00}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\tilde{\theta}(t)^{l_1},$$
$$|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}\left(\frac{1}{t}\right)^{pl_2},$$

for all $(t,x,\xi) \in Z_{int}(N)$ and for some $C_{\alpha\beta} > 0$ where $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}^n$.

Definition 3.3.3. *$G^{m_1,m_2}\{l_1,l_2,l_3,l_4;\tilde{\gamma},p\}_{ext,N}(\omega,g_{\Phi,k})$ for $l_j\in\mathbb{R}, j=1,\dots,4$ and $p\in[0,1)$ is the space of all t-dependent symbols $a=a(t,x,\xi)$ in $C^1((0,T];G^{m_1,m_2}(\omega,g_{\Phi,k}))$ satisfying*

$$|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)|\leq C_{\alpha\beta}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}\left(\frac{1}{t}\right)^{l_1+p(l_2+|\beta|)}\tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)}$$

for all $(t,x,\xi)\in Z_{ext}(N)$ and for some $C_{\alpha\beta}>0$ where $\alpha,\beta\in\mathbb{N}_0^n$.

Given a t-dependent global symbol $a(t,x,\xi)$, we can associate a pseudodifferential operator $Op(a)=a(t,x,D_x)$ to $a(t,x,\xi)$ by the following oscillatory integral

$$\begin{aligned}a(t,x,D_x)u(t,x)&=\iint\limits_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi}a(t,x,\xi)u(t,y)dy\,\bar{d}\xi\\&=(2\pi)^{-n}\int\limits_{\mathbb{R}^n}e^{ix\cdot\xi}a(t,x,\xi)\hat{u}(t,\xi)\,\bar{d}\xi,\end{aligned}$$

where $\bar{d}\xi=(2\pi)^{-n}d\xi$ and $\hat{u}$ is the Fourier transform of u in the space variable.

We denote the class of operators with symbols in $G^{m_1,m_2}(\omega,g_{\Phi,k}^{\rho,r})$ by $OPG^{m_1,m_2}(\omega,g_{\Phi,k}^{\rho,r})$. We refer to [58, Section 1.2 & 3.1] and [24] for the calculus of such operators. The calculus for the operators with symbols of form $a(t,x,\xi)=a_1(t,x,\xi)+a_2(t,x,\xi)$ such that

$$\begin{aligned}&a_1\in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta_1\}_{int,N_1}(\omega,g_{\Phi,k}),\\&a_2\in G^{m_1,m_2}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta_2\}_{ext,N_2}(\omega,g_{\Phi,k}),\end{aligned}$$

for $N_1\geq N_2$, is given in Section A.1 of Appendix A.

3.4 Global Well-Posedness

We first give the proof of Theorem 3.1.1 when (3.1.10) is satisfied and the case for (3.1.10*) follows in similar lines. There are three key steps in the proof. First, we factorize the operator $P(t,x,\partial_t,D_x)$ in (3.1.7). To this end, we begin with modifying the coefficients of the principal part by performing an excision so that the resulting coefficients are regular near $t=0$. Second, we reduce the original Cauchy problem to a Cauchy problem for a first order system (with respect to ∂_t). Lastly, using sharp Gårding's inequality we arrive at the L^2-wellposedness of a related auxiliary Cauchy problem, which gives wellposedness of the original problem in the Sobolev spaces $\mathcal{H}_{\Phi,k}^{s,\mu,\tilde{\gamma}}$.

3.4.1 Factorization

From (3.1.10), we observe that $a(t,x,\xi)$ is L^1 integrable in t. More precisely, $a(t,x,\xi)$ is sublogarithmically bounded at $t=0$, i.e.,

$$|a(t,x,\xi)|\leq C\omega(x)^2\langle\xi\rangle_k^2\tilde{\theta}(t),\quad C>0,\tag{3.4.1}$$

where $\tilde{\theta}(t)$ is as in (3.3.1). We modify the symbol a in $Z_{int}(2)$, by defining

$$\tilde{a}(t,x,\xi)=\varphi(t\Phi(x)\langle\xi\rangle_k)\omega(x)^2\langle\xi\rangle_k^2+(1-\varphi(t\Phi(x)\langle\xi\rangle_k))a(t,x,\xi)\tag{3.4.2}$$

for $\varphi \in C^\infty(\mathbb{R})$, $0 \le \varphi \le 1$, $\varphi = 1$ in $[0,1]$, $\varphi = 0$ in $[2,+\infty)$. Note that

$$(a-\tilde{a}) \in G^{2,2}\{1,1;\delta_1\}_{int,2}(\omega, g_{\Phi,k}) \text{ and } (a-\tilde{a}) \sim 0 \text{ in } Z_{ext}(2).$$

This implies that $t^{\delta_1}(a-\tilde{a})$ for $t \in [0,T]$ is a bounded and continuous family in $G^{2,2}(\omega, g_{\Phi,k})$. Observe that $a-\tilde{a}$ is L^1 integrable in t, i.e.,

$$\begin{aligned}\int_0^T |(a-\tilde{a})(t,x,\xi)|dt &\le \kappa_0\omega(x)^2\langle\xi\rangle_k^2 \int_0^{2/\Phi(x)\langle\xi\rangle_k} \tilde{\theta}(t)dt \\ &\le \kappa_0\omega(x)\langle\xi\rangle_k(\ln(1+\Phi(x)\langle\xi\rangle_k))^{\tilde{\gamma}}.\end{aligned} \tag{3.4.3}$$

Let $\tau(t,x,\xi) = \sqrt{\tilde{a}(t,x,\xi)}$ and $\delta = \max\{\delta_1,\delta_2\}$. It is easy to note that

i) $\tau(t,x,\xi)$ is G_Φ-elliptic symbol of order $(1,1)$ i.e. there is $C > 0$ such that all $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ we have

$$|\tau(t,x,\xi)| \ge C\omega(x)\langle\xi\rangle_k.$$

ii) $\tau(t,x,\xi) \in G^{1,1}\{0,0;\tilde{\gamma},0\}_{int,2}(\omega, g_{\Phi,k}) + G^{1,1}\{0,0,1,1;\tilde{\gamma},\delta_1\}_{ext,1}(\omega, g_{\Phi,k})$.

iii) By definition

$$\partial_t\tau(t,x,\xi) = \frac{1}{2\tau}\left[\Phi(x)\langle\xi\rangle_k\varphi'(t\Phi(x)\langle\xi\rangle_k)(\omega(x)^2\langle\xi\rangle_k^2 - a) + (1-\varphi(t\Phi(x)\langle\xi\rangle_k))\partial_t a\right].$$

Hence,

$$\begin{array}{ll}\partial_t\tau \sim 0 & \text{in } Z_{int}(1), \\ \partial_t\tau \in G^{2,2}\{1,1;\tilde{\gamma},\delta_1\}_{int,2}(\Phi, g_{\Phi,k}) & \text{in } Z_{int}(2)\setminus Z_{int}(1), \\ \partial_t\tau \in G^{1,1}\{1,0,1,1;\tilde{\gamma},\delta\}_{ext,1}(\omega, g_{\Phi,k}) & \text{in } Z_{ext}(1).\end{array}$$

To be precise, there are $C_0, C_{\alpha\beta} > 0$ such that for $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and $|\alpha| \ge 0, |\beta| > 0$ we have in $Z_{int}(1)$,

$$\left.\begin{aligned}|\partial_\xi^\alpha D_x^\beta \partial_t\tau(t,x,\xi)| &\sim \chi_{int}(1)\, 0 \\ |\partial_t\tau(t,x,\xi)| &\le C_0\left(\chi_{int}(2)-\chi_{int}(1)\right)\Phi(x)^2\langle\xi\rangle_k^2\,\tilde{\theta}(t),\end{aligned}\right\}$$

in $Z_{int}(2)\setminus Z_{int}(1)$,

$$|\partial_\xi^\alpha D_x^\beta \partial_t\tau(t,x,\xi)| \le C_{\alpha\beta}\left(\chi_{int}(2)-\chi_{int}(1)\right)\Phi(x)^2\langle\xi\rangle_k^2\,\frac{1}{t^{\delta_1}},$$

in $Z_{ext}(1)$,

$$\left.\begin{aligned}|\partial_t\tau(t,x,\xi)| &\le C_0\,\chi_{ext}(1)\langle\xi\rangle_k\omega(x)\frac{\tilde{\theta}(t)}{t}, \\ |\partial_\xi^\alpha D_x^\beta \partial_t\tau(t,x,\xi)| &\le C_{\alpha\beta}\,\chi_{ext}(1)\langle\xi\rangle_k^{1-|\alpha|}\omega(x)\Phi(x)^{-|\beta|}\frac{\tilde{\theta}(t)^{1-\frac{1}{\tilde{\gamma}}}}{t}\frac{\tilde{\theta}(t)^{|\alpha|+|\beta|}}{t^{\delta|\beta|}}.\end{aligned}\right\}$$

Here $\chi_{int}(N_1)$ and $\chi_{ext}(N_2)$ are the indicator functions for the regions $Z_{int}(N_1)$ and $Z_{ext}(N_2)$, respectively. From the properties (i-iii) of τ and by the definition of $\tilde{a}$ in (3.4.2), we have the following two lemmas.

Lemma 3.4.1. *Let $\varepsilon, \varepsilon'$ be such that $0 < \varepsilon < \varepsilon' < 1 - \delta$. Then,*

i) $\tau \in C([0,T]; G^{1+\varepsilon,1}(\omega\Phi^{\varepsilon}, g_{\Phi,k}^{(1,\delta_1),(1-\delta_1,0)})$,

ii) $\tau^{-1} \in C([0,T]; G^{-1,-1}(\omega, g_{\Phi,k}^{(1,\delta_1),(1-\delta_1,0)})$,

iii) $t^{1-\varepsilon}\partial_t\tau(t,\cdot,\cdot) \in G^{1+\varepsilon',1}(\omega\Phi^{\varepsilon'}, g_{\Phi,k}^{(1,\delta),(1-\delta,0)})$, *for all* $t \in [0,T]$.

Proof. The first two claims follow from Proposition A.1.1 while the third from the observation that in $Z_{ext}(1)$

$$t^{1-\varepsilon}\left(\frac{\tilde{\theta}(t)^{1+|\alpha|+|\beta|}}{t}\right) \leq \frac{1}{t^{\varepsilon'}} \leq \left(\Phi(x)\langle\xi\rangle_k\right)^{\varepsilon'}.$$

□

Lemma 3.4.2. *Let ε be such that $0 < \varepsilon < 1 - \delta$. Then,*

i) $t^{1-\varepsilon}(\tilde{a}(t,x,D_x) - \tau(t,x,D_x)^2) \in C\Big([0,T]; OPG^{1,1}(\omega, g_{\Phi,k}^{(1,\delta_1),(1-\delta_1,0)})\Big)$,

ii) $t^{1-\varepsilon}(a(t,x,D_x) - \tilde{a}(t,x,D_x)) \in C\Big([0,T]; OPG^{1,1}(\omega, g_{\phi,k})\Big)$.

Proof. The proof is a consequence of the fact that $t^{1-\varepsilon}\tilde{\theta}(t)^{1+|\alpha|+|\beta|}$ is bounded and continuous for all $t \in [0,T]$ and for all $\alpha, \beta \in \mathbb{N}_0^n$. □

We have the following factorization for the operator $P(t,x,\partial_t,D_x)$

$$P(t,x,\partial_t,D_x) = (\partial_t - i\tau(t,x,D_x))(\partial_t + i\tau(t,x,D_x)) + (a-\tilde{a})(t,x,D_x) + a_1(t,x,D_x)$$

where the operator $a_1(t,x,D_x)$ is such that, for $t \in [0,T]$,

$$a_1 = -i[\partial_t, \tau] + \tilde{a} - \tau^2 + b \text{ and } t^{1-\varepsilon}a_1(t,x,D_x) \in OPG^{1+\varepsilon',1}(\omega\Phi^{\varepsilon'}, g_{\Phi,k}^{(1,\delta),(1-\delta,0)}).$$

We define the functions $\psi_0(t,x,\xi)$ and $\psi_1(t,x,\xi)$ in $L^1([0,T]; C^\infty(\mathbb{R}^{2n}))$ as

$$\begin{aligned} \psi_0(t,x,\xi) &= C_1\varphi(t\Phi(x)\langle\xi\rangle_k)\omega(x)\langle\xi\rangle_k\tilde{\theta}(t), \\ \psi_1(t,x,\xi) &= C_2\left(\varphi(t\Phi(x)\langle\xi\rangle_k)\omega(x)\langle\xi\rangle_k\tilde{\theta}(t) + (1-\varphi(t\Phi(x)\langle\xi\rangle_k))\frac{1}{t}\right), \end{aligned} \tag{3.4.4}$$

for some $C_1, C_2 > 0$, such that

$$\frac{|(a-\tilde{a})(t,x,\xi)|}{\omega(x)\langle\xi\rangle_k} \leq \psi_0(t,x,\xi) \text{ and } \frac{a_1(t,x,\xi)}{\omega(x)\langle\xi\rangle_k} \leq \psi_1(t,x,\xi).$$

Let $\psi = \psi_0 + \psi_1$. We observe that $t^{1-\varepsilon}\psi \in C\left([0,T]; G^{\varepsilon',\varepsilon'}(\Phi, g_{\phi,k})\right)$ and $\partial_\xi^\alpha \partial_x^\beta \psi(t,x,\xi)$ is supported in $Z_{int}(2)$ for $|\alpha|+|\beta|>0$. Hence,

$$\begin{aligned}\int_0^T |\psi(t,x,\xi)|dt &\le C\left(\omega(x)\langle\xi\rangle_k \int_0^{2/\Phi(x)\langle\xi\rangle_k} \tilde{\theta}(t)dt + \int_{1/\Phi(x)\langle\xi\rangle_k}^T \frac{\tilde{\theta}(t)^{1-\frac{1}{\tilde{\gamma}}}}{t} dt\right) \\ &\le C(\ln(1+\Phi(x)\langle\xi\rangle_k))^{\tilde{\gamma}},\end{aligned} \tag{3.4.5}$$

and for $|\alpha|+|\beta|>0$,

$$\int_0^t |\partial_\xi^\alpha D_x^\beta \psi(r,x,\xi)|dr \le C\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{\tilde{\gamma}}\chi_{int}(2).$$

The function ψ is used for making a change of variable while arriving at an energy estimate.

3.4.2 Reduction to a First Order Pseudodifferential System

Next we obtain a first order 2×2 pseudodifferential system equivalent to the operator P by generalizing the procedure used in [9]. To achieve this, we introduce the change of variables $U = U(t,x) = (u_1(t,x), u_2(t,x))^T$, where

$$\begin{cases} u_1(t,x) = (\partial_t + i\tau(t,x,D_x))u(t,x), \\ u_2(t,x) = \omega(x)\langle D_x\rangle_k u(t,x) - H(t,x,D_x)u_1, \end{cases}$$

and the operator H with the symbol $\sigma(H)(t,x,\xi)$ is such that

$$\sigma(H)(t,x,\xi) = -\frac{i}{2}\omega(x)\langle\xi\rangle_k \frac{\Big(1-\varphi\Big(t\Phi(x)\langle\xi\rangle_k/3\Big)\Big)}{\tau(t,x,\xi)}.$$

Note that by the definition of H, supp $\sigma(H) \cap$ supp $\sigma(a-\tilde{a}) = \emptyset$ and we have

$$\begin{aligned} &\sigma(2iH(t,x,D_x)\circ\tau(t,x,D_x)) \sim 0, \quad \text{in } Z_{int}(3), \\ &\sigma(2iH(t,x,D_x)\circ\tau(t,x,D_x)) = \omega(x)\langle\xi\rangle_k(1+\sigma(K_1)), \quad \text{in } Z_{ext}(3), \end{aligned}$$

where $\sigma(K_1) \in G^{-1,-1}\{0,0;\tilde{\gamma},\delta_1\}_{int,6}(\omega, g_{\Phi,k}) + G^{-1,-1}\{0,1,2,1;\tilde{\gamma},\delta_1\}_{ext,3}(\omega, g_{\Phi,k})$. Then, the equation $Pu = f$ is equivalent to the first order 2×2 system:

$$\begin{aligned} LU &= (\partial_t + \mathcal{D} + A_0 + A_1)U = F, \\ U(0,x) &= (f_2 + i\tau(0,x,D_x)f_1, \Phi(x)\langle D_x\rangle f_1)^T, \end{aligned} \tag{3.4.6}$$

where

$$\begin{aligned} F &= (f(t,x), -H(t,x,D_x)f(t,x))^T, \\ \mathcal{D} &= \operatorname{diag}(-i\tau(t,x,D_x), i\tau(t,x,D_x)), \\ A_0 &= \begin{pmatrix} B_0H & B_0 \\ -HB_0H & HB_0 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_1 & B_0 \\ -\mathcal{R}_3 & \mathcal{R}_2 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} B_1H & B_1 \\ B_2 & i[M,\tau]M^{-1} - HB_1 \end{pmatrix}. \end{aligned}$$

The operators M, M^{-1}, B_0 and B_1 are as follows

$$\begin{aligned}
&M = \omega(x)\langle D_x\rangle_k, \quad M^{-1} = \langle D_x\rangle_k^{-1}\omega(x)^{-1},\\
&B_0 = (a(t,x,D_x) - \tilde{a}(t,x,D_x))\langle D_x\rangle_k^{-1}\omega(x)^{-1},\\
&B_1 = (-i\partial_t\tau(t,x,D_x) + \tilde{a}(t,x,D_x) - \tau(t,x,D_x)^2 + b(t,x,D_x))\langle D_x\rangle_k^{-1}\omega(x)^{-1},\\
&B_2 = 2iH\tau - M + i[M,\tau]M^{-1}H + i[\tau,H] - HB_1H + \partial_t H.
\end{aligned}$$

By the definition of operator H, we have $B_0H = \mathcal{R}_1, HB_0 = \mathcal{R}_2,\ HB_0H = \mathcal{R}_3$ for $\mathcal{R}_j \in G^{-\infty,-\infty}(\omega, g_{\Phi,k}), j = 1,2,3,$ and the operator $2iH\tau - M$ is such that

$$\sigma(2iH\tau - M) = \begin{cases} -\omega(x)\langle\xi\rangle_k, & \text{in } Z_{int}(3),\\ \omega(x)\langle\xi\rangle_k\sigma(K_1), & \text{in } Z_{ext}(3). \end{cases}$$

The symbols of operators $\mathcal{D}, A_0$ and A_1 are such that

$$\left.\begin{aligned}
\sigma(\mathcal{D}) &\in G^{1,1}\{0,0;\tilde{\gamma},0\}_{int,2}(\omega,g_{\Phi,k}) + G^{1,1}\{0,0,1,1;\tilde{\gamma},\delta_1\}_{ext,1}(\omega,g_{\Phi,k})\\
\sigma(A_0) &\in G^{1,1}\{1,1;\tilde{\gamma},\delta_1\}_{int,2}(\omega,g_{\Phi,k}) + G^{-\infty,-\infty}\{0,0,0,0;\tilde{\gamma},0\}_{ext,3}(\omega,g_{\Phi,k}),\\
\sigma(A_1) &\in G^{1,1}\{0,0;\tilde{\gamma},0\}_{int,6}(\omega,g_{\Phi,k}) + G^{0,0}\{1,0,1,1;\tilde{\gamma},\delta\}_{ext,1}(\omega,g_{\Phi,k})\\
&\qquad + G^{0,0}\{0,1,2,1;\tilde{\gamma},\delta\}_{ext,3}(\omega,g_{\Phi,k})
\end{aligned}\right\} \tag{3.4.7}$$

and thus, by Propositions A.1.1 - A.1.2 and Remarks A.1.2 - A.1.3, for every $\varepsilon < 1-\delta$,

$$\left.\begin{aligned}
&t^{1-\varepsilon}\sigma(A_0(t)) \in C\left([0,T]; G^{1,1}(\omega, g_{\Phi,k})\right),\\
&t^{1-\varepsilon}\sigma(A_1(t)) \in C\left([0,T]; G^{\varepsilon',\varepsilon'}(\omega, g_{\Phi,k}^{(1,\delta),(1-\delta,0)})\right).
\end{aligned}\right\} \tag{3.4.8}$$

As in (3.4.4), one can define positive functions

$$\tilde{\psi}_0, \tilde{\psi}_1 \in L^1([0,T]; C^\infty(\mathbb{R}^n)) \cap C^1((0,T]; C^\infty(\mathbb{R}^n)), \text{ where}$$

$$\left.\begin{aligned}
&\tilde{\psi}_0(t,x,\xi) = C_0\varphi(t\Phi(x)\langle\xi\rangle_k/3)\tilde{\theta}(t)\omega(x)\langle\xi\rangle_k,\\
&\tilde{\psi}_1(t,x,\xi) = C_1\left(\varphi(t\Phi(x)\langle\xi\rangle_k/3)\tilde{\theta}(t)\omega(x)\langle\xi\rangle_k + (1-\varphi(t\Phi(x)\langle\xi\rangle_k))\frac{\tilde{\theta}(t)^{\frac{\tilde{\gamma}-1}{\tilde{\gamma}}}}{t}\right),
\end{aligned}\right\} \tag{3.4.9}$$

for an appropriate choice of $C_0, C_1 > 0$, satisfying the estimates

$$|\sigma(A_0)| \le \tilde{\psi}_0 \quad \text{and} \quad |\sigma(A_1)| \le \tilde{\psi}_1.$$

The function $\tilde{\psi} = \tilde{\psi}_0 + \tilde{\psi}_1$ satisfies

$$\begin{aligned}
&\int_0^T |\tilde{\psi}(t,x,\xi)|dt \le \kappa_{00}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{\tilde{\gamma}},\\
&\int_0^t |\partial_\xi^\alpha D_x^\beta\tilde{\psi}(r,x,\xi)|dr \le \kappa_{\alpha\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{\tilde{\gamma}}\chi_{int}(6),
\end{aligned} \tag{3.4.10}$$

for $|\alpha| + |\beta| > 0$.

3.4.3 Energy Estimate

In this section, we prove the estimate (3.1.11). Note that it is sufficient to consider the case $s = (0, 0)$ as the operator $\Phi(x)^{s_2}\langle D\rangle_k^{s_1} L\langle D\rangle_k^{-s_1}\Phi(x)^{-s_2}$, $s = (s_1, s_2)$, satisfies the same hypotheses as L.

In the following, we establish some lower bounds for the operator $\mathcal{D} + A_0 + A_1$. The symbol $d(t, x, \xi)$ of the operator $\mathcal{D}(t) + \mathcal{D}^*(t)$ is such that

$$d \in G^{0,0}\{0, 0; \tilde{\gamma}, 0\}_{int,2}(\omega, g_{\Phi,k}) + G^{0,0}\{0, 0, 1, 1; \tilde{\gamma}, \delta_1\}_{ext,1}(\omega, g_{\Phi,k}).$$

It follows from Proposition A.1.1 and Remark A.1.4 that $t^{1-\varepsilon}d \in C([0, T]; G^{0,0}(\omega, g_{\Phi,k})$. Thus

$$2\,\mathrm{Re}\langle\mathcal{D}U, U\rangle_{L^2} \geq -\frac{C}{t^{1-\varepsilon}}\langle U, U\rangle_{L^2}, \quad C > 0. \tag{3.4.11}$$

We perform a change of variable, which allows us to control lower order terms. We set

$$V_1(t, x) = e^{-\int_0^t \tilde{\psi}(r,x,D_x)dr}U(t, x), \tag{3.4.12}$$

where $\tilde{\psi}(t, x, \xi)$ is as in (3.4.9). From (A.1.6) and (A.1.7), we observe that the operator $e^{\pm\int_0^t \tilde{\psi}(r,x,D_x)dr}$ is a pseudodifferential operator whose order is arbitrarily small if $\tilde{\gamma} \in (0, 1)$ and finite if $\tilde{\gamma} = 1$. Applying Lemma A.1.7 to the identity operator we see that

$$\begin{aligned} e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}e^{-\int_0^t \tilde{\psi}(r,x,D_x)dr} &= I + K_2^{(1)}(t, x, D_x),\\ e^{-\int_0^t \tilde{\psi}(r,x,D_x)dr}e^{\int_0^t \tilde{\psi}(r,x,D_x)dr} &= I + K_2^{(2)}(t, x, D_x), \end{aligned} \tag{3.4.13}$$

where for every $\tilde{\varepsilon} \ll 1$ and $j = 1, 2$,

$$\begin{aligned}(\ln(1 + \Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}\sigma(K_2^{(j)}) \in &G^{(-1+\tilde{\varepsilon})e}\{0, 0; \tilde{\gamma}, 0\}_{int,6}(\Phi, g_{\Phi,k})\\ &+ G^{-e}\{0, 0, 0, 0; \tilde{\gamma}, 0\}_{ext,1}(\Phi, g_{\Phi,k}).\end{aligned}$$

By Proposition A.1.2, the symbol of operator $K_2^{(j)}$ is in $G^{(-1+\varepsilon)e}(\omega; g_{\Phi,k})$. We choose $k > k_1$ for large k_1 so that the operator norm of $K_2^{(j)}$, $j = 1, 2$ is strictly lesser than 1 and the existence of

$$(I + K_2^{(j)}(t, x, D_x))^{-1} = \sum_{l=0}^{\infty}(-1)^j K_2^{(j)}(t, x, D_x)^l, \quad j = 1, 2, \tag{3.4.14}$$

is guaranteed. The equation (3.4.12) implies that

$$\left.\begin{aligned} U(0, x) &= V_1(0, x),\\ \|U(t, \cdot)\|_{\Phi,k;(0,0),-\kappa_0,\tilde{\gamma}} &\leq 2\|V_1(t, \cdot)\|_{L^2}\ ,\ \kappa_0 > 0\ ,\ 0 < t \leq T,\\ U(t, x) &= (1 + K_2^{(1)}(t, x, D_x))^{-1}e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}V_1(t, x). \end{aligned}\right\} \tag{3.4.15}$$

Here κ_0 is same as $\kappa_{\alpha\beta}$ appearing in (3.4.10) for $\alpha = \beta = 0$. For L as in (3.4.6), we have

$$LU = (\partial_t + \mathcal{D} + A)(1 + K_2^{(1)}(t, x, D_x))^{-1}e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}V_1 = F,\ \ A = A_0 + A_1.$$

Note that

$$\begin{aligned}
&\partial_t(1+K_2^{(1)})^{-1}e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}V_1 \\
&\quad= \left(\partial_t(1+K_2^{(1)})^{-1}\right)e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}V_1 + (1+K_2^{(1)})^{-1}e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}\tilde{\psi}(t,x,D_x)V_1 \\
&\qquad+ (1+K_2^{(1)})^{-1}e^{\int_0^t \tilde{\psi}(r,x,D_x)dr}\partial_t V_1.
\end{aligned} \tag{3.4.16}$$

Observe the third term on the RHS of the above expression. To obtain a new first order pseudodifferential system in ∂_t, we first apply $e^{-\int_0^t \tilde{\psi}(r,x,D_x)dr}(1+K_2^{(1)})$ on the left of L. In the light of (3.4.13) and (3.4.16), the resulting operator will have a term of the form $(1+K_2^{(2)})\partial_t$. Hence, we apply $(1+K_2^{(2)})^{-1}$ on the resulting operator to obtain a first order pseudodifferential system, formally equivalent to the one in (3.4.6), $L_1V_1 = F_1$ where

$$\left.\begin{aligned}
&L_1 = \partial_t + \mathcal{D} + \tilde{\psi}I + A + R_1,\ A = A_0 + A_1, \\
&F_1 = (1+K_2^{(2)}(t,x,D_x))^{-1}e^{-\int_0^t \tilde{\psi}(r,x,D_x)dr}(1+K_2^{(1)}(t,x,D_x))F.
\end{aligned}\right\}$$

Here the operator $\mathcal{D}, A_0, A_1$ are as in (3.4.6). Noting (3.4.7) we apply Lemma A.1.7. This yields for an arbitrary small $\tilde{\varepsilon} > 0$,

$$\begin{aligned}
&(\ln(1+\Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}\sigma(R_1) \\
&\quad\in G^{\tilde{\varepsilon},1}\{1,1;\tilde{\gamma},\delta_1\}_{int,6}(\omega\Phi^{-1+\tilde{\varepsilon}}, g_{\Phi,k}) + G^{0,1}\{0,0,1,1;\tilde{\gamma},\delta\}_{ext,1}(\omega\Phi^{-1}, g_{\Phi,k}) \\
&\qquad+ G^{-1,1}\{1,0,1,1;\tilde{\gamma},\delta\}_{ext,1}(\Phi^{-1}, g_{\Phi,k}) + G^{-1,1}\{0,1,2,1;\tilde{\gamma},\delta\}_{ext,3}(\Phi^{-1}, g_{\Phi,k}).
\end{aligned}$$

Using the compensation procedure outlined in Remark A.1.4, one can show that

$$t^{1-\varepsilon}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}\sigma(R_1) \in C([0,T]; G^{0,0}(\Phi; g_{\Phi,k}^{(1,\delta),(1-\delta,0)}), \quad 0<\varepsilon<1-\delta.$$

For an appropriate choice of $C_0, C_1 > 0$ in the definition of $\tilde{\psi}$ as in (3.4.9-3.4.10), we observe that $\tilde{\psi}I + A$ satisfies

$$\begin{aligned}
&2\tilde{\psi}I + \sigma(A{+}A^*) \geq 0, \\
&t^{1-\varepsilon}(\tilde{\psi}I + \sigma(A)) \in C([0,T]; G^{\varepsilon',1}(\Phi^{\varepsilon'}, g_{\Phi,k}^{(1,\delta),(1-\delta,0)})).
\end{aligned}$$

Here $A = A_0 + A_1$ with A_0 and A_1 as in (3.4.6)-(3.4.8). We now apply sharp Gårding inequality (see [43, Theorem 18.6.14] to the operators $2\tilde{\psi}_0 I + A_0$ and $2\tilde{\psi}_1 I + A_1$ separately. The symbols of these operators are governed by the metrics $g_{\Phi,k}$ and $g_{\Phi,k}^{(1,\delta),(1-\delta,0)}$ respectively where the respective Planck functions are $h(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$ and $\tilde{h}(x,\xi) = \Phi(x)^{-1+\delta}\langle\xi\rangle_k^{-1+\delta}$. Notice that the symbol of A_0 has the weight function $\omega(x)\langle\xi\rangle_k$ while the Planck function of the governing metric is given by $h(x,\xi)$. Hence, for the application of sharp Gårding inequality, we need

$$\omega(x) \lesssim \Phi(x).$$

Ensuring this yields

$$2\,\mathrm{Re}\langle(\psi I + A)V_1, V_1\rangle_{L^2} \geq -Ct^{-1+\varepsilon}\langle V_1, V_1\rangle_{L^2}\ ,\ C>0. \tag{3.4.17}$$

As for the operator R_1, since the symbol $t^{1-\varepsilon}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}R_1$ is uniformly bounded, for a large choice of κ_1, the application of sharp Gårding inequality yields

$$2\,\mathrm{Re}\langle R_1V_1,V_1\rangle_{L^2}\geq -\frac{\kappa_1}{t^{1-\varepsilon}}\big(2\,\mathrm{Re}\langle(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}V_1,V_1\rangle_{L^2}+\|V_1\|_{L^2}\big). \tag{3.4.18}$$

We make a further change of variable

$$V_2(t,x)=e^{-\mu(t)(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}}V_1(t,x),\qquad \mu(t)=\kappa_1t^{\varepsilon}/\varepsilon, \tag{3.4.19}$$

where κ_1 is the constant as in (3.4.18). Let

$$e^{\pm\mu(t)(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}}e^{\mp\mu(t)(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}}=I+K_3^{(\pm)}(t,x,D_x),$$

where $K_3^{(\pm)}(t,x,\xi)\in C([0,T];G^{-1,-1}(\omega,g_{\Phi,k}))$. As in (3.4.14), we choose $k>k_2$, k_2 large, so that $(I+K_3^{(\pm)}(t,x,D_x))^{-1}$ exists. From now on we fix k such that $k>\max\{k_1,k_2\}$. Further, note that

$$\left.\begin{aligned}V_2(0,x)&=U(0,x),\\ \|U(t,\cdot)\|_{\Phi,k;(0,0),-\Lambda(t),\tilde{\gamma}}&\leq 2^{\mu(T)+1}\|V_2(t,\cdot)\|_{L^2},\quad \Lambda(t)=\kappa_0+\kappa_1t^{\varepsilon}/\varepsilon\ ,\ 0<t\leq T,\\ V_1(t,x)&=(1+K_3^{(+)}(t,x,D_x))^{-1}e^{\mu(t)(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}}V_2(t,x).\end{aligned}\right\} \tag{3.4.20}$$

This implies that $LU=F$ if and only if $L_2V_2=F_2$ where

$$\left.\begin{aligned}L_2&=\partial_t+\mathcal{D}+(\tilde{\psi}I+A)+(\kappa_1t^{-1+\varepsilon}(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}+R_1)+R_2\\ F_2&=(I+K_3^{(-)}(t,x,D_x))^{-1}e^{-\mu(t)(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}}(I+K_3^{(+)}(t,x,D_x))F_1\end{aligned}\right\} \tag{3.4.21}$$

and the operator R_2 is such that its symbol is in

$$G^{0,0}\{1,1;\tilde{\gamma},\delta_1\}_{int,6}(\Phi,g_{\Phi,k})+G^{0,0}\{1,0,1,1;\tilde{\gamma},\delta\}_{ext,1}(\Phi,g_{\Phi,k}),$$

in other words $t^{1-\varepsilon}R_2(t,x,\xi)\in C([0,T];G^{0,0}(\Phi;g_{\Phi,k}^{(1,\delta),(1-\delta,0)})$. From (3.4.11), (3.4.17), (3.4.18) and noting the fact that the operator $t^{1-\varepsilon}R_2$ is uniformly bounded in $L^2(\mathbb{R}^n)$ for $0\leq t\leq T$, it follows that

$$2\,\mathrm{Re}\langle\mathcal{K}V_2,V_2\rangle_{L^2}\geq-\frac{C}{t^{1-\varepsilon}}\langle V_2,V_2\rangle_{L^2},\quad C>0, \tag{3.4.22}$$

where $\mathcal{K}=\mathcal{D}+(\psi I+A)+(\kappa_1t^{-1+\varepsilon}(\ln(1+\Phi(x)\langle D_x\rangle_k))^{\tilde{\gamma}}+R_1)+R_2$. From (3.4.21) and (3.4.22), we have

$$\partial_t\|V_2\|_{L^2}^2\leq C(t^{-1+\varepsilon}\|V_2\|_{L^2}^2+\|F_2\|_{L^2}^2).$$

Considering the above inequality as a differential inequality, we apply Gronwalls lemma and obtain that

$$\|V_2(t,\cdot)\|_{L^2}^2\leq e^{Ct^{\varepsilon}/\varepsilon}\left(\|V_2(0,\cdot)\|_{L^2}^2+\int_0^t\|F_2(\tau,\cdot)\|_{L^2}d\tau\right),$$

for $t \in [0,T]$. In other words, from (3.4.20),

$$\|U(t,\cdot)\|^2_{\Phi,k;s,-\Lambda(t),\tilde{\gamma}} \leq C' e^{CT^\varepsilon/\varepsilon} \left(\|U(0,\cdot)\|^2_{\Phi,k;s} + \int_0^t \|F(\tau,\cdot)\|_{\Phi,k;s,-\Lambda(\tau),\tilde{\gamma}} \, d\tau \right).$$

Returning to our original solution $u = u(t,x)$, we obtain that

$$\sum_{j=0}^{1} \|\partial_t^j u(t,\cdot)\|_{\Phi,k;s+(1-j)e,-\Lambda(t),\tilde{\gamma}} \\ \leq C' e^{CT^\varepsilon/\varepsilon} \left(\sum_{j=1}^{2} \|f_j\|_{\Phi;s+(2-j)e} + \int_0^t \|f(\tau,\cdot)\|_{\Phi,k;s,-\Lambda(\tau),\tilde{\gamma}} \, d\tau \right).$$

This means that the original problem (3.1.7) is well-posed for $u = u(t,x)$, with

$$u \in C\Big([0,T]; \mathcal{H}^{s+e,-\Lambda(t),\tilde{\gamma}}_{\Phi,k}\Big) \cap C^1\Big([0,T]; \mathcal{H}^{s,-\Lambda(t),\tilde{\gamma}}_{\Phi,k}\Big).$$

This shows that when (3.1.10) is satisfied one has an arbitrarily small loss if $\tilde{\gamma} \in (0,1)$ and finite loss if $\tilde{\gamma} = 1$. When (3.1.10*) is satisfied instead of (3.1.10), the proof follows in similar lines. Note that the condition (3.1.10*) suggests that the the coefficients are bounded in t. This implies that the majorizing functions in (3.4.9) are zero order symbols in both x and ξ as $\delta_3 \in [0,1)$ in (3.1.10*). Implying that the elliptic operators in the changes of variable as in (3.4.12) and (3.4.19) are order zero pseudodifferential operators and hence one can take $\Lambda(t) = 0$ in the above discussion. This suggests that there is no loss in regularity index. This concludes the proof.

3.5 Anisotropic Cone Condition

Existence and uniqueness follow from the a priori estimate established in the previous section. It now remains to prove the existence of cone of dependence.

We note here that the L^1 integrability of the singularity plays a crucial role in arriving at the finite propagation speed. The implications of the discussion in [68, Section 2.3 & 2.5] to the global setting suggest that if the Cauchy data in (3.1.7) is such that $f \equiv 0$ and f_1, f_2 are supported in the ball $|x| \leq R$, then the solution to Cauchy problem (3.1.7) is supported in the ball $|x| \leq R + c^*\omega(x)\tilde{\theta}(t)t$. Note that the support of the solution increases as $|x|$ increases since $\omega(x)$ is monotone increasing function of $|x|$. Recall that $\tilde{\theta}(t) = (\ln(1+1/t))^{\tilde{\gamma}}$. The quantity $t\tilde{\theta}(t)$ is bounded in $[0,T]$. The constant c^* is such that the quantity $c^*\omega(x)\tilde{\theta}(t)$ dominates the characteristic roots, i.e.,

$$c^* = \sup\left\{ \sqrt{a(t,x,\xi)}\,\omega(x)^{-1}\tilde{\theta}(t)^{-1} : (t,x,\xi) \in [0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi,\ |\xi| = 1 \right\}. \tag{3.5.1}$$

Here $a(t,x,\xi)$ is as in (3.1.8).

In the following we prove the cone condition for the Cauchy problem (3.1.7) as in [76, Section 3.11]. Let $K(x^0,t^0)$ denote the cone with the vertex (x^0,t^0):

$$K(x^0,t^0) = \{(t,x) \in [0,T] \times \mathbb{R}^n : |x - x^0| \leq c^*\omega(x)\tilde{\theta}(t^0 - t)(t^0 - t)\}.$$

Observe that the slope of the cone is anisotropic, that is, it varies with both x and t.

Proposition 3.5.1. *The Cauchy problem (3.1.7) has a cone dependence, that is, if*

$$f\big|_{K(x^0,t^0)} = 0, \quad f_i\big|_{K(x^0,t^0)\cap\{t=0\}} = 0, \; i = 1, 2, \tag{3.5.2}$$

then

$$u\big|_{K(x^0,t^0)} = 0. \tag{3.5.3}$$

Proof. Consider $t^0 > 0$, $c^* > 0$ and assume that (3.5.2) holds. We define a set of operators $P_\varepsilon(t, x, \partial_t, D_x), 0 \le \varepsilon \le \varepsilon_0$ by means of the operator $P(t, x, \partial_t, D_x)$ in (3.1.7) as follows

$$P_\varepsilon(t, x, \partial_t, D_x) = P(t + \varepsilon, x, \partial_t, D_x), \; t \in [0, T - \varepsilon_0], x \in \mathbb{R}^n,$$

and $\varepsilon_0 < T - t^0$, for a fixed and sufficiently small ε_0. For these operators we consider Cauchy problems

$$\begin{aligned} P_\varepsilon v_\varepsilon &= f, & & t \in [0, T - \varepsilon_0], \; x \in \mathbb{R}^n, \\ \partial_t^{k-1} v_\varepsilon(0, x) &= f_k(x), & & k = 1, 2. \end{aligned}$$

Note that $v_\varepsilon(t, x) = 0$ in $K(x^0, t^0)$ and v_ε satisfies an a priori estimate (3.1.11) for all $t \in [0, T - \varepsilon_0]$. Further, we have

$$\begin{aligned} P_{\varepsilon_1}(v_{\varepsilon_1} - v_{\varepsilon_2}) &= (P_{\varepsilon_2} - P_{\varepsilon_1})v_{\varepsilon_2}, & & t \in [0, T - \varepsilon_0], \; x \in \mathbb{R}^n, \\ \partial_t^{k-1}(v_{\varepsilon_1} - v_{\varepsilon_2})(0, x) &= 0, & & k = 1, 2. \end{aligned}$$

Since our operator is of second order, for the sake of simplicity we denote $b_j(t, x)$, the coefficients of lower order terms, as $a_{0,j}(t, x), 1 \le j \le n$, and $b_{n+1}(t, x)$ as $a_{0,0}(t, x)$. Let $a_{i,0}(t, x) = 0, \; 1 \le i \le n$. Substituting $s - e$ for s in the a priori estimate, we obtain

$$\begin{aligned} &\sum_{j=0}^{1} \|\partial_t^j (v_{\varepsilon_1} - v_{\varepsilon_2})(t, \cdot)\|_{\Phi,k;s-(j+\Lambda(t))e} \\ &\le C \int_0^t \|(P_{\varepsilon_2} - P_{\varepsilon_1})v_{\varepsilon_2}(\tau, \cdot)\|_{\Phi,k;s-e-\Lambda(\tau)e} \, d\tau \\ &\le C \int_0^t \sum_{i,j=0}^{n} \|(a_{i,j}(\tau + \varepsilon_1, x) - a_{i,j}(\tau + \varepsilon_2, x))D_{ij}v_{\varepsilon_2}(\tau, \cdot)\|_{\Phi,k;s-e-\Lambda(\tau)e} \, d\tau, \end{aligned} \tag{3.5.4}$$

where $D_{00} = I, D_{i0} = 0, i \neq 0, D_{0j} = \partial_{x_j}, j \neq 0$ and $D_{ij} = \partial_{x_i}\partial_{x_j}, i, j \neq 0$. Using the Taylor series approximation in τ variable, we have

$$\begin{aligned} |a_{i,j}(\tau + \varepsilon_1, x) - a_{i,j}(\tau + \varepsilon_2, x)| &= \Big| \int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} (\partial_t a_{i,j})(r, x) dr \Big| \\ &\le \omega(x)^2 \Big| \int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} \frac{\tilde{\theta}(t)^{1-\frac{1}{\tilde{\gamma}}}}{r} dr \Big| \\ &\le \omega(x)^2 |E(\tau, \varepsilon_1, \varepsilon_2)|, \end{aligned} \tag{3.5.5}$$

where

$$E(\tau, \varepsilon_1, \varepsilon_2) = \left(\ln\left(1 + \frac{\varepsilon_1 - \varepsilon_2}{\tau + \varepsilon_2}\right)\right)^{\tilde{\gamma}}.$$

Note that $\omega(x) \lesssim \Phi(x)$ and $E(\tau, \varepsilon, \varepsilon) = 0$. Then right-hand side of the inequality in (3.5.4) is dominated by

$$C \int_0^t |E(\tau, \varepsilon_1, \varepsilon_2)| \|v_{\varepsilon_2}(\tau, \cdot)\|_{\Phi,k;s+(1-\Lambda(\tau))e}\, d\tau,$$

where C is independent of ε. By definition, E is L^1-integrable in τ.

The sequence v_{ε_k}, $k = 1, 2, \dots$ corresponding to the sequence $\varepsilon_k \to 0$ is in the space

$$C\Big([0, T^*]; H^{s-\nu e,}_{\Phi,k}\Big) \bigcap C^1\Big([0, T^*]; H^{s-e-\nu e,}_{\Phi,k}\Big), \quad T^* > 0,$$

for an arbitrarily small $\nu > 0$ and $u = \lim_{k\to\infty} v_{\varepsilon_k}$ in the above space and hence, in $\mathcal{D}'(K(x^0, t^0))$. In particular,

$$\langle u, \varphi \rangle = \lim_{k\to\infty} \langle v_{\varepsilon_k}, \varphi \rangle = 0, \ \forall \varphi \in \mathcal{D}(K(x^0, t^0))$$

gives (3.5.3) and completes the theorem. □

3.6 Existence of Counterexample

Let us consider a Cauchy problem of the form

$$\begin{aligned} &\partial_t^2 u(t,x) + c(t)A(x, D_x)u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \\ &u(0,x) = 0, \quad \partial_t u(0,x) = f(x), \end{aligned} \tag{3.6.1}$$

where $A(x, D_x) = \langle x\rangle(I - \triangle_x)\langle x\rangle$ is a G-elliptic (see Definition 2.1.1), positive, self-adjoint operator with the domain $D(A) = \{u \in L^2(\mathbb{R}^n) : Au \in L^2(\mathbb{R}^n)\}$ and the propagation speed $c(t)$ is in $C([0,T]) \cap C^1((0,T])$. In order to show that there exists a propagation speed $c(t)$ for which the Cauchy problem (3.6.1) has infinite loss of regularity (decay and derivatives), we extend the techniques developed by Ghisi and Gobbino [35, Section 4] to a global setting.

Let us first define the following special class propagation speeds.

Definition 3.6.1. *We denote $\mathcal{C}^{(1)}(\mu_1, \mu_2, \theta)$ as the set of functions $c \in C([0,T]) \cap C^1((0,T])$ that satisfy the following growth estimates*

$$0 < \mu_1 \le c(t) \le \mu_2, \quad t \in [0,T], \tag{3.6.2}$$

$$|c'(t)| \le C\frac{\theta(t)}{t}, \quad t \in (0,T], \tag{3.6.3}$$

for a positive and monotone decreasing function $\theta : (0,T] \to (0, +\infty)$ satisfying

$$\lim_{t\to 0^+} \theta(t) = +\infty. \tag{3.6.4}$$

The set $\mathcal{C}^{(1)}(\mu_1, \mu_2, \theta)$ is a complete metric space with respect to the metric

$$d_1(c_1, c_2) := \sup_{t\in(0,T)} |c_1(t) - c_2(t)| + \sup_{t\in(0,T)} \left\{ \frac{t^2}{\theta(t)} |c_1'(t) - c_2'(t)| \right\}.$$

A sequence c_n converges to some c_∞ with respect to the metric d_1 if and only if $c_n \to c_\infty$ uniformly in $[0,T]$ and for every $\tau \in (0,T)$, $c_n' \to c_\infty'$ uniformly in $[\tau, T]$.

Definition 3.6.2. *We call $\mathcal{D}(\mu_1, \mu_2)$ the set of functions $c : [0,T] \to [\mu_1, \mu_2]$ for which there exists two real numbers $T_1 \in (0,T)$ and $\mu_3 \in (\mu_1, \mu_2)$ such that $c(t) = \mu_3$ for every $t \in [0, T_1]$.*

For the sake of simplicity, let us denote $\mathcal{C}^{(1)}(\mu_1, \mu_2, \theta)$ and $\mathcal{D}(\mu_1, \mu_2)$ by $\mathcal{C}^{(1)}$ and $\mathcal{D}$, respectively.

Remark 3.6.1. *From [35, Proposition 4.7], we have that $\mathcal{D} \cap \mathcal{C}^{(1)}$ is dense in $\mathcal{C}^{(1)}$. The weight factor $\frac{t^2}{\theta(t)}$ appearing in the definition of the metric d_1 plays a crucial role in proving the above denseness result.*

The main aim of this section is to prove the following result.

Theorem 3.6.1. *The interior of the set of all $c \in \mathcal{C}^{(1)}(\mu_1, \mu_2, \theta)$ for which the Cauchy problem (3.6.1) exhibits an infinite loss of regularity is nonempty.*

Since the operator A is positive and G-elliptic, and the symbol

$$\sigma(A^\kappa) \sim \langle x \rangle^{2\kappa} \langle \xi \rangle^{2\kappa} + \text{ lower order terms}, \quad \text{for } \kappa \in \mathbb{R},$$

the Sobolev spaces associated to the Cauchy problem (3.6.1) are $H^{2\kappa,2\kappa}_{\langle x \rangle}$. We characterize the Sobolev spaces $H^{2m,2m}_{\langle x \rangle}(\mathbb{R}^n), m \in \mathbb{Z}$, using the spectral theorem [58, Theorem 4.2.9] for pseudodifferential operators on $\mathbb{R}^n$. The theorem guarantees the existence of an orthonormal basis $(e_i(x))_{i=1}^\infty, e_i \in \mathcal{S}(\mathbb{R}^n)$, of $L^2(\mathbb{R}^n)$ and a nondecreasing sequence $(\lambda_i)_{i=1}^\infty$ of nonnegative real numbers diverging to $+\infty$ such that

$$Ae_i(x) = \lambda_i^2 e_i(x). \tag{3.6.5}$$

Using λ_is we identify $v(x) \in H^{2m,2m}_{\langle x \rangle}(\mathbb{R}^n)$ with a sequence (v_i) in weighted ℓ^2, where $v_i = \langle v, e_i \rangle_{L^2}$. One can prove the following proposition using Riesz representation theorem showing the correspondence between $H^{2m,2m}_{\langle x \rangle}(\mathbb{R}^n)$ and a weighted ℓ^2 space.

Proposition 3.6.2. *Let (v_i) be a sequence of real numbers and $m \in \mathbb{Z}$. Then*

$$\sum_{i=1}^\infty v_i e_i(x) \in H^{2m,2m}_{\langle x \rangle}(\mathbb{R}^n) \quad \textit{if and only if} \quad \sum_{i=1}^\infty \lambda_i^{2m} v_i^2 < +\infty.$$

Proof. Let m be a positive integer and $v = \sum_{i=1}^\infty v_i e_i(x)$. Suppose $v \in H^{2m,2m}_{\langle x \rangle}$. This implies that

$$\|A^m v\|_{L^2} = \left\| A^m \sum_{i=1}^\infty v_i e_i(x) \right\|_{L^2} = \left\| \sum_{i=1}^\infty v_i \lambda_i^\alpha e_i(x) \right\|_{L^2} = \sum_{i=1}^\infty v_i^2 \lambda_i^{2\alpha} < +\infty.$$

This is yields the desired result for positive m.

Now, let $w = \sum_{i=1}^{\infty} w_i e_i(x)$ be such that $\sum_{i=1}^{\infty} \lambda_i^{-2m} w_i^2 < +\infty$. Then,

$$\langle w, v\rangle = \sum_{i=1}^{\infty} w_i v_i \le \left(\sum_{i=1}^{\infty} \lambda_i^{-2\alpha} w_i^2\right)^{1/2} \left(\sum_{i=1}^{\infty} \lambda_i^{2\alpha} v_i^2\right)^{1/2} < +\infty.$$

Implying $w \in H_{\langle x\rangle}^{-2m,-2m}$ as $v \in H_{\langle x\rangle}^{2m,2m}$.

Conversely, let $w \in H_{\langle x\rangle}^{-2m,-2m}$. As $H_{\langle x\rangle}^{-2m,-2m}$ is dual of $H_{\langle x\rangle}^{2m,2m}$, the Riesz representation theorem shows that there is a $u = \sum_i u_i e_i(x) \in H_{\langle x\rangle}^{2m,2m}$ such that

$$\langle w, v\rangle = \langle u, v\rangle_{H_{\langle x\rangle}^{2m,2m}} = \langle A^m u, A^m v\rangle_{L^2} = \sum_{i=1}^{\infty} \lambda_i^{4m} u_i v_i.$$

Implying $w = \sum_{i=1}^{\infty} w_i e_i(x) = \sum_{i=1}^{\infty} \lambda_i^{4m} u_i e_i(x)$. Since $v \in H_{\langle x\rangle}^{2m,2m}$, it follows that

$$\sum_{i=1}^{\infty} \lambda_i^{-2m} w_i^2 = \sum_{i=1}^{\infty} \lambda_i^{2m} u_i^2 < +\infty,$$

as claimed. □

The solution to (3.6.1) is $u(t,x) = \sum_{i=1}^{\infty} u_i(t) e_i(x)$ where the functions $u_i(t)$ satisfy the decoupled system of ODEs

$$\begin{aligned} &u_i''(t) + c(t)\lambda_i^2 u_i(t) = 0, \quad i \in \mathbb{N},\ t \in [0,T], \\ &u_i(0) = 0, \quad u_i'(0) = f_i, \end{aligned} \tag{3.6.6}$$

for $\partial_t u(0,x) = f(x) = \sum_{i=1}^{\infty} f_i e_i(x)$.

Definition 3.6.3. *(Infinite loss of regularity) We say that the solution to (3.6.1) experiences infinite loss of regularity if the initial velocity $f \in H_{\langle x\rangle}^{2m,2m}$ for all $m \in \mathbb{Z}^+$ but $(u, \partial_t u) \notin H_{\langle x\rangle}^{-2\tilde{m}+1,-2\tilde{m}+1} \times H_{\langle x\rangle}^{-2\tilde{m},-2\tilde{m}}$ for any $\tilde{m} \in \mathbb{Z}^+$ and for $t \in (0,T]$.*

Following the terminology of Ghisi and Gobbino [35], we now introduce special classes of propagation speeds: universal and asymptotic activators. Let $\phi : (0,+\infty) \to (0,+\infty)$ be a function.

Definition 3.6.4. *A universal activator of the sequence (λ_i) with rate ϕ is a propagation speed $c \in L^1((0,T))$ such that the corresponding sequence $(u_i(t))$ of solutions to*

$$u_i''(t) + c(t)\lambda_i^2 u_i(t) = 0, \quad u_i(0) = 0, \quad u_i'(0) = 1, \tag{3.6.7}$$

satisfies

$$\limsup_{i\to+\infty} \left(|u_i'(t)|^2 + \lambda_i^2 |u_i(t)|^2\right) \exp\left(-\phi(\lambda_i)\right) \ge 1, \quad \forall t \in (0,T]. \tag{3.6.8}$$

Then the solution u to problem (3.6.1) is given by

$$u(t,x) = \sum_{i=1}^{\infty} f_i u_i(t) e_i(x).$$

Definition 3.6.5. *A family of asymptotic activators with rate ϕ is a family of propagation speeds $\{c_\lambda(t)\} \subseteq L^1((0,T))$ with the property that, for every $\delta \in (0,T)$, there exist two positive constants M_δ and λ_δ such that the corresponding family $\{u_\lambda(t)\}$ of solutions to*

$$u''_\lambda(t) + c_\lambda(t)\lambda^2 u_\lambda(t) = 0, \quad u_\lambda(0) = 0, \quad u'_\lambda(0) = 1, \tag{3.6.9}$$

satisfies

$$|u'_\lambda(t)|^2 + \lambda^2 |u_\lambda(t)|^2 \geq M_\delta \exp(2\phi(\lambda)), \quad \forall t \in [\delta, T], \quad \forall \lambda \geq \lambda_\delta. \tag{3.6.10}$$

One can note that (3.6.8) is a qualitative statement which appears to be weaker than the quantitative estimate (3.6.10). But a careful observation shows that (3.6.8) is stronger as it concerns the family of equations (3.6.7), where the propagation speed is the same for every i, while (3.6.10) concerns the family of equations (3.6.9), where the propagation speed depends on λ. Following result by Ghisi and Gobbino establishes a crucial connection between universal activators and infinite loss of regularity.

Proposition 3.6.3. *(Ghisi & Gobbino [35]) By considering a subsequence of $(\lambda_i)_{i=1}^\infty$ (as in (3.6.5)) if required, let us assume, without loss of generality that*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < +\infty.$$

Let $c \in L^1((0,T])$ be a universal activator of the sequence (λ_i) with the rate ϕ. If ϕ is such that

$$\lim_{\rho \to +\infty} \frac{\phi(\rho)}{\ln \rho} = +\infty, \tag{3.6.11}$$

then the solutions to problem (3.6.1) exhibit an infinite loss of regularity according to Definition 3.6.3.

The following result establishes a passage from asymptotic to universal activators using the Baire category theorem.

Proposition 3.6.4. *(Ghisi & Gobbino [35]) Let $\phi : (0,+\infty) \to (0,+\infty)$ be a function such that $\phi(\rho) \to +\infty$ as $\rho \to +\infty$. Let $\mathcal{C} \subseteq L^1((0,T_0))$ be a subset that is a complete metric space with respect to some metric $d_{\mathcal{C}}$ with the property that convergence with respect to $d_{\mathcal{C}}$ implies convergence in $L^1((0,T])$. Let there be a subset $\mathcal{D} \subseteq \mathcal{C}$, dense with respect to the metric $d_{\mathcal{C}}$, such that for every $c \in \mathcal{D}$ there exists a family of asymptotic activators $(c_\lambda) \subseteq \mathcal{C}$ with rate ϕ such that $c_\lambda \to c$, always with respect to the metric $d_{\mathcal{C}}$. Then, for every unbounded sequence (λ_i) of positive real numbers, the set of elements in $\mathcal{C}$ that are universal activators of the sequence (λ_i) with rate ϕ is nonempty and residual in $\mathcal{C}$.*

The outline of the proof of Theorem 3.6.1 is as follows. Let $\mathcal{C}$ denote the set $\mathcal{C}^{(1)}$ and $d_{\mathcal{C}}$ denote the respective metric d_1. Due to the denseness of the set of initially constant functions, $\mathcal{D}$ in $\mathcal{C}$, for every $c \in \mathcal{D}$ there exists a family of asymptotic activators $(c_\lambda) \subseteq \mathcal{C}$ with rate ϕ such that $c_\lambda \to c$ with respect to $d_{\mathcal{C}}$. The existence of families of asymptotic activators converging to elements of a dense set implies the existence of a residual set of universal activators. Since the problem (3.6.1) exhibits an infinite loss of regularity whenever $c(t)$ is a universal activator, construction of couterexample amounts to the existence of such asymptotic activators. Once the asymptotic activators are constructed and if ϕ is such that $\phi(\rho) \to +\infty$ as $\rho \to +\infty$, then Proposition 3.6.4 guarantees that the set of elements in $\mathcal{C}$ that are universal activators of the sequence (λ_i) with rate ϕ is residual in $\mathcal{C}$. In addition, if the function ϕ satisfies (3.6.11), then by Proposition 3.6.3 we can show that for each of the universal activator $c(t)$ of sequence (λ_n) with rate ϕ the solution to problem (3.6.1) exhibits infinite loss of regularity. Thus, we are left with the construction of asymptotic activators with rate ϕ satisfying (3.6.11).

Proof. (Proof of Theorem 3.6.1) We consider T_1 and γ such that $0 < T_1 < T$ and $0 < \mu_1 < \gamma^2 < \mu_2$, and define a initially constant speed $c_* : [0, T] \to [\mu_1, \mu_2]$ such that

$$c_*(t) = \gamma^2, \quad \forall t \in [0, T_1] .$$

For every large enough real number λ, let a_λ and b_λ be real numbers such that

$$a_\lambda := \frac{2\pi}{\gamma\lambda} \left\lfloor \lambda^{1/4} \right\rfloor, \qquad b_\lambda := \frac{2\pi}{\gamma\lambda} \left\lfloor \lambda^{1/2} \right\rfloor .$$

where $\lfloor \alpha \rfloor$ stands for integer part of a real number α. Observe that

$$0 < a_\lambda < 2a_\lambda < \frac{b_\lambda}{2} < b_\lambda < T_1, \qquad \frac{\gamma\lambda a_\lambda}{2\pi} \in \mathbb{N} \quad \text{and} \quad \frac{\gamma\lambda b_\lambda}{2\pi} \in \mathbb{N}. \tag{3.6.12}$$

Let us choose a cutoff function $\tilde{\nu} : \mathbb{R} \to \mathbb{R}$ of class C^∞ such that $0 \le \tilde{\nu}(r) \le 1$, $\tilde{\nu}(r) = 0$ for $r \le 0$ and $\tilde{\nu}(r) = 1$ for $r \ge 1$. Setting $\theta_\lambda := \min\{\theta(b_\lambda), \ln \lambda\}$, we define $\varepsilon_\lambda : [0, T] \to \mathbb{R}$ as

$$\varepsilon_\lambda(t) := \begin{cases} 0 & \text{if } t \in [0, a_\lambda] \cup [b_\lambda, T_0] \\ \frac{\theta_\lambda}{t} & \text{if } t \in [2a_\lambda, b_\lambda/2] \\ \frac{\theta_\lambda}{t} \cdot \tilde{\nu}\left(\frac{t - a_\lambda}{a_\lambda}\right) & \text{if } t \in [a_\lambda, 2a_\lambda] \\ \frac{\theta_\lambda}{t} \cdot \tilde{\nu}\left(\frac{2(b_\lambda - t)}{b_\lambda}\right) & \text{if } t \in [b_\lambda/2, b_\lambda] \end{cases} \tag{3.6.13}$$

Using the functions $c_*(t)$ and $\varepsilon_\lambda(t)$ we define $c_\lambda : [0, T] \to \mathbb{R}$ as

$$c_\lambda(t) := c_*(t) - \frac{\varepsilon_\lambda(t)}{4\gamma\lambda} \sin(2\gamma\lambda t) - \frac{\varepsilon'_\lambda(t)}{8\gamma^2\lambda^2} \sin^2(\gamma\lambda t) - \frac{\varepsilon_\lambda(t)^2}{64\gamma^4\lambda^2} \sin^4(\gamma\lambda t). \tag{3.6.14}$$

By Propositions 4.8 and 4.9 in [35], $(c_\lambda(t))$ is a family of asymptotic activators with rate

$$\phi(\lambda) := \frac{\theta_\lambda}{32\gamma^2} \ln\left(\frac{\left\lfloor \lambda^{1/2} \right\rfloor}{\left\lfloor \lambda^{1/4} \right\rfloor}\right),$$

and $d_1(c_\lambda, c_*) \to 0$ as $\lambda \to +\infty$. Since c_* is a generic element of a dense subset, we see that these universal activators cause an infinite loss of regularity.

□

3.7 Discussion

In this chapter we have shown that when the singularity is at most logarithmic, the solution experiences at most finite loss in regularity index. We now comment on the optimality of the loss.

Is the logarithmic blow-up ($q = \tilde{\gamma} = 1, p = 0$ in Definition 1.2.3) a threshold for the finite loss? Yes. In fact, in Theorem 3.6.1 we have shown through an example that when the blow-up rate of the first t-derivative exceeds $O(t^{-1})$, one indeed encounters infinite loss of regularity. The techniques used in this chapter in fact suggest that the loss of infinite order is quite expected.

Suppose that the propagation speed $c(t)$ satisfies the estimate (3.6.3), then the following observation

$$(\ln(1/t))^{-1} \int_t^T |c'(t)| dt \to +\infty \quad \text{as} \quad t \to 0^+,$$

along with the region definitions given in Section 3.2 imply that the averaged behavior of the majorizing function $\tilde{\psi}$ in (3.4.9) is given by

$$(\ln(1 + \Phi(x)\langle\xi\rangle_k))^{-1} \int_0^T |\tilde{\psi}(t, x, \xi)| dt \to +\infty \quad \text{as} \quad |x| + |\xi| \to \infty.$$

This suggests that the function spaces involved in the change of variable (3.4.12) are of exponential order in both x and D_x. For example, if $\theta(t) = \ln(1/t)$, then the function spaces are of the form

$$\left\{ v \in L^2(\mathbb{R}^n) : e^{\kappa(\ln(1+\Phi(x)\langle D_x\rangle_k)^2} u \in L^2(\mathbb{R}^n) \right\},$$

and the loss is quantified in these spaces. Thus, knowing the nature of $\theta(t)$ allows one to quantify the infinite loss of regularity.

Chapter 4

Oscillations and Strong Blow-up: $q = 1$ Case

> When a problem of partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable inequality and much of our new knowledge is, in fact, essentially contained in such inequalities.
>
> — Lars Gårding

The study of global well-posedness and regularity issues in case of oscillatory coefficients presents new difficulties from the point of view of the associated pseudodifferential calculus and energy estimates. The study is complete only in the case of at most slow oscillations in time and $B^\infty(\mathbb{R}^n)$ regularity in space, see [69]. Though certain special cases of fast oscillations are studied in [50], [16] and [75] via the construction of a parametrix, Colombini, Del Santo and Reissig remark in [16] that the case of fast oscillations remains still open, let alone the case of very fast oscillations. They also show that one cannot expect C^∞ well-posedness in case of very fast oscillations.

In this chapter, we settle the well-posedness issue for the case of oscillatory behavior in time and at most polynomial growth in space using the method of energy estimates. In order to show the generality of our methodolgy, we allow the coefficients to be either oscillatory or blowing-up near $t = 0$. Since the case of at most logarithmic blow-up is already treated in the previous chapter, we consider the case of strong blow-up in this chapter. We construct a loss operator depending on the behavior in x and the singularity in t. The order of this operator determines the quantity of loss in the regularity index.

Singular Behavior	**Loss of Regularity Index**
Very Slow Oscillations	Zero
Slow Oscillations	Arbitrarily small
Fast Oscillations	Finite
Very Fast Oscillations ($q = 1$)	Infinite
Strong Blow-up	Infinite

Table 4.1: Quantity of loss in the regularity index depending on the singular behavior

4.1 Introduction and Statement of Main Results

Let us consider the prototypical Cauchy problem:

$$\left.\begin{array}{l}\partial_t^2 u - a(t,x)\Delta_x u = 0, \quad (t,x) \in [0,T]\times\mathbb{R}^n,\\ u(0,x) = u_1(x), \quad \partial_t u(0,x) = u_2(x),\end{array}\right\} \tag{4.1.1}$$

where the coefficient $a(t,x)$ is in $C^2((0,T];C^\infty(\mathbb{R}^n))$ and satisfies the following estimates

$$a(t,x) \geq C_0\,\omega(x)^2, \tag{4.1.2}$$

$$|\partial_x^\beta a(t,x)| \leq C_\beta^{(1)}\,\tilde{\theta}(t)\,\omega(x)^2\Phi(x)^{-|\beta|}, \tag{4.1.3}$$

$$|\partial_x^\beta \partial_t a(t,x)| \leq C_\beta^{(2)}\,\frac{\theta(t)}{t}\,\omega(x)^2\Phi(x)^{-|\beta|}, \tag{4.1.4}$$

$$|\partial_x^\beta \partial_t^2 a(t,x)| \leq C_\beta^{(3)}\left(\frac{\theta(t)}{t}\right)^2 e^{\psi(t)}\omega(x)^2\Phi(x)^{-|\beta|}, \tag{4.1.5}$$

where $(t,x)\in(0,T]\times\mathbb{R}^n$, $\beta\in\mathbb{N}_0^n$, $C_0, C_\beta^{(j)} > 0, j = 1,2,3$. Here $\tilde{\theta},\theta,\psi : (0,+\infty)\to(0,+\infty)$ are some positive nonincreasing smooth functions such that $\tilde{\theta}(t),\theta(t),\psi(t)\geq 1$.

Remark 4.1.1. *From (4.1.4), we have the following estimate*

$$|\partial_x^\beta a(T,x) - \partial_x^\beta a(t,x)| \leq \int_t^T |\partial_x^\beta\partial_s a(s,x)|ds \leq C_\beta\omega(x)^2\Phi(x)^{-|\beta|}\int_t^T \frac{\theta(s)}{s}ds.$$

Since, with respect to the t-variable, $\partial_x^\beta a(t,x)$ is in $C^1([T_0,T])$ for any $T_0>0$, we have $|\partial_x^\beta a(T,x)| \leq C\omega(x)^2\Phi(x)^{-|\beta|}$. Implying

$$|\partial_x^\beta a(t,x)| \leq C_\beta\omega(x)^2\Phi(x)^{-|\beta|}\int_t^T\frac{\theta(s)}{s}ds.$$

Hence, we define $\tilde{\theta}(t)$ as

$$\tilde{\theta}(t) = \begin{cases} C\int_t^T \frac{\theta(s)}{s}ds, & \text{when } a(t,\cdot) \text{ is unbounded}\\ 1, & \text{otherwise,}\end{cases} \tag{4.1.6}$$

for some $C>0$. The definition (4.1.6) of $\tilde{\theta}$ suggests that it grows at least as $|\ln t|$ near $t=0$ and $|\tilde{\theta}(t)'| = \frac{\theta(t)}{t}$ when the coefficients are unbounded in t.

Below are certain examples of $a(t,x)$ satisfying (4.1.2) - (4.1.5). Let $n=1$, $\kappa_1,\kappa_2\in[0,1]$ and T be sufficiently small.

Example 4.1.1. $a(t,x) = 4\langle x\rangle^{2\kappa_1}\left(2+\sin\left(\langle x\rangle^{1-\kappa_2}\right)\right)c(t)$ *where*

$$c(t) = 2 + e^{-|\ln t|^{1-\alpha}}\sin\left(|\ln t|^{2\alpha}e^{|\ln t|^{1-\alpha}}\right), \qquad \text{for some } \alpha\in(0,1).$$

Here $\omega(x) = 2\langle x\rangle^{\kappa_1}, \Phi(x) = \langle x\rangle^{\kappa_2}, \tilde{\theta}(t) = 3, \theta(t) = |\ln t|^\alpha$, and $\psi(t) = |\ln t|^{1-\alpha}$.

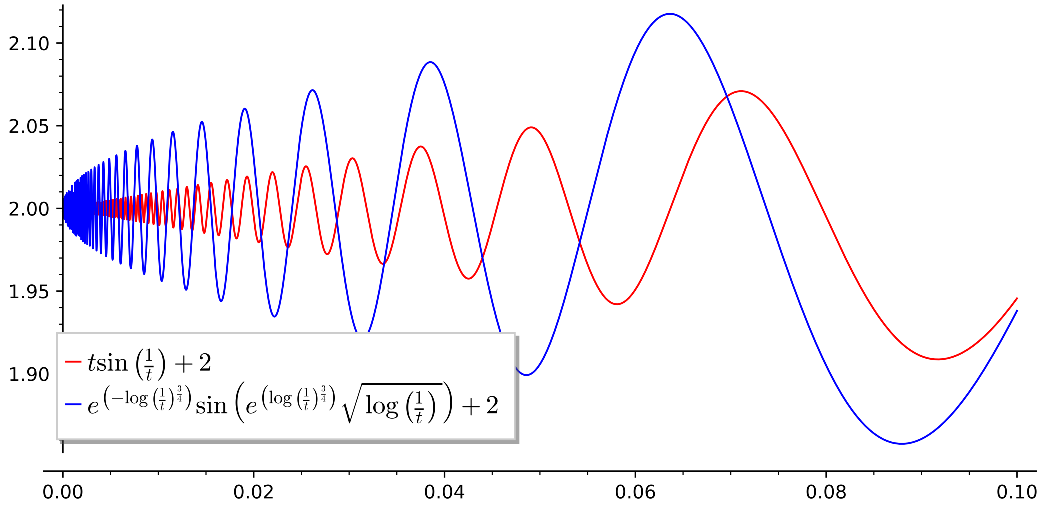

Figure 4.1: Behavior w.r.t time variable for the Examples 4.1.1 and 4.1.2

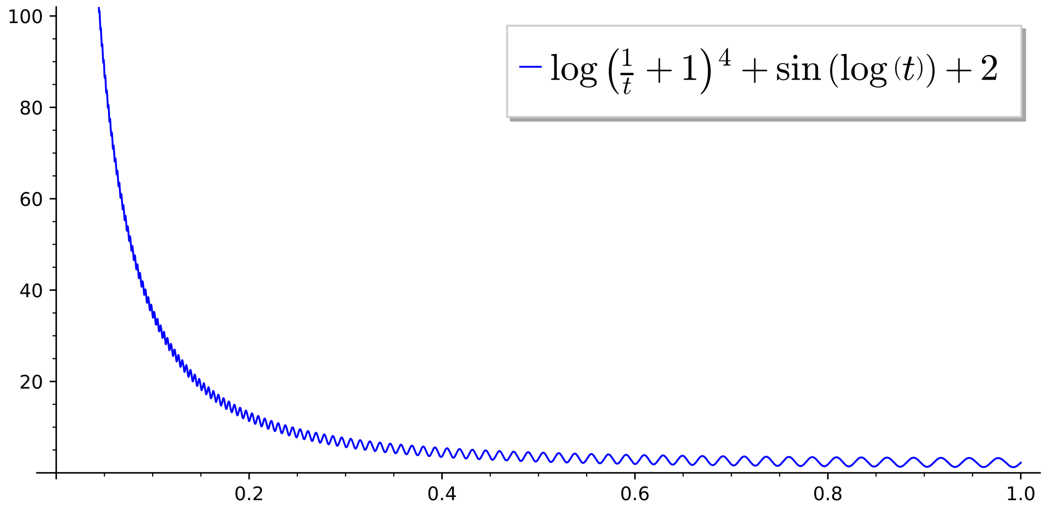

Figure 4.2: Plot of Example 4.1.3

Example 4.1.2. *$a(t,x) = 9\langle x\rangle^2(2+t\sin(1/t))$. Here $\omega(x) = 3\langle x\rangle, \Phi(x) = \langle x\rangle, \tilde{\theta}(t) = \theta(t) = 1$ and $\psi(t) = \ln\left(1+\frac{1}{t}\right)$.*

Example 4.1.3. *$a(t,x) = 2+(\ln(1+1/t))^4 + \sin(\ln t)$. Here $\omega(x) = \Phi(x) = 1, \tilde{\theta}(t) = (\ln(1+1/t))^4, \theta(t) = (\ln(1+1/t))^3$, and $\psi(t) = 1$.*

We define a function $\vartheta : (0,+\infty) \to (0,+\infty)$ as

$$\vartheta\left(\frac{1}{t}\right) := \theta(t)(\tilde{\theta}(t)+\psi(t)). \tag{4.1.7}$$

This function plays a crucial role in performing conjugation and in defining Sobolev spaces. The rate of growth of ϑ defines the quantity of the loss in regularity. For the purpose of pseudodifferential calculus in our context, we need ϑ to satisfy the following estimate

$$\left|\frac{d^j}{dr^j}\vartheta(r)\right| \leq C_j\frac{\vartheta(r)}{r^j}, \quad r \in \mathbb{R}^+ \tag{4.1.8}$$

for some $C_j > 0$. Note that the above estimate is natural for logarithmic-type functions.

The goals of this chapter are as follows:

1. Analyze the loss of regularity when

$$\vartheta(1/t) \leq C_0^* \ln\left(1+\frac{1}{t}\right). \tag{4.1.9}$$

   This is the case for very slow to fast oscillations. When the coefficients are only dependent on time, the well-posedness issue is addressed using the energy methods by Ghisi and Gobbino [35] via the spectral theory of self-adjoint operators and by Reissig [69, Theorem 8] via Fourier transformation with respect to x. In this chapter, we extend the results to a global setting using pseudodifferential calculus and energy methods.

2. Analyze the loss of regularity when (4.1.9) is violated i.e.,

$$\left.\begin{aligned} \lim_{t\to 0^+}\frac{\vartheta(1/t)}{|\ln t|} = \infty, \qquad \text{and} \\ C_1^*\left(\ln\left(1+\frac{1}{t}\right)\right)^{\varrho_1} \leq \vartheta\left(\frac{1}{t}\right) \leq C_2^*\left(\ln\left(1+\frac{1}{t}\right)\right)^{\varrho_2}, \end{aligned}\right\} \tag{4.1.10}$$

   for some $1 < \varrho_1 \leq \varrho_2$. This is the case for very fast oscillations ($q = 1$) and strong blow-up (at most as $|\ln t|^r, r > 1$). When the coefficients are independent of x, Colombini et al. [16, Theorem 1.2 and 1.4] have shown that one can not expect C^∞ well-posedness in the case of *very fast oscillations*. In this chapter, we establish global well-posedness for the both the cases and quantify the infinite loss of regularity using infinite order pseudodifferential operators.

To handle the singular behavior in the global setting we propose a new localization technique on the extended phase space. We employ a diagonalization procedure to arrive at an equivalent first order system whose symbols contain singularities localized in certain regions of the extended phase space. This helps in arriving at an appropriate loss operator for conjugation so that one can microlocally compensate the loss of regularity. The loss operator is of the form

$$e^{\nu(t)\Theta(x,D_x)}, \tag{4.1.11}$$

where $\nu \in C([0,T]) \cap C^1((0,T])$ and $\Theta(x,\xi) = \sigma(\Theta(x,D_x))$ is defined as

$$\Theta(x,\xi) := \vartheta(\Phi(x)\langle\xi\rangle_k) = \theta\big(h(x,\xi)\big)\big(\tilde{\theta}(h(x,\xi)) + \psi(h(x,\xi))\big) \tag{4.1.12}$$

When (4.1.10) is satisfied, the operator in (4.1.11) is of infinite order in both x and D_x. The operator $\Theta(x,D_x)$ explains the quantity of the loss by linking it to the metric on the phase space and the singular behavior while $\nu(t)$ gives a scale for the loss. The symbol of the operator arising after the conjugation is governed by a metric $\tilde{g}_{\Phi,k}$ that is conformally equivalent to the initial metric $g_{\Phi,k}$. The metric $\tilde{g}_{\Phi,k}$ is of the form

$$\tilde{g}_{\Phi,k} = \Theta(x,\xi)^2 g_{\Phi,k}. \tag{4.1.13}$$

This is discussed in Section 4.4.

4.1.1 Sobolev Spaces

Following are the Sobolev spaces defined using the loss operator.

Definition 4.1.1. *The Sobolev space $\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)$ for $s=(s_1,s_2)\in\mathbb{R}^2$ and $\delta\in\mathbb{R}$ is defined as*

$$\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n) = \{v\in L^2(\mathbb{R}^n) : e^{\delta\Theta(x,D_x)}\Phi(x)^{s_2}\langle D_x\rangle_k^{s_1} v \in L^2(\mathbb{R}^n)\}, \tag{4.1.14}$$

equipped with the norm $\|v\|_{\Phi,k;\Theta,\delta,s} = \|e^{\delta\Theta(\cdot,D)}\Phi(\cdot)^{s_2}\langle D\rangle_k^{s_1} v\|_{L^2}$.

Here the operator $\Theta(x,D_x)$ is as in (4.1.12). When ϑ satisfies the estimate in (4.1.9), the operator $e^{\delta\Theta(x,D_x)}$ is a finite order pseudodifferential operator. In that case, the Sobolev spaces $\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)$ are of the form given by the following definition.

Definition 4.1.2. *The Sobolev space $H^s_{\Phi,k}(\mathbb{R}^n)$ for $s=(s_1,s_2)\in\mathbb{R}^2$ is defined as*

$$H^s_{\Phi,k}(\mathbb{R}^n) = \{v\in L^2(\mathbb{R}^n) : \Phi(x)^{s_2}\langle D_x\rangle_k^{s_1} v\in L^2(\mathbb{R}^n)\}, \tag{4.1.15}$$

equipped with the norm $\|v\|_{\Phi,k;s} = \|\Phi(\cdot)^{s_2}\langle D\rangle_k^{s_1} v\|_{L^2}$.

Remark 4.1.2. *(Relation between $\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)$ and $H^s_{\Phi,k}(\mathbb{R}^n)$)*

1. *If ϑ is a bounded function then we have the equivalence $H^s_{\Phi,k}(\mathbb{R}^n)\equiv\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)$, as $\Theta(x,\xi)$ is a bounded function in both x and ξ.*
2. *If $\lim\limits_{t\to0^+}\frac{\vartheta(1/t)}{|\ln t|}=0$, $H^{s+\varepsilon e}_{\Phi,k}(\mathbb{R}^n)\subseteq\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)\subseteq H^{s-\varepsilon e}_{\Phi,k}(\mathbb{R}^n)$ for every $\varepsilon>0$.*
3. *If $\vartheta(1/t)\equiv C_0\ln(1+1/t)$ for some $C_0>0$, then $H^{s+C_0\delta e}_{\Phi,k}(\mathbb{R}^n)\equiv\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)$. Here $\Theta(x,\xi)=C_0\ln(1+\Phi(x)\langle\xi\rangle_k)$.*

4.1.2 Main Results

Let us generalize the problem (4.1.1) and consider

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x), & (t,x)\in(0,T]\times\mathbb{R}^n,\\ u(0,x)=f_1(x),\quad \partial_t u(0,x)=f_2(x) \end{cases} \tag{4.1.16}$$

with the strictly hyperbolic operator $P(t,x,D_t,D_x) = -D_t^2 + a(t,x,D_x) + b(t,x,D_x)$ where

$$a(t,x,\xi)=\sum_{j,l=1}^n a_{j,l}(t,x)\xi_j\xi_l \quad\text{and}\quad b(t,x,\xi)=i\sum_{j=1}^n b_j(t,x)\xi_j + b_{n+1}(t,x). \tag{4.1.17}$$

The matrix $(a_{j,l}(t,x))$ is real symmetric with $a_{j,l} \in C^2((0,T];C^\infty(\mathbb{R}^n))$ and the lower order coefficients $b_j \in C([0,T];C^\infty(\mathbb{R}^n))$. The assumptions on P are as follows

$$a(t,x,\xi) \geq C_0\langle\xi\rangle_k^2\omega(x)^2, \tag{4.1.18}$$

$$|\partial_\xi^\alpha\partial_x^\beta b(t,x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_k^{1-|\alpha|}\omega(x)\Phi(x)^{-|\beta|}, \tag{4.1.19}$$

for $(t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and

$$|\partial_\xi^\alpha\partial_x^\beta a(t,x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_k^{2-|\alpha|}\omega(x)^2\Phi(x)^{-|\beta|}\tilde{\theta}(t), \tag{4.1.20}$$

$$|\partial_\xi^\alpha\partial_x^\beta\partial_t a(t,x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_k^{2-|\alpha|}\omega(x)^2\Phi(x)^{-|\beta|}\frac{\theta(t)}{t}, \tag{4.1.21}$$

$$|\partial_\xi^\alpha\partial_x^\beta\partial_t^2 a(t,x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle_k^{2-|\alpha|}\omega(x)^2\Phi(x)^{-|\beta|}\frac{\theta(t)^2}{t^2}e^{\psi(t)}, \tag{4.1.22}$$

for $(t,x,\xi) \in (0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \alpha,\beta \in \mathbb{N}_0^n$.

Following are the main results of this chapter whose proofs are presented in Section 4.5. Let $e = (1,1)$.

Theorem 4.1.1 (zero, arbitrarily small or finite loss). *Consider the strictly hyperbolic Cauchy problem (4.1.16) satisfying the conditions (4.1.18) - (4.1.22) and (4.1.9). Let the initial datum f_j belong to $H^{s+(2-j)e}_{\Phi,k}(\mathbb{R}^n)$, $j = 1,2$ and the right hand side $f \in C([0,T];H^s_{\Phi,k}(\mathbb{R}^n))$. Then, for every $\varepsilon \in (0,1)$ there exist $\kappa_0,\kappa_1 > 0$ such that for every $s \in \mathbb{R}^2$ there is a unique global in time solution*

$$u \in \bigcap_{j=0}^{1} C^{1-j}\Big([0,T];\mathcal{H}^{s+je,-\kappa(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\Big),$$

where

$$\kappa(t) = \begin{cases} \kappa_0 + \kappa_1 t^\varepsilon/\varepsilon, & \text{when } \tilde{\theta} \text{ is unbounded} \\ \kappa_0 + \kappa_1 t, & \text{otherwise.} \end{cases}$$

More specifically, the solution satisfies the a-priori estimate

$$\begin{aligned} C\sum_{j=0}^{1}\|\partial_t^j u(t,\cdot)\|_{\Phi,k;\Theta,-\kappa(t),s+(1-j)e} &\leq \sum_{j=1}^{2}\|f_j\|_{\Phi,k;s+(2-j)e} \\ &\quad + \int_0^t \|f(\tau,\cdot)\|_{\Phi,k;\Theta,-\kappa(\tau),s}\,d\tau \end{aligned} \tag{4.1.23}$$

for $0 \leq t \leq T,\ C = C_s > 0$.

In view of the Remark 4.1.2, we see that when $\theta(t),\tilde{\theta}(t)$ and $\psi(t)$ are all bounded i.e., $\Theta(x,\xi) \sim 1$, we have no loss. When $\vartheta(1/t) \sim \ln(1+1/t)$ i.e., $\Theta(x,\xi) \sim \ln(1+\Phi(x)\langle\xi\rangle_k)$, we have at most finite loss of regularity. In between both the cases, the loss is arbitrarily small. The above result not only extends [69, Theorem 8] to the case of coefficients depending on x and unbounded in t but also to a global setting and hence settles the well-posedness issue for the oscillatory behavior case.

Theorem 4.1.2 (Infinite Loss). *Consider the strictly hyperbolic Cauchy problem (4.1.16) satisfying the conditions (4.1.18) - (4.1.22) and (4.1.10) with generic constant $C_{\alpha\beta}$ replaced by $CC'_{|\alpha|}K'_{|\beta|}$ as in (4.3.6). Let the initial datum f_j belong to $\mathcal{H}^{s+(2-j)e,\delta_1}_{\Phi,k;\Theta}(\mathbb{R}^n)$, $\delta_1 > 0, j = 1,2$ and the right hand side $f \in C([0,T];\mathcal{H}^{s,\delta_2}_{\Phi,k;\Theta}(\mathbb{R}^n)), \delta_2 > 0$. Then, for every $\varepsilon \in (0,1)$ there exist $\delta^*, \tilde{\kappa}_0, \tilde{\kappa}_1 > 0$ such that for every $s \in \mathbb{R}^2$ there exists a unique solution*

$$u \in \bigcap_{j=0}^{1} C^{1-j}\Big([0,T^*];\mathcal{H}^{s+je,\tilde{\kappa}(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\Big),$$

where

$$\tilde{\kappa}(t) = \begin{cases} \tilde{\kappa}_0 + \tilde{\kappa}_1(T_0{}^{\varepsilon} - t^{\varepsilon})/\varepsilon, & \text{when } \tilde{\theta} \text{ is unbounded} \\ \tilde{\kappa}_0 + \tilde{\kappa}_1(T_0 - t), & \text{otherwise,} \end{cases}$$

and $T_0 = \min\{\delta^*, \delta_1, \delta_2\}$. More specifically, the solution satisfies the a-priori estimate

$$\begin{aligned} C\sum_{j=0}^{1} \|\partial_t^j u(t,\cdot)\|_{\Phi,k;\Theta,\tilde{\kappa}(t),s+(1-j)e} \leq & \sum_{j=1}^{2} \|f_j\|_{\Phi,k;\Theta,\tilde{\kappa}(0),s+(2-j)e} \\ & + \int_0^t \|f(\tau,\cdot)\|_{\Phi,k;\Theta,\tilde{\kappa}(\tau),s}\, d\tau \end{aligned} \tag{4.1.24}$$

for $0 \leq t \leq T^*,\ C > 0$.

4.2 Subdivision of the Phase Space

We divide the extended phase space, $J = [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, into three regions using the Planck function, $h(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$, and the functions $\tilde{\theta}, \theta$ and ψ which specify the order of singularity. Let us define $t_{x,\xi}$ and $\tilde{t}_{x,\xi}$ for a fixed (x,ξ) as

$$\begin{aligned} t_{x,\xi} &= N\, h(x,\xi)\theta(h(x,\xi)), \qquad \text{and} \\ \tilde{t}_{x,\xi} &= N\, h(x,\xi)\tilde{\theta}(h(x,\xi))\theta(h(x,\xi))e^{\psi(h(x,\xi))}, \end{aligned}$$

where N is the positive integer. For a fixed (x,ξ) we split the time interval as

$$[0,T] = [0,t_{x,\xi}] \cup (t_{x,\xi},\tilde{t}_{x,\xi}] \cup (\tilde{t}_{x,\xi},T]$$

and define the regions as below:

$$\begin{aligned} Z_{int}(N) &= \{(t,x,\xi) \in J : 0 \leq t \leq t_{x,\xi}\}, \\ Z_{mid}(N) &= \{(t,x,\xi) \in J : t_{x,\xi} < t \leq \tilde{t}_{x,\xi}\}, \\ Z_{ext}(N) &= \{(t,x,\xi) \in J : \tilde{t}_{x,\xi} < t\}. \end{aligned}$$

Note that for all $(x,\xi) \in \mathbb{R}^{2n}$,

$$h(x,\xi) \leq t_{x,\xi} \leq \tilde{t}_{x,\xi} \leq T. \tag{4.2.1}$$

Remark 4.2.1. *When $\tilde{\theta}(t)$ and $\psi(t)$ are bounded functions we see that $t_{x,\xi} \sim \tilde{t}_{x,\xi}$. In such a case, given (x,ξ) we split the time interval and the extended phase space as*

$$\begin{aligned}[0,T] &= [0, t_{x,\xi}] \cup (t_{x,\xi}, T], \\ J &= Z_{int}(N) \cup Z_{ext}(N).\end{aligned}$$

We do not need the region $Z_{mid}(N)$. An example of such a case is - $\theta(t) \sim (\ln(1 + 1/t))^{\tilde{\gamma}}, \tilde{\theta}(t) = \psi(t) = 1, \tilde{\gamma} \in [0, +\infty)$. In [16] where fast oscillating coefficients depending only on t are dealt, the authors subdivide the extended phase space into two regions - $Z_{int}(N)$ and $Z_{ext}(N)$ using the time splitting point $t_{x,\xi}$.

4.3 Parameter Dependent Global Symbol Classes

In this section, we define parameter dependent global symbol classes whose geometry is governed by the metrics $g_{\Phi,k}$ and $\tilde{g}_{\Phi,k}$ as in (4.1.13). Let $m_j \in \mathbb{R}$ for $j = 1, \dots, 6$.

Definition 4.3.1. *$G^{m_1,m_2}_{\Phi,k}(m_1, m_2)$ is the space of all functions $a = a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying the symbolic estimate*

$$|\partial_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle_k^{m_1 - |\alpha|} \omega(x)^{m_2} \Phi(x)^{-|\beta|},$$

for constants $C_{\alpha\beta} > 0$ and all $\alpha, \beta \in \mathbb{N}_0^n$.

We denote by $G^{-\infty}$ the class of symbols in

$$\bigcap_{m_1, m_2 \in \mathbb{R}} G^{m_1,m_2}_{\Phi,k}(m_1, m_2).$$

Note that $C_{\alpha\beta}(> 0)$ is a generic constant.

Definition 4.3.2. *$G^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k})$ is the space of all functions $a = a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying the symbolic estimate*

$$|\partial_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle_k^{m_1 - |\alpha|} \omega(x)^{m_2} \Phi(x)^{-|\beta|} \Theta(x,\xi)^{(|\alpha| + |\beta|)},$$

for constants $C_{\alpha\beta} > 0$ and all $\alpha, \beta \in \mathbb{N}_0^n$.

Definition 4.3.3. *$G^{m_1,m_2}\{m_3\}(\omega, g_{\Phi,k})^{(1)}_N$ is the space of all functions $a(t,x,\xi)$ in $C^2((0,T]; C^\infty(\mathbb{R}^{2n}))$ satisfying*

$$|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle_k^{m_1 - |\alpha|} \omega(x)^{m_2} \Phi(x)^{-|\beta|} \tilde{\theta}(t)^{m_3} \tag{4.3.1}$$

for constants $C_{\alpha\beta} > 0$ and for all $(t,x,\xi) \in Z_{int}(N)$ and all $\alpha, \beta \in \mathbb{N}_0^n$.

Definition 4.3.4. *$G^{m_1,m_2}\{m_3, m_4, m_5\}(\omega, g_{\Phi,k})^{(2)}_N$ is the space of all functions $a(t,x,\xi) \in C^2((0,T]; C^\infty(\mathbb{R}^{2n}))$ satisfying the symbolic estimate*

$$|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle_k^{m_1 - |\alpha|} \omega(x)^{m_2} \Phi(x)^{-|\beta|} \left(\frac{\theta(t)}{t} \right)^{m_3} \tilde{\theta}(t)^{m_4 + m_5(|\alpha| + |\beta|)}. \tag{4.3.2}$$

for constants $C_{\alpha\beta} > 0$ and for all $(t,x,\xi) \in Z_{mid}(N)$ and all $\alpha, \beta \in \mathbb{N}_0^n$.

Definition 4.3.5. *$G^{m_1,m_2}\{m_3,m_4,m_5,m_6\}(\omega,g_{\Phi,k})^{(3)}_N$ for $m_3\geq m_4$, is the space of all functions $a(t,x,\xi)\in C^2((0,T];C^\infty(\mathbb{R}^{2n}))$ satisfying the symbolic estimate*

$$|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)|\leq C_{\alpha\beta}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}\left(\frac{\theta(t)}{t}\right)^{m_3}e^{m_4\psi(t)}\tilde{\theta}(t)^{m_5+m_6(|\alpha|+|\beta|)}. \quad (4.3.3)$$

for constants $C_{\alpha\beta}>0$ and for all $(t,x,\xi)\in Z_{ext}(N)$ and all $\alpha,\beta\in\mathbb{N}_0^n$.

Similar to Definition 4.3.2, we can define $G^{m_1,m_2}\{m_3\}(\omega,\tilde{g}_{\Phi,k})^{(1)}_N$, $G^{m_1,m_2}\{m_3,m_4,m_5\}(\omega,\tilde{g}_{\Phi,k})^{(2)}_N$ and $G^{m_1,m_2}\{m_3,m_4,m_5,m_6\}(\omega,\tilde{g}_{\Phi,k})^{(3)}_N$.

Remark 4.3.1. *When $\tilde{\theta}(t)$ is a bounded function, we have*

$$\left.\begin{aligned} G^{m_1,m_2}\{m_3\}(\omega,g_{\Phi,k})^{(1)}_N &\equiv G^{m_1,m_2}\{0\}(\omega,g_{\Phi,k})^{(1)}_N,\\ G^{m_1,m_2}\{m_3,m_4,m_5\}(\omega,g_{\Phi,k})^{(2)}_N &\equiv G^{m_1,m_2}\{m_3,0,0\}(\omega,g_{\Phi,k})^{(2)}_N,\\ G^{m_1,m_2}\{m_3,m_4,m_5,m_6\}(\omega,g_{\Phi,k})^{(3)}_N &\equiv G^{m_1,m_2}\{m_3,m_4,0,0\}(\omega,g_{\Phi,k})^{(3)}_N. \end{aligned}\right\} \quad (4.3.4)$$

Given a t-dependent global symbol $a(t,x,\xi)$, we can associate a pseudodifferential operator $Op(a)=a(t,x,D_x)$ to $a(t,x,\xi)$ by the following oscillatory integral

$$\begin{aligned} a(t,x,D_x)u(t,x) &= \iint\limits_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}a(t,x,\xi)u(t,y)dyd\xi\\ &= \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}a(t,x,\xi)\hat{u}(t,\xi)đ\xi. \end{aligned}$$

where $đ\xi=(2\pi)^{-n}d\xi$.

As for the calculus of symbol classes $G^{m_1,m_2}(\omega,g_{\Phi,k})$, we refer to [58, Section 1.2 & 3.1]. The calculus for the operators with symbols in the additive form

$$\left.\begin{aligned} &a(t,x,\xi)=a_1(t,x,\xi)+a_2(t,x,\xi)+a_3(t,x,\xi), \quad \text{for}\\ &a_1\in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{m}_3\}(\omega,g_{\Phi,k})^{(1)}_N,\\ &a_2\in G^{m_1',m_2'}\{m_3',m_4',m_5'\}(\omega,g_{\Phi,k})^{(2)}_N\\ &a_3\in G^{m_1,m_2}\{m_3,m_4,m_5,m_6\}(\omega,g_{\Phi,k})^{(3)}_N \end{aligned}\right\} \quad (4.3.5)$$

is given in Section A.2 of Appendix A. The calculus for the $\tilde{g}_{\Phi,k}$ versions of the symbol classes follows in similar lines. This requires that the function $\Theta(x,\xi)$ is sub-additive and sub-multiplicative both x and ξ variables separately i.e.,

$$\begin{aligned} \Theta(x+y,\xi)&\leq C\big(\Theta(x,\xi)+\Theta(y,\xi)\big), &\quad \Theta(x+y,\xi)&\leq C\Theta(x,\xi)\Theta(y,\xi),\\ \Theta(x,\xi+\eta)&\leq C\big(\Theta(x,\xi)+\Theta(x,\eta)\big), &\quad \Theta(x,\xi+\eta)&\leq C\Theta(x,\xi)\Theta(x,\eta). \end{aligned}$$

In fact, the sub-multiplicative property can be derived from the sub-additivity as $\Theta\geq 1$.

While dealing with the case of infinite order loss, we need to keep track of the weight sequences with respect to both x and ξ. To this end we replace the generic constant $C_{\alpha\beta}$ by $CC'_{|\alpha|}K'_{|\beta|}$ such that

$$\inf_{j\in\mathbb{N}}\frac{C'_jK'_j}{(\Phi(x)\langle\xi\rangle_k)^j}\lesssim e^{-\delta_0\Theta(x,\xi)}, \quad \delta_0>0. \quad (4.3.6)$$

The calculus of the operators with symbols governed by such weight sequences can be developed in similar lines to the calculi given in Section A.2 of Appendix A and Section A.3 of Appendix Awhich are based on the standard techniques from the book [58, Section 6.3].

4.4 Conjugation by an Infinite Order Pseudodifferential Operator

In this section, we perform a conjugation of operators with symbols of the form (4.3.5) by $e^{\nu(t)\Theta(x,D_x)}$. Here we assume that $\nu(t)$ is a continuous function for $t\in[0,T]$. When $e^{\nu(t)\Theta(x,D_x)}$ is an infinite order pseudodifferential operator, we need to consider an appropriate weight sequence so that the conjugation is well-defined. For this reason one can replace the generic contant $C_{\alpha\beta}$ appearing in the definitions of the symbol classes with $CC'_{|\alpha|}K'_{|\beta|}$ satisfying the condition (4.3.6).

The following proposition gives an upper bound on the function $\nu(t)$ for the conjugation to be well defined.

Theorem 4.4.1. *Consider a symbol $a(t,x,\xi)$ as in (4.3.5) where the generic constant $C_{\alpha\beta}$ in the symbol estimates (4.3.1)- (4.3.3) is replaced by $CC'_{|\alpha|}K'_{|\beta|}$ satisfying the condition (4.3.6). Let $\nu=\nu(t)\in C([0,T])\cap C^1((0,T])$. Then, there exists $\delta^*>0$ such that for $\nu(t)>0$ with $\nu(t)<\delta^*$,*

$$e^{\nu(t)\Theta(x,D_x)}a(t,x,D_x)e^{-\nu(t)\Theta(x,D_x)}=a(t,x,D)+\sum_{j=1}^{3}r_\nu^{(j)}(t,x,D_x), \tag{4.4.1}$$

where $r_\nu^{(j)}(t,x,D_x), j=1,2,3,$ are such that

$$\left.\begin{array}{r}\Theta(x,\xi)^{-1}r_\nu^{(1)}(t,x,\xi)\in L^\infty\left([0,T];G^{-\infty,l_2^*-1}(\Phi,\tilde g_{\Phi,k})\right)\\ \Theta(x,\xi)^{-1}r_\nu^{(2)}(t,x,\xi)\in L^\infty\left([0,T];G^{l_1^*-1,-\infty}(\Phi,\tilde g_{\Phi,k})\right)\end{array}\right\}, \quad \text{if } \tilde\theta \text{ is bounded,}$$
$$\left.\begin{array}{r}t^{1-\varepsilon}\Theta(x,\xi)^{-1}r_\nu^{(1)}(t,x,\xi)\in C\left([0,T];G^{-\infty,l_2^*-1}(\Phi,\tilde g_{\Phi,k})\right)\\ t^{1-\varepsilon}\Theta(x,\xi)^{-1}r_\nu^{(2)}(t,x,\xi)\in C\left([0,T];G^{l_1^*-1,-\infty}(\Phi,\tilde g_{\Phi,k})\right)\end{array}\right\}, \quad \text{otherwise,}$$

while $\Theta(x,\xi)^{-1}r_\nu^{(3)}(t,x,\xi)\in L^\infty([0,T];G^{-\infty})$ for every $\varepsilon\in(0,1)$ and $l_i^*=\max\{\tilde m_i,m_i'+m_3',m_i+m_3\}, i=1,2.$

To prove Theorem 4.4.1, we need the following lemma, which can be given an inductive proof.

Lemma 4.4.2. *Let $\delta\neq 0$. Then, for every $\alpha,\beta\in\mathbb{Z}_+^n$, we have*

$$\partial_x^\beta\partial_\xi^\alpha e^{\delta\Theta(x,\xi)}\leq (C\delta)^{|\alpha|+|\beta|}\alpha!\beta!e^{\delta\Theta(x,\xi)}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\Theta(x,\xi)^{|\alpha|+|\beta|}.$$

Proof of Theorem 4.4.1. Throughout this proof we write ν in place of $\nu(t)$ for the sake of simplicity of the notation. Let $a_{\nu,\Theta}(t,x,\xi)$ be the symbol of the operator

$$\exp\{\nu\Theta(x,D_x)\}a(t,x,D_x)\exp\{-\nu\Theta(x,D_x)\}.$$

Then $a_{\nu,\Theta}(t,x,\xi)$ can be written in the form of an oscillatory integral (for example, see [51, Chapter 2]) as follows:

$$a_{\nu,\Theta}(t,x,\xi) = \int \cdots \int e^{-iy\cdot\eta} e^{-iz\cdot\zeta} e^{\nu\Theta(x,\xi+\zeta+\eta)} a(t,x+z,\xi+\eta) \times e^{-\nu\Theta(x+y,\xi)} dz\, đ\zeta\, dy\, đ\eta, \tag{4.4.2}$$

Taylor expansions of $\exp\{\nu\Theta(x,\xi)\}$ in the first and second variables, respectively, are

$$e^{-\nu\Theta(x+y,\xi)} = e^{-\nu\Theta(x,\xi)} + \sum_{j=1}^{n} \int_0^1 y_j \partial_{w'_j} e^{-\nu\Theta(w',\xi)}\Big|_{w'=x+\theta_1 y} d\theta_1, \text{ and}$$
$$e^{\nu\Theta(x,\xi+\zeta+\eta)} = e^{\nu\Theta(x,\xi)} + \sum_{i=1}^{n} \int_0^1 (\zeta_i+\eta_i) \partial_{w_i} e^{\nu\Theta(x,w)}\Big|_{w=\xi+\theta_2(\eta+\zeta)} d\theta_2.$$

We can write $a_{\nu,\Theta}$ as

$$a_{\nu,\Theta}(t,x,\xi) = a(t,x,\xi) + \sum_{l=1}^{3} r_\nu^{(l)}(t,x,\xi) \quad \text{where}$$

$$r_\nu^{(l)}(x,\xi) = \int \cdots \int e^{-iy\cdot\eta} e^{-iz\cdot\zeta} I_l a(t,x+z,\xi+\eta) dz\, đ\zeta\, dy\, đ\eta,$$

and I_l, $l = 1,2,3$ are as follows:

$$I_1 = e^{\nu\Theta(x,\xi)} \sum_{j=1}^{n} \int_0^1 y_j \partial_{w'_j} e^{-\nu\Theta(w',\xi)}\Big|_{w'=x+\theta_1 y} d\theta_1,$$
$$I_2 = e^{-\nu\Theta(x,\xi)} \sum_{i=1}^{n} \int_0^1 (\zeta_i+\eta_i) \partial_{w_i} e^{\nu\Theta(x,w)}\Big|_{w=\xi+\theta_2(\zeta+\eta)} d\theta_2,$$
$$I_3 = \left(\sum_{i=1}^{n} \int_0^1 (\zeta_i+\eta_i) \partial_{w_i} e^{\nu\Theta(x,w)}\Big|_{w=\xi+\theta_2(\zeta+\eta)} d\theta_2 \right) \times \left(\sum_{j=1}^{n} \int_0^1 y_j \partial_{w'_j} e^{-\nu\Theta(w',\xi)}\Big|_{w'=x+\theta_1 y} d\theta_1 \right).$$

Denote the indicator functions for the regions $Z_{int}(N)$, $Z_{mid}(N)$ and $Z_{ext}(N)$ by χ_1, χ_2 and χ_3,respectively. Let $m_1^* = \max\{\tilde{m}_1, m_1', m_1\}$ and $m_2^* = \max\{\tilde{m}_2, m_2', m_2\}$.

We will now determine the growth estimate for $r_\nu^{(1)}(t,x,\xi)$ using integration by parts (see for example, [70, Section 1.4]). For $\alpha,\beta,\kappa\in(\mathbb{Z}_0^+)^n$ and $l\in\mathbb{Z}^+$ we have

$$\begin{aligned}
&\partial_\xi^\alpha\partial_x^\beta r_\nu^{(1)}(t,x,\xi)\\
&=\sum_{j=1}^n\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int y^{-\kappa}y^{\kappa}e^{-iy\cdot\eta}e^{-iz\cdot\zeta}(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{\xi_j}a)(t,x+z,\xi+\eta)\\
&\quad\times\int_0^1\partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w'_j}e^{\nu(\Theta(x,\xi)-\Theta(w',\xi))}\Big|_{w'=x+\theta_1 y}d\theta_1 dz\,đ\zeta\, dy\, đ\eta\\
&=\sum_{j=1}^n\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int y^{-\kappa}e^{-iy\cdot\eta}e^{-iz\cdot\zeta}\langle\eta\rangle_k^{-2l}\langle y\rangle_k^{-2l}\langle z\rangle_k^{-2l}\langle D_\zeta\rangle_k^{2l}\langle\zeta\rangle_k^{-2l}\\
&\quad\times\langle D_\eta\rangle_k^{2l}\langle D_z\rangle_k^{2l}D_\eta^\kappa(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{\xi_j}a)(t,x+z,\xi+\eta)\\
&\quad\times\int_0^1\langle D_y\rangle_k^{2l}\partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w'_j}e^{\nu(\Theta(x,\xi)-\Theta(w',\xi))}\Big|_{w'=x+\theta_1 y}d\theta_1 dz\,đ\zeta\, dy\, đ\eta.
\end{aligned}$$

Using the easy to show inequality $\Phi(x+z)^r\le 2^{|r|}\Phi(x)^r\Phi(z)^{|r|},\forall r\in\mathbb{R}$,

$$\begin{aligned}
\partial_{w'_j}e^{\nu(\Theta(x,\xi)-\Theta(w',\xi))}\Big|_{w'=x+\theta_1 y}&\le E_1(t,x,y,\xi)\Phi(x+\theta_1 y)^{-1}\Theta(x+\theta_1 y,\xi)\\
&\le CE_1(t,x,y,\xi)\Phi(x)^{-1}\Phi(y)\Theta(x,\xi)\Theta(y,\xi).
\end{aligned}$$

where $E_1(t,x,y,\xi)=\exp\{\nu(\Theta(x,\xi)-\Theta(x+\theta_1 y,\xi))\}$. From the following estimation

$$\begin{aligned}
\Big|\sum_{j=0}^{2l}\partial_{y_i}^j e^{\nu(\Theta(x,\xi)-\Theta(w',\xi))}\Big|_{w'=x+\theta_1 y}\Big|&\le CE_1(t,x,y,\xi)\sum_{j=0}^{2l}\left(\frac{\Theta(x+\theta_1 y,\xi)}{\Phi(x+\theta_1 y)}\right)^j\\
&\le CE_1\sum_{j=0}^{2l}\left(\frac{(\ln(2\Phi(x+\theta_1 y)\langle\xi\rangle_k))^{\varrho_2}}{\Phi(x+\theta_1 y)}\right)^j\\
&=CE_1\sum_{j=0}^{2l}\left(\frac{(\ln(2\Phi(x+\theta_1 y)))^j}{\Phi(x+\theta_1 y)^{j/\varrho_2}}+\frac{(\ln\langle\xi\rangle_k)^j}{\Phi(x+\theta_1 y)^{j/\varrho_2}}\right)^{\varrho_2}\\
&\le CE_1(\ln\langle\xi\rangle_k)^{2\varrho_2 l},
\end{aligned}$$

we have

$$\langle D_y\rangle_k^{2l}E_1(t,x,y,\xi)\le CE_1(t,x,y,\xi)(\ln\langle\xi\rangle_k)^{2\varrho_2 l}.$$

Let

$$\begin{aligned}
G(t,x,\xi)&=\chi_1\omega(x)^{\tilde m_2}\langle\xi\rangle_k^{\tilde m_1}\tilde\theta(t)^{\tilde m_3}+\chi_2\omega(x)^{m'_2}\langle\xi\rangle_k^{m'_1}\left(\frac{\theta(t)}{t}\right)^{m'_3}\tilde\theta(t)^{m'_4+m'_5(|\alpha|+|\beta|+1+4l)}\\
&\quad+\chi_3\omega(x)^{m_2}\langle\xi\rangle_k^{m_1}\left(\frac{\theta(t)}{t}\right)^{m_3}e^{m_4\psi(t)}\tilde\theta(t)^{m_5+m_6(|\alpha|+|\beta|+1+4l)}.
\end{aligned}$$

Note that for $|y|\ge 1$ we have $\langle y\rangle\le\sqrt{2}|y|$ and in the case $|y|<1$ we have $\langle y\rangle<\sqrt{2}$.

Using these estimates along with the fact that $\langle y\rangle^{-|\kappa|} \le \Phi(y)^{-|\kappa|}$ we have

$$\begin{aligned}|\partial_\xi^\alpha \partial_x^\beta r_\nu^{(1)}(t,x,\xi)| \le\, & CC'_{|\alpha|+1}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}G(t,x,\xi)\\ & \times \sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int \omega(z)^{|m_2'|}\Phi(z)^{|\beta'|}\Phi(y)^{1+|\beta''|}\\ & \times \Theta(y,\xi)^{1+|\beta''|+|\alpha''|}\langle\eta\rangle_k^{|m_1'-1-|\alpha'||+|\kappa|-2l}(\ln(\langle\xi\rangle_k))^{2\varrho_2 l}\tilde\theta(t)^\kappa\\ & \times \frac{C'_{|\kappa|}K'_{|\kappa|}}{(\Phi(y)\langle\xi\rangle_k)^{|\kappa|}}E_1(t,x,y,\xi)\langle z\rangle_k^{-2l}\langle y\rangle_k^{-2l}\langle\zeta\rangle_k^{-2l}dz\,đ\zeta\, dy\, đ\eta.\end{aligned}$$

Given α, β and κ, we choose l such that $2l > n + \max\{m_1^*, m_2^*\} + |\alpha| + |\beta| + |\kappa|$. So that

$$\begin{aligned}|\partial_\xi^\alpha \partial_x^\beta r_\nu^{(1)}(t,x,\xi)| \le\, & CC'_{|\alpha|+1}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}G(t,x,\xi)\\ & \times \tilde\theta(t)^\kappa\int\Big((\ln(\langle\xi\rangle_k))^{2\mu l}\Theta(y,\xi)^{1+|\beta''|+|\alpha''|}\\ & \times \frac{C'_{|\kappa|}K'_{|\kappa|}}{(\Phi(y)\langle\xi\rangle_k)^{|\kappa|}}E_1(t,x,y,\xi)\Big)dy.\end{aligned}$$

Noting the inequality (4.3.6), we have $(\ln(\langle\xi\rangle_k))^{2\varrho_2 l}\Theta(y,\xi)^{1+|\beta''|+|\alpha''|}e^{-\delta_0\Theta(y,\xi)} \le Ce^{-\frac{\delta_0}{2}\Theta(y,\xi)}$. Thus,

$$\begin{aligned}|\partial_\xi^\alpha \partial_x^\beta r_\nu^{(1)}(t,x,\xi)| \le\, & CC'_{|\alpha|+1}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}G(t,x,\xi)\\ & \times \int \exp\Big\{\nu(\Theta(x,\xi)-\Theta(x+\theta_1 y,\xi)) - \frac{\delta_0}{2}\Theta(y,\xi)\Big\}dy.\end{aligned}$$

Since

$$\begin{aligned}\Theta(x,\xi)-\Theta(x+\theta_1 y,\xi) &\le \Theta(x,\xi) - (\Theta(x,\xi)-\Theta(\theta_1 y,\xi))\\ &\le \Theta(\theta_1 y,\xi) \le \Theta(y,\xi),\end{aligned} \tag{4.4.3}$$

and δ_0 is independent of ν, there exists $\delta_1^* > 0$ (in fact $\delta_1^* = \frac{\delta_0}{2}$) such that, for $\nu(t) < \delta_1^*$ and ϱ_1 as in (4.1.10) we obtain the estimate

$$\begin{aligned}|\partial_\xi^\alpha \partial_x^\beta r_\nu^{(1)}(t,x,\xi)| \le\, & CC'_{|\alpha|+1}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}G(t,x,\xi)\\ & \times e^{-\delta_0'(\ln\langle\xi\rangle_k)^{\varrho_1}}, \quad \delta_0' > 0.\end{aligned} \tag{4.4.4}$$

In view of Propositions (A.2.1)-(A.2.3), we have

$$\begin{aligned}\Theta(x,\xi)^{-1}r_\nu^{(1)} &\in L^\infty\left([0,T]; G^{-\infty,l_2^*-1}(\Phi,\tilde g_{\Phi,k})\right), && \text{if } \tilde\theta \text{ is bounded}\\ t^{1-\varepsilon}\Theta(x,\xi)^{-1}r_\nu^{(1)} &\in C\left([0,T]; G^{-\infty,l_2^*-1}(\Phi,\tilde g_{\Phi,k})\right), && \text{otherwise}\end{aligned}$$

for $l_2^* = \max\{\tilde m_2, m_2' + m_3', m_2 + m_3\}$.

In a similar fashion, we will determine the growth estimate for $r_\nu^{(2)}(t,x,\xi)$. Let

$\alpha,\beta,\kappa\in(\mathbb{Z}_0^+)^n$ and $l\in\mathbb{Z}^+$. Then

$$\begin{aligned}
&\partial_\xi^\alpha\partial_x^\beta r_\nu^{(2)}(t,x,\xi)\\
&=\sum_{i=1}^{n}\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int \eta^{-\kappa}\eta^{\kappa}e^{-iy\cdot\eta}\zeta^{-\kappa}\zeta^{\kappa}e^{-iz\cdot\zeta}\langle z\rangle_k^{-2l}\langle\eta\rangle_k^{-2l}\langle D_y\rangle_k^{2l}\\
&\quad\times\langle y\rangle_k^{-2l}\langle\zeta\rangle_k^{-2l}\langle D_z\rangle_k^{2l}\langle D_\zeta\rangle_k^{2l}\langle D_\eta\rangle_k^{2l}(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{x_i}a)(t,x+z,\xi+\eta)\\
&\quad\times\int_0^1\partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w_i}e^{\nu(\Theta(x,w)-\Theta(x,\xi))}\Big|_{w=\xi+\theta_2(\eta+\zeta)}d\theta_2\,dz\,đ\zeta\,dy\,đ\eta,\\
&=\sum_{i=1}^{n}\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int \eta^{-\kappa}e^{-iy\cdot\eta}\zeta^{-\kappa}e^{-iz\cdot\zeta}D_y^\kappa D_z^\kappa\langle z\rangle_k^{-2l}\langle\eta\rangle_k^{-2l}\langle D_y\rangle_k^{2l}\\
&\quad\times\langle y\rangle_k^{-2l}\langle\zeta\rangle_k^{-2l}\langle D_z\rangle_k^{2l}\langle D_\zeta\rangle_k^{2l}\langle D_\eta\rangle_k^{2l}(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{x_i}a)(t,x+z,\xi+\eta)\\
&\quad\times\int_0^1\partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w_i}e^{\nu(\Theta(x,w)-\Theta(x,\xi))}\Big|_{w=\xi+\theta_2(\eta+\zeta)}d\theta_2\,dz\,đ\zeta\,dy\,đ\eta.
\end{aligned}$$

Let $E_2(t,x,\xi,\eta,\zeta)=\exp\{\nu(\Theta(x,\xi+\theta_2(\eta+\zeta))-\Theta(x,\xi))\}$. We have

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta r_\nu^{(2)}(t,x,\xi)|&\le CC'_{|\alpha|+1}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}G(t,x,\xi)\\
&\quad\times\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int\omega(z)^{|m_2'|}\Phi(z)^{|\beta'|+1}\langle\eta\rangle_k^{|m_1'-|\alpha'||}\\
&\quad\times\langle\eta\rangle_k^{-2l}\langle\zeta\rangle_k^{-2l}\langle\eta+\zeta\rangle_k^{1+|\alpha''|}\left(\frac{C'_{|\kappa|}K'_{|\kappa|}}{(\Phi(x)\langle\zeta\rangle_k\langle\eta\rangle_k)^{|\kappa|}}\right)\tilde\theta(t)^\kappa\\
&\quad\times\langle z\rangle_k^{-2l+|\kappa|}\langle y\rangle_k^{-2l-|\kappa|}E_2(t,x,\xi,\eta,\zeta)dz\,đ\zeta\,dy\,đ\eta,
\end{aligned}$$

where we have noted the estimate

$$\langle D_\zeta\rangle_k^{2l}\langle D_\eta\rangle_k^{2l}E_2(t,x,\xi,\eta,\zeta)\le CE_2(t,x,\xi,\eta,\zeta)(\ln\Phi(x))^{4\varrho_2 l},$$

as in the case of $r_\nu^{(1)}$. In this case we choose l such that $2l>2(n+1)+\max\{m_1^*,m_2^*\}+|\alpha|+|\beta|+|\kappa|$. Noting $(\langle\eta\rangle_k\langle\zeta\rangle_k)^{-1}\le\langle\zeta+\eta\rangle_k^{-1}$ and $(\ln\Phi(x))^{4\varrho_2 l}e^{-\delta_0\Theta(x,\eta+\zeta)}\le e^{-\frac{\delta_0}{2}\Theta(x,\eta+\zeta)}$, from (4.3.6) we get

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta r_\nu^{(2)}(t,x,\xi)|&\le CC'_{|\alpha|+1}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}\\
&\quad\times G(t,x,\xi)\tilde\theta(t)^\kappa\int\int(\langle\eta\rangle_k\langle\zeta\rangle_k)^{-2(n+1)}\\
&\quad\times\exp\left\{\nu(\Theta(x,\xi+\eta+\zeta)-\Theta(x,\xi))-\frac{\delta_0}{2}\Theta(x,\eta+\zeta))\right\}đ\zeta\,đ\eta.
\end{aligned}$$

Since $\Theta(x,\xi+\eta+\zeta)-\Theta(x,\xi)\le\Theta(x,\eta+\zeta)$ and and δ_0 is independent of ν, there exists $\delta_2^*>0$ (in fact $\delta_2^*=\frac{\delta_0}{2}$) such that, for $\nu(t)<\delta_2^*$ we obtain the estimate

$$\begin{aligned}
|\partial_\xi^\alpha\partial_x^\beta r_\nu^{(2)}(t,x,\xi)|&\le CC'_{|\alpha|+2}K'_{|\beta|+1}\Theta(x,\xi)^{1+|\alpha|+|\beta|}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}G(t,x,\xi)\\
&\quad\times e^{-\delta_0''(\ln\Phi(x))^{\varrho_1}},\quad \delta_0''>0.
\end{aligned}$$

In view of Propositions (A.2.1)-(A.2.3), we have

$$\begin{aligned} \Theta(x,\xi)^{-1} r_\nu^{(2)} &\in L^\infty\left([0,T]; G^{l_1^*-1,-\infty}(\Phi, \tilde{g}_{\Phi,k})\right), && \text{if } \tilde{\theta} \text{ is bounded} \\ t^{1-\varepsilon}\Theta(x,\xi)^{-1} r_\nu^{(2)} &\in C\left([0,T]; G^{l_1^*-1,-\infty}(\Phi, \tilde{g}_{\Phi,k})\right), && \text{otherwise} \end{aligned}$$

for $l_1^* = \max\{\tilde{m}_1, m_1' + m_3', m_1 + m_3\}$.

By similar techniques used in the case of $r_\nu^{(1)}$ and $r_\nu^{(2)}$, one can show that $r_\nu^{(3)} \in C([0,T]; G^{-\infty})$. Taking $\delta^* = \min\{\delta_1^*, \delta_2^*\}$, proves the theorem. □

Remark 4.4.1. *When $\tilde{\theta}$ is bounded, we get a more refined symbol estimate for the remainder $r_\nu^{(1)}$ as seen from (4.4.4), suggesting that*

$$\begin{aligned} \Theta^{-1} r_\nu^{(1)} \in\ & G^{-\infty,1}\{0\}(\omega^{\tilde{m}_2}\Phi^{-1}, \tilde{g}_{\Phi,k})_N^{(1)} \cap G^{-\infty,1}\{m_3', 0, 0\}(\omega^{m_2'}\Phi^{-1}, \tilde{g}_{\Phi,k})_N^{(2)} \\ & \cap G^{-\infty,1}\{m_3, m_4, 0, 0\}(\omega^{m_2}\Phi^{-1}, \tilde{g}_{\Phi,k})_N^{(3)} \end{aligned}$$

Similar is the case for the remainder $r_\nu^{(2)}$.

Substituting $a(t,x,D_x)$ in Theorem 4.4.1 with identity operator I and taking k sufficiently large, one can easily arrive at the following two corollaries.

Corollary 4.4.1. *There exists a $k^* > 1$ such that for $k \geq k^*$*

$$e^{\pm\nu(t)\Theta(x,D_x)} e^{\mp\nu(t)\Theta(x,D_x)} = I + R^{(\pm)}(t,x,D_x)$$

where $I + R^{(\pm)}$ are invertible operators with $\Theta(x,\xi)^{-1}\sigma(R^{(\pm)}) \in C([0,T]; G^{-1,-1}(\omega, \tilde{g}_{\Phi,k}))$.

Corollary 4.4.2. *Let $0 \leq \varepsilon \leq \varepsilon' < \delta^*$ where δ^* is as in Theorem 4.4.1. Then*

$$e^{\varepsilon\Theta(x,D_x)} e^{-\varepsilon'\Theta(x,D_x)} = e^{(\varepsilon-\varepsilon')\Theta(x,D_x)}\left(I + \hat{R}(x,D_x)\right)$$

where $\Theta(x,\xi)^{-1}\sigma(\hat{R}) \in G^{-1,-1}(\omega, \tilde{g}_{\Phi,k})$ and for sufficiently large k, $I + \hat{R}$ is invertible.

We use the above corollaries to prove the continuity of the operator $e^{\varepsilon\Theta(x,D_x)}$ on the spaces $\mathcal{H}^{s,\varepsilon'}_{\Phi,k;\Theta}(\mathbb{R}^n)$. The following proposition is helpful in making change of variable in Section 4.5.4.

Proposition 4.4.3. *The operator $e^{\varepsilon\Theta(x,D_x)} : \mathcal{H}^{s,\varepsilon'}_{\Phi,k;\Theta}(\mathbb{R}^n) \to \mathcal{H}^{s,\varepsilon'-\varepsilon}_{\Phi,k;\Theta}(\mathbb{R}^n)$ is continuous for $k \geq k_0$ and $0 \leq \varepsilon \leq \varepsilon' < \delta^*$ where k_0 sufficiently large and δ^* is as in Thereom 4.4.1.*

Proof. Consider w in $\mathcal{H}^{s,\varepsilon'}_{\Phi,k;\Theta}(\mathbb{R}^n)$. From Corollaries 4.4.1 and 4.4.2, we have

$$\begin{aligned} e^{-\varepsilon'\Theta(x,D_x)} e^{\varepsilon'\Theta(x,D_x)} &= I + R_1(x,D_x), \\ e^{\varepsilon\Theta(x,D_x)} e^{-\varepsilon'\Theta(x,D_x)} &= e^{(\varepsilon-\varepsilon')\Theta(x,D_x)}(I + R_2(x,D_x)), \\ e^{(\varepsilon'-\varepsilon)\Theta(x,D_x)} e^{-(\varepsilon'-\varepsilon)\Theta(x,D_x)} &= I + R_3(x,D_x). \end{aligned}$$

where $\Theta(x,\xi)^{-1}\sigma(R_j) \in G^{-1,-1}(\omega, \tilde{g}_{\Phi,k})$. For $k \geq k_0$, k_0 sufficiently large, the operators $I + R_j(x,D_x), j = 1,2,3$ are invertible. Then, one can write

$$e^{\varepsilon\Theta(x,D_x)} w = e^{\varepsilon(\Theta(x,D_x)}\left(e^{-\varepsilon'\Theta(x,D_x)} e^{\varepsilon'\Theta(x,D_x)} - R_1\right) w.$$

This implies that

$$e^{\varepsilon\Theta(x,D_x)}(I + R_1)w = e^{(\varepsilon-\varepsilon')\Theta(x,D_x)}(I + R_2)e^{\varepsilon'\Theta(x,D_x)}w. \tag{4.4.5}$$

From (4.4.5), we have

$$e^{(\varepsilon'-\varepsilon)\Theta(x,D_x)}e^{\varepsilon\Theta(x,D_x)}(I + R_1)w = (I + R_3)(I + R_2)e^{\varepsilon'\Theta(x,D_x)}w.$$

Note that $(I + R_j), j = 1, 2, 3$, are bounded and invertible operators. Substituting $w = (I + R_1)^{-1}v$ and taking L^2 norm on both sides of the above equation yields

$$\|e^{\varepsilon\Theta(x,D_x)}v\|_{\Phi,k;\Theta,\varepsilon'-\varepsilon,s} \leq C_1\|(I + R_1)^{-1}v\|_{\Phi,k;\Theta,\varepsilon',s} \leq C_2\|v\|_{\Phi,k;\Theta,\varepsilon',s} ,$$

for all $v \in \mathcal{H}^{s,\varepsilon'}_{\Phi,k;\Theta}(\mathbb{R}^n)$ and for some $C_1, C_2 > 0$. This proves the proposition. □

4.5 Global Well-Posedness

In this section, we give proofs of the main results. There are four key steps in the proofs of Theorem 4.1.1 and Theorem 4.1.2. First, we employ the diagonalization procedure available in the literature [76, 50] to arrive at a first order system with diagonalized principal part. Second, we localize the singularity to the regions $Z_{mid}(N)$ and $Z_{ext}(N)$. Third, using the locazation achieved in the previous step we define a function that majorizes the symbol with singularity. This helps in making appropriate change of variable to handle the singularity. Lastly, using sharp Gårding's inequality we arrive at an energy estimate that proves well-posedness of the problem in the Sobolev spaces $\mathcal{H}^{s,\delta}_{\Phi,k;\Theta}(\mathbb{R}^n)$.

4.5.1 A First Order System with a Diagonalized Principal Part

Let $\Lambda(x, D_x) = \Phi(x)\langle D_x\rangle_k$ and $\tilde{\Lambda}(x, D_x)$ be such that $\sigma(\tilde{\Lambda}) = \Phi(x)^{-1}\langle\xi\rangle_k^{-1}$. Observe that

$$\tilde{\Lambda}(x, D_x)\Lambda(x, D_x) = I + K(x, D_x)$$

where $\sigma(K) \in G^{m_1,m_2}_{\Phi,k}(-1, -1)$. We choose $k > k_0$ for large k_0 so that the operator norm of K is strictly lesser than 1. This guarantees

$$(I + K(x, D_x))^{-1} = \sum_{j=0}^{\infty}(-1)^j K(t, x, D_x)^j \in G^{m_1,m_2}_{\Phi,k}(0, 0).$$

The transformation $U = (U_1, U_2)^T = (\Lambda(x, D_x)u, D_t u)^T$ transfers the Cauchy problem (4.1.16) to

$$D_t U - AU = F, \quad U(0, x) = \begin{pmatrix} \Lambda(x, D_x)f_1(x) \\ -if_2(x) \end{pmatrix} \tag{4.5.1}$$

where $F := (0, -f)^T$ and

$$A := \begin{pmatrix} 0 & \Lambda(x, D_x) \\ (a(t, x, D_x) + b(t, x, D_x))\,(I + K(x, D_x))^{-1}\tilde{\Lambda}(x, D_x) & 0 \end{pmatrix}.$$

Consider a function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(s) \equiv 1$ for $|s| \leq 1, \chi(s) \equiv 0$ for $|s| \geq 2$ and $0 \leq \chi(s) \leq 1$. Let $N > k$ in the definition of $t_{x,\xi}$ and $\tilde{t}_{x,\xi}$, where k is appropriately chosen later in the discussion. We define the functions

$$\lambda_j(t,x,\xi) = d_j\chi(Z_{int}(N))\omega(x)\langle\xi\rangle_k + \chi(Z_{ext}(N))\tau_j(t,x,\xi), \quad j = 1,2$$

where $d_2 = -d_1$ is a positive constant and

$$\tau_j(t,x,\xi) = (-1)^j\sqrt{a(t,x,\xi)}, \quad a(t,x,\xi) := \sum_{l,m=1}^{n} a_{l,m}(t,x)\xi_l\xi_m.$$

Note that $a \in G^{2,2}\{1\}(\omega, g_{\Phi,k})_N^{(1)} \cap G^{2,2}\{0,1,0\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{2,2}\{0,0,1,0\}(\omega, g_{\Phi,k})_N^{(3)}$. Let χ_1, χ_2 and χ_3 be the indicator functions for the regions $Z_{int}(2N)$, $Z_{mid}(N)$ and $Z_{ext}(N)$, respectively. Observe that

i) $\lambda_j(t,x,\xi)$ is G_ω-elliptic symbol i.e. for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$

$$|\lambda_j(t,x,\xi)| \geq C\omega(x)\langle\xi\rangle_k,$$

for some $C > 0$ independent of k.

ii) $\lambda_j \in G^{1,1}\{0\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{1,1}\{0,1,1\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{1,1}\{0,0,1,1\}(\omega, g_{\Phi,k})_N^{(3)}$. More precisely, for $|\alpha| + |\beta| > 0$,

$$\left.\begin{aligned} |\lambda_j(t,x,\xi)| &\leq C_0\omega(x)\langle\xi\rangle_k\left\{\chi_1 + (\chi_2+\chi_3)\,\tilde{\theta}(t)\right\} \\ |\partial_\xi^\alpha\partial_x^\beta\lambda_j(t,x,\xi)| &\leq C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}\left\{\chi_1 + (\chi_2+\chi_3)\,\tilde{\theta}(t)^{|\alpha|+|\beta|}\right\} \end{aligned}\right\} \tag{4.5.2}$$

iii) $\partial_t\lambda_j \in G^{1,1}\{1\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{1,1}\{1,0,1\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{1,1}\{1,0,0,1\}(\omega, g_{\Phi,k})_N^{(3)}$. More precisely, for $|\alpha| + |\beta| > 0$,

$$\left.\begin{aligned} \partial_t\lambda_j &\sim 0 \quad \text{in } Z_{int}(N) \\ |\partial_t\lambda_j(t,x,\xi)| &\leq C_0\omega(x)\langle\xi\rangle_k\left\{\chi_1\tilde{\theta}(t) + (\chi_2+\chi_3)\,\frac{\theta(t)}{t}\right\} \\ |\partial_\xi^\alpha\partial_x^\beta\partial_t\lambda_j(t,x,\xi)| &\leq C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}\left\{\chi_1\tilde{\theta}(t) + (\chi_2+\chi_3)\,\frac{\theta(t)}{t}\tilde{\theta}(t)^{|\alpha|+|\beta|}\right\} \end{aligned}\right\} \tag{4.5.3}$$

We begin the diagonalization procedure by defining the matrix pseudodifferential operators $\mathcal{N}_1(t,x,D_x)$ and $\tilde{\mathcal{N}_1}(t,x,D_x)$ with symbols

$$\mathcal{N}_1(t,x,D_x) = \begin{pmatrix} I & I \\ \lambda_1(t,x,D_x)\tilde{\Lambda}(x,D_x) & \lambda_2(t,x,D_x)\tilde{\Lambda}(x,D_x) \end{pmatrix}, \quad \text{and}$$

$$\tilde{\mathcal{N}_1}(t,x,D_x) = \frac{1}{2}\Lambda(x,D_x)\tilde{\lambda}_2(t,x,D_x)I\begin{pmatrix} \lambda_2(t,x,D_x)\tilde{\Lambda}(x,D_x) & -I \\ -\lambda_1(t,x,D_x)\tilde{\Lambda}(x,D_x) & I \end{pmatrix},$$

where $\sigma\big(\tilde{\lambda}_2(t,x,D_x)\big) = \lambda_2(t,x,\xi)^{-1}$. Notice that $\mathcal{N}_1$ and $\tilde{\mathcal{N}}_1$ are elliptic and satisfy

$$\mathcal{N}_1(t,x,D_x)\tilde{\mathcal{N}}_1(t,x,D_x) = I + K_1(t,x,D_x)$$

with $\sigma(K_1)$ in

$$G^{-1,-1}\{0\}(\omega, g_{\Phi,k})_N^{(1)} \cap G^{-1,-1}\{0,2,1\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{-1,-1}\{0,0,2,1\}(\omega, g_{\Phi,k})_N^{(3)}.$$

From Propositions A.2.1-A.2.3 and Remark A.2.1, $\sigma(K_1) \in G^{-1+\varepsilon,-1+\varepsilon}(\Phi, g_{\Phi,k})$. We choose $k > k_1$ for large k_1 so that the operator norm of K_1 is strictly lesser than 1. This guarantees

$$(I + K_1(t,x,D_x))^{-1} = \sum_{j=0}^{\infty}(-1)^j K_1(t,x,D_x)^j \in C([0,T]; OPG_{\Phi,k}^{m_1,m_2}(0,0)).$$

We make the following change of variable

$$V_1(t,x) = \tilde{\mathcal{N}}_1(t,x,D_x)U(t,x). \tag{4.5.4}$$

Implying

$$\begin{aligned} D_tV_1 &= \tilde{\mathcal{N}}_1 D_tU + D_t\tilde{\mathcal{N}}_1 U \\ &= \tilde{\mathcal{N}}_1 A(I+K_1)^{-1}\mathcal{N}_1 V_1 + D_t\tilde{\mathcal{N}}_1(I+K_1)^{-1}\mathcal{N}_1 V_1 + F_1. \end{aligned}$$

Here $F_1 = \tilde{\mathcal{N}}_1 F$ and

- $\tilde{\mathcal{N}}_1 A(I+K_1)^{-1}\mathcal{N}_1 = A_1 + A_2 + A_3$ where

$$\begin{aligned} &\sigma(A_1) = diag(\lambda_1(t,x,\xi), \lambda_2(t,x,\xi)), \\ &\sigma(A_2) \in G^{1,1}\{1\}(\omega, g_{\Phi,k})_{2N}^{(1)}, \quad \sigma(A_2) \equiv 0 \text{ in } Z_{mid}(N) \cup Z_{ext}(N), \\ &\sigma(A_3) \in G^{0,0}\{0\}(\omega, g_{\Phi,k})_N^{(1)} \cap G^{0,0}\{0,0,0\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{0,0}\{0,0,0,0\}(\omega, g_{\Phi,k})_N^{(3)} \end{aligned}$$

- $D_t\tilde{\mathcal{N}}_1(I+K_1)^{-1}\mathcal{N}_1 = -\tilde{\mathcal{N}}_1 D_t\mathcal{N}_1 + B_0$, where $B_0 = D_tK_1 + D_t\tilde{\mathcal{N}}_1\left(\sum_{j=1}^{\infty}(-K_1)^j\right)\mathcal{N}_1$,

$$\begin{aligned} \sigma(\tilde{\mathcal{N}}_1)\sigma\left(D_t\mathcal{N}_1\right) &= \frac{1}{2}\begin{pmatrix} \frac{\Lambda(x,\xi)}{\lambda_1(t,x,\xi)} D_t \frac{\lambda_1(t,x,\xi)}{\Lambda(x,\xi)} & \frac{-\Lambda(x,\xi)}{\lambda_2(t,x,\xi)} D_t \frac{\lambda_2(t,x,\xi)}{\Lambda(x,\xi)} \\ \frac{-\Lambda(x,\xi)}{\lambda_1(t,x,\xi)} D_t \frac{\lambda_1(t,x,\xi)}{\Lambda(x,\xi)} & \frac{\Lambda(x,\xi)}{\lambda_2(t,x,\xi)} D_t \frac{\lambda_2(t,x,\xi)}{\Lambda(x,\xi)} \end{pmatrix}, \\ \sigma(\tilde{\mathcal{N}}_1 D_t\mathcal{N}_1) &\in G^{0,0}\{1\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{0,0}\{1,0,1\}(\omega, g_{\Phi,k})_N^{(2)} \\ &\quad \cap G^{0,0}\{1,0,0,1\}(\omega, g_{\Phi,k})_N^{(3)}, \\ \sigma(B_0) &\in G^{-1,-1}\{1\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{-1,-1}\{1,2,1\}(\omega, g_{\Phi,k})_N^{(2)} \\ &\quad \cap G^{-1,-1}\{1,0,2,1\}(\omega, g_{\Phi,k})_N^{(3)}, \end{aligned}$$

  and $\sigma(\tilde{\mathcal{N}}_1 D_t\mathcal{N}_1) = \sigma(B_0) = 0$ in $Z_{int}(N)$ by (4.5.3).

In summary, we have the following proposition.

Proposition 4.5.1. *The solution $U(t,x)$ to the first order system (4.5.1) is given by*

$$U(t,x) = (I + K_1(t,x,D_x))^{-1}\mathcal{N}_1(t,x,D_x)V_1(t,x)$$

where $V_1(t,x)$ is the solution to the following system

$$(D_t - \mathcal{D} + P_1 + P_2 + Q_1)V_1 = F_1,\ V_1(0,x) = \tilde{\mathcal{N}}_1(0,x,D_x)U(0,x) \tag{4.5.5}$$

The matrix pseudodifferential operators $\mathcal{D}, P_1, P_2, Q$ possess the following properties:

- *The operator $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$ is such that*

$$\begin{aligned}
&\sigma(\mathcal{D}_1) = diag\{\lambda_1(t,x,\xi), \lambda_2(t,x,\xi)\}\,,\\
&\sigma(\mathcal{D}_1) \in G^{1,1}\{0\}(\omega, g_{\Phi,k})^{(1)}_{2N} \cap G^{1,1}\{0,1,1\}(\omega, g_{\Phi,k})^{(2)}_{N}\\
&\qquad\qquad \cap G^{1,1}\{0,0,1,1\}(\omega, g_{\Phi,k})^{(3)}_{N},\\
&\sigma(\mathcal{D}_2) = \frac{1}{2}diag\left\{\frac{\Lambda(x,\xi)}{\lambda_1(t,x,\xi)}D_t\frac{\lambda_1(t,x,\xi)}{\Lambda(x,\xi)}, \frac{\Lambda(x,\xi)}{\lambda_2(t,x,\xi)}D_t\frac{\lambda_2(t,x,\xi)}{\Lambda(x,\xi)}\right\},\\
&\sigma(\mathcal{D}_2) \in G^{0,0}\{1\}(\omega, g_{\Phi,k})^{(1)}_{2N} \cap G^{0,0}\{1,0,1\}(\omega, g_{\Phi,k})^{(2)}_{N}\\
&\qquad\qquad \cap G^{0,0}\{1,0,0,1\}(\omega, g_{\Phi,k})^{(3)}_{N},\\
&\sigma(\mathcal{D}_2) = 0 \text{ in } Z_{int}(N).
\end{aligned}$$

- *P_1 is diagonal while P_2 is anti-diagonal and $\sigma(P_2) = 0$ in $Z_{int}(N)$.*
- $\sigma\left(P_1\right) \in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_{N} \cap G^{0,0}\{0,0,0\}(\omega, g_{\Phi,k})^{(2)}_{N} \cap G^{0,0}\{0,0,0,0\}(\omega, g_{\Phi,k})^{(3)}_{N}.$
- $\sigma\left(P_2\right) \in G^{0,0}\{1\}(\omega, g_{\Phi,k})^{(1)}_{2N} \cap G^{0,0}\{1,0,1\}(\omega, g_{\Phi,k})^{(2)}_{N} \cap G^{0,0}\{1,0,0,1\}(\omega, g_{\Phi,k})^{(3)}_{N}.$
- $\sigma(Q_1) \in G^{1,1}\{1\}(\omega, g_{\Phi,k})^{(1)}_{2N}, \sigma(Q) \equiv 0$ *in* $Z_{mid}(2N) \cup Z_{ext}(2N)$.

4.5.2 Localization in Z_{ext}

The main goal of this section is to localize the singularity arising from the first t-derivative of coefficients to $Z_{mid}(N)$ and the one from the second t-derivative to $Z_{ext}(N)$. We will not directly localize the first order system in Proposition 4.5.1 but a modified one after an application of a suitable transformation. To this end, consider the elliptic pseudodifferential operators $\mathcal{N}_2, \tilde{\mathcal{N}}_2$ with symbols

$$\sigma(\mathcal{N}_2) = \sigma(\tilde{\mathcal{N}}_2)^{-1} = \begin{pmatrix} \left(\frac{\lambda_1(t,x,\xi)}{\Lambda(x,\xi)}\right)^{1/2} & 0 \\ 0 & \left(\frac{\lambda_2(t,x,\xi)}{\Lambda(x,\xi)}\right)^{1/2} \end{pmatrix},$$

$\sigma(\mathcal{N}_2), \sigma(\tilde{\mathcal{N}}_2) \in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_{N} \cap G^{0,0}\{0,1,1\}(\omega, g_{\Phi,k})^{(2)}_{N} \cap G^{0,0}\{0,0,1,1\}(\omega, g_{\Phi,k})^{(3)}_{N}.$
Note that the symbols are constant in $Z_{int}(N)$. Let

$$\mathcal{N}_2(t,x,D_x)\tilde{\mathcal{N}}_2(t,x,D_x)(t,x,D_x) = I + K_2(t,x,D_x),$$

with $\sigma(K_2)$ in

$$G^{-1,-1}\{0\}(\omega,g_{\Phi,k})_N^{(1)}\cap G^{-1,-1}\{0,2,1\}(\omega,g_{\Phi,k})_N^{(2)}\cap G^{-1,-1}\{0,0,2,1\}(\omega,g_{\Phi,k})_N^{(3)}.$$

We choose $k>k_2$ for large k_2 so that the operator norm of K_2 is strictly lesser than 1. This guarantees

$$(I+K_2(t,x,D_x))^{-1}=\sum_{j=0}^{\infty}(-1)^jK_2(t,x,D_x)^j\in C([0,T];OPG^{m_1,m_2}_{\Phi,k}(0,0)).$$

We make the following change of variable

$$V_2(t,x)=\tilde{\mathcal{N}}_2(t,x,D_x)V_1(t,x).$$

Implying

$$\begin{aligned} D_tV_2&=\tilde{\mathcal{N}}_2D_tV_1+D_t\tilde{\mathcal{N}}_2V_1\\ &=\left(\tilde{\mathcal{N}}_2(\mathcal{D}_1+\mathcal{D}_2-P_1-P_2-Q)+D_t\tilde{\mathcal{N}}_2\right)(I+K_2)^{-1}\mathcal{N}_2V_2+F_2, \end{aligned}$$

where $F_2=\tilde{\mathcal{N}}_2F_1$. Observe that $\sigma(\tilde{\mathcal{N}}_2)\sigma(\mathcal{D}_2)+\sigma(D_t\tilde{\mathcal{N}}_2)=0$ as

$$\sigma(\tilde{\mathcal{N}}_2)\sigma(\mathcal{D}_2)=-\sigma(D_t\tilde{\mathcal{N}}_2)=\begin{pmatrix}\frac{1}{2}\frac{\Lambda(x,\xi)^{1/2}}{\lambda_1(t,x,\xi)^{3/2}}D_t\lambda_1(t,x,\xi) & 0\\ 0 & \frac{1}{2}\frac{\Lambda(x,\xi)^{1/2}}{\lambda_2(t,x,\xi)^{3/2}}D_t\lambda_2(t,x,\xi)\end{pmatrix}.$$

Here

$$\begin{aligned}\sigma(\tilde{\mathcal{N}}_2\mathcal{D}_2+D_t\tilde{\mathcal{N}}_2)\in G^{-1,-1}\{2\}(\omega,g_{\Phi,k})_{2N}^{(1)}&\cap G^{-1,-1}\{1,2,1\}(\omega,g_{\Phi,k})_N^{(2)}\\ &\cap G^{-1,-1}\{1,0,2,1\}(\omega,g_{\Phi,k})_N^{(3)}.\end{aligned}$$

To be precise, due to the presence of $D_t\lambda_j, j=1,2,$ in $\sigma(D_t\tilde{\mathcal{N}}_2)$ the singularity of order 2 in $Z_{int}(2N)$ appears only in the region $Z_{int}(2N)/Z_{int}(N)$ as $D_t\lambda_j=0$ in $Z_{int}(N)$. From the estimate (4.5.3) Remark A.2.1,

$$\begin{aligned}\sigma(\tilde{\mathcal{N}}_2\mathcal{D}_2+D_t\tilde{\mathcal{N}}_2)\in G^{0,0}\{0\}(\omega,g_{\Phi,k})_{2N}^{(1)}&\cap G^{0,0}\{1,0,0\}(\omega,g_{\Phi,k})_N^{(2)}\\ &\cap G^{-1,-1}\{1,0,2,1\}(\omega,g_{\Phi,k})_N^{(3)}.\end{aligned}$$

Let

$$\left.\begin{aligned}\tilde{P}_1&=\tilde{\mathcal{N}}_2P_1(I+K_2)^{-1}\mathcal{N}_2+\mathcal{D}_1-\tilde{\mathcal{N}}_2\mathcal{D}_1(I+K_2)^{-1}\mathcal{N}_2,\\ \tilde{P}_2&=\tilde{\mathcal{N}}_2P_2(I+K_2)^{-1}\mathcal{N}_2,\\ \tilde{Q}_1&=\tilde{\mathcal{N}}_2Q_1(I+K_2)^{-1}\mathcal{N}_2,\\ \tilde{Q}_2&=-\big(\tilde{\mathcal{N}}_2\mathcal{D}_2+D_t\tilde{\mathcal{N}}_2\big)(I+K_2)^{-1}\mathcal{N}_2.\end{aligned}\right\}\tag{4.5.6}$$

Here $\sigma(\tilde{P}_1)$ is in

$$G^{0,0}\{0\}(\omega,g_{\Phi,k})_N^{(1)}\cap G^{0,0}\{0,2,1\}(\omega,g_{\Phi,k})_N^{(2)}\cap G^{0,0}\{0,0,2,1\}(\omega,g_{\Phi,k})_N^{(3)}.$$

By Propositions A.2.1-A.2.3 and Remark A.2.1, $t^{1-\varepsilon}\sigma(\tilde{P}_1) \in G^{m_1,m_2}_{\Phi,k}(0,0)$ for every $\varepsilon \in (0,1)$. It is easy to see that $\tilde{P}_1$ is of diagonal structure while $\tilde{P}_2$ is of anti-diagonal and

$$\begin{aligned}
\sigma(\tilde{P}_1) &\in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{0,0}\{0,2,1\}(\omega, g_{\Phi,k})^{(2)}_N \\
&\quad \cap G^{0,0}\{0,0,2,1\}(\omega, g_{\Phi,k})^{(3)}_N, \\
\sigma(\tilde{P}_2) &\in G^{0,0}\{1\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{0,0}\{1,0,1\}(\omega, g_{\Phi,k})^{(2)}_N \\
&\quad \cap G^{0,0}\{1,0,0,1\}(\omega, g_{\Phi,k})^{(3)}_N, \\
\sigma(\tilde{Q}_1) &\in G^{1,1}\{1\}(\omega, g_{\Phi,k})^{(1)}_N, \sigma(\tilde{Q}_1) \equiv 0 \text{ in } Z_{mid}(2N) \cup Z_{ext}(2N), \\
\sigma(\tilde{Q}_2) &\in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_{2N} \cap G^{0,0}\{1,0,0\}(\omega, g_{\Phi,k})^{(2)}_N \\
&\quad \cap G^{-1,-1}\{1,0,2,1\}(\omega, g_{\Phi,k})^{(3)}_N.
\end{aligned}$$

Summarizing the above discussion we have the following proposition.

Proposition 4.5.2. *The solution $V_1(t,x)$ to first order system (4.5.5) is given by*

$$V_1(t,x) = (I + K_2(t,x,D_x))^{-1}\mathcal{N}_2(t,x,D_x)V_2(t,x)$$

where $V_2(t,x)$ satisfies the following system

$$(D_t - \mathcal{D}_1 + \tilde{P}_1 + \tilde{P}_2 + \tilde{Q}_1 + \tilde{Q}_2)V_2 = F_2,\ V_2(0,x) = \tilde{\mathcal{N}}_2(0,x,D_x)V_1(0,x). \tag{4.5.7}$$

The matrix pseudodifferential operators $\tilde{P}_1, \tilde{P}_2, \tilde{Q}_1, \tilde{Q}_2$ are as in (4.5.6) and $\mathcal{D}_1$ is as in Proposition 4.5.1.

Next, we aim to localize the singularity. Let

$$\sigma\left(\tilde{P}_2\right) = \begin{pmatrix} 0 & p_{12} \\ p_{21} & 0 \end{pmatrix}.$$

We define the elliptic pseudodifferential operators $\mathcal{N}_3(t,x,D_x)$ and $\tilde{\mathcal{N}}_3(t,x,D_x)$ with symbols

$$\sigma(\mathcal{N}_3(t,x,D_x)) = \sigma(\tilde{\mathcal{N}}_3(t,x,D_x))^{-1} = I + \eta(t,x,\xi)$$

where

$$\eta(t,x,\xi) := \left(1 - \chi(t/\tilde{t}_{x,\xi})\right)\begin{pmatrix} 0 & \frac{p_{12}}{2\lambda_1} \\ \frac{p_{21}}{2\lambda_2} & 0 \end{pmatrix} \in G^{-1,-1}\{1,0,0,1\}(\omega, g_{\Phi,k})^{(3)}_N.$$

The form of $\sigma(\tilde{\mathcal{N}}_3(t,x,D_x))$ is

$$\sigma(\tilde{\mathcal{N}}_3(t,x,D_x)) = \frac{1}{1-r}\begin{pmatrix} 1 & -(1-\chi)\frac{p_{12}}{2\lambda_1} \\ -(1-\chi)\frac{p_{21}}{2\lambda_2} & 1 \end{pmatrix},\ r = \left(1 - \chi(t/\tilde{t}_{x,\xi})\right)^2 \frac{p_{12}p_{21}}{4\lambda_1\lambda_2}.$$

The symbol $\sigma(\tilde{\mathcal{N}}_3(t,x,D_x))$ is well defined as we see that

$$|r(t,x,\xi)| \leq \frac{C}{\Phi(x)^2\langle\xi\rangle_k^2}\left(\frac{\theta(t)}{t}\right)^2 \leq \frac{C}{N^2} \text{ in } Z_{ext}(N)$$

and a large N ensures that $|r(t,x,\xi)| \le 1/2$ in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$. Let

$$\mathcal{N}_3(t,x,D_x)\tilde{\mathcal{N}}_3(t,x,D_x) = I + K_3(t,x,D_x),$$

with $\sigma(K_3) = 0$ in $Z_{int}(N)\cup Z_{mid}(N)$ and $\sigma(K_3)\in G^{-2,-2}\{1,0,0,1\}(\omega,g_{\Phi,k})^{(3)}_N$. From Propositions A.2.1-A.2.3 and Remark A.2.1, $\sigma(K_3)G^{-1+\varepsilon,-1+\varepsilon}(\Phi,g_{\Phi,k})$. We choose $k>k_3$ for large k_3 so that the operator norm of K_3 is strictly lesser than 1. This guarantees

$$(I+K_3(t,x,D_x))^{-1} = \sum_{j=0}^{\infty}(-1)^j K_3(t,x,D_x)^j \in C([0,T];OPG^{m_1,m_2}_{\Phi,k}(0,0)).$$

We make a following change of variable

$$V_3(t,x) = \tilde{\mathcal{N}}_3(t,x,D_x)V_2(t,x)$$

Implying

$$\begin{aligned} D_tV_3 &= \tilde{\mathcal{N}}_3 D_t V_2 + D_t\tilde{\mathcal{N}}_3 V_2 \\ &= \left(\tilde{\mathcal{N}}_3(\mathcal{D}_1 - \tilde{P}_1 - \tilde{P}_2 - \tilde{Q}_1 - \tilde{Q}_2) + D_t\tilde{\mathcal{N}}_3\right)(I+K_3)^{-1}\mathcal{N}_3V_3 + F_3 \end{aligned}$$

where $F_3 = \tilde{\mathcal{N}}_3F_2$. Let us write

$$\tilde{\mathcal{N}}_3(\mathcal{D}_1-\tilde{P}_2)(I+K_3)^{-1}\mathcal{N}_3 = \tilde{\mathcal{N}}_3(\mathcal{D}_1-\tilde{P}_2)\mathcal{N}_3 + \tilde{\mathcal{N}}_3(\mathcal{D}_1-\tilde{P}_2)\left(\sum_{j=1}^{\infty}(-K_3)^j\right)\mathcal{N}_3.$$

We see that

$$\begin{aligned} &\sigma(\tilde{\mathcal{N}}_3)\sigma(\mathcal{D}_1)\sigma(\mathcal{N}_3) \\ &= \frac{1}{1-r}\begin{pmatrix} 1 & -(1-\chi)\frac{p_{12}}{2\lambda_1} \\ -(1-\chi)\frac{p_{21}}{2\lambda_2} & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 & (1-\chi)\frac{p_{12}}{2\lambda_1} \\ (1-\chi)\frac{p_{21}}{2\lambda_2} & 1 \end{pmatrix} \\ &= \frac{1}{1-r}\left\{\mathcal{D}_1 + (1-\chi)^2\frac{p_{12}p_{21}}{4\lambda_2}\mathrm{diag}\{1,-1\} + (1-\chi)\tilde{P}_2\right\} \\ &= \mathcal{D}_1 + (1-\chi)\tilde{P}_2 + \frac{r}{1-r}\left(\mathcal{D}_1+\tilde{P}_2\right) + \frac{1}{1-r}(1-\chi)^2\frac{p_{12}p_{21}}{4\lambda_2}\mathrm{diag}\{1,-1\} \\ &= \mathcal{D}_1 + (1-\chi)\tilde{P}_2 \mod G^{-1,-1}\{2,0,2,1\}(\omega,g_{\Phi,k})^{(3)}_N. \end{aligned}$$

Similarly,

$$\begin{aligned} &\sigma(\tilde{\mathcal{N}}_3)\sigma(\tilde{P}_2)\sigma(\mathcal{N}_3) \\ &= \frac{1}{1-r}\begin{pmatrix} 1 & -(1-\chi)\frac{p_{12}}{2\lambda_1} \\ -(1-\chi)\frac{p_{21}}{2\lambda_2} & 1 \end{pmatrix}\begin{pmatrix} 0 & p_{12} \\ p_{21} & 0 \end{pmatrix}\begin{pmatrix} 1 & (1-\chi)\frac{p_{12}}{2\lambda_1} \\ (1-\chi)\frac{p_{21}}{2\lambda_2} & 1 \end{pmatrix} \\ &= \tilde{P}_2 + \frac{r}{1-r}\tilde{P}_2 - \frac{1}{1-r}\left((1-\chi)\frac{p_{12}p_{21}}{2\lambda_1} - \frac{p_{12}p_{21}}{2\lambda_2}\right)I - \frac{(1-\chi)}{1-r}\begin{pmatrix} 0 & \frac{p_{12}^2p_{21}}{4\lambda_1^2} \\ \frac{p_{12}p_{21}^2}{4\lambda_2^2} & 0 \end{pmatrix} \\ &= \tilde{P}_2 \mod G^{-1,-1}\{2,0,2,1\}(\omega,g_{\Phi,k})^{(3)}_N. \end{aligned}$$

Thus $\sigma(\tilde{\mathcal{N}}_3)\left(\sigma(\mathcal{D}_1)-\sigma(\tilde{P}_2)\right)\sigma(\mathcal{N}_3)=\mathcal{D}_1-\chi\tilde{P}_2 \mod G^{-1,-1}\{2,0,2,1\}(\omega,g_{\Phi,k})^{(3)}_N$. Note that $\sigma(\chi\tilde{P}_2)=0$ in $Z_{int}(N)$ and

$$\sigma(\chi\tilde{P}_2)\in G^{0,0}\{1\}(\omega,g_{\Phi,k})^{(1)}_{2N}\cap G^{0,0}\{1,0,1\}(\omega,g_{\Phi,k})^{(2)}_{2N},\ \sigma(\chi\tilde{P}_2)\equiv 0 \text{ in } Z_{ext}(N).$$

From the structure of $\tilde{P}_2$ and the estimate on the second derivative in time of the characteristics,

$$\sigma(D_t\tilde{\mathcal{N}}_3)\in G^{-1,-1}\{2,1,2,1\}(\omega,g_{\Phi,k})^{(3)}_N,\ \sigma(D_t\tilde{\mathcal{N}}_3)\equiv 0 \text{ in } Z_{int}(N)\cup Z_{mid}(N).$$

Let

$$\begin{aligned}
P_3&=\tilde{\mathcal{N}}_3\tilde{P}_1(I+K_3)^{-1}\mathcal{N}_3,\\
P_4&=\tilde{\mathcal{N}}_3\left(\tilde{P}_2+\tilde{Q}_2-\mathcal{D}_1\right)(I+K_3)^{-1}\mathcal{N}_3+\mathcal{D}_1-D_t\tilde{\mathcal{N}}_3.\\
Q_3&=\tilde{\mathcal{N}}_3\tilde{Q}_1(I+K_3)^{-1}\mathcal{N}_3.
\end{aligned}\tag{4.5.8}$$

Since

$$G^{-1,-1}\{2,0,2,1\}(\omega,g_{\Phi,k})^{(3)}_N\subset G^{-1,-1}\{2,1,2,1\}(\omega,g_{\Phi,k})^{(3)}_N,$$

it is easy to see that

$$\begin{aligned}
\sigma(P_3)&\in G^{0,0}\{0\}(\omega,g_{\Phi,k})^{(1)}_N\cap G^{0,0}\{0,2,1\}(\omega,g_{\Phi,k})^{(2)}_N\\
&\qquad\cap G^{0,0}\{0,0,2,1\}(\omega,g_{\Phi,k})^{(3)}_N,\\
\sigma(P_4)&\in G^{0,0}\{1\}(\omega,g_{\Phi,k})^{(1)}_N\cap G^{0,0}\{1,0,1\}(\omega,g_{\Phi,k})^{(2)}_N\\
&\qquad\cap G^{-1,-1}\{2,1,2,1\}(\omega,g_{\Phi,k})^{(3)}_N,\\
\sigma(Q_3)&\in G^{1,1}\{1\}(\omega,g_{\Phi,k})^{(1)}_N,\sigma(Q_3)\equiv 0 \text{ in } Z_{mid}(2N)\cup Z_{ext}(2N).
\end{aligned}\tag{4.5.9}$$

Let us summarize the above discussion in the following proposition.

Proposition 4.5.3. *The solution $V_2(t,x)$ to the first order system (4.5.7) is given by*

$$V_2(t,x)=(I+K_3(t,x,D_x))^{-1}\mathcal{N}_3(t,x,D_x)V_3(t,x)$$

where $V_3(t,x)$ is solution to first order system

$$(D_t-\mathcal{D}_1+P_3+P_4+Q_3)V_3=F_3,\ V_3(0,x)=\tilde{\mathcal{N}}_3(0,x,D_x)V_2(0,x)\tag{4.5.10}$$

The matrix pseudodifferential operator $\mathcal{D}_1$ is as in Proposition 4.5.1 while P_3,P_4,Q_3 are as in (4.5.8)-(4.5.9).

Remark 4.5.1. *The localization technique in our case has led to such a factorization where the regularity in the t-variable was lost along the way. Infact, after the localization procedure no new control of the t-derivative of the symbol P_4 was available.*

Remark 4.5.2. *Let us explain the philosophy of our approach. We use a careful amalgam of a localization technique on the extended phase space (defined in Section 4.2) and the diagonalization procedure already available in the literature (see [76, 50]) to handle the singularity. Note that we have restricted the singularity arising from the first t-derivative to $Z_{mid}(N)$ and the one from the second t-derivative to $Z_{ext}(N)$. This kind of localization of the singularities allows one to come with a function (see (4.5.11)) with a good estimate as in (4.5.13) that majorizes the symbol of the operator P_4+Q_3. It is woth noting that we do not need the so called "perfect diagnalization" in our analysis.*

4.5.3 An Upper Bound for the Lower Order Terms with Singularity

In this section, we define a function that majorizes $\sigma(P_4) + \sigma(Q_3)$. Consider a smooth function $\mathfrak{M}_1(t, x, \xi)$ of the form

$$\begin{aligned}\mathfrak{M}_1(t, x, \xi) = \kappa\Big(\chi(t/t_{x,\xi})\omega(x)\langle\xi\rangle_k\tilde{\theta}(t) + (1 - \chi(t/t_{x,\xi})\Big(\chi(t/\tilde{t}_{x,\xi})\frac{\theta(t)}{t} \\ + \frac{(1 - \chi(t/\tilde{t}_{x,\xi}))}{\Phi(x)\langle\xi\rangle_k}\frac{\theta(t)^2}{t^2}e^{\psi(t)}\tilde{\theta}(t)^2\Big)\Big)\end{aligned} \tag{4.5.11}$$

where $\kappa > 0$ is chosen in such way that we have

$$|\sigma(P_4)| + |\sigma(Q_3)| \leq \mathfrak{M}_1.$$

From Propositions A.2.1 - A.2.3, $t^{1-\varepsilon}\mathfrak{M}_1 \in C([0, T]; G^{1,1}(\Phi, g_{\Phi,k}))$, for every $\varepsilon \in (0, 1)$.

When $\tilde{\theta}(t)$ is unbounded near $t = 0$, we use the following estimate

$$\int_0^t \tilde{\theta}(s)ds \lesssim t\tilde{\theta}(t). \tag{4.5.12}$$

Observe that the estimate (4.5.12) is natural in the context of logarithmic-type functions. Using the estimate (4.5.12), we readily have

$$\begin{aligned}\int_0^{2t_{x,\xi}} |\partial_\xi^\alpha D_x^\beta \mathfrak{M}_1(s, x, \xi)|ds &\leq \kappa_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}\int_0^{2t_{x,\xi}} \tilde{\theta}(t)dt \\ &\leq \kappa_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}2t_{x,\xi}\tilde{\theta}(2t_{x,\xi}) \\ &\leq \kappa_{\alpha\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\tilde{\theta}(h)\theta(h).\end{aligned}$$

Similarly using the definitions of $t_{x,\xi}$ and $\tilde{t}_{x,\xi}$, we have the following estimates

$$\begin{aligned}\int_{t_{x,\xi}}^{2\tilde{t}_{x,\xi}} |\partial_\xi^\alpha D_x^\beta \mathfrak{M}_1(s, x, \xi)|ds &\leq \kappa_{\alpha\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\left|\int_{t_{x,\xi}}^{2\tilde{t}_{x,\xi}} \frac{\theta(s)}{s}ds\right| \\ &\leq \kappa_{\alpha\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\theta(h)\left|\int_{t_{x,\xi}}^{2\tilde{t}_{x,\xi}} \frac{1}{s}ds\right| \\ &\leq \kappa_{\alpha\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\theta(h)(\ln 2 + |\ln\tilde{\theta}(h)| + \psi(h)),\end{aligned}$$

$$\begin{aligned}\int_{\tilde{t}_{x,\xi}}^{T} |\partial_\xi^\alpha D_x^\beta \mathfrak{M}_1(s, x, \xi)|ds &\leq \kappa_{\alpha\beta}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}\left|\int_{\tilde{t}_{x,\xi}}^{T} \frac{\theta(s)^2}{s^2}e^{\psi(s)}\tilde{\theta}(s)^2ds\right| \\ &\leq \kappa_{\alpha\beta}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}\theta(h)^2\tilde{\theta}(h)^2e^{\psi(h)}\left|\int_{\tilde{t}_{x,\xi}}^{T} \frac{1}{s^2}ds\right| \\ &\leq \kappa_{\alpha\beta}\Phi(x)^{-1-|\beta|}\langle\xi\rangle_k^{-1-|\alpha|}\theta(h)^2\tilde{\theta}(h)^2e^{\psi(h)}\frac{\Phi(x)\langle\xi\rangle_k}{N\tilde{\theta}(h)\theta(h)e^{\psi(h)}} \\ &\leq \kappa_{\alpha\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\tilde{\theta}(h)\theta(h).\end{aligned}$$

Thus we have,

$$\int_0^T |D_x^\beta \partial_\xi^\alpha \mathfrak{M}_1(t,x,\xi)| dt \le \kappa_{\alpha\beta} \Theta(x,\xi)\Phi(x)^{-|\beta|}\langle \xi \rangle^{-|\alpha|}. \tag{4.5.13}$$

The function $\mathfrak{M}_1$ plays an important role in performing conjugation and thus in quantifying the loss of regularity.

4.5.4 At most Finite Loss of Regularity via Energy Estimate

In this section, we derive an energy estimate when (4.1.9) is satisfied and show that the loss is finite. Consider the operator L defined by

$$L = D_t - \mathcal{D}_1 + P_3 + P_4 + Q_3,$$

the matrix pseudodifferential operators $\mathcal{D}, P_3, P_4, Q_3$ are as in the Proposition 4.5.3. Then the first order system (4.5.10) is equivalent to

$$LV_3 = F_3, \; V_3(0,x) = \tilde{\mathcal{N}}_3(0,x,D_x)V_2(0,x). \tag{4.5.14}$$

Note that to prove the estimate (4.1.23), it is sufficient to consider the case $s = (0,0)$ as the operator $\Phi(x)^{s_2}\langle D \rangle_k^{s_1} L \langle D \rangle_k^{-s_1}\Phi(x)^{-s_2}$, where $s = (s_1, s_2)$ is the index of the weighted Sobolev space, satisfies the same hypotheses as L.

We perform a change of variable, which allows us to control lower order terms. We set

$$V_4(t,x) = W_1(t,x,D_x)V_3(t,x), \tag{4.5.15}$$

where W_1 is a finite order pseudodifferential operator with $\sigma(W_1)(t,x,\xi) = e^{-\int_0^t \mathfrak{M}_1(r,x,\xi)dr}$ for $\mathfrak{M}_1$ is as in (4.5.11). We have $V_4(0,x) = V_3(0,x)$ and for $0 < t \le T$

$$||V_3(t)||_{\Phi,k;\Theta,-\kappa_0,0e} \le C||V_4(t)||_{L^2},$$

where $\kappa_0 = \kappa_{00}C_0^* > 0$. Here κ_{00} is the constant $\kappa_{\alpha\beta}$ in (4.5.13) with $\alpha = \beta = 0$ and C_0^* is as in (4.1.9). Let $\tilde{W}_1(t,x,D_x)$ be such that $\sigma(\tilde{W}_1) = \exp\left(\int_0^t \mathfrak{M}_1(r,x,\xi)dr\right)$. Then

$$\begin{aligned} \tilde{W}_1(t,x,D_x)W_1(t,x,D_x) &= I + K_4^{(1)}(t,x,D_x), \\ W_1(t,x,D_x)\tilde{W}_1(t,x,D_x) &= I + K_4^{(2)}(t,x,D_x) \end{aligned}$$

where $\Theta(x,\xi)\sigma(K_4^{(l)}) \in C([0,T]; G^{-1,-1}(\omega, \tilde{g}_{\Phi,k})), l = 1,2$. We choose $k > k_4$ for large k_4 so that the operator norm of $K_4^{(l)}$ is strictly lesser than 1. This guarantees

$$(I + K_4^{(l)}(t,x,D_x))^{-1} = \sum_{j=0}^{\infty}(-1)^j K_4^{(l)}(t,x,D_x)^j \in C([0,T]; OPG^{0,0}(\omega, \tilde{g}_{\Phi,k})), \; l = 1,2.$$

For the sake of simplicity, let us denote the operators $\left(I + K_4^{(1)}\right)^{-1}$ and $\left(I + K_4^{(2)}\right)^{-1}$ by $\mathcal{K}_1$ and $\tilde{\mathcal{K}}_1$, respectively.

In the following we arbitrarily fix $\varepsilon \in (0,1)$ when $\tilde{\theta}$ is unbounded while for the bounded case we set $\varepsilon = 1$. We see that the pseudodifferential system (4.5.14) is equivalent to

$$\tilde{L}(t,x,D_t,D_x)V_4(t,x) = F_4(t,x), \tag{4.5.16}$$

where $\tilde{L} = L - i\mathfrak{M}_1(t,x,D_x)I + P_5(t,x,D_x)$, $F_4 = W_1(t,x,D_x)F_3$ and

$$\begin{aligned}\sigma(P_5) = \tilde{\mathcal{K}}_1 W_1 \mathcal{K}_1^{-1}\left(\left(P_3+P_4+Q_3-\mathcal{D}_1\right)\mathcal{K}_1 + D_t\mathcal{K}_1\right)\tilde{W}_1 - (P_3+P_4+Q_3-\mathcal{D}_1)\\ + (\tilde{\mathcal{K}}_1 W_1 D_t \tilde{W}_1 + i\mathfrak{M}_1 I),\end{aligned}$$

where $t^{1-\varepsilon}\Theta(x,\xi)^{-1}\sigma(P_5) \in C([0,T]; G^{0,0}(\omega,\tilde{g}_{\Phi,k})$. Let $\kappa_1 > 0$ be such that

$$|\sigma(P_5)| \leq \mathfrak{M}_2(t,x,\xi) = \kappa_1 t^{-1+\varepsilon}\Theta(x,\xi). \tag{4.5.17}$$

We make a further change of variable

$$V_5(t,x) = W_2(t,x,D_x)V_4(t,x) \tag{4.5.18}$$

where W_2 is a finite order pseudodifferential operator with $\sigma(W_2)(t,x,\xi) = e^{-\kappa_1 \frac{t^\varepsilon}{\varepsilon}\Theta(x,\xi)}$. We have $V_5(0,x) = V_4(0,x)$ and for $0 < t \leq T$

$$||V_4(t)\|_{\Phi,k;\Theta,-\kappa_1^*,0e} \leq C||V_5(t)||_{L^2},$$

where $\kappa_1^*(t) = \kappa_1\frac{t^\varepsilon}{\varepsilon}$. Let $\tilde{W}_2(t,x,D_x)$ be such that $\sigma(\tilde{W}_2) = e^{\kappa_1\frac{t^\varepsilon}{\varepsilon}\Theta(x,\xi)}$. Then

$$\begin{aligned}\tilde{W}_2(t,x,D_x)W_2(t,x,D_x) &= I + K_5^{(1)}(t,x,D_x),\\ W_2(t,x,D_x)\tilde{W}_2(t,x,D_x) &= I + K_5^{(2)}(t,x,D_x)\end{aligned}$$

where $\Theta(x,\xi)^{-1}\sigma(K_5^{(l)}) \in C([0,T]; G^{-1,-1}(\omega,\tilde{g}_{\Phi,k})), l = 1,2$. We choose $k > k_5$ for large k_5 so that the operator norm of $K_5^{(l)}$ is strictly lesser than 1. This guarantees

$$(I + K_5^{(l)}(t,x,D_x))^{-1} = \sum_{j=0}^{\infty}(-1)^j K_5^{(l)}(t,x,D_x)^j \in C([0,T]; OPG^{0,0}(\omega,\tilde{g}_{\Phi,k})),\ l = 1,2.$$

Let us denote the operators $\left(I + K_5^{(1)}\right)^{-1}$ and $\left(I + K_5^{(2)}\right)^{-1}$ by $\mathcal{K}_2$ and $\tilde{\mathcal{K}}_2$, respectively.

We see that the pseudodifferential system (4.5.16) is equivalent to

$$\left(\tilde{L}(t,x,D_t,D_x) - i\mathfrak{M}_2(t,x,D_x)I + P_6(t,x,D_x)\right)V_5(t,x) = F_5(t,x), \tag{4.5.19}$$

where $F_5 = W_2(t,x,D_x)F_4$ and

$$\begin{aligned}\sigma(P_6) = \tilde{\mathcal{K}}_2 W_2 \mathcal{K}_2^{-1}\left(\left(P_3+P_4+Q_3+P_5-\mathcal{D}_1\right)\mathcal{K}_2 + D_t\mathcal{K}_2\right)\tilde{W}_2\\ - (P_3+P_4+Q_3+P_5-\mathcal{D}_1) + (\tilde{\mathcal{K}}_2 W_2 D_t\tilde{W}_2 + i\mathfrak{M}_2 I),\end{aligned}$$

where $t^{1-\varepsilon}\sigma(P_6) \in C([0,T]; G^{0,0}(\omega,\tilde{g}_{\Phi,k})$. Let us write down the first order system (4.5.19) explicitly as below

$$\partial_t V_5 = \left(i\mathcal{D}_1 - ((\mathfrak{M}_1 + \mathfrak{M}_2)I + iP_4 + iP_5) - iP_3 - iP_6\right)V_5 + iF_5,$$

Here, the diagonal matrix operator, $i\mathcal{D}$ is of the form

$$i\mathcal{D}_1 = \operatorname{diag}\{i\lambda_1, i\lambda_2\}.$$

Observe that the symbol $d(t,x,\xi)$ of the operator $i\mathcal{D}_1 - i\mathcal{D}_1^*$ is such that

$$d \in G^{0,0}\{0\}(\omega, g_{\Phi,k})_N^{(1)} \cap G^{0,0}\{0,1,1\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{0,0}\{0,0,1,1\}(\omega, g_{\Phi,k})_N^{(3)}.$$

It follows from Propositions A.2.1-A.2.3 and Remark A.2.1 that

$$t^{1-\varepsilon} d \in C([0,T]; G_{\Phi,k}^{m_1,m_2}(0,0)).$$

Hence, by Calderón-Vaillancourt theorem,

$$2\operatorname{Re}\langle i\mathcal{D}_1 V_5, V_5\rangle \leq \frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle,\ C>0. \tag{4.5.20}$$

Further, by the choice of the functions, $\mathfrak{M}_1(t,x,\xi) + \mathfrak{M}_2(t,x,\xi)$

$$\operatorname{Re}((\mathfrak{M}_1(t,x,\xi) + \mathfrak{M}_2(t,x,\xi))I + i\sigma(P_4) + i\sigma(Q_3) + i\sigma(P_5)) \geq 0.$$

Assuming

$$\omega(x) \lesssim \Phi(x),$$

we apply sharp Gårding inequality (see Thereom B.0.1 in Appendix B and [43, Theorem 18.6.14]) to $2\operatorname{Re}(\mathfrak{M}_1 I + iP_4 + iQ_3)$ with the metric $g_{\Phi,k}$ and Planck function $h(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$ and to $2\operatorname{Re}(\mathfrak{M}_2 I + iP_5)$ with the metric $\tilde{g}_{\Phi,k}$ and Planck function $\tilde{h}(x,\xi) = \Theta(x,\xi)(\Phi(x)\langle\xi\rangle_k)^{-1}$. We obtain

$$2\operatorname{Re}\langle((\mathfrak{M}_1+\mathfrak{M}_2)I + iP_4 + iQ_3 + iP_5)V_5, V_5\rangle_{L^2} \geq -\frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle_{L^2},\ C>0. \tag{4.5.21}$$

Since $t^{1-\varepsilon}\sigma(iP_3 + iP_6)$ is uniformly bounded, by Calderon-Vaillancourt theorem we have

$$-2\operatorname{Re}\langle i(P_3+P_6)V_5, V_5\rangle \leq \frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle. \tag{4.5.22}$$

From (4.5.20)-(4.5.22) it follows that

$$2\operatorname{Re}\langle(i\mathcal{D}_1 - ((\mathfrak{M}_1+\mathfrak{M}_2)I + iP_4 + iQ_3 + iP_5) - iP_6 - iP_3)V_5, V_5\rangle_{L^2} \leq \frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle_{L^2}.$$

This yields

$$\partial_t ||V_5(t,\cdot)||_{L^2}^2 \leq C(t^{-1+\varepsilon}\|V_5(t,\cdot)\|_{L^2}^2 + \|F_5(t,\cdot)\|_{L^2}^2),\ t\in[0,T].$$

Considering the above inequality as a differential inequality, we apply Gronwall's lemma and obtain that

$$\|V_5(t,\cdot)\|_{L^2}^2 \leq C_\varepsilon\left(\|V_5(0,\cdot)\|_{L^2}^2 + \int_0^t \|F_5(\tau,\cdot)\|_{L^2}^2\, d\tau\right), \quad t\in[0,T].$$

If $\tilde{\theta}$ is unbounded $C_\varepsilon = C' e^{T^\varepsilon/\varepsilon}$ for a fixed $\varepsilon\in(0,1)$, else $C_\varepsilon = C''$ for some $C', C'' > 0$.

This proves well-posedness of the auxiliary Cauchy problem (4.5.19). Note that the solution V_3 to (4.5.14) belongs to $C\left([0,T];\mathcal{H}_{\Phi,k;\Theta}^{s,\kappa(t)}(\mathbb{R}^n)\right)$, $\kappa(t) = \kappa_0 + \kappa_1\frac{t^\varepsilon}{\varepsilon}$. Returning to our original solution $u = u(t,x)$ we obtain the estimate (4.1.23) with

$$u \in C\left([0,T];\mathcal{H}_{\Phi,k;\Theta}^{s+e,\kappa(t)}(\mathbb{R}^n)\right)\bigcap C^1\left([0,T];\mathcal{H}_{\Phi,k;\Theta}^{s,\kappa(t)}(\mathbb{R}^n)\right).$$

Depending the order of the operator $\Theta(x,D_x)$ we have zero, arbitrarily small or finite loss. This concludes the proof of Theorem 4.1.1.

4.5.5 Infinite Loss of Regularity via Energy Estimate

In this section, we derive an energy estimate when (4.1.10) is satisfied and show that the loss is infinite. Consider the Cauchy problem (4.5.14). Note that to prove the estimate (4.1.24), it is sufficient to consider the case $s = (0,0)$ as the operator $\Phi(x)^{s_2}\langle D\rangle_k^{s_1} L \langle D\rangle_k^{-s_1}\Phi(x)^{-s_2}$, where $s = (s_1, s_2)$ is the index of the Sobolev space, satisfies the same hypotheses as L.

We perform following change of variable

$$V_4(t,x) = W_1(t,x,D_x)V_3(t,x), \tag{4.5.23}$$

where W_1 is a infinite order pseudodifferential operator with

$$\sigma(W_1)(t,x,\xi) = \exp\Big(\int_t^{T_1^*} \mathfrak{M}_1(r,x,\xi)dr\Big)$$

for $\mathfrak{M}_1$ is as in (4.5.11). We have $V_4(0,x) = W_1(0,x,D_x)V_3(0,x)$ and for $0 < t \leq T$

$$||V_3(t)\|_{\Phi,k;\Theta,\tilde{\kappa}_0,0e} \leq C||V_4(t)||_{L^2},$$

where $C > 0$ and $e = (1,1)$. Here $\tilde{\kappa}_0$ is the constant $\kappa_{\alpha\beta}$ in (4.5.13) with $\alpha = \beta = 0$. Let $\tilde{W}_1(t,x,D_x)$ be such that $\sigma(\tilde{W}_1) = \exp\Big(-\int_t^{T_1^*}\mathfrak{M}_1(r,x,\xi)dr\Big)$. Then, by Theorem 4.4.1,

$$\begin{aligned} \tilde{W}_1(t,x,D_x)W_1(t,x,D_x) &= I + K_4^{(1)}(t,x,D_x),\\ W_1(t,x,D_x)\tilde{W}_1(t,x,D_x) &= I + K_4^{(2)}(t,x,D_x) \end{aligned}$$

where $\Theta(x,\xi)^{-1}\sigma(K_4^{(l)}) \in C([0,T]; G^{-1,-1}(\omega,\tilde{g}_{\Phi,k})), l = 1,2$. We choose $k > k_4$ for large k_4 so that the operator norm of $K_4^{(l)}$ is strictly lesser than 1. This guarantees the existence of $(I + K_4^{(l)}(t,x,D_x))^{-1}, l = 1,2$. For the sake of simplicity, let us denote the operators $\big(I + K_4^{(1)}\big)^{-1}$ and $\big(I + K_4^{(2)}\big)^{-1}$ by $\mathcal{K}_1$ and $\tilde{\mathcal{K}}_1$, respectively.

In the following we arbitrarily fix $\varepsilon \in (0,1)$ when $\tilde{\theta}$ is unbounded while for the bounded case we set $\varepsilon = 1$. We see that the pseudodifferential system (4.5.10) is equivalent to

$$\tilde{L}(t,x,D_t,D_x)V_4(t,x) = F_4(t,x), \tag{4.5.24}$$

where $\tilde{L} = L - i\mathfrak{M}_1(t,x,D_x)I + P_5(t,x,D_x)$, $F_4 = W_1(t,x,D_x)F_3$ and

$$\begin{aligned}\sigma(P_5) = \tilde{\mathcal{K}}_1 W_1 \mathcal{K}_1^{-1}\left((P_3 + P_4 + Q_3 - \mathcal{D}_1)\mathcal{K}_1 + D_t\mathcal{K}_1\right)\tilde{W}_1 + (\tilde{\mathcal{K}}_1 W_1 D_t \tilde{W}_1 + i\mathfrak{M}_1 I)\\ - (P_3 + P_4 + Q_3 - \mathcal{D}_1).\end{aligned}$$

From Theorem 4.4.1, $t^{1-\varepsilon}\Theta(x,\xi)^{-1}\sigma(P_5) \in C([0,T_1^*]; G^{0,0}(\Phi,\tilde{g}_{\Phi,k})$. T_1^* is chosen in such a way that all the above conjugations with operator W_1 are valid in view of Theorem 4.4.1.

Let $\tilde{\kappa}_1 > 0$ be such that

$$|\sigma(P_5)| \leq \mathfrak{M}_2(t,x,\xi) = \tilde{\kappa}_1 t^{-1+\varepsilon}\Theta(x,\xi). \tag{4.5.25}$$

We make a further change of variable

$$V_5(t,x) = W_2(t,x,D_x)V_4(t,x), \tag{4.5.26}$$

where W_2 is an infinite order pseudodifferential operator with

$$\sigma(W_2)(t,x,\xi) = \exp\left\{\int_t^{T_2^*} \mathfrak{M}_2(s,x,\xi)ds\right\}.$$

We have $V_5(0,x) = W_2(0,x,D_x)V_4(0,x)$ and

$$||V_4(t)\|_{\Phi,k;\Theta,\kappa_1^*,0e} \leq C||V_5(t)||_{L^2},$$

where $\kappa_1^*(t) = \tilde{\kappa}_1(T_2^{*\varepsilon} - t^\varepsilon)/\varepsilon$. Let $\tilde{W}_2(t,x,D_x)$ be such that $\sigma(\tilde{W}_2) = e^{-\int_t^{T_2^*} \mathfrak{M}_2(s,x,\xi)ds}$. Then, by Theorem 4.4.1,

$$\tilde{W}_2(t,x,D_x)W_2(t,x,D_x) = I + K_5^{(1)}(t,x,D_x),$$
$$W_2(t,x,D_x)\tilde{W}_2(t,x,D_x) = I + K_5^{(2)}(t,x,D_x),$$

where $\Theta(x,\xi)^{-1}\sigma(K_5^{(l)}) \in C([0,T];G^{-1,-1}(\Phi,\tilde{g}_{\Phi,k})), l = 1,2$. We choose $k > k_5$ for large k_5 so that the operator norm of $K_5^{(l)}$ is strictly lesser than 1. This guarantees the existence of $(I+K_5^{(l)}(t,x,D_x))^{-1}, l = 1,2$. Let us denote the operators $\big(I+K_5^{(1)}\big)^{-1}$ and $\big(I+K_5^{(2)}\big)^{-1}$ by $\mathcal{K}_2$ and $\tilde{\mathcal{K}}_2$, respectively.

We see that the pseudodifferential system (4.5.24) is equivalent to

$$\big(\tilde{L}(t,x,D_t,D_x) - i\mathfrak{M}_2(t,x,D_x)I + P_6(t,x,D_x)\big)V_5(t,x) = F_5(t,x), \tag{4.5.27}$$

where $F_5 = W_2(t,x,D_x)F_4$ and

$$\begin{aligned}\sigma(P_6) = \tilde{\mathcal{K}}_2 W_2 \mathcal{K}_2^{-1}\left(\big(P_3 + P_4 + Q_3 + P_5 - \mathcal{D}_1\big)\mathcal{K}_2 + D_t\mathcal{K}_2\right)\tilde{W}_2 \\ - (P_3 + P_4 + Q_3 + P_5 - \mathcal{D}_1) + (\tilde{\mathcal{K}}_2 W_2 D_t \tilde{W}_2 + i\mathfrak{M}_2 I),\end{aligned}$$

with $t^{1-\varepsilon}\sigma(P_6) \in C([0,T_2^*];G^{0,0}(\omega,\tilde{g}_{\Phi,k})$. T_2^* is chosen in such a way that all the above conjugations with operator W_2 are valid in view of Theorem 4.4.1.

Let us write down the first order system (4.5.27) explicitly as below

$$\partial_t V_5 = (i\mathcal{D}_1 - (\mathfrak{M}_1 I + iP_4 + iQ_3 + \mathfrak{M}_2 I + iP_5) - iP_6 - iP_3)\,V_5 + iF_5.$$

Here, the diagonal matrix operator, $i\mathcal{D}_1$ is of the form

$$i\mathcal{D}_1 = \operatorname{diag}\{i\lambda_1, i\lambda_2\}.$$

Observe that the symbol $d(t,x,\xi)$ of the operator $i\mathcal{D}_1 - i\mathcal{D}_1^*$ is such that

$$d \in G^{0,0}\{0\}(\omega,g_{\Phi,k})_N^{(1)} \cap G^{0,0}\{0,1,1\}(\omega,g_{\Phi,k})_N^{(2)} \cap G^{0,0}\{0,0,1,1\}(\omega,g_{\Phi,k})_N^{(3)}.$$

It follows from Propositions A.2.1-A.2.3 and Remark A.2.1 that

$$t^{1-\varepsilon}d \in C([0,T];G_{\Phi,k}^{m_1,m_2}(0,0)).$$

Hence, by Calderon-Vaillancourt theorem ,

$$2\,\mathrm{Re}\langle i\mathcal{D}_1 V_5, V_5\rangle \leq \frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle,\ C > 0. \tag{4.5.28}$$

Further, by the choice of the functions, $\mathfrak{M}_1(t,x,\xi) + \mathfrak{M}_2(t,x,\xi)$

$$\operatorname{Re}((\mathfrak{M}_1(t,x,\xi) + \mathfrak{M}_2(t,x,\xi))I + i\sigma(P_4) + i\sigma(Q_3) + i\sigma(P_5)) \geq 0.$$

Assuming

$$\omega(x) \lesssim \Phi(x),$$

we apply sharp Gårding inequality (see Thereom B.0.1 in Appendix B and [43, Theorem 18.6.14]) to $2\operatorname{Re}(\mathfrak{M}_1 I + iP_4 + iQ_3)$ with the metric $g_{\Phi,k}$ and Planck function $h(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$ and to $2\operatorname{Re}(\mathfrak{M}_2 I + iP_5)$ with the metric $\tilde{g}_{\Phi,k}$ and Planck function $\tilde{h}(x,\xi) = \Theta(x,\xi)(\Phi(x)\langle\xi\rangle_k)^{-1}$. We obtain

$$2\operatorname{Re}\langle((\mathfrak{M}_1 + \mathfrak{M}_2)I + iP_4 + iQ_3 + iP_5)V_5, V_5\rangle_{L^2} \geq -\frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle_{L^2},\ C > 0. \tag{4.5.29}$$

Since $t^{1-\varepsilon}\sigma(iP_3 + iP_6)$ is uniformly bounded, by Calderon-Vaillancourt theorem we have

$$-2\operatorname{Re}\langle i(P_3 + P_6)V_5, V_5\rangle \leq \frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle. \tag{4.5.30}$$

Let $T^* = \min\{T_1^*, T_2^*, \delta_1, \delta_2\}$ where δ_1 and δ_2 are related to the initial datum and the right hand side of the Cauchy problem (4.1.16). From (4.5.28)-(4.5.30) it follows that

$$2\operatorname{Re}\langle(i\mathcal{D}_1 - ((\mathfrak{M}_1 + \mathfrak{M}_2)I + iP_4 + iQ_3 + iP_5) - iP_6 - iP_3)V_5, V_5\rangle_{L^2} \leq \frac{C}{t^{1-\varepsilon}}\langle V_5, V_5\rangle_{L^2}.$$

This yields

$$\partial_t||V_5(t,\cdot)||_{L^2}^2 \leq C(t^{-1+\varepsilon}\|V_5(t,\cdot)\|_{L^2}^2 + \|F_5(t,\cdot)\|_{L^2}^2),\ t \in [0,T^*].$$

Considering the above inequality as a differential inequality, we apply Gronwall's lemma and obtain that

$$\|V_5(t,\cdot)\|_{L^2}^2 \leq C_\varepsilon\left(\|V_5(0,\cdot)\|_{L^2}^2 + \int_0^t \|F_5(\tau,\cdot)\|_{L^2}^2\,d\tau\right), \quad t \in [0,T^*].$$

If $\tilde{\theta}$ is unbounded $C_\varepsilon = C'e^{T^\varepsilon/\varepsilon}$ for a fixed $\varepsilon \in (0,1)$, else $C_\varepsilon = C''$ for some $C', C'' > 0$.

This proves well-posedness of the auxiliary Cauchy problem (4.5.27). Note that the solution V_3 to (4.5.10) belongs to $C\left([0,T];\mathcal{H}^{s,\tilde{\kappa}(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\right)$, $\tilde{\kappa}(t) = \tilde{\kappa}_0 + \kappa_1^*(t)$. Returning to our original solution $u = u(t,x)$ we obtain the estimate (4.1.24) with

$$u \in C\left([0,T];\mathcal{H}^{s+e,\tilde{\kappa}(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\right)\bigcap C^1\left([0,T];\mathcal{H}^{s,\tilde{\kappa}(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\right).$$

This proves Theorem 4.1.2.

4.6 Anisotropic Cone Condition

Existence and uniqueness follow from the a priori estimate established in the previous section. It now remains to prove the existence of cone of dependence.

Cone condition in this section follows from same arguments used in Section 3.5. Note that the function $\tilde{\theta}$ as in (4.1.20) is such that the quantity $t\tilde{\theta}(t)$ is bounded in $[0,T]$. The constant c^* is such that the quantity $c^*\omega(x)\tilde{\theta}(t)$ dominates the characteristic roots, i.e.,

$$c^* = \sup\left\{\sqrt{a(t,x,\xi)}\omega(x)^{-1}\tilde{\theta}(t)^{-1} : (t,x,\xi) \in [0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi,\ |\xi| = 1\right\}, \tag{4.6.1}$$

for $a(t,x,\xi)$ as in (4.1.17). In the following we prove the cone condition for the Cauchy problem (4.1.16). Let $K(x^0,t^0)$ denote the cone with the vertex (x^0,t^0):

$$K(x^0,t^0) = \{(t,x) \in [0,T] \times \mathbb{R}^n : |x - x^0| \leq c^*\omega(x)\tilde{\theta}(t^0 - t)(t^0 - t)\}.$$

Observe that the slope of the cone is anisotropic, that is, it varies with both x and t.

Proposition 4.6.1. *The Cauchy problem (4.1.16) has a cone dependence, that is, if*

$$f\big|_{K(x^0,t^0)} = 0, \quad f_i\big|_{K(x^0,t^0)\cap\{t=0\}} = 0,\ i = 1,2, \tag{4.6.2}$$

then

$$u\big|_{K(x^0,t^0)} = 0. \tag{4.6.3}$$

Proof. The proof follows in similar to the one for Proposition 3.5.1. Let us define P_ε and v_ε as in the proof of Proposition 3.5.1, and denote $b_j(t,x)$, the coefficients of lower order terms, as $a_{0,j}(t,x), 1 \leq j \leq n$, and $b_{n+1}(t,x)$ as $a_{0,0}(t,x)$. Let $a_{i,0}(t,x) = 0,\ 1 \leq i \leq n$. Following the energy estimate (4.1.23), we obtain in place of (3.5.4) the following estimate

$$\begin{aligned}
&\sum_{j=0}^{1} \|\partial_t^j(v_{\varepsilon_1} - v_{\varepsilon_2})(t,\cdot)\|_{\Phi,k;\Theta,-\kappa(t),s-je} \\
&\leq C\int_0^t \|(P_{\varepsilon_2} - P_{\varepsilon_1})v_{\varepsilon_2}(\tau,\cdot)\|_{\Phi,k;\Theta,-\kappa(\tau),s-e}\,d\tau \\
&\leq C\int_0^t \sum_{i,j=0}^{n} \|(a_{i,j}(\tau+\varepsilon_1,x) - a_{i,j}(\tau+\varepsilon_2,x))D_{ij}v_{\varepsilon_2}(\tau,\cdot)\|_{\Phi,k;\Theta,-\kappa(\tau),s-e}\,d\tau,
\end{aligned} \tag{4.6.4}$$

where $D_{00} = I, D_{i0} = 0, i \neq 0, D_{0j} = \partial_{x_j}, j \neq 0$ and $D_{ij} = \partial_{x_i}\partial_{x_j}, i,j \neq 0$. Similar estimate holds if we had used the a priori estimate (4.1.24) instead of (4.1.23).

Using the Taylor series approximation in τ variable, we have

$$\begin{aligned}
|a_{i,j}(\tau+\varepsilon_1,x) - a_{i,j}(\tau+\varepsilon_2,x)| &= \Big|\int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} (\partial_t a_{i,j})(r,x)dr\Big| \\
&\leq \omega(x)^2\Big|\int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} \frac{\theta(r)}{r}dr\Big| \\
&\leq \omega(x)^2|E(\tau,\varepsilon_1,\varepsilon_2)|,
\end{aligned}$$

where
$$E(\tau, \varepsilon_1, \varepsilon_2) = \frac{1}{2}\left(\ln\left(1 + \frac{\varepsilon_1 - \varepsilon_2}{\tau + \varepsilon_2}\right)\right)^{\varrho_2+1}.$$

Note that $\omega(x) \lesssim \Phi(x)$ and $E(\tau, \varepsilon, \varepsilon) = 0$. Then right-hand side of the inequality in (4.6.4) is dominated by

$$C\int_0^t |E(\tau, \varepsilon_1, \varepsilon_2)| \|v_{\varepsilon_2}(\tau, \cdot)\|_{\Phi,k;\Theta,-\kappa(\tau),s+e}\, d\tau,$$

in the case of finite loss of regularity while for the case of infinite loss by

$$C\int_0^t |E(\tau, \varepsilon_1, \varepsilon_2)| \|v_{\varepsilon_2}(\tau, \cdot)\|_{\Phi,k;\Theta,\tilde{\kappa}(\tau),s+e}\, d\tau,$$

where C is independent of ε. By definition, E is L^1-integrable in τ.

The sequence v_{ε_k}, $k = 1, 2, \ldots$ corresponding to the sequence $\varepsilon_k \to 0$ is in the space

$$C\Big([0, T^*]; \mathcal{H}^{s,-\kappa(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\Big) \bigcap C^1\Big([0, T^*]; \mathcal{H}^{s-e,-\kappa(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\Big), \quad T^* > 0,$$

or

$$C\Big([0, T^*]; \mathcal{H}^{s,\tilde{\kappa}(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\Big) \bigcap C^1\Big([0, T^*]; \mathcal{H}^{s-e,\tilde{\kappa}(t)}_{\Phi,k;\Theta}(\mathbb{R}^n)\Big), \quad T^* > 0,$$

depending on the loss and $u = \lim\limits_{k\to\infty} v_{\varepsilon_k}$ in the above space and hence, in $\mathcal{D}'(K(x^0, t^0))$. In particular,

$$\langle u, \varphi\rangle = \lim_{k\to\infty} \langle v_{\varepsilon_k}, \varphi\rangle = 0, \ \forall\varphi \in \mathcal{D}(K(x^0, t^0))$$

gives (4.6.3) and completes the theorem. □

Remark 4.6.1. *It is worth noting that while the regularity of the solution is dictated by $\Phi(x)$, the cone condition and there by the support of the solution is controlled by the weight function $\omega(x)$.*

4.7 Existence of Counterexamples

Let us consider a Cauchy problem of the form

$$\begin{aligned} &\partial_t^2 u(t, x) + c(t)A(x, D_x)u = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \\ &u(0, x) = 0, \quad \partial_t u(0, x) = f(x), \end{aligned} \tag{4.7.1}$$

where $A(x, D_x) = \langle x\rangle(I - \triangle_x)\langle x\rangle$ is a G-elliptic, positive, self-adjoint operator with the domain $D(A) = \{u \in L^2(\mathbb{R}^n) : Au \in L^2(\mathbb{R}^n)\}$ and the propagation speed $c(t)$ is in $C([0, T]) \cap C^1((0, T])$. In order to show that there exists a propagation speed $c(t)$ for which the Cauchy problem (4.7.1) has infinite loss of regularity (decay and derivatives), we extend the techniques developed by Ghisi and Gobbino [35, Section 4] to a global setting.

Let us first define the following special class propagation speeds.

Definition 4.7.1. *We denote $\mathcal{C}^{(2)}(\mu_1, \mu_2, \theta, \psi)$ as the set of functions $c \in C([0,T]) \cap C^2((0,T])$ that satisfy the following growth estimates*

$$0 < \mu_1 \le c(t) \le \mu_2, \qquad t \in [0,T], \tag{4.7.2}$$

$$|c'(t)| \le C\frac{\theta(t)}{t}, \qquad t \in (0,T], \tag{4.7.3}$$

$$|c''(t)| \le C\frac{\theta(t)^2}{t^2}e^{\psi(t)}, \qquad t \in (0,T], \tag{4.7.4}$$

for positive and monotone decreasing functions $\theta, \psi : (0,T] \to (0, +\infty)$ satisfying

$$\lim_{t\to 0^+} \frac{\theta(t)\psi(t)}{|\ln t|} = +\infty. \tag{4.7.5}$$

The set $\mathcal{C}^{(2)}(\mu_1, \mu_2, \theta, \psi)$ is a complete metric space with respect to the metric

$$\begin{aligned} d_2(c_1, c_2) := \sup_{t\in(0,T)} |c_1(t) - c_2(t)| &+ \sup_{t\in(0,T)} \left\{ \frac{t^2}{\theta(t)} |c_1'(t) - c_2'(t)| \right\} \\ &+ \sup_{t\in(0,T)} \left\{ \frac{t^3 e^{-\psi(t)}}{\theta(t)^2} |c_1''(t) - c_2''(t)| \right\}. \end{aligned}$$

A sequence c_n converges to c_∞ with respect to metric d_2 if and only if $c_n \to c_\infty$ uniformly in $[0,T]$, and for every $\tau \in (0,T)$, $c_n' \to c_\infty'$ and $c_n'' \to c_\infty''$uniformly in $[\tau, T]$.

Let $\mathcal{D}(\mu_1, \mu_2)$ be the set of initially constant functions as defined in Definition 3.6.2. For the sake of simplicity, let us denote $\mathcal{C}^{(2)}(\mu_1, \mu_2, \theta, \psi)$ and $\mathcal{D}(\mu_1, \mu_2)$ by $\mathcal{C}^{(2)}$, and $\mathcal{D}$, respectively.

Remark 4.7.1. *From [35, Proposition 4.7], we have that $\mathcal{D} \cap \mathcal{C}^{(2)}$ is dense in $\mathcal{C}^{(2)}$. The weight factors $\frac{t^2}{\theta(t)}$ and $\frac{t^3 e^{-\psi(t)}}{\theta(t)^2}$ appearing in the definition of the metric d_2 plays a crucial role in proving the above denseness result.*

The main aim of this section is to prove the following result.

Theorem 4.7.1. *The interior of the set of all $c \in \mathcal{C}^{(2)}(\mu_1, \mu_2, \theta, \psi)$ for which the Cauchy problem (4.7.1) exhibits an infinite loss of regularity is nonempty.*

The proof of the above theorem follows from the same arguments used in Section 3.5. We recall the defintions of universal and asymptotic activators as in Definitions 3.6.4 and 3.6.5 and also the notion of infinite loss of regularity as in Definition 3.6.3. We note the Propositions 3.6.2, 3.6.3 and 3.6.4 for $\mathcal{C} = \mathcal{C}^{(2)}$ and $d_{\mathcal{C}} = d_2$. In order to prove Theorem 4.7.1, we need to just construct asymptotic activators with rate ϕ satisfying (3.6.11).

Proof. (Proof of Theorem 4.7.1) Consider T_1, γ and a initially constant speed $c_*(t)$ as in the proof of Theorem 3.6.1. We set

$$\Gamma_\lambda := \frac{\theta(\sqrt{\lambda})\psi(\sqrt{\lambda})}{\ln \lambda},$$

$$\psi_\lambda := \min\left\{ \frac{1}{8}\ln\lambda, \frac{1}{4}\psi\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{1}{4}\ln\Gamma_\lambda \right\}.$$

For every large enough real number λ, let a_λ and b_λ be real numbers such that

$$a_\lambda := \frac{2\pi}{\gamma\lambda}\lfloor \ln\lambda \exp(\psi_\lambda)\rfloor, \qquad b_\lambda := \frac{2\pi}{\gamma\lambda}\lfloor \ln\lambda \exp(2\psi_\lambda)\rfloor. \tag{4.7.6}$$

where $\lfloor\alpha\rfloor$ stands for integer part of a real number α. Observe that a_λ and b_λ satisfy the estimates as in (3.6.12). Let us choose a cutoff function $\tilde{\nu}:\mathbb{R}\to\mathbb{R}$ of class C^∞ such that $0\le\tilde{\nu}(r)\le 1$, $\tilde{\nu}(r)=0$ for $r\le 0$ and $\tilde{\nu}(r)=1$ for $r\ge 1$. With a_λ and b_λ as in (4.7.6) and $\theta_\lambda=\min\{\theta(b_\lambda),\ln\lambda\}$, we define ε_λ and $c_\lambda(t)$ as in (3.6.13) and (3.6.14). By [35, Propositions 4.8-4.9], $(c_\lambda(t))$ is a family of asymptotic activators with rate

$$\phi(\lambda) := \frac{\theta_\lambda}{32\gamma^2}\ln\left(\frac{a_\lambda}{b_\lambda}\right),$$

with a_λ and b_λ as in (4.7.6) and $d_2(c_\lambda, c_*)\to 0$ as $\lambda\to+\infty$. Since c_* is a generic element of a dense subset, we see that these universal activators cause an infinite loss of regularity.

□

4.8 Discussion

In this chapter we have shown that when the coefficients are at most fast oscillating in t, the solution experiences at most finite loss in regularity index and for the cases of very fast oscillations and strong blow-up, in general, the solution experiences infinite loss. We now comment on some related issues.

Is the case of fast oscillation a threshold for the finite loss? Yes. In fact, in Theorem 4.7.1 we have shown through an example that when we go beyond fast oscillations, solution indeed encounters infinite loss of regularity.

How optimal is the subdivision of the phase space in this chapter? We note that subdividing the extended phase space into three regions is indeed necessary to deal with oscillatory coefficients see for instance, Examples 4.1.1 and 4.1.2. But the case of strong blow-up, we do not need the estimate on the second t-derivative as considered in this Chapter and it is enough to consider only two regions defined by the time splitting point

$$t_{x,\xi} = Nh(x,\xi)$$

for a fixed (x,ξ) and a positive integer N. The techniques used in the Chapter 3 along with the relations (4.1.6) and (4.5.12) suggest that the loss operator is of the form

$$e^{\nu(t)\Theta(x,D_x)}, \text{ where } \Theta(x,\xi)=\tilde{\theta}(Nh(x,\xi)).$$

Chapter 5

Very Fast Oscillations: $q > 1$ Case

> One of the difficulties in the theory of partial differential operators arises from the loss of many derivatives of solutions.
>
> — Kajitani and Nishitani, The Hyperbolic Cauchy Problem

The oscillatory behavior ranging from very slow oscillations to very fast oscillations with $q = 1$ was dealt in the previous chapter. It now remains to investigate the case of very fast oscillations when $q > 1$ in Definition 1.2.2. In order to present a generic theory in this case, we consider m^{th} order strictly hyperbolic equations with coefficients polynomially bound in x and with their t-derivative of order $\mathrm{O}(t^{-q})$, where $q \in \big(1, \frac{3}{2}\big)$ without any growth conditions on the second t-derivatives. We demonstrate that the solution experiences an infinite loss in regularity index in relation to the initial datum defined in a Sobolev space tailored to the metric and the order of the singularity.

5.1 Introduction and Statement of Main Result

Let us consider the prototypical Cauchy problem:

$$\left.\begin{aligned} &\partial_t^2 u - a(t,x)\Delta_x u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n \\ &u(0,x) = u_0(x)\ \partial_t u(0,x) = u_1(x). \end{aligned}\right\} \tag{5.1.1}$$

where the coefficient $a(t,x)$ is in $C\big([0,T]; C^\infty(\mathbb{R}^n)\big) \cap C^1\big((0,T]; C^\infty(\mathbb{R}^n)\big)$ and satisfies the following conditions

$$\left.\begin{aligned} a(t,x) &\geq C_0 \omega(x)^2, \\ |\partial_x^\beta \partial_t a(t,x)| &\leq C_\beta \omega(x)^2 \Phi(x)^{-|\beta|} \frac{1}{t^q}, \end{aligned}\right\} \tag{5.1.2}$$

where $C_0, C_\beta > 0$, $\beta \in \mathbb{N}_0^n$ and $q > 1$. Note that the blow-up rate of second t-derivative is not prescribed as in Definition 1.2.2. In view of the conditions (5.1.2), the very fast oscillatory behavior is treated as a specific case in this chapter. An example of such a coefficient $a(t,x)$ is given below.

Example 5.1.1. *Let $\kappa \in [0,1]$ and $a(t,x) = \langle x\rangle^{2\kappa}\left(2+\sin\left(\langle x\rangle^{1-\kappa}\right)\right)c(t)$, where*

$$c(t) = \begin{cases} 1, & \text{if } t = 0, \\ 1 + t\sin\left(\frac{1}{t^{4/3}}\right), & \text{if } t \in (0, 0.5]. \end{cases}$$

Here $\omega(x) = \Phi(x) = \langle x\rangle^\kappa$ and $q = \frac{4}{3}$. Observe that the second t-derivative of $c(t)$ blows up as $O(t^{-3})$ and $3 > 2q$.

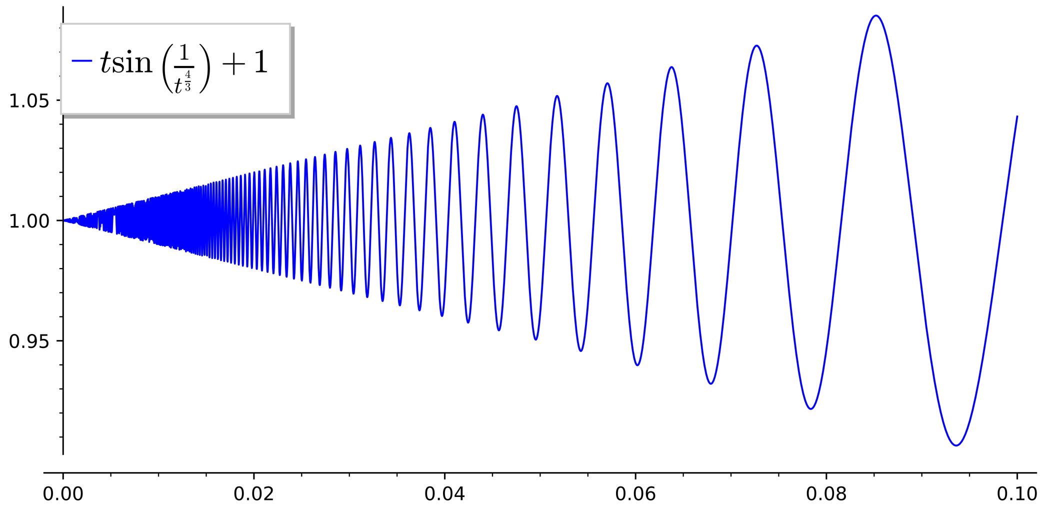

Figure 5.1: Behavior w.r.t time variable for the Example 5.1.1

Cicognani [9] studied the Cauchy problem (5.1.1) with $a(t,x)$ in $C([0,T];G^\sigma(\mathbb{R}^n)) \cap C^1((0,T];G^\sigma(\mathbb{R}^n))$, $1 < \sigma < q/(q-1)$, and satisfying (5.1.2) with $\omega(x) = \Phi(x) = 1$, $q > 1$. The author reports Gevrey-Sobolev well-posedness for the Cauchy problem (5.1.1) with an infinite loss of derivatives. In this chapter, we consider generic sublinear weights $\Phi(x)$ and $\omega(x)$ governing the growth rate of coefficients with respect to x and the singularity of order $O(t^{-q})$, $1 \leq q < \frac{3}{2}$, in the estimates (5.1.2) .

One of the key steps in our analysis that helps in dealing with low-regularity in t is the conjugation by a loss operator which is an infinite order pseudodifferential operator of the form

$$e^{\Lambda(t)(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}},$$

where $\Lambda(t)$ is a continuous function for $t \in [0,T], T > 0$ and $\langle D_x\rangle_k = (k^2 - \Delta_x)^{1/2}$. Here, the symbol of the operator $\Phi(x)\langle D_x\rangle_k$ is given by $h(x,\xi)^{-1} = \Phi(x)\langle\xi\rangle_k$ where $h(x,\xi)$ is the Planck function related to the metric $g_{\Phi,k}$ in (2.1.2). Note that, in the literature (see [2]), some authors have used an infinite order pseudodifferential operators of the form $e^{\Lambda(t)(\langle x\rangle^{1/\sigma} + \langle D\rangle^{1/\sigma})}$ for conjugation. We report (see Theorem 5.3.1) that the metric governing the decay estimates of the symbol of an operator arising after the conjugation changes. In our case, this metric is of the form

$$\tilde{g}_{\Phi,k} = (\Phi(x)\langle\xi\rangle_k)^{\frac{2}{\sigma}} g_{\Phi,k}, \tag{5.1.3}$$

We demonstrate this in Section 5.3.

We report that the solution experiences infinite loss in regularity index in relation to the Cauchy data defined in Sobolev spaces tailored to the loss operator, $e^{\Lambda(t)(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}}$. The Sobolev spaces needed for our analysis are defined in the following section.

5.1.1 Sobolev Spaces

We now introduce the Sobolev spaces suitable for our analysis that are tailored to the metric $g_{\Phi,k}$ and the order of singularity.

Definition 5.1.1. *The Sobolev space $H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n)$ for $\sigma > 2$, $\varepsilon \geq 0$ and $s = (s_1, s_2) \in \mathbb{R}^2$ is defined as*

$$H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : \Phi(x)^{s_2}\langle D\rangle_k^{s_1} \exp\{\varepsilon(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}\}v \in L^2(\mathbb{R}^n)\}, \tag{5.1.4}$$

equipped with the norm $\|v\|_{\Phi,k;s,\varepsilon,\sigma} = \|\Phi(\cdot)^{s_2}\langle D\rangle_k^{s_1} \exp\{\varepsilon(\Phi(\cdot)\langle D\rangle_k)^{1/\sigma}\}v\|_{L^2}$. The operator $\exp\{\varepsilon(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}\}$ is an infinite order pseudodifferential operator with the Fourier multiplier $\exp\{\varepsilon(\Phi(x)\langle \xi\rangle_k)^{1/\sigma}\}$.

The spaces $H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n)$ and $H^{-s,-\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n)$ are dual to each other. Let $s' = (s_1', s_2') \in \mathbb{R}^2$, $\varepsilon' \geq 0$ and $\sigma' > 2$. We have that $H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n) \subset H^{s',\varepsilon',\sigma'}_{\Phi,k}(\mathbb{R}^n)$ if $\sigma \leq \sigma'$, $\varepsilon' \leq \varepsilon$, $s_j' \leq s_j, j = 1, 2$.

Definition 5.1.2. *The function space $\mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n), \sigma \geq 3$ is a set of functions $v \in C^\infty(\mathbb{R}^n)$ that satisfy*

$$\|e^{a(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}} v(x)\|_{L^2} \leq C$$

for some positive constants a and C.

The function space $\mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n)$ and its dual $\mathcal{M}^{\sigma}_{\Phi,k}{}'(\mathbb{R}^n)$ are related to the Sobolev spaces as follows

$$\mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n) = \bigcup_{\varepsilon>0}\bigcap_{s\in\mathbb{R}^2} H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{M}^{\sigma}_{\Phi,k}{}'(\mathbb{R}^n) = \bigcap_{\varepsilon>0}\bigcup_{s\in\mathbb{R}^2} H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n).$$

We can relate these spaces to the Gelfand-Shilov spaces. Let us denote the Gelfand-Shilov space of indices $\mu, \nu > 0$ as $\mathcal{S}^\mu_\nu(\mathbb{R}^n)$. We refer to [58, Section 6.1] for the definition and properties of Gelfand-Shilov spaces. Note that

$$k^{-1/\sigma}(\Phi(x)\langle\xi\rangle_k)^{1/\sigma} \leq \frac{1}{2}(\Phi(x)^{2/\sigma} + \langle\xi\rangle^{2/\sigma}) \leq \frac{1}{2}(\langle x\rangle^{2/\sigma} + \langle\xi\rangle^{2/\sigma}).$$

Thus, we have the inclusion

$$\mathcal{S}^{\frac{\sigma}{2}}_{\frac{\sigma}{2}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n).$$

Further, if $\Phi(x) = \langle x\rangle$ we have

$$\mathcal{S}^{\frac{\sigma}{2}}_{\frac{\sigma}{2}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{\sigma}_{\Phi(x)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}^{\sigma}_{\sigma}(\mathbb{R}^n).$$

In the pseudodifferential calculus, the transposition, composition and construction of parametrix are done modulo an operator that maps $\mathcal{M}^{\sigma}_{\Phi,k}{}'(\mathbb{R}^n)$ to $\mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n)$. This is detailed in Section A.3. A study of $\mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n)$ and its dual from an abstract viewpoint is our future work, see Chapter 7.

5.1.2 Main Result

Let us generalize the Cachy problem (5.1.1) and consider

$$\left.\begin{aligned} &P(t,x,\partial_t,D_x)u(t,x)=f(t,x), && (t,x)\in[0,T]\times\mathbb{R}^n,\\ &\partial_t^{j-1}u(0,x)=f_j(x), && j=1,\dots,m \end{aligned}\right\} \tag{5.1.5}$$

where the operator $P(t,x,\partial_t,D_x)$ is given by

$$P=\partial_t^m-\sum_{j=0}^{m-1}\Big(A_{m-j}(t,x,D_x)+B_{m-j}(t,x,D_x)\Big)\partial_t^j \quad \text{with}$$
$$A_{m-j}(t,x,D_x)=\sum_{|\alpha|+j=m}a_{j,\alpha}(t,x)D_x^\alpha \quad \text{and}$$
$$B_{m-j}(t,x,D_x)=\sum_{|\alpha|+j<m}b_{j,\alpha}(t,x)D_x^\alpha,$$

using the usual multi-index notation we denote $D_x^\alpha=(-i)^\alpha\partial_x^\alpha$. The operator P in (5.1.5) is said to be strictly hyperbolic operator if the symbol of the principal part

$$\begin{aligned} P_m(t,x,i\tau,\xi)&=(i\tau)^m-\sum_{j=0}^{m-1}A_{m-j}(t,x,\xi)(i\tau)^j\\ &=(i\tau)^m-\sum_{j=0}^{m-1}\sum_{|\alpha|+j=m}a_{j,\alpha}(t,x)\xi^\alpha(i\tau)^j \end{aligned}$$

has purely imaginary characteristic roots $i\tau_j(t,x,\xi)$, $j=1,\dots,m$ where $\tau_j(t,x,\xi)$ is a real-valued, simple function in t and positively homogeneous of degree 1 for $\xi\neq 0$ in $\mathbb{R}^n$. These roots are numbered and arranged so that

$$\begin{aligned} \tau_1(t,x,\xi)<\tau_2(t,x,\xi)<\cdots<\tau_m(t,x,\xi), \quad \text{and}\\ C\omega(x)\langle\xi\rangle\le|\tau_j(t,x,\xi)|, \end{aligned} \tag{5.1.6}$$

for some $C>0$, for all $t\in[0,T]$, $x,\xi\in\mathbb{R}^n$ and $1\le j\le m$. We assume $a_{j,\alpha}\in C([0,T];C^\infty(\mathbb{R}^n))\cap C^1((0,T];C^\infty(\mathbb{R}^n))$ satisfy

$$\left.\begin{aligned} |D_x^\beta a_{j,\alpha}(t,x)|&\le C^{|\beta|}\beta!^\sigma\omega(x)^{m-j}\Phi(x)^{-|\beta|}, && (t,x)\in[0,T]\times\mathbb{R}^n,\\ |D_x^\beta\partial_t a_{j,\alpha}(t,x)|&\le C^{|\beta|}\beta!^\sigma\omega(x)^{m-j}\Phi(x)^{-|\beta|}\frac{1}{t^q}, && (t,x)\in(0,T]\times\mathbb{R}^n, \end{aligned}\right\} \tag{5.1.7}$$

for $3\le\sigma<q/(q-1)$ and the coefficients of the lower order terms, $b_{j,\alpha}\in C^1([0,T];C^\infty(\mathbb{R}^n))$ satisfy

$$|D_x^\beta b_{j,\alpha}(t,x)|\le C^{|\beta|}\beta!^\sigma\omega(x)^{m-j-1}\Phi(x)^{-|\beta|}, \quad (t,x)\in[0,T]\times\mathbb{R}^n, \quad C>0. \tag{5.1.8}$$

Remark 5.1.1. *Observe that we have assumed $3\le\sigma<q/(q-1)$ where as in [9], it is $1<\sigma<q/(q-1)$. The increase in the lower bound for σ is due to two factors:*

(*i*) *the uncertainty principle which applied to the metric* $\tilde{g}_{\Phi,k}$ *gives* $\sigma > 2$,

(*ii*) *the application of sharp Gårding inequality in our context dictates that* $\sigma \geq 3$; *this is discussed in Section 5.5.3.*

Due to this increment in σ, *we have* $q \in \left[1, \frac{3}{2}\right)$.

Theorem 5.1.1. *Consider the strictly hyperbolic Cauchy problem (5.1.5) satisfying the following conditions:*

i) The coefficients $a_{j,\alpha}$ *of the principal part satisfy (5.1.7) and the coefficients* $b_{j,\alpha}$ *satisfy (5.1.8).*

ii) The initial data f_j *belongs to* $H^{s+(m-j)e,\Lambda_1,\sigma}_{\Phi,k}, \Lambda_1 > 0$ *for* $j = 1, \cdots, m$, $e = (1, 1)$.

iii) The right hand side $f \in C([0, T]; H^{s,\Lambda_2,\sigma}_{\Phi,k}), \Lambda_2 > 0$.

Then, there exist a continuous function $\Lambda(t)$ *and* $\Lambda_0 > 0$, *such that there is a unique solution*

$$u \in \bigcap_{j=0}^{m-1} C^{m-1-j}\left([0, T]; H^{s+je,\Lambda(t),\sigma}_{\Phi,k}\right)$$

for $\Lambda(t) < \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$. *More specifically, for a sufficiently large* λ *and* $\delta \in (0, 1)$, *we have the a priori estimate*

$$\sum_{j=0}^{m-1} \|\partial_t^j u(t, \cdot)\|_{\Phi,k;s+(m-1-j)e,\Lambda(t),\sigma} \leq C\left(\sum_{j=1}^{m} \|f_j\|_{\Phi,k;s+(m-j)e,\Lambda(0),\sigma} + \int_0^t \|f(\tau, \cdot)\|_{\Phi,k;s,\Lambda(\tau),\sigma}\, d\tau\right) \tag{5.1.9}$$

for $\Lambda(t) = \frac{\lambda}{\delta}(T^\delta - t^\delta)$, $0 \leq t \leq T < (\delta\Lambda^*/\lambda)^{1/\delta}$, $C = C_s > 0$ *and* $\Lambda^* = \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$.

The constants Λ_0 and δ in the above theorem are the constants in Theorem 5.3.1 and equation (5.4.1), respectively.

5.2 Global Symbol Classes

Definition 5.2.1. $G^{m_1,m_2}(\omega, g_{\Phi,k})$ *is the space of all functions* $p = p(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ *satisfying*

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)| < C_{\alpha\beta}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}, \tag{5.2.1}$$

for $C_{\alpha\beta} > 0$ *and for all multi-indices* α *and* β.

We need the following symbol classes with Gevrey regularity. Let μ, ν be real numbers with $\mu \geq 1$, $\nu \geq 1$.

Definition 5.2.2. *We denote by $AG^{m_1,m_2}_{\mu,\nu}(\omega, g_{\Phi,k})$ the Banach space of all symbols $p(x,\xi) \in G^{m_1,m_2}_{\Phi,k}$ such that the constant $C_{\alpha\beta} > 0$ in (5.2.1) is of the form,*

$$C_{\alpha\beta} = B\ C^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu},$$

for some $B > 0$ independent of α and β, and $C > 0$.

After the conjugation by infinite order pseudodifferential operator, $e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}}$, the growth estimates for the lower order terms are governed by the metric $\tilde{g}_{\Phi,k}$ given in (5.1.3). We will now define the symbol classes associated with this metric.

Definition 5.2.3. *For every $\sigma \geq 3$, we denote by $G^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ the space of all functions $p(x,\xi) \in C^{\infty}(\mathbb{R}^{2n})$ satisfying*

$$|\partial_\xi^\alpha D_x^\beta p(x,\xi)| < C_{\alpha\beta}\langle\xi\rangle_k^{m_1-\gamma|\alpha|+|\beta|/\sigma}\omega(x)^{m_2}\Phi(x)^{-\gamma|\beta|+|\alpha|/\sigma}, \tag{5.2.2}$$

for $C_{\alpha\beta} > 0$, $\gamma = 1 - \frac{1}{\sigma}$ and for all multi-indices α and β.

Moreover, we shall need the following Gevrey variant of the above symbol class.

Definition 5.2.4. *We denote by $AG^{m_1,m_2}_{\sigma;\mu,\nu}(\omega, g_{\Phi,k})$ the Banach space of all symbols $p(x,\xi) \in G^{m_1,m_2}_{\Phi,\sigma}$ such that the constant $C_{\alpha\beta} > 0$ in (5.2.2) is of the form,*

$$C_{\alpha,\beta} = B\ C^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu},$$

for some $C > 0$ and $B > 0$ independent of α and β.

Inspired from [2], we introduce the following symbol class in order to deal with the symbols which are polynomial in ξ and Gevrey of order $\sigma \geq 3$ with respect to x. Here we impose analytic estimates with respect to ξ on an exterior domain of $\mathbb{R}^{2n}$. A suitable class for our purpose is defined as follows.

Definition 5.2.5. *We shall denote by $AG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ the space of all symbols $p(x,\xi) \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ satisfying the following condition: there exist positive constants B, C such that*

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{\Phi(x)\langle\xi\rangle_k \geq B|\alpha|^\sigma} C^{-|\alpha|-|\beta|}(\alpha!)^{-1}(\beta!)^{-\sigma}\langle\xi\rangle_k^{-m_1+|\alpha|} \times \omega(x)^{-m_2}\Phi(x)^{|\beta|}|D_\xi^\alpha\partial_x^\beta p(x,\xi)| < +\infty.$$

The following inclusions hold:

$$AG^{m_1,m_2}_{1,\sigma}(\omega, g_{\Phi,k}) \subset AG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k}) \subset AG^{m_1,m_2}_{\sigma,\sigma}(\omega, g_{\Phi,k}) \subset AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k}).$$

Given a symbol $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$, we can consider the associated pseudodifferential operator $P = p(x, D_x)$ defined by the following oscillatory integral

$$\begin{aligned} Pu(x) &= \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}p(x,\xi)u(y)dy\,đ\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)đ\xi, \end{aligned} \tag{5.2.3}$$

where $u \in \mathcal{S}(\mathbb{R}^n)$ and $đ\xi = (2\pi)^{-n} d\xi$. We shall denote by $OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$, the space of all operators of the form (5.2.3) defined by a symbol $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$. In Section A.3 of Appendix A we give the calculi of the operators in $OPAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ and $OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ which can be easily constructed using the standard arguments given in [58, Section 6.3] and [2, Appendix A].

5.3 Conjugation by an Infinite Order Pseudodifferential Operator

In this section, we perform a conjugation of operators from $OPAG^{m_1,m_2}_{\sigma,\sigma}(\omega, g_{\Phi,k})$ by $e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}}, \sigma \geq 3$. Here we assume that $\Lambda(t)$ is a continuous function for $t \in [0,T]$. The following proposition gives an upper bound on the function $\Lambda(t)$ for the conjugation to be well defined.

Theorem 5.3.1. *Let $p \in AG^{m_1,m_2}_{\sigma,\sigma}(\omega, g_{\Phi,k})$ for some $\sigma \geq 3$, $m = (m_1, m_2) \in \mathbb{R}^2$ and $\Lambda = \Lambda(t)$ be a positive continuous function of $t \in [0,T]$. Then, there exists $\Lambda_0 > 0$ such that for $\Lambda(t) < \Lambda_0$,*

$$e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} p(x,D) e^{-\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} = p(x,D) + \sum_{k=1}^{3} r^{(k)}_{\Lambda}(t,x,D_x), \tag{5.3.1}$$

where

$$\begin{aligned} r^{(1)}_{\Lambda}(t,x,D_x) &\in C([0,T]; AG^{-\infty,1}_{\sigma;\sigma,\sigma}(\omega^{m_2}\Phi^{-\gamma}, \tilde{g}_{\Phi,k})),\\ r^{(2)}_{\Lambda}(t,x,D_x) &\in C([0,T]; AG^{m_1-\gamma,-\infty}_{\sigma;\sigma,\sigma}(\Phi, \tilde{g}_{\Phi,k})),\\ r^{(3)}_{\Lambda}(t,x,D_x) &\in C([0,T]; AG^{-\infty,-\infty}_{\sigma;\sigma,\sigma}(\Phi, \tilde{g}_{\Phi,k})), \end{aligned}$$

for $\gamma = 1 - \frac{1}{\sigma}$.

To prove the Theorem 5.3.1, we need the following lemma, which can be given an inductive proof.

Lemma 5.3.2. *Let $\varepsilon \neq 0$, $\sigma \geq 3$. Then, for every $\alpha, \beta \in \mathbb{Z}^n_+$, we have*

$$\partial^\beta_x \partial^\alpha_\xi e^{\varepsilon(\Phi(x)\langle\xi\rangle_k)^{1/\sigma}} \leq (C\varepsilon)^{|\alpha|+|\beta|} \alpha!\beta! \Phi(x)^{-\gamma|\beta|+|\alpha|/\sigma} \langle\xi\rangle_k^{-\gamma|\alpha|+|\beta|/\sigma} e^{\varepsilon(\Phi(x)\langle\xi\rangle_k)^{1/\sigma}}.$$

Proof of Theorem 5.3.1. Throughout this proof we write Λ in place of $\Lambda(t)$ and denote $\Phi(x)\langle\xi\rangle_k$ by $\psi(x,\xi)$ for the sake of simplicity of notation. Let $p_{\Lambda,\sigma}(x,\xi)$ be the symbol of the operator

$$e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} p(x,D) e^{-\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}}.$$

Then $p_{\Lambda,\sigma}(x,\xi)$ can be written in the form of an oscillatory integral as follows:

$$\begin{aligned} p_{\Lambda,\sigma}(x,\xi) = \int \cdots \int e^{-iy\cdot\eta} e^{-iz\cdot\zeta} e^{\Lambda\psi(x,\xi+\zeta+\eta)^{1/\sigma}} p(x+z,\xi+\eta) \\ \times e^{-\Lambda\psi(x+y,\xi)^{1/\sigma}} dz\, đ\zeta\, dy\, đ\eta, \end{aligned} \tag{5.3.2}$$

Taylor expansions of $\exp\{\Lambda\psi(x,\xi)^{1/\sigma}\}$ in the first and second variables, respectively, are

$$e^{-\Lambda\psi(x+y,\xi)^{1/\sigma}} = e^{-\Lambda\psi(x,\xi)^{1/\sigma}} + \sum_{j=1}^{n}\int_0^1 y_j \partial_{w'_j} e^{-\Lambda\psi(w',\xi)^{1/\sigma}}\Big|_{w'=x+\theta_1 y} d\theta_1, \text{ and}$$
$$e^{\Lambda\psi(x,\xi+\zeta+\eta)^{1/\sigma}} = e^{\Lambda\psi(x,\xi)^{1/\sigma}} + \sum_{i=1}^{n}\int_0^1 (\zeta_i+\eta_i)\partial_{w_i} e^{\Lambda\psi(x,w)^{1/\sigma}}\Big|_{w=\xi+\theta_2(\eta+\zeta)} d\theta_2.$$

We can write $p_{\Lambda,\sigma}$ as

$$p_{\Lambda,\sigma}(x,\xi) = p(x,\xi) + \sum_{l=1}^{3} r_\Lambda^{(l)}(t,x,\xi) \quad \text{where}$$

$$r_\Lambda^{(l)}(x,\xi) = \int\cdots\int e^{-iy\cdot\eta}e^{-iz\cdot\zeta} I_l p(x+z,\xi+\eta)dz\,đ\zeta\,dy\,đ\eta,$$

and I_l, $l=1,2,3$ are as follows:

$$I_1 = e^{\Lambda\psi(x,\xi)^{1/\sigma}}\sum_{j=1}^{n}\int_0^1 y_j\partial_{w'_j}e^{-\Lambda\psi(w',\xi)^{1/\sigma}}\Big|_{w'=x+\theta_1 y}d\theta_1,$$
$$I_2 = e^{-\Lambda\psi(x,\xi)^{1/\sigma}}\sum_{i=1}^{n}\int_0^1 (\zeta_i+\eta_i)\partial_{w_i}e^{\Lambda\psi(x,w)^{1/\sigma}}\Big|_{w=\xi+\theta_2(\zeta+\eta)}d\theta_2 \quad \text{and}$$
$$I_3 = \left(\sum_{i=1}^{n}\int_0^1 (\zeta_i+\eta_i)\partial_{w_i}e^{\Lambda\psi(x,w)^{1/\sigma}}\Big|_{w=\xi+\theta_2(\zeta+\eta)}d\theta_2\right) \times \left(\sum_{j=1}^{n}\int_0^1 y_j\partial_{w'_j}e^{-\Lambda\psi(w',\xi)^{1/\sigma}}\Big|_{w'=x+\theta_1 y}d\theta_1\right).$$

We will now determine the growth estimate for $r_\Lambda^{(1)}(t,x,\xi)$ using integration by parts [37]. For $\alpha,\beta,\kappa\in\mathbb{Z}_+^n$ and $l\in\mathbb{Z}_+$ we have

$$\begin{aligned}
&\partial_\xi^\alpha\partial_x^\beta r_\Lambda^{(1)}(t,x,\xi)\\
&=\sum_{j=1}^{n}\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int y^{-\kappa}y^{\kappa}e^{-iy\cdot\eta}e^{-iz\cdot\zeta}(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{\xi_j}p)(x+z,\xi+\eta)\\
&\quad\times\int_0^1\partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w'_j}e^{\Lambda(\Phi(x)^{1/\sigma}-\Phi(w')^{1/\sigma})\langle\xi\rangle_k^{1/\sigma}}\Big|_{w'=x+\theta_1 y}d\theta_1\,dz\,đ\zeta\,dy\,đ\eta\\
&=\sum_{j=1}^{n}\sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int y^{-\kappa}e^{-iy\cdot\eta}e^{-iz\cdot\zeta}\langle\eta\rangle_k^{-2l}\langle z\rangle_k^{-2l}\langle D_\zeta\rangle_k^{2l}\langle\zeta\rangle_k^{-2l}\\
&\quad\times\langle D_z\rangle_k^{2l}D_\eta^\kappa(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{\xi_j}p)(x+z,\xi+\eta)\\
&\quad\times\int_0^1\langle D_y\rangle_k^{2l}\partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w'_j}e^{\Lambda(\Phi(x)^{1/\sigma}-\Phi(w')^{1/\sigma})\langle\xi\rangle_k^{1/\sigma}}\Big|_{w'=x+\theta_1 y}d\theta_1\,dz\,đ\zeta\,dy\,đ\eta.
\end{aligned}$$

Let $E_1(t,x,y,\xi) = \exp\{\Lambda(\Phi(x)^{1/\sigma} - \Phi(x+\theta_1 y)^{1/\sigma})\langle\xi\rangle_k^{1/\sigma}\}$. Note that for $|y| \geq 1$ we have $\langle y\rangle \leq \sqrt{2}|y|$ and in the case $|y| < 1$ we have $\langle y\rangle < \sqrt{2}$. Using these estimates along with the fact that $\langle y\rangle^{-|\kappa|} \leq \Phi(y)^{-|\kappa|}$ we have

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(1)}(t,x,\xi)| \leq\ & C_1^{|\alpha|+|\beta|+2}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+|\alpha|/\sigma}\langle\xi\rangle_k^{m_1-\gamma-\gamma|\alpha|+|\beta|/\sigma} \\
& \times \sum_{\beta'+\beta''\leq\beta}\sum_{\alpha'+\alpha''\leq\alpha}\int\cdots\int \Phi(z)^{|m_2-|\beta'||}\Phi(y)^{\gamma|\beta''|+|\alpha''|/\sigma} \\
& \times \langle\eta\rangle_k^{|m_1-1-|\alpha'||+\gamma|\alpha''|+|\beta''|/\sigma+|\kappa|-2l}|\kappa|!^\sigma\left(\frac{C_1 2^\sigma}{\Phi(y)\langle\xi\rangle_k}\right)^{|\kappa|} \\
& \times \langle\xi\rangle_k^{2l/\sigma}E_1(t,x,y,\xi)\langle z\rangle_k^{-2l}\langle\zeta\rangle_k^{-2l}dz\,đ\zeta\,dy\,đ\eta.
\end{aligned}$$

Given α, β and κ, we choose l such that $2l > n + \max\{m_1, m_2\} + |\alpha| + |\beta| + |\kappa|$. So that

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(1)}(t,x,\xi)| \leq\ & C_1^{|\alpha|+|\beta|+2}\langle\xi\rangle_k^{m_1-\gamma-\gamma|\alpha|+(|\beta|+2l)/\sigma}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+|\alpha|/\sigma} \\
& \times \int |\kappa|!^\sigma\left(\frac{C_1 2^\sigma}{\Phi(y)\langle\xi\rangle_k}\right)^{|\kappa|}\Phi(y)^{2l}E_1(x,y,\xi)dy.
\end{aligned}$$

Noting the inequality (see [58, Lemma 6.3.10])

$$\inf_{j\in\mathbb{Z}_+} j!^\sigma\left(\frac{C_1 2^\sigma}{\Phi(y)\langle\xi\rangle_k}\right)^j \leq C' e^{-c_1(\Phi(y)\langle\xi\rangle_k)^{1/\sigma}},$$

for some positive constants C' and c_1 where C' depends only on C_1 and c_1 on n and C_1, we have

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(1)}(t,x,\xi)| \ \leq\ \ & C^{|\alpha|+|\beta|+2}\langle\xi\rangle_k^{m_1-\gamma-\gamma|\alpha|+|\beta|/\sigma}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+|\alpha|/\sigma} \\
& \times \int e^{-c_1\Phi(y)^{1/\sigma}\langle\xi\rangle_k^{1/\sigma}}\Phi(y)^{2l}E_1(x,y,\xi)dy.
\end{aligned}$$

Let $l' \in \mathbb{Z}^+$ such that $\frac{l'}{\sigma} \geq l$. Then, we have $e^{-c_1\Phi(y)^{1/\sigma}\langle\xi\rangle_k^{1/\sigma}}\Phi(y)^{2l'/\sigma} \leq (2l')!\ e^{-\frac{c_1}{2}\Phi(y)^{1/\sigma}\langle\xi\rangle_k^{1/\sigma}}$. Hence,

$$\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(1)}(t,x,\xi)| \\
&\qquad \leq C_1^{|\alpha|+|\beta|+2}\langle\xi\rangle_k^{m_1-\gamma-\gamma|\alpha|+|\beta|/\sigma}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+|\alpha|/\sigma} \\
&\qquad\quad \times \int \exp\left\{(\Lambda\Phi(x)^{1/\sigma} - \Lambda\Phi(x+\theta_1 y)^{1/\sigma} - \frac{c_1}{2}\Phi(y)^{1/\sigma})\langle\xi\rangle_k^{1/\sigma}\right\}dy.
\end{aligned}$$

For $|x| \leq |y|$, clearly $\Phi(x)^{1/\sigma} - \Phi(x+\theta_1 y)^{1/\sigma} \leq \Phi(y)^{1/\sigma}$. For $|x| \geq |y|$, we have

$$\begin{aligned}
\Phi(x)^{1/\sigma} - \Phi(x+\theta_1 y)^{1/\sigma} &\leq \Phi(x)^{1/\sigma} - (\Phi(x) - \Phi(\theta_1 y))^{1/\sigma} \\
&\leq \Phi(x)^{1/\sigma} - (\Phi(x)^{1/\sigma} - \Phi(\theta_1 y)^{1/\sigma}) \leq \Phi(y)^{1/\sigma}.
\end{aligned} \tag{5.3.3}$$

Since c_1 is independent of Λ, there exists $\Lambda^{(1)} > 0$ (in fact, $\Lambda^{(1)} = c_1/2$) such that, for $\Lambda = \Lambda(t) < \Lambda^{(1)}$ we obtain the estimate

$$|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(1)}(t,x,\xi)| \leq C^{|\alpha|+|\beta|+2}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+|\alpha|/\sigma}e^{-\frac{c_1}{8}\langle\xi\rangle_k^{1/\sigma}}$$

Thus $r_\Lambda^{(1)} \in C\big([0,T]; AG^{-\infty,1}_{\sigma;\sigma,\sigma}(\omega^{m_2}\Phi^{-\gamma}, \tilde{g}_{\Phi,k})\big)$.

In a similar fashion, we will determine the growth estimate for $r_\Lambda^{(2)}(t,x,\xi)$. Let $\alpha,\beta,\kappa \in \mathbb{Z}_+^n$ and $l \in \mathbb{Z}_+$. Then

$$
\begin{aligned}
&\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(2)}(t,x,\xi)\\
&= \sum_{i=1}^n \sum_{\beta'+\beta''\le\beta} \sum_{\alpha'+\alpha''\le\alpha} \int\cdots\int \eta^{-\kappa}\eta^{\kappa} e^{-iy\cdot\eta}\zeta^{-\kappa}\zeta^{\kappa}e^{-iz\cdot\zeta}\langle z\rangle_k^{-2l}\langle\eta\rangle_k^{-2l}\langle D_y\rangle_k^{2l}\\
&\quad\times \langle y\rangle_k^{-2l}\langle D_\zeta\rangle_k^{2l}\langle D_\eta\rangle_k^{2l}(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{x_i}p)(x+z,\xi+\eta)\\
&\quad\times \int_0^1 \partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w_i} e^{\Lambda\Phi(x)^{1/\sigma}(\langle w\rangle_k^{1/\sigma}-\langle\xi\rangle_k^{1/\sigma})}\Big|_{w=\xi+\theta_2(\eta+\zeta)} d\theta_2\, dz\, đ\zeta\, dy\, đ\eta,
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(2)}(t,x,\xi)\\
&= \sum_{i=1}^n \sum_{\beta'+\beta''\le\beta} \sum_{\alpha'+\alpha''\le\alpha} \int\cdots\int \eta^{-\kappa} e^{-iy\cdot\eta}\zeta^{-\kappa}e^{-iz\cdot\zeta}D_y^\kappa D_z^\kappa\langle z\rangle_k^{-2l}\langle\eta\rangle_k^{-2l}\langle D_y\rangle_k^{2l}\\
&\quad\times \langle y\rangle_k^{-2l}\langle D_\zeta\rangle_k^{2l}\langle D_\eta\rangle_k^{l}(\partial_\xi^{\alpha'}\partial_x^{\beta'}D_{x_i}p)(x+z,\xi+\eta)\\
&\quad\times \int_0^1 \partial_\xi^{\alpha''}\partial_x^{\beta''}\partial_{w_i} e^{\Lambda\Phi(x)^{1/\sigma}(\langle w\rangle_k^{1/\sigma}-\langle\xi\rangle_k^{1/\sigma})}\Big|_{w=\xi+\theta_2(\eta+\zeta)} d\theta_2\, dz\, đ\zeta\, dy\, đ\eta.
\end{aligned}
$$

Let $E_2(t,x,\xi,\eta,\zeta) = \exp\{\Lambda\Phi(x)^{1/\sigma}(\langle\xi+\theta_2(\eta+\zeta)\rangle_k^{1/\sigma} - \langle\xi\rangle_k^{1/\sigma})\}$. Using the easy to show inequality $\Phi(x+z)^s \le 2^{|s|}\Phi(x)^s\Phi(z)^{|s|}, \forall s\in\mathbb{R}$, we have

$$
\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(2)}(t,x,\xi)| &\le C_2^{|\alpha|+|\beta|+2}\langle\xi\rangle_k^{m_1-\gamma-\gamma|\alpha|+|\beta|/\sigma}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+|\alpha|/\sigma}\\
&\quad\times \sum_{\beta'+\beta''\le\beta}\sum_{\alpha'+\alpha''\le\alpha}\int\cdots\int \Phi(z)^{|m_2-|\beta'||+1}\langle\eta\rangle_k^{|m_1-|\alpha'||}\\
&\quad\times \langle\eta\rangle_k^{-2l}\langle\eta+\zeta\rangle_k^{\gamma(1+|\alpha''|)+|\beta''|/\sigma}|\kappa|!^\sigma\Big(\frac{C_2 2^\sigma}{\Phi(x)\langle\zeta\rangle_k\langle\eta\rangle_k}\Big)^{|\kappa|}\\
&\quad\times \langle z\rangle_k^{-2l+|\kappa|}\langle y\rangle_k^{-2l-|\kappa|}E_2(t,x,\xi,\eta,\zeta)\, dz\, đ\zeta\, dy\, đ\eta.
\end{aligned}
$$

In this case we choose l such that $2l > 2(n+1)+\max\{m_1,m_2\}+|\alpha|+|\beta|+|\kappa|$. Noting that $(\langle\eta\rangle_k\langle\zeta\rangle_k)^{-1} \le \langle\zeta+\eta\rangle_k^{-1}$ and

$$
\inf_{j\in\mathbb{Z}_+} j!^\sigma\Big(\frac{C_2 2^\sigma}{\Phi(x)\langle\zeta+\eta\rangle_k}\Big)^j \le C' e^{-c_2(\Phi(x)\langle\zeta+\eta\rangle_k)^{1/\sigma}},
$$

for some $c_2>0$. Thus we have

$$
\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(2)}(t,x,\xi)|\\
&\le C^{|\alpha|+|\beta|+2}\langle\xi\rangle_k^{m_1-\gamma-\gamma|\alpha|+|\beta|/\sigma}\omega(x)^{m_2}\Phi(x)^{-\gamma-\gamma|\beta|+(|\alpha|+l)/\sigma}\int\int\langle\eta\rangle_k^{-2(n+1)}\\
&\quad\times \exp\{\Phi(x)^{1/\sigma}(\Lambda\langle\xi+(\eta+\zeta)\rangle_k^{1/\sigma} - \Lambda\langle\xi\rangle_k^{1/\sigma} - \frac{c_2}{2}\langle\zeta+\eta\rangle_k^{1/\sigma})\}\, đ\zeta\, đ\eta.
\end{aligned}
$$

For $\langle \xi+\eta+\zeta\rangle_k \le 3\langle \eta+\zeta\rangle_k$, we have $|\langle \xi+\eta+\zeta\rangle_k^{1/\sigma} - \langle \xi\rangle_k^{1/\sigma}| \le 3\langle \eta+\zeta\rangle_k^{1/\sigma}$. For $\langle \xi+\eta+\zeta\rangle_k \ge 3\langle \eta+\zeta\rangle_k$, that is, $\langle \xi\rangle_k \ge 2\langle \eta+\zeta\rangle_k$, we have

$$\langle \xi+\eta+\zeta\rangle_k^{1/\sigma} - \langle \xi\rangle_k^{1/\sigma} \le |\eta+\zeta|(\langle \xi\rangle_k - \langle \eta+\zeta\rangle_k)^{\frac{1}{\sigma}-1} \le \langle \eta+\zeta\rangle_k^{1/\sigma}. \tag{5.3.4}$$

Since c_2 is independent of Λ, there exists $\Lambda^{(2)} > 0$ (in fact, $\Lambda^{(2)} = c_2/12$) such that, for $\Lambda = \Lambda(t) < \Lambda^{(2)}$ we obtain the estimate

$$|\partial_\xi^\alpha \partial_x^\beta r_\Lambda^{(2)}(t,x,\xi)| \le C^{|\alpha|+|\beta|+2} \langle \xi\rangle_k^{m_1-\gamma-\gamma|\beta|+|\alpha|/\sigma} e^{-\frac{c_2}{8}\Phi(x)^{1/\sigma}}.$$

Thus $r_\Lambda^{(2)} \in C([0,T]; AG^{m_1-\gamma,-\infty}_{\sigma;\sigma,\sigma}(\Phi,\tilde{g}_{\Phi,k}))$. By similar techniques used in the case of $r_\Lambda^{(1)}$ and $r_\Lambda^{(2)}$, one can show that $r_\Lambda^{(3)} \in C([0,T]; AG^{-\infty,-\infty}_{\sigma;\sigma,\sigma}(\Phi,\tilde{g}_{\Phi,k}))$. Taking $\Lambda_0 = \min\{\Lambda^{(1)},\Lambda^{(2)}\}$, proves the theorem. □

Remark 5.3.3. *1. The conjugation of Theorem 5.3.1 can also be performed by starting with a symbol $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$.*

2. If $\Phi(x) \equiv C$ for some $C \ge 1$, then the proof of Theorem 5.3.1 takes simpler form as in [47, Proposition 2.3]. In such case, for $C_0 = C^{1/\sigma}$ we have

$$e^{\Lambda(t)C_0\langle D_x\rangle^{1/\sigma}}\, p(x,D_x)\, e^{-\Lambda(t)C_0\langle D_x\rangle^{1/\sigma}} = p(x,D_x) + r_\Lambda(t,x,D_x),$$

where $r_\Lambda(t,x,\xi)$ is in the Hörmander class $S^{m_1-\gamma}_{\gamma,0}$, $\gamma = 1-\frac{1}{\sigma}$, for each t.

Next, we prove two corollaries of Theorem 5.3.1 which will be helpful in making change of variables in the proof of the main result.

Corollary 5.3.4. *There exists $k^* > 1$ such that for $k \ge k^*$,*

$$e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{-\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} = I + R(t,x,D_x) \tag{5.3.5}$$

$$e^{-\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} = I + \tilde{R}(t,x,D_x) \tag{5.3.6}$$

where $I+R$ and $I+\tilde{R}$ are invertible operators with $R,\tilde{R} \in C([0,T]; OPAG^{-\gamma e}_{\sigma;\sigma,\sigma}(\Phi,\tilde{g}_{\Phi,k}))$.

Proof. The equation (5.3.5) can be derived by an application of Theorem 5.3.1 with $p(x,D) \equiv I \in OPAG^{0,0}_{\sigma}(\omega, g_{\Phi,k})$, where I is an identity operator. This yields R as in (5.3.5). We can estimate the operator norm of $R(t,x,D_x)$ by $C_1k^{-\gamma}$. Choosing $k \ge k_1$, where k_1 is sufficiently large, ensures that the operator norm of $R(t,x,D_x)$ is strictly lesser than 1. This guarantees the existence of

$$(I+R(t,x,D_x))^{-1} = \sum_{j=0}^{\infty}(-R(t,x,D_x))^j.$$

As for the equation (5.3.6), we follow the same procedure given in the proof of Theorem 5.3.1 with inequality (5.3.3) replaced with

$$\begin{aligned} -\Phi(x)^{1/\sigma} + \Phi(x+\theta_1 y)^{1/\sigma} &\le -\Phi(x)^{1/\sigma} + (\Phi(x)+\Phi(\theta_1 y))^{1/\sigma} \\ &\le -\Phi(x)^{1/\sigma} + \Phi(x)^{1/\sigma} + \Phi(\theta_1 y))^{1/\sigma} \le \Phi(y))^{1/\sigma}, \end{aligned}$$

for $x,y \in \mathbb{R}^n$. This yields (5.3.6) with $\tilde{R} \in C([0,T]; OPAG^{-\gamma e}_{\sigma;\sigma,\sigma}(\Phi,\tilde{g}_{\Phi,k}))$. Choosing $k \ge k_2$ for k_2 sufficiently large, guarantees that $I+\tilde{R}$ is invertible. Taking $k^* = \max\{k_1,k_2\}$ proves the corollary. □

If we take $\Lambda(t) = \frac{\lambda}{\delta}(T^\delta - t^\delta)$ for $\lambda > 0$ and $\delta \in (0,1)$, then we easily have

$$\begin{aligned} e^{\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}}\, \partial_t\, e^{-\Lambda(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} w(t,x) \\ &= (I+R)\left(\partial_t w(t,x) - \Lambda'(t)(\Phi(x)\langle D\rangle_k)^{1/\sigma} w(t,x)\right) \\ &= (I+R)\Big(\partial_t + \frac{\lambda}{t^{1-\delta}}(\Phi(x)\langle D\rangle_k)^{1/\sigma}\Big) w(t,x), \end{aligned}$$

where the operator $R(t,x,D_x)$ is in $C([0,T]; OPAG^{-\gamma e}_{\sigma;\sigma,\sigma}(\Phi, \tilde{g}_{\Phi,k}))$. As in Corollary 5.3.4, the operator $I + R(t,x,D_x)$ is invertible for sufficiently large k. In the proof of the main result, we choose λ appropriately so that we can apply sharp Gårding inequality to prove the a priori estimate (5.1.9).

Corollary 5.3.5. *Let $0 \le \varepsilon \le \varepsilon' < \Lambda_0$ where Λ_0 is as in Thereom 5.3.1. Then*

$$e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{-\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} = e^{(\varepsilon-\varepsilon')(\Phi(x)\langle D\rangle_k)^{1/\sigma}}(I + \hat{R}(x,D_x)), \tag{5.3.7}$$

where $\hat{R} \in OPAG^{-\gamma e}_{\sigma;\sigma,\sigma}(\Phi, \tilde{g}_{\Phi,k})$ and for sufficiently large k, $I + \hat{R}$ is invertible.

Proof. The equation (5.3.7) can be derived by an easy extension of Theorem 5.3.1. For this we replace the inequality (5.3.3) with

$$\begin{aligned} \varepsilon\Phi(x)^{1/\sigma} - \varepsilon'\Phi(x+\theta_1 y)^{1/\sigma} &\le \varepsilon\Phi(x)^{1/\sigma} - \varepsilon'(\Phi(x) - \Phi(\theta_1 y))^{1/\sigma} \\ &\le \varepsilon\Phi(x)^{1/\sigma} - \varepsilon'(\Phi(x)^{1/\sigma} - \Phi(\theta_1 y)^{1/\sigma}) \\ &\le (\varepsilon - \varepsilon')\Phi(x)^{1/\sigma} + \varepsilon'\Phi(y)^{1/\sigma} \end{aligned} \tag{5.3.8}$$

when $|x| \ge |y|$ and $\varepsilon\Phi(x)^{1/\sigma} - \varepsilon'\Phi(x+\theta_1 y)^{1/\sigma} \le \varepsilon\Phi(x)^{1/\sigma} + \varepsilon'\Phi(y)^{1/\sigma} - \varepsilon'\Phi(x)^{1/\sigma}$ when $|x| \le |y|$ and the inequality (5.3.4) with

$$\begin{aligned} \varepsilon\langle\xi+\eta+\zeta\rangle_k^{1/\sigma} - \varepsilon'\langle\xi\rangle_k^{1/\sigma} &\le \varepsilon\langle\xi\rangle_k^{1/\sigma} + \varepsilon\langle\eta+\zeta\rangle_k^{1/\sigma} - \varepsilon'\langle\xi\rangle_k^{1/\sigma} \\ &\le (\varepsilon-\varepsilon')\langle\xi\rangle_k^{1/\sigma} + \varepsilon\langle\eta+\zeta\rangle_k^{1/\sigma}, \end{aligned} \tag{5.3.9}$$

for $\xi, \eta, \zeta \in \mathbb{R}^n$. □

We use the above corollaries to prove the continuity of an infinite order operator $e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}}$ on the spaces $H^{s,\varepsilon',\sigma}_{\Phi,k}$.

Proposition 5.3.3. *The operator $e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} : H^{s,\varepsilon',\sigma}_{\Phi,k} \to H^{s,\varepsilon'-\varepsilon,\sigma}_{\Phi,k}$ is continuous for $k \ge k_0$ and $0 \le \varepsilon \le \varepsilon' < \Lambda_0$ where k_0 sufficiently large and Λ_0 is as in Theorem 5.3.1.*

Proof. Consider w in $H^{s,\varepsilon',\sigma}_{\Phi,k}$. From Corollaries 5.3.4 and 5.3.5, we have

$$\begin{aligned} e^{-\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} &= I + R_1(x,D_x), \\ e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{-\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} &= e^{(\varepsilon-\varepsilon')(\Phi(x)\langle D\rangle_k)^{1/\sigma}}(I + R_2(x,D_x)), \\ e^{(\varepsilon'-\varepsilon)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{-(\varepsilon'-\varepsilon)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} &= I + R_3(x,D_x). \end{aligned}$$

where $R_1, R_2, R_3 \in OPAG^{-\gamma e}_{\sigma;\sigma,\sigma}(\Phi, \tilde{g}_{\Phi,k})$. For $k \geq k_0$, k_0 sufficiently large, the operators $I + R_j(x, D_x), j = 1, 2, 3$ are invertible. Then, one can write

$$e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} w = e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} \left(e^{-\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} - R_1 \right) w.$$

This implies that

$$e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} (I + R_1) w = e^{(\varepsilon-\varepsilon')(\Phi(x)\langle D\rangle_k)^{1/\sigma}} (I + R_2) e^{\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} w. \tag{5.3.10}$$

From (5.3.10), we have

$$e^{(\varepsilon'-\varepsilon)(\Phi(x)\langle D\rangle_k)^{1/\sigma}} e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} (I + R_1) w = (I + R_3)(I + R_2) e^{\varepsilon'(\Phi(x)\langle D\rangle_k)^{1/\sigma}} w.$$

Note that $(I + R_j), j = 1, 2, 3$, are bounded and invertible operators. Substituting $w = (I + R_1)^{-1} v$ and taking L^2 norm on both sides of the above equation yields

$$\| e^{\varepsilon(\Phi(x)\langle D\rangle_k)^{1/\sigma}} v \|_{\Phi,k;s,\varepsilon'-\varepsilon,\sigma} \leq C_1 \| (I + R_1)^{-1} v \|_{\Phi,k;s,\varepsilon',\sigma} \leq C_2 \| v \|_{\Phi,k;s,\varepsilon',\sigma},$$

for all $v \in H^{s,\varepsilon',\sigma}_{\Phi,k}$ and for some $C_1, C_2 > 0$. This proves the proposition. $\square$

5.4 Subdivision of the Phase Space

We divide the extended phase space into two regions using the Planck function, $h(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$. For the sake of subdivision we define $t_{x,\xi}$, for a fixed (x,ξ), as the solution to the equation

$$t^q = N\, h(x,\xi),$$

where N is the positive constant and q is the given order of singularity. Since $3 \leq \sigma < q/(q-1)$, we consider $\delta \in (0,1)$ such that

$$\frac{1}{\sigma} = \frac{q - 1 + \delta}{q} = 1 - \frac{1-\delta}{q}. \tag{5.4.1}$$

Denote $\gamma = 1 - \frac{1}{\sigma}$. Using $t_{x,\xi}$ and the notation $J = [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ we define the interior region

$$\begin{aligned} Z_{int}(N) &= \{(t,x,\xi) \in J : 0 \leq t \leq t_{x,\xi},\ |x| + |\xi| > N\} \\ &= \{(t,x,\xi) \in J : t^{1-\delta} \leq N^\gamma\ \ h(x,\xi)^\gamma,\ |x| + |\xi| > N\}, \end{aligned} \tag{5.4.2}$$

and the exterior region

$$\begin{aligned} Z_{ext}(N) &= \{(t,x,\xi) \in J : t_{x,\xi} \leq t \leq T,\ |x| + |\xi| > N\} \\ &= \{(t,x,\xi) \in J : t^{1-\delta} \geq N^\gamma\ \ h(x,\xi)^\gamma,\ |x| + |\xi| > N\}. \end{aligned} \tag{5.4.3}$$

The utility of these regions lies in decomposing our operator into, mainly, two operators. The first operator has a high-order in (x,ξ) but excludes the singularity at $t = 0$ and the second operator has a singularity at $t = 0$ but is of lower-order in (x,ξ).

5.5 Global Well-Posedness

In this section, we give a proof of the main result. There are three key steps in the proof. First, we factorize the operator $P(t, x, \partial_t, D_x)$. To this end, we begin with regularizing the characteristic roots of the principal symbol of the operator. Second, we reduce the operator P to a pseudodifferential system of first order. Lastly, we perform a conjugation to deal with the low-regularity in t. Using sharp Gårdings inequality we arrive at L^2-well-posedness of a related auxiliary Cauchy problem, which gives well-posedness of the original problem in the Sobolev spaces $H^{s,\varepsilon,\sigma}_{\Phi,k}$, $3 \leq \sigma < q/(q-1)$.

5.5.1 Factorization

We are interested in a factorization of the operator $P(t, x, \partial_t, D_x)$. Formally, this leads to

$$\begin{aligned} P(t, x, \partial_t, D_x) &= (\partial_t - i\tau_m(t, x, D_x)) \cdots (\partial_t - i\tau_1(t, x, D_x)) \\ &\quad + \sum_{k=0}^{m-1} R_k(t, x, D_x)\partial_t^k \end{aligned} \tag{5.5.1}$$

where

$$\begin{aligned} \tau_j &\in C([0, T]; AG^e_\sigma(\omega, g_{\Phi,k})) \cap C^1((0, T]; AG^e_\sigma(\omega, g_{\Phi,k})), \\ t^q\partial_t\tau_j &\in C([0, T]; AG^e_\sigma(\omega, g_{\Phi,k})) \cap C((0, T]; AG^e_\sigma(\omega, g_{\Phi,k})). \end{aligned}$$

Since the operators $\tau_j(t, x, D_x)$ are not differentiable with respect to t at $t = 0$, we use regularized roots $\lambda_j(t, x, D_x)$ in (5.5.1) instead of $\tau_j(t, x, D_x)$ for $j = 1, \cdots, m$. For this purpose we extend the roots on $(T, \infty]$ by setting

$$\tau_j(t, x, \xi) = \tau_j(T, x, \xi) \quad \text{when } t > T.$$

Then we define the regularized root $\lambda_j(t, x, \xi)$ as

$$\lambda_j(t, x, \xi) = \int_{\mathbb{R}} \tau_j\left(t - h(x, \xi)s, x, \xi\right)\rho(s)ds \tag{5.5.2}$$

where ρ is compactly supported smooth function in $\mathcal{S}^1_1(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \rho(s)ds = 1$ and $0 \leq \rho(s) \leq 1$ with supp $\rho(s) \subset \mathbb{R}_{<0}$. Then

$$\begin{aligned} (\lambda_j - \tau_j)(t, x, \xi) &= \int (\tau_j(t - h(x, \xi)s, x, \xi) - \tau_j(t, x, \xi))\rho(s)ds \\ &= \frac{1}{h(x, \xi)} \int (\tau_j(s, x, \xi) - \tau_j(t, x, \xi))\rho((t - s)h(x, \xi)^{-1})ds. \end{aligned}$$

It is easy to see that

$$\begin{cases} \lambda_j - \tau_j \in L^1([0, T]; AG^e_\sigma(\omega, g_{\Phi,k})) \cap C((0, T]; AG^e_\sigma(\omega, g_{\Phi,k})) \\ \partial_t^j\lambda_j \in L^1([0, T]; AG^{j+1,1}_\sigma(\omega\Phi^j, g_{\Phi,k})) \cap C([0, T]; AG^{j+1,1}_\sigma(\omega\Phi^j, g_{\Phi,k})), \end{cases} \tag{5.5.3}$$

and

$$\begin{cases} t^q(\lambda_j - \tau_j) \in C([0, T]; AG^{0,0}_\sigma(\omega, g_{\Phi,k})) \\ t^q\partial_t^j\lambda_j \in C([0, T]; AG^{je}_\sigma(\omega, g_{\Phi,k})), \quad j \in \mathbb{N}. \end{cases} \tag{5.5.4}$$

We define the operator

$$\tilde{P}(t,x,\partial_t,D_x) = (\partial_t - i\lambda_m(t,x,D_x))\cdots(\partial_t - i\lambda_1(t,x,D_x)).$$

By (5.5.3) and (5.5.4) one has the following factorization of the operator P

$$P(t,x,\partial_t,D_x) = \tilde{P}(t,x,\partial_t,D_x) + R(t,x,\partial_t,D_x),$$

where

$$R(t,x,\partial_t,D_x) = \sum_{j=0}^{m-1} R_j(t,x,D_x)\partial_t^j$$

such that for $j = 0,\dots,m-1$,

$$R_j \in L^1([0,T]; AG_\sigma^{(m-j)e}(\Phi, g_{\Phi,k})) \cap C([0,T]; AG_\sigma^{(m-j)e}(\Phi, g_{\Phi,k})), \quad \text{and} \tag{5.5.5}$$

$$t^q R_j \in C([0,T]; AG_\sigma^{(m-1-j)e}(\Phi, g_{\Phi,k})). \tag{5.5.6}$$

To determine the precise Gevrey regularity for $R_j(t,x,\xi)$, we consider the regions, $Z_{int}(N)$ and $Z_{ext}(N)$, separately. In $Z_{int}(N)$, we have $(\Phi(x)\langle\xi\rangle_k)^\gamma \leq \frac{N^\gamma}{t^{1-\delta}}$. Using this and (5.5.5), we can write

$$\begin{aligned} |\partial_\xi^\alpha D_x^\beta R_j(t,x,\xi)| &\leq C^{|\alpha|+|\beta|}\beta!^\sigma\alpha!\Phi(x)^{m-\gamma-j-|\beta|}\langle\xi\rangle_k^{m-\gamma-j-|\alpha|}(\Phi(x)\langle\xi\rangle_k)^\gamma \\ &\leq C_1^{|\alpha|+|\beta|}\beta!^\sigma\alpha!\frac{N^\gamma}{t^{1-\delta}}\Phi(x)^{m-\gamma-j-|\beta|}\langle\xi\rangle_k^{m-\gamma-j-|\alpha|} \end{aligned}$$

Similarly, in $Z_{ext}(N)$, we have $t^{q/\sigma} \geq (N\,h(x,\xi))^{\frac{1}{\sigma}}$ and

$$\frac{1}{t^q} = \frac{1}{t^{1-\delta}}\frac{1}{t^{q/\sigma}} \leq \frac{1}{t^{1-\delta}}\left(\frac{\Phi(x)\langle\xi\rangle_k}{N}\right)^{1/\sigma}$$

Using this and (5.5.6), we have

$$\begin{aligned} |\partial_\xi^\alpha D_x^\beta R_j(t,x,\xi)| &\leq C^{|\alpha|+|\beta|}\beta!^\sigma\alpha!\frac{1}{t^q}\Phi(x)^{m-1-j-|\beta|}\langle\xi\rangle_k^{m-1-j-|\alpha|} \\ &\leq C_1^{|\alpha|+|\beta|}\beta!^\sigma\alpha!\frac{N^{-1/\sigma}}{t^{1-\delta}}\Phi(x)^{m-\gamma-j-|\beta|}\langle\xi\rangle_k^{m-\gamma-j-|\alpha|} \end{aligned}$$

Hence, we have $t^{1-\delta}R_j \in C([0,T]; AG_\sigma^{(m-\gamma-j)e}(\Phi, g_{\Phi,k}))$ for $j = 0,\cdots,(m-1)$.

5.5.2 Reduction to First Order Pseudodifferential System

We will now reduce the operator P to an equivalent first order pseudodifferential system. The procedure is similar to the one used in [12, Section 4.2 & 4.3]. To achieve this, we introduce the change of variables $U = U(t,x) = (u_0(t,x),\cdots,u_{m-1}(t,x))^T$, where

$$\begin{cases} u_0(t,x) = \omega(x)^{m-1}\langle D_x\rangle_k^{m-1}u(t,x), \\ u_j(t,x) = \omega(x)^{m-1-j}\langle D_x\rangle_k^{m-1-j}(\partial_t - i\lambda_j(t,x,D_x))\cdots(\partial_t - i\lambda_1(t,x,D_x))u(t,x), \end{cases}$$

for $j = 1, \cdots m-1$. Then, $Pu = f$ is equivalent to

$$(\partial_t - A_1(t,x,D_x) + A_2(t,x,D_x))U(t,x) = F(t,x),$$

where $F(t,x) = (0,\dots,0,f(t,x))^T$,

$$A_1(t,x,D_x) = \begin{pmatrix} i\lambda_1(t,x,D_x) & \omega(x)\langle D_x\rangle_k & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \omega(x)\langle D_x\rangle_k \\ 0 & \dots & \dots & 0 & i\lambda_m(t,x,D_x) \end{pmatrix},$$

and $A_2(t,x,D_x) = \{a^{(2)}_{i,j}(t,x,D_x)\}_{1\le i,j\le m}$ is a matrix of lower order terms with $a^{(2)}_{i,j} \in AG^{0,0}_\sigma(\Phi, g_{\Phi,k})$ for $i = 1,\dots,m-1$ and $j = 1,\dots,m$, and $t^{1-\delta}a^{(2)}_{m,j} \in C([0,T]; AG^{\frac{1}{\sigma}e}_\sigma(\Phi, g_{\Phi,k}))$ for $j = 1,\dots,m$.

Consider the $T(t,x,\xi) = \{\beta_{p,q}(t,x,\xi)\}_{0\le p,q\le m-1}$, where

$$\begin{aligned} \beta_{p,q}(t,x,\xi) &= 0, \qquad p \ge q; \\ \beta_{p,q}(t,x,\xi) &= \frac{(1-\varphi_1(\Phi(x)\langle\xi\rangle))(\omega(x)\langle\xi\rangle_k)^{q-p}}{d_{p,q}(t,x,\xi)}, \qquad p < q; \\ d_{p,q}(t,x,\xi) &= \prod_{r=p+1}^{q} i\Big(\lambda_{q+1}(t,x,\xi) - \lambda_r(t,x,\xi)\Big), \end{aligned}$$

where $\varphi_1 \in C_0^\infty(\mathbb{R})$, $\varphi_1(r) = 1$ for $|r| \le M$, for a large parameter M. Note that the matrix $T(t,x,\xi)$ is nilpotent. We define $H(t,x,D_x)$ and $\tilde{H}(t,x,D_x)$ to be pseudodifferential operators with symbols

$$\begin{aligned} H(t,x,\xi) &= I + T(t,x,\xi), \text{ and} \\ \tilde{H}(t,x,\xi) &= I + \sum_{j=1}^{m-1}(-1)^j T^j(t,x,\xi), \end{aligned} \tag{5.5.7}$$

respectively.

Proposition 5.5.1. *For the operators $H(t,x,D_x)$ and $\tilde{H}(t,x,D_x)$, the following assertions hold true*

i) $H(t,x,D_x)$ and $\tilde{H}(t,x,D_x)$ are in $C\Big([0,T]; OPAG^{0,0}_\sigma(\Phi, g_{\Phi,k})\Big)$.

ii) The compositions of $H(t,x,D_x)$ and $\tilde{H}(t,x,D_x)$ satisfy

$$\begin{cases} H(t,x,D_x)\circ\tilde{H}(t,x,D_x) = I + K_1(t,x,D_x), \\ \tilde{H}(t,x,D_x)\circ H(t,x,D_x) = I + K_2(t,x,D_x), \end{cases} \tag{5.5.8}$$

for $K_j(t,x,D_x) \in C\Big([0,T]; OPAG^{-e}_\sigma(\Phi, g_{\Phi,k})\Big)$, $j = 1, 2$.

iii) The operator $t^q(\partial_t H)(t,x,D_x)$ *belongs to* $C\Big([0,T]; OPAG^{0,0}_\sigma(\Phi, g_{\Phi,k})\Big)$.

iv) The operator $t^{1-\delta}(\partial_t H)(t,x,D_x)$ *belongs to* $C\Big([0,T]; OPAG^{\frac{1}{\sigma}e}_\sigma(\Phi, g_{\Phi,k})\Big)$.

Note that $I + K_j(t,x,D_x)$ is invertible for sufficiently large M. Since $K_j(t,x,D_x) \in C\Big([0,T]; OPAG^{-e}_\sigma(\Phi, g_{\Phi,k})\Big)$, we can estimate the operator norm of K by CM^{-1} where M is as in the definition of $H(t,x,D_x)$. We choose M sufficiently large so that the operator norm of $K(t,x,D_x)$ is strictly lesser than 1. This implies that $I + K_j(t,x,D_x)$ is invertible and

$$(I + K_j(t,x,D_x))^{-1} = \sum_{l=0}^{\infty}(-K_j(t,x,D_x))^j \in C\Big([0,T]; OPAG^{0,0}_\sigma(\Phi, g_{\Phi,k})\Big).$$

We perform the described change of variable by setting

$$\hat{U} = \hat{U}(t,x) = \tilde{H}(t,x,D_x)U(t,x).$$

The above equation and (5.5.8) imply that

$$U(t,x) = (I + K_1(t,x,D_x))^{-1}H(t,x,D_x)\hat{U}(t,x).$$

Noting this fact, we obtain (similar to [12, Section 4.3]) the first order system equivalent to (5.1.5) :

$$(\partial_t - A_3(t,x,D_x) + A_4(t,x,D_x))\hat{U}(t,x) = F_1(t,x),$$

where $F_1(t,x) = \mathcal{K}_2(t,x,D_x)^{-1}\tilde{H}(t,x,D_x)\mathcal{K}_1(t,x,D_x)F(t,x)$ for $\mathcal{K}_j(t,x,D_x) = (I + K(t,x,D_x))$, $j = 1,2$, and the operators A_3 and A_4 are as follows

$$\begin{aligned} A_3 &= \mathcal{K}_2^{-1}\tilde{H}\mathcal{K}_1 A_1 \mathcal{K}_1^{-1}H, \\ A_4 &= \mathcal{K}_2^{-1}\tilde{H}\mathcal{K}_1 A_2 \mathcal{K}_1^{-1}H + \mathcal{K}_2^{-1}\tilde{H}\mathcal{K}_1(\partial_t\mathcal{K}_1^{-1})H + \mathcal{K}_2^{-1}\tilde{H}\partial_t H. \end{aligned}$$

We can write

$$\begin{aligned} A_3(t,x,D_x) &= \mathcal{D}(t,x,D_x) + \tilde{A}_3(t,x,D_x) \\ A(t,x,D_x) &= A_4(t,x,D_x) - \tilde{A}_3(t,x,D_x) \end{aligned}$$

where $\mathcal{D} = \operatorname{diag}(i\lambda_1(t,x,D_x), \dots, i\lambda_m(t,x,D_x))$, and $A(t,x,D_x)$ contains the lower order terms whose symbol is such that

$$t^{1-\delta}A \in C([0,T]; AG^{\frac{1}{\sigma}e}_\sigma(\Phi, g_{\Phi,k})).$$

Then, $Pu = f$ is equivalent to

$$L_1\hat{U} = (\partial_t - \mathcal{D} + A)\hat{U} = F_1(t,x). \tag{5.5.9}$$

We prove the a priori estimate (5.1.9) by proving that

$$\|\hat{U}(t)\|_{\Phi,k;s,\Lambda(t),\sigma} \leq C\Big(\|\hat{U}(0)\|_{\Phi,k;s,\Lambda(0),\sigma} + \int_0^t \|F_1(\tau,\cdot)\|_{\Phi,k;s,\Lambda(\tau),\sigma} d\tau\Big),$$

where $\Lambda(t) = \frac{\lambda}{\delta}(T^\delta - t^\delta)$ for sufficiently large λ.

It is sufficient to prove the above estimate for $s = (0,0)$ since the operator $L_2 = \Phi(x)^{s_2}\langle D_x\rangle_k^{s_1} L_1 \langle D_x\rangle_k^{-s_1}\Phi(x)^{-s_2}$ satisfies the same hypotheses as L_1. That is, for $V(t,x) = \Phi(x)^{s_2}\langle D_x\rangle_k^{s_1}\hat{U}(t,x)$, $L_1\hat{U} = F_1$ implies $L_2 V = F_2$ where $F_2(t,x) = \Phi(x)^{s_2}\langle D_x\rangle_k^{s_1}F_1(t,x)$. So, assuming $s = (0,0)$ we let $L_2 = L_1 = \partial_t - \mathcal{D} + A$.

To deal with the low-regularity in t, we introduce the following change of variable

$$W(t,x) = e^{\Lambda(t)(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}}V(t,x). \tag{5.5.10}$$

This implies that $V(t,x) = (I+\tilde{R}(t,x,D_x))^{-1}e^{-\Lambda(t)(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}}W(t,x)$. Here $I+\tilde{R}(t,x,D_x)$ and $I + R(t,x,D_x)$ are invertible operators as in Corollary 5.3.4. Let us denote $I + R(t,x,D_x)$, $I+\tilde{R}(t,x,D_x)$ and $e^{\pm\Lambda(t)(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}}$ by $\mathcal{R}(t,x,D_x)$, $\tilde{\mathcal{R}}(t,x,D_x)$ and $E^{(\pm)}(t,x,D_x)$, respectively. Then $Pu = f$ is equivalent to $L_3 W = F_3$ where

$$L_3 = \partial_t - \mathcal{D} + \Big(B + \frac{\lambda}{t^{1-\delta}}(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}\Big),$$

$F_3(t,x) = \mathcal{R}^{-1}E^{(+)}\tilde{\mathcal{R}}F_2(t,x)$ and the operator $B(t,x,D_x)$ is given by

$$B = \mathcal{R}^{-1}E^{(+)}\Big(\tilde{\mathcal{R}}(\partial_t\tilde{\mathcal{R}}^{-1}) + \tilde{\mathcal{R}}A\tilde{\mathcal{R}}^{-1}\Big)E^{(-)} - \big(\mathcal{R}^{-1}E^{(+)}\tilde{\mathcal{R}}\mathcal{D}\tilde{\mathcal{R}}^{-1}E^{(-)} - \mathcal{D}\big).$$

Observe that from Theorem 5.3.1 and from the Cauchy data given in conditions (ii) and (iii) of Theorem 5.1.1, we need $\Lambda(t) < \Lambda^* = \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$. This implies $T < \big(\frac{\delta}{\lambda}\Lambda^*\big)^{1/\delta}$. Then, we have $t^{1-\delta}B \in C([0,T]; OPAG^{\frac{1}{\sigma}e}_{\sigma;\sigma,\sigma}(\Phi, g_{\Phi,k}))$. The estimate (5.1.9) on the solution u can be established by proving that the function $W(t,x)$ satisfies the a priori estimate

$$\|W(t)\|^2_{L^2} \le C\Big(\|W(0)\|^2_{L^2} + \int_0^t \|F_3(\tau,\cdot)\|_{L^2}d\tau\Big), \quad t \in [0,T],\ C > 0. \tag{5.5.11}$$

5.5.3 Energy Estimate

Observe that we have $L_3 W = F_3$ whenever $L_1\hat{U} = F_1$ and $\|W(t)\|_{L^2} = \|\hat{U}(t)\|_{\Phi,k;s,\Lambda(t),\sigma}$. Moreover, the problem $L_3 W = F_3$ is equivalent to an auxiliary problem

$$\partial_t W = \mathcal{D}W - \Big(\frac{\lambda}{t^{1-\delta}}(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}W + BW\Big) + F_3(t,x),$$

for $(t,x) \in (0,T] \times \mathbb{R}^n$, with initial conditions

$$W(0,x) = (w_0(x), \dots, w_{m-1}(x))^T, \quad \text{where}$$

$$\begin{aligned} w_j(x) =& e^{\Lambda(0)(\Phi(x)\langle D_x\rangle_k)^{1/\sigma}}\Phi(x)^{s_2}\langle D_x\rangle^{s_1}\tilde{H}(0,x,D_x)\Phi(x)^{m-1-j}\langle D_x\rangle^{m-1-j} \\ &\times (\partial_t - i\lambda_j(0,x,D_x))\cdots(\partial_t - i\lambda_1(0,x,D_x))u(0,x) \end{aligned}$$

for $j = 0, \dots, m-1$. To prove (5.5.11), let us consider

$$\begin{aligned}\partial_t \|W(t)\|_{L^2}^2 &= 2\,\mathrm{Re}\langle \partial_t W, W\rangle_{L^2} \\ &= 2\,\mathrm{Re}\langle \mathcal{D}W, W\rangle_{L^2} - 2\,\mathrm{Re}\Big\langle \Big(\frac{\lambda}{t^{1-\delta}}(\Phi(x)\langle D_x\rangle_k)^{1/\sigma} + B\Big)W, W\Big\rangle_{L^2} \\ &\quad + 2\,\mathrm{Re}\langle F_3, W\rangle. \end{aligned} \tag{5.5.12}$$

Since $\mathcal{D}$ is diagonal with purely imaginary entries, we have

$$\mathrm{Re}\langle \mathcal{D}W, W\rangle_{L^2} \leq C_1 \|W(t)\|_{L^2}. \tag{5.5.13}$$

Also, note that $t^{1-\delta} h(x,D_x)^{1/\sigma} B(t,x,D_x) \in C([0,T]; OPAG^{0,0}_{\sigma;\sigma,\sigma}(\Phi, g_{\Phi,k}))$. We choose λ sufficiently large so that we can apply sharp Gårding inequality, see [43, Theorem 18.6.14], for the metric $\tilde{g}_{\Phi,k}$ given in (5.1.3) with the Planck function $\tilde{h}(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{\frac{1}{\sigma}-\gamma}$. It is important to note that the application of sharp Gårding inequality requires $\sigma \geq 3$. This yields

$$\mathrm{Re}\Big\langle \Big(\frac{\lambda}{t^{1-\delta}}(\Phi(x)\langle D_x\rangle_k)^{1/\sigma} + B\Big)W, W\Big\rangle_{L^2} \geq -C_2\|W\|_{L^2}, \quad C_2 > 0. \tag{5.5.14}$$

From (5.5.12), (5.5.13) and (5.5.14) we have

$$\frac{d}{dt}\|W(t)\|_{L^2}^2 \leq C\|W(t)\|_{L^2} + C\|F_3(t,\cdot)\|_{L^2}.$$

Considering the above inequality as a differential inequality, we apply Gronwall's lemma and obtain that

$$\|W(t)\|_{L^2}^2 \leq C\|W(0)\|_{L^2}^2 + C\int_0^t \|F_3(\tau,\cdot)\|_{L^2}^2 d\tau.$$

This proves well-posedness of the auxiliary Cauchy problem. Note that the solution $\hat{U}$ to (5.5.9) belongs to $C([0,T]; H^{s,\Lambda(t),\sigma}_{\Phi,k})$. Returning to our original solution $u = u(t,x)$ we obtain the estimate (D.3.7) with

$$u \in \bigcap_{j=0}^{m-1} C^{m-1-j}([0,T]; H^{s+ej,\Lambda(t),\sigma}_{\Phi,k}).$$

This concludes the proof. □

5.6 Anisotropic Cone Condition

Let $K(x^0, t^0)$ denote the cone with the vertex (x^0, t^0):

$$K(x^0,t^0) = \{(t,x) \in [0,T]\times\mathbb{R}^n : |x - x^0| \leq c\omega(x)(t^0 - t)\},$$

where $c > 0$. The cone $K(x^0,t^0)$ has a slope $c\omega(x)$ which governs the speed of the growth of the cone. Note that the speed is anisotropic, that is, it varies with x. For a given (t,x), it is well known in the literature (see [76, Section 3.11]) that the influence of the vertex of cone is carried farther by the dominating characteristic root of the principal symbol of the operator P in (5.1.5). So, the constant c is determined by the characteristic roots as

$$c=\sup\Big\{|\tau_k(t,x,\xi)|\omega(x)^{-1}:k=1,\dots,m\Big\}, \tag{5.6.1}$$

where supremum is taken over the set $\{(t,x,\xi)\in[0,T]\times\mathbb{R}^n_x\times\mathbb{R}^n_\xi:|\xi|=1\}$. Note that the speed of growth of the cone increases as $|x|$ increases since ω is monotone increasing function of $|x|$. In the following we prove the cone condition for the Cauchy problem (5.1.5).

Proposition 5.6.1. *The Cauchy problem (5.1.5) has a cone dependence, that is, if*

$$f\big|_{K(x^0,t^0)}=0,\quad f_i\big|_{K(x^0,t^0)\cap\{t=0\}}=0,\ i=1,\dots,m \tag{5.6.2}$$

then

$$u\big|_{K(x^0,t^0)}=0, \tag{5.6.3}$$

provided that c is as in (5.6.1).

Proof. Consider $t^0>0$, $c>0$ and assume that (5.6.2) holds. We define a set of operators $P_\varepsilon(t,x,\partial_t,D_x), 0\le\varepsilon\le\varepsilon_0$ by means of the operator $P(t,x,\partial_t,D_x)$ in (5.1.5) as follows

$$P_\varepsilon(t,x,\partial_t,D_x)=P(t+\varepsilon,x,\partial_t,D_x),\ t\in[0,T-\varepsilon_0],x\in\mathbb{R}^n,$$

and $\varepsilon_0<T-t^0$, for a fixed and sufficiently small ε_0. For these operators we consider Cauchy problems

$$P_\varepsilon v_\varepsilon=f,\qquad t\in[0,T-\varepsilon_0],\ x\in\mathbb{R}^n \tag{5.6.4}$$

$$\partial_t^{k-1}v_\varepsilon(0,x)=f_k(x),\qquad k=1,\dots,m. \tag{5.6.5}$$

Note that $v_\varepsilon(t,x)=0$ in $K(x^0,t^0)$ and v_ε satisfies the a priori estimate (5.1.9) for all $t\in[0,T-\varepsilon_0]$. Further, we have

$$P_{\varepsilon_1}(v_{\varepsilon_1}-v_{\varepsilon_2})=(P_{\varepsilon_2}-P_{\varepsilon_1})v_{\varepsilon_2},\qquad t\in[0,T-\varepsilon_0],\ x\in\mathbb{R}^n \tag{5.6.6}$$

$$\partial_t^{k-1}(v_{\varepsilon_1}-v_{\varepsilon_2})(0,x)=0,\qquad k=1,\dots,m. \tag{5.6.7}$$

For the sake of simplicity we denote $b_{j,\alpha}(t,x)$, the coefficients of lower order terms, as $a_{j,\alpha}$ for $j+|\alpha|<m$. Substituting $s-e$ for s in the a priori estimate, we obtain

$$\begin{aligned}
&\sum_{j=0}^{m-1}\|\partial_t^j(v_{\varepsilon_1}-v_{\varepsilon_2})(t,\cdot)\|_{\Phi,k;s+(m-2-j)e,\Lambda(t),\sigma}\\
&\le C\int_0^t\|(P_{\varepsilon_2}-P_{\varepsilon_1})v_{\varepsilon_2}(\tau,\cdot)\|_{\Phi,k;s-e,\Lambda(\tau),\sigma}\,d\tau\\
&\le C\int_0^t\sum_{\substack{j+|\alpha|\le m\\ j<m}}\|(a_{j,\alpha}(\tau+\varepsilon_1,x)-a_{j,\alpha}(\tau+\varepsilon_2,x))\partial_t^jD_x^\alpha v_{\varepsilon_2}(\tau,\cdot)\|_{\Phi,k;s-e,\Lambda(\tau),\sigma}\,d\tau.
\end{aligned} \tag{5.6.8}$$

Using the Taylor series approximation, we have

$$\begin{aligned}|a_{j,\alpha}(\tau+\varepsilon_1,x)-a_{j,\alpha}(\tau+\varepsilon_2,x)| &= \Big|\int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1}(\partial_t a_{j,\alpha})(r,x)dr\Big| \\ &\le \omega(x)^{m-j}\Big|\int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1}\frac{dr}{r^q}\Big| \\ &\le \omega(x)^{m-j}|E_q(\tau,\varepsilon_1,\varepsilon_2)|,\end{aligned}$$

where

$$E_q(\tau,\varepsilon_1,\varepsilon_2)=\frac{1}{q-1}\frac{(\tau+\varepsilon_1)^{q-1}-(\tau+\varepsilon_2)^{q-1}}{((\tau+\varepsilon_2)(\tau+\varepsilon_1))^{q-1}}$$

Note that $E_q(\tau,\varepsilon,\varepsilon)=0$. Then right-hand side of the inequality in (5.6.8) is dominated by

$$C\int_0^t |E_q(\tau,\varepsilon_1,\varepsilon_2)|\sum_{j=0}^{m-1}\|(\partial_t^j v_{\varepsilon_2})(\tau,\cdot)\|_{\Phi,k;s+(m-1-j)e,\Lambda(\tau),\sigma}\,d\tau,$$

where C is independent of ε. By definition, E_q is L_1-integrable in τ.

The sequence v_{ε_k}, $k=1,2,\dots$ corresponding to the sequence $\varepsilon_k\to 0$ is in the space

$$\bigcap_{j=0}^{m-1} C^{m-1-j}\Big([0,T^*];H_{\Phi,k}^{s+(m-2-j)e,\Lambda(t),\sigma}\Big),\quad T^*>0$$

and $u=\lim_{k\to\infty}v_{\varepsilon_k}$ in the above space and hence, in $\mathcal{D}'(K(x^0,t^0))$. In particular,

$$\langle u,\varphi\rangle=\lim_{k\to\infty}\langle v_{\varepsilon_k},\varphi\rangle=0,\ \forall\varphi\in\mathcal{D}(K(x^0,t^0)).$$

gives (5.6.3) and completes the theorem. □

Chapter 6

Very Strong Blow-up

> In many circumstances in modern analysis, contrary to the usual point of view, the operation of integration proves a much simpler one than the operation of derivation.
>
> — Hadamard

The goal of this chapter is to establish global well-posedness and loss of regularity for singular hyperbolic equations with coefficients having very strong blow-up rate in time and polynomial growth in space. In the previous chapters we dealt with mild, logarithmic and strong blow-up. In this chapter we deal with blow-up rate given by $p \in \left[0, \frac{1}{2}\right), q \in \left(1, \frac{3}{2}\right)$ in Definition 1.2.3.

6.1 Introduction and Statement of Main Result

Let us consider a prototypical strictly hyperbolic Cauchy problem:

$$\begin{cases} \partial_t^2 u - a(t,x)\partial_x^2 u + \displaystyle\sum_{\substack{j,l=0\\ j+l\leq 1}}^{1} b_{j,l}(t,x)\partial_x^j\partial_t^l u = f(t,x), \quad (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x). \end{cases} \tag{6.1.1}$$

The operator coefficients are in $L^1\left((0,T];C^\infty(\mathbb{R})\right) \cap C^1\left((0,T];C^\infty(\mathbb{R})\right)$ and the singular behavior of the above Cauchy problem is described by the following estimates

$$\left.\begin{aligned} |\partial_x^\beta \partial_t a(t,x)| &\leq C_\beta^{(1)}\ \omega(x)^2 \Phi(x)^{-|\beta|}\ \frac{1}{t^q}, \\ |\partial_x^\beta a(t,x)| &\leq C_\beta^{(2)}\ \omega(x)^2 \Phi(x)^{-|\beta|}\ \frac{1}{t^p}, \\ |\partial_x^\beta b_{j,l}(t,x)| &\leq C_\beta^{(3)}\ \omega(x)^j \Phi(x)^{-|\beta|}\ \frac{1}{t^r}, \end{aligned}\right\} \tag{6.1.2}$$

with $\beta \in \mathbb{N}_0, C_\beta^{(i)} > 0, i = 1,2,3,\ q \in (1,\infty), p \in [0,1), p \leq q-1$ and $r \in [0,1)$. The functions $\omega(x)$ and $\Phi(x)$ are positive monotone increasing in $|x|$ such that $1 \leq \omega(x) \lesssim$

$\Phi(x) \lesssim \langle x \rangle = (1+|x|^2)^{1/2}$. They specify the structure of the differential equation in the space variable.

An example of a coefficient $a(t,x)$ satisfying (6.1.2) is given below.

Example 6.1.1. *Let $T = 1$, $\kappa_1 \in [0,1]$ and $\kappa_2 \in (0,1]$ such that $\kappa_1 \leq \kappa_2$. Then,*

$$a(t,x) = \langle x \rangle^{2\kappa_1}\left(2+\cos\langle x \rangle^{1-\kappa_2}\right)\left(\frac{1}{t^{1/4}}\left(2+\sin\left(\frac{1}{t^{1/8}}\right)\right)\right)$$

satisfies the estimates (6.1.2) for $\omega(x) = \langle x \rangle^{\kappa_1}$, $\Phi(x) = \langle x \rangle^{\kappa_2}$, $p = \frac{1}{4}$, $q = \frac{11}{8}$. The example shows that singular coefficients can also have infinitely many oscillations near $t = 0$.

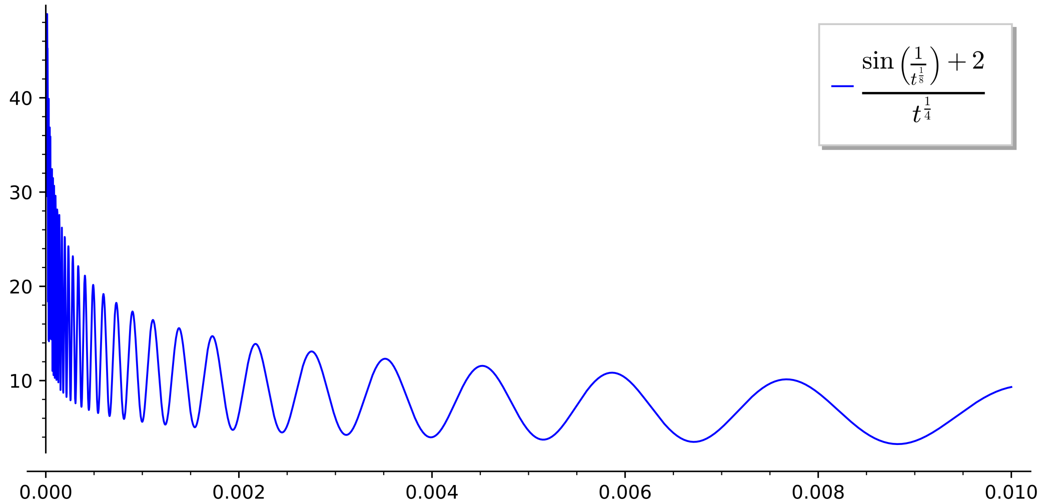

Figure 6.1: Behavior w.r.t time variable for the Example 6.1.1

Colombini et al. [15] considered the Cauchy problem (6.1.1) with operator coefficients independent of x and singular behavior prescribed by the parameters $p \in [0,1), q \in (1,\infty)$ and $r = 0$. They report well-posedness in Gevrey space $G^s, 1 \leq s < \frac{q-p}{q-1}$, with infinite loss of derivatives. In the previous chapter we studied the case of $p = 0, q \in \left(1, \frac{3}{2}\right)$ and $r = 0$ with generic structure functions ω and Φ in (6.1.2) and report infinite loss of both derivatives and decay.

In this chapter our interest is in the operator coefficients with polynomial growth in x and very strong blow-rate. In particular, our interest is $p \in \left[0, \frac{1}{2}\right), q \in \left(1, \frac{3}{2}\right), p \leq q-1, r \in [0,1)$ and polynomial growth in x prescribed by $\omega(x), \Phi(x)$ in (6.1.2).

In order to microlocally compensate the infinite loss of regularity, we conjugate a first order system related to the Cauchy problem by a loss operator which is an infinite order pseudodifferential operator of the form

$$e^{\Lambda(t)\Theta(x,D_x)}. \tag{6.1.3}$$

Here $\Lambda \in C([0,T])$ and the symbol of the operator $\Theta(x,D_x)$ is given by $h(x,\xi)^{-1/\sigma} = (\Phi(x)\langle\xi\rangle_k)^{1/\sigma}$ where $h(x,\xi)$ is the Planck function related to the metric $g_{\Phi,k}$ in (2.1.2) and $3 \leq \sigma < (q-p)/(q-1)$. The operator $\Theta(x, D_x)$ explains the quantity of the loss by linking it to the metric on the phase space and the singular behavior while $\Lambda(t)$ gives a scale for the loss.

Our methodology relies upon two important techniques: the subdivision of the extended phase space into two regions and conjugation by a loss operator. Both these techniques result in the change of the metric governing the operator where the new metric is conformally equivalent to the one in (2.1.2). As seen in (6.4.3), the characteristic roots corresponding to the operator P showcase a stronger singular behavior compared to the principal symbol. Due to the subdivision of the phase space, this results in the change of the metric as demonstrated in Lemma 6.4.1. This metric is of the form

$$\tilde{g}^{(1)}_{\Phi,k} = (\Phi(x)\langle\xi\rangle_k)^{2\delta'} g_{\Phi,k}, \tag{6.1.4}$$

where $\delta' = \frac{p}{q-p}$. On the other hand, Theorem 5.3.1 in the previous chapter suggests that the conjugation by the loss operator changes the metric to

$$\tilde{g}^{(2)}_{\Phi,k} = (\Phi(x)\langle\xi\rangle_k)^{2/\sigma} g_{\Phi,k}. \tag{6.1.5}$$

As our approach to establish well-posedness is based on the energy estimates, we consider the metric $\tilde{g}^{(2)}_{\Phi,k} = \tilde{g}^{(1)}_{\Phi,k} \vee \tilde{g}^{(2)}_{\Phi,k}$ for the application of the sharp Gårding inequality [43, Theorem 18.6.14].

We report that the solution experiences an infinite loss of regularity index in relation to the initial datum in Sobolev spaces $H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n)$ $\left(\sigma > 2, \varepsilon \geq 0,\ s=(s_1,s_2) \in \mathbb{R}^2\right)$ that are defined in Section 5.1.1 of the previous chapter.

6.1.1 Main Result

Let us generalize the problem in (6.1.1) and consider

$$\begin{cases} P(t,x,\partial_t,D_x)u(t,x) = f(t,x), \qquad D_x = -i\nabla_x,\ (t,x) \in (0,T] \times \mathbb{R}^n, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x), \end{cases} \tag{6.1.6}$$

with the strictly hyperbolic operator $P(t,x,\partial_t,D_x) = \partial_t^2 + b_0(t,x)\partial_t + a(t,x,D_x) + b(t,x,D_x)$ where

$$a(t,x,\xi) = \sum_{i,j=1}^n a_{i,j}(t,x)\xi_i\xi_j \quad \text{and} \quad b(t,x,\xi) = i\sum_{j=1}^n b_j(t,x)\xi_j + b_{n+1}(t,x). \tag{6.1.7}$$

Here, the matrix $(a_{i,j}(t,x))$ is real symmetric for all $(t,x) \in (0,T]\times\mathbb{R}^n$, $a_{i,j} \in L^1((0,T];C^\infty(\mathbb{R}^n)) \cap C^1((0,T];C^\infty(\mathbb{R}^n))$ and $b_j \in L^1((0,T];C^\infty(\mathbb{R}^n))$. We have the following assumptions on $a(t,x,\xi)$, $b(t,x,\xi)$ and $b_j(t,x), j = 0, n+1$:

$$\left.\begin{aligned} a(t,x,\xi) &\geq C_0\omega(x)^2\langle\xi\rangle_k^2, \quad C_0 > 0, \\ |\partial_\xi^\alpha\partial_x^\beta a(t,x,\xi)| &\leq C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma \frac{1}{t^p}\omega(x)^2\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{2-|\alpha|}, \\ |\partial_\xi^\alpha\partial_x^\beta\partial_t a(t,x,\xi)| &\leq C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma \frac{1}{t^q}\omega(x)^2\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{2-|\alpha|}, \\ |\partial_\xi^\alpha\partial_x^\beta b(t,x,\xi)| &\leq C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma \frac{1}{t^r}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}, \\ |\partial_x^\beta b_j(t,x)| &\leq C^{|\beta|}(\beta!)^\sigma \frac{1}{t^r}\Phi(x)^{-|\beta|}, \quad j = 0, n+1, \end{aligned}\right\} \tag{6.1.8}$$

$q \in \left(1, \frac{3}{2}\right), p \leq q-1, p \in \left[0, \frac{1}{2}\right), r \in \left[0, 1\right), 3 \leq \sigma < (q-p)/(q-1)$ and $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$. Note that $C > 0$ is a generic constant.

We now state the main result of this chapter. Let $e = (1, 1)$.

Theorem 6.1.1. *Consider the strictly hyperbolic Cauchy problem (6.1.6) satisfying the conditions in (6.1.8). Let the initial data f_j belong to $H_{\Phi,k}^{s+(2-j)e,\Lambda_1,\sigma}$ and the right hand side $f \in C([0,T]; H_{\Phi,k}^{s,\Lambda_2,\sigma})$, $\Lambda_j > 0, j = 1, 2$. Then, there exist a continuous function $\Lambda(t)$ and positive constants Λ_0 and δ^*, such that there is a unique solution*

$$u \in C\left([0,T]; H_{\Phi,k}^{s+e,\Lambda(t),\sigma}\right) \bigcap C^1\left([0,T]; H_{\Phi,k}^{s,\Lambda(t),\sigma}\right),$$

for $\Lambda(t) < \Lambda^* = \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$. More specifically, the solution satisfies an a priori estimate

$$\begin{aligned}\sum_{j=0}^{1} \|\partial_t^j u(t,\cdot)\|_{\Phi,k;s+(1-j)e,\Lambda(t),\sigma} \\ \leq C\left(\sum_{j=1}^{2} \|f_j\|_{\Phi,k;s+(2-j)e,\Lambda(0),\sigma} + \int_0^t \|f(\tau,\cdot)\|_{\Phi,k;s,\Lambda(\tau),\sigma}\, d\tau\right)\end{aligned} \tag{6.1.9}$$

where $0 \leq t \leq T \leq (\delta^*\Lambda^*/\lambda)^{1/\delta^*}$, $C = C_s > 0$ and $\Lambda(t) = \frac{\lambda}{\delta^*}\left(T^{\delta^*} - t^{\delta^*}\right)$ for a sufficiently large λ.

Remark 6.1.1. *Observe that we have $3 \leq \sigma < (q-p)/(q-1)$ where as in [15, Theorem 2], it is $1 \leq \sigma < (q-p)/(q-1)$. The increase in the lower bound for σ is due to the application of sharp Gårding inequality in our context that dictates $\sigma \geq 3$. This is discussed in Section 6.4.3. Due to this increment in σ, we have $q \in \left(1, \frac{3}{2}\right)$.*

6.2 Subdivision of the Phase Space

We subdivide the extended phase space $J = [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, where $T > 0$, into two regions using the Planck function $h(x,\xi) = (\Phi(x)\langle\xi\rangle_k)^{-1}$ of the metric $g_{\Phi,k}$ in (2.1.2). We use these regions in the proof of Theorem 6.1.1 (see Section 6.4.1) to handle the low regularity in t. To this end we define the time splitting point $t_{x,\xi}$, for a fixed (x,ξ), as the solution to the equation

$$t^{q-p} = Nh(x,\xi),$$

where N is the positive constant chosen appropriately later. Since $3 \leq \sigma < \frac{q-p}{q-1}$, we consider $\delta \in (0,1)$ such that

$$\frac{1}{\sigma} = \frac{q-1+\delta}{q-p}. \tag{6.2.1}$$

Implying

$$\gamma := 1 - \frac{1}{\sigma} = \frac{1-\delta-p}{q-p}. \tag{6.2.2}$$

Using $t_{x,\xi}$ and (6.2.2) we define the interior region

$$\begin{aligned} Z_{int}(N) &= \{(t,x,\xi) \in J : 0 \leq t \leq t_{x,\xi},\ |x|+|\xi| > N\} \\ &= \{(t,x,\xi) \in J : t^{1-\delta-p} \leq N^{\gamma} h(x,\xi)^{\gamma},\ |x|+|\xi| > N\} \end{aligned} \tag{6.2.3}$$

and the exterior region

$$\begin{aligned} Z_{ext}(N) &= \{(t,x,\xi) \in J : t_{x,\xi} < t \leq T,\ |x|+|\xi| > N\} \\ &= \{(t,x,\xi) \in J : t^{1-\delta-p} > N^{\gamma} h(x,\xi)^{\gamma},\ |x|+|\xi| > N\}. \end{aligned} \tag{6.2.4}$$

We use these regions to define the parameter dependent global symbol classes in Section 6.3.

6.3 Parameter Dependent Global Symbol Classes

We now define certain parameter dependent global symbols that are associated with the study of the Cauchy problem (6.1.6). Let $m = (m_1, m_2) \in \mathbb{R}^2$. Consider the metrics $g_{\Phi,k}$, $\tilde{g}^{(1)}_{\Phi,k}$ and $\tilde{g}^{(2)}_{\Phi,k}$ as in (2.1.2) (6.1.4) and (6.1.5). These metrics can be related to the metrics of the form (2.1.3) where $(\rho, \tilde{\rho})$ are $((1,0),(1,0))$, $((1-p/(q-p), p/(q-p)), (1-p/(q-p), p/(q-p)))$ and $((1-1/\sigma, 1/\sigma), (1-1/\sigma, 1/\sigma))$.

Definition 6.3.1. *$G^{m_1,m_2}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$ is the space of all functions $a \in C^{\infty}(\mathbb{R}^{2n})$ satisfying*

$$|\partial^{\alpha}_{\xi} D^{\beta}_{x} a(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle_k^{m_1-\rho_1|\alpha|+\rho_2|\beta|} \omega(x)^{m_2} \Phi(x)^{-\tilde{\rho}_1|\beta|+\tilde{\rho}_2|\alpha|} \tag{6.3.1}$$

Since $g_{\Phi,k} \leq \tilde{g}^{(1)}_{\Phi,k} \leq \tilde{g}^{(2)}_{\Phi,k}$, we have

$$G^{m_1,m_2}(\omega, g_{\Phi,k}) \subset G^{m_1,m_2}(\omega, \tilde{g}^{(1)}_{\Phi,k}) \subset G^{m_1,m_2}(\omega, \tilde{g}^{(2)}_{\Phi,k}).$$

Let $\mu \geq 1$ and $\nu \geq 1$.

Definition 6.3.2. *$AG^{m_1,m_2}_{\mu,\nu}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$ is the space of all functions $a \in C^{\infty}(\mathbb{R}^{2n})$ satisfying (6.3.1) with $C_{\alpha\beta} = C^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu}$ for some $C > 0$.*

Definition 6.3.3. *$AG^{m_1,m_2}_{\sigma}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$ is the space of all functions $a \in C^{\infty}(\mathbb{R}^{2n})$ satisfying (6.3.1) when $h(x,\xi) \leq C_1|\alpha|^{-\sigma}$ with $C_{\alpha\beta} = C_2^{|\alpha|+|\beta|}(\alpha!)(\beta!)^{\sigma}$ for some positive constants $C_1, C_2 > 0$*

We denote the set of operators with symbols in $G^{m_1,m_2}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$ and $AG^{m_1,m_2}_{\sigma}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$ by $OPG^{m_1,m_2}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$ and $OPAG^{m_1,m_2}_{\sigma}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})$, respectively. As far as the calculi of these pseudodifferential operators are concerned we refer to Sections A.2 and A.3 of Appendix A.

In our analysis, we require the following conjugation result.

Proposition 6.3.1. *Let $e^{\Lambda(t)\Theta(x,D_x)}$ be as in (6.1.3) and $a(x,\xi) \in AG^{m_1,m_2}_{\sigma,\sigma}(\omega, \tilde{g}^{(1)}_{\Phi,k})$. Then, there exists $\Lambda_0 > 0$ such that for $\Lambda(t) > 0$ with $\Lambda(t) < \Lambda_0$,*

$$e^{\Lambda(t)\Theta(x,D_x)} a(x,D_x) e^{-\Lambda(t)\Theta(x,D_x)} = a(x,D_x) + \sum_{j=1}^{3} r^{(j)}_{\Lambda}(t,x,D_x)$$

where the symbols of $r_\Lambda^{(j)}(t,x,D_x)$ for $j=1,2,3$ are in $C([0,T];AG_{\sigma,\sigma}^{-\infty,1}(\omega^{m_2}\Phi^{-\gamma},\tilde{g}^{(2)}_{\Phi,k}))$, $C([0,T];AG_{\sigma,\sigma}^{m_1-\gamma,-\infty}(\omega,\tilde{g}^{(2)}_{\Phi,k}))$, and $C([0,T];AG_{\sigma,\sigma}^{-\infty,-\infty}(\omega,\tilde{g}^{(2)}_{\Phi,k}))$, respectively.

Proof. Noting the fact that $\tilde{g}^{(1)}_{\Phi,k}\leq\tilde{g}^{(2)}_{\Phi,k}$, the proof follows in similar lines to Theorem 5.3.1 in the previous chapter. □

Observe that the derivatives of $\sqrt{a(t,x,\xi)}$, characteristic roots of operator P in (6.1.6) show stronger singular behavior compared to $a(t,x,\xi)$ due to the singularity. Thus, to handle the singular behavior of the characteristics, we have the following symbol classes.

Definition 6.3.4. *$AG_\sigma^{m_1,m_2}\{l;\delta_1\}_N^{(1)}(\omega,g^{\rho,\tilde{\rho}}_{\Phi,k})$ for $l\in\mathbb{R}$ and $\delta_1\in[0,1)$is the space of all functions $a\in C^1((0,T];G^{m_1,m_2}(\omega,g^{\rho,\tilde{\rho}}_{\Phi,k}))$ satisfying*

$$|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)|\leq C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma\langle\xi\rangle_k^{m_1-\rho_1|\alpha|+\rho_2|\beta|}\omega(x)^{m_2}\Phi(x)^{-\tilde{\rho}_1|\beta|+\tilde{\rho}_2|\alpha|}\left(\frac{1}{t}\right)^{\delta_1 l},\quad(6.3.2)$$

for all $(t,x,\xi)\in Z_{int}(N)$ and for some $C>0$ where $\alpha,\beta\in\mathbb{N}_0^n$.

Definition 6.3.5. *$AG_\sigma^{m_1,m_2}\{l_1,l_2,l_3;\delta_1,\delta_2\}_N^{(2)}(\omega,g^{\rho,\tilde{\rho}}_{\Phi,k})$ for $l_1.l_3\in\mathbb{R},l_2\in\{0,1\}$ and $\delta_1\in[0,1),\delta_2\in(1,3/2)$ is the space of all functions $a\in C^1((0,T];G^{m_1,m_2}(\omega,g^{\rho,\tilde{\rho}}_{\Phi,k}))$ satisfying*

$$\begin{aligned}|\partial_\xi^\alpha D_x^\beta a(t,x,\xi)|\leq{}& C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma\langle\xi\rangle_k^{m_1-\rho_1|\alpha|+\rho_2|\beta|}\omega(x)^{m_2}\\&\Phi(x)^{-\tilde{\rho}_1|\beta|+\tilde{\rho}_2|\alpha|}\left(\frac{1}{t}\right)^{\delta_1(l_1+l_2(|\alpha|+|\beta|))+\delta_2 l_3}\end{aligned}\qquad(6.3.3)$$

for all $(t,x,\xi)\in Z_{ext}(N)$ and for some $C_{\alpha\beta}>0$ where $\alpha,\beta\in\mathbb{N}_0^n$.

Given a t-dependent global symbol $a(t,x,\xi)$, we can associate a pseudodifferential operator $Op(a)=a(t,x,D_x)$ to $a(t,x,\xi)$ by the following oscillatory integral

$$\begin{aligned}a(t,x,D_x)u(t,x)&=\iint\limits_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi}a(t,x,\xi)u(t,y)dy\,đ\xi\\&=(2\pi)^{-n}\int\limits_{\mathbb{R}^n}e^{ix\cdot\xi}a(t,x,\xi)\hat{u}(t,\xi)đ\xi,\end{aligned}$$

where $đ\xi=(2\pi)^{-n}d\xi$ and $\hat{u}$ is the Fourier transform of u in the space variable. The calculus for the operators with symbols of form $a(t,x,\xi)=a_1(t,x,\xi)+a_2(t,x,\xi)$ such that

$$\begin{aligned}&a_1\in AG_\sigma^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l};\delta_1\}_{N_1}^{(1)}(\omega,g^{\rho,\tilde{\rho}}_{\Phi,k}),\\&a_2\in AG_\sigma^{m_1,m_2}\{l_1,l_2,l_3;\delta_1,\delta_2\}_{N_2}^{(1)}(\omega,g^{\rho,\tilde{\rho}}_{\Phi,k}),\end{aligned}$$

for $N_1\geq N_2$, can be readily built by following the similar standard arguments given in Sections A.2 and A.3 of Appendix A.

6.4 Global Well-Posedness

In this section, we give a proof of the main result, Theorem 6.1.1. There are three key steps in the proof. First, we factorize the operator $P(t, x, \partial_t, D_x)$. To this end, we begin with modifying the coefficients of the principal part by performing an excision so that the resulting coefficients are regular at $t = 0$. Second, we reduce the original Cauchy problem to a Cauchy problem for a first order system (with respect to ∂_t). Lastly, using sharp Gårdings inequality we arrive at L^2 well-posedness of a related auxiliary Cauchy problem, which gives well-posedness of the original problem in the Sobolev spaces $H^{s,\varepsilon,\sigma}_{\Phi,k}$.

6.4.1 Factorization

Consider the operator $a(t, x, D_x)$ defined in (6.1.6). We modify its symbol $a(t, x, \xi)$ in $Z_{int}(2)$, by defining

$$\tilde{a}(t, x, \xi) = \varphi(t\Phi(x)\langle\xi\rangle_k)\omega(x)^2\langle\xi\rangle_k^2 + (1 - \varphi(t\Phi(x)\langle\xi\rangle_k))a(t, x, \xi) \tag{6.4.1}$$

for $\varphi \in C^\infty(\mathbb{R})$, $0 \le \varphi \le 1$, $\varphi = 1$ in $[0, 1]$, $\varphi = 0$ in $[2, +\infty)$. Note that $(a - \tilde{a}) \in AG^{2,2}_\sigma\{1; p\}_{int,2}(\omega, g_{\Phi,k})$ and $(a-\tilde{a}) \sim 0$ in $Z_{ext}(2)$. This implies that $t^p(a-\tilde{a})$ for $t \in [0, T]$ is a bounded and continuous family in $AG^{2,2}_\sigma(\omega, g_{\Phi,k})$. Observe that $a-\tilde{a}$ is L^1 integrable in t, i.e.,

$$\begin{aligned}\int_0^T |(a - \tilde{a})(t, x, \xi)|dt &\le \kappa_0'\omega(x)^2\langle\xi\rangle_k^2 \int_0^{(2/\Phi(x)\langle\xi\rangle_k)^{1/(q-p)}} \frac{1}{t^p}dt \\ &\le (\Phi(x)\langle\xi\rangle_k)^{2-(1-p)/(q-p)}.\end{aligned} \tag{6.4.2}$$

as $\omega(x) \lesssim \Phi(x)$.

Let $\tau(t, x, \xi) = \sqrt{\tilde{a}(t, x, \xi)}$. Denote the indicator functions for the regions $Z_{int}(N_1)$ and $Z_{ext}(N_2)$ by $\chi_1(N_1)$ and $\chi_2(N_2)$, respectively. It is easy to note that

i) $\tau(t, x, \xi)$ is G_ω-elliptic symbol of order $(1, 1)$ i.e. there is $C > 0$ such that for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ we have

$$|\tau(t, x, \xi)| \ge C\omega(x)\langle\xi\rangle_k.$$

ii) $\tau \in AG^{1,1}_\sigma\{0; 0\}^{(1)}_1(\omega, g_{\Phi,k}) + AG^{1,1}_\sigma\{1/2, 1, 0; p, 0\}^{(2)}_1(\omega, g_{\Phi,k})$. More precisely, for $|\alpha| + |\beta| > 0$,

$$\left.\begin{aligned}|\tau(t, x, \xi)| &\le C_0\omega(x)\langle\xi\rangle_k \left(\chi_1(1) + \chi_2(1)t^{-p/2}\right), \\ |\partial_\xi^\alpha D_x^\beta\tau(t, x, \xi)| &\le C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|} \left(\chi_1(1) + \chi_2(1)t^{-p(|\alpha|+|\beta|)}\right).\end{aligned}\right\} \tag{6.4.3}$$

iii) $\partial_t\tau$ is such that for $|\alpha| + |\beta| > 0$ we have

$$\begin{aligned}|\partial_t\tau(t, x, \xi)| &\sim 0 \text{ in } Z_{int}(1), \\ |\partial_t\tau(t, x, \xi)| &\le C_0\, \omega(x)\langle\xi\rangle_k \left(\chi_1(2)\omega(x)\langle\xi\rangle_k t^{-p} + \chi_2(1)t^{-q}\right), \\ |\partial_\xi^\alpha D_x^\beta\partial_t\tau(t, x, \xi)| &\le C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|} \left(\chi_1(2)\omega(x)\langle\xi\rangle_k + \chi_2(1)t^{-q}\right) t^{-p(|\alpha|+|\beta|)}.\end{aligned}$$

By the definition of the time splitting point and the subdivision of the phase space, we see that

$$\left.\begin{aligned}&|\partial_t\tau(t,x,\xi)|\sim 0 \text{ in } Z_{int}(1),\\ &|\partial_\xi^\alpha D_x^\beta\partial_t\tau(t,x,\xi)|\le C_{\alpha\beta}\chi_1(2)\omega(x)\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1-|\alpha|}t^{-q}t^{-p(|\alpha|+|\beta|)},\end{aligned}\right\}$$

for $|\alpha|+|\beta|\ge 0$. Hence, $\partial_t\tau\sim 0$ in $Z_{int}(1)$ and $\partial_t\tau\in AG_\sigma^{1,1}\{0,1,1;p,q\}_1^{(2)}(\omega,g_{\Phi,k})$.

From the above properties of τ and by the definition of $\tilde a$ in (6.4.1), we have the following two lemmas.

Lemma 6.4.1. *Let $\tilde g_{\Phi,k}^{(1)})$ be as in (6.1.4) and δ as in (6.2.2). Then,*

i) $\tau\in L^\infty\left([0,T];AG_\sigma^{1+\delta'/2,1}(\omega\Phi^{\delta'/2},\tilde g_{\Phi,k}^{(1)})\right)$,

ii) $t^{1-\frac{p}{2}}\tau\in C\left([0,T];AG_\sigma^{1,1}(\omega,\tilde g_{\Phi,k}^{(1)})\right)$,

iii) $\tau^{-1}\in C\big([0,T];AG_\sigma^{-1,-1}(\omega,\tilde g_{\Phi,k}^{(1)})\big)$,

iv) $t^{1-\delta}\partial_t\tau\in C\big([0,T];AG_\sigma^{1+1/\sigma,1}(\omega\Phi^{1/\sigma},\tilde g_{\Phi,k}^{(1)})\big)$.

Proof. The first claim follows from (6.4.3) and the observation that in $Z_{ext}(1)$

$$\left(\frac{1}{t}\right)^p\le\left(\frac{\Phi(x)\langle\xi\rangle_k}{N}\right)^{p/(q-p)},\tag{6.4.4}$$

while the second and third claims are straight forward consequences of (6.4.3) and (6.4.4). The fourth claim follows from (6.4.4) and the following estimate in $Z_{ext}(1)$

$$\frac{1}{t^q}=\frac{1}{t^{1-\delta}}\frac{1}{t^{(q-p)/\sigma}}\le\frac{1}{t^{1-\delta}}\left(\frac{\Phi(x)\langle\xi\rangle_k}{N}\right)^{1/\sigma}.$$

□

Lemma 6.4.2. *Let δ be as in (6.2.2). Then,*

i) $t^{1-\delta}(a(t,x,D_x)-\tilde a(t,x,D_x))\in C\Big([0,T];OPAG_\sigma^{1+1/\sigma,1+1/\sigma}(\omega,g_{\phi,k})\Big)$,

ii) $\tilde a(t,x,D_x)-\tau(t,x,D_x)^2\in L^\infty\Big([0,T];OPAG_\sigma^{1,1}(\omega,\tilde g_{\Phi,k}^{(1)})\Big)$,

iii) $t^rb(t,x,D_x)\in C\Big([0,T];OPAG_\sigma^{1,1}(\omega,g_{\phi,k})\Big)$

Proof. The proof is a consequence of the fact that in $Z_{int}(2)$

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha(a-\tilde a)(t,x,\xi)|&\le C_{\alpha\beta}\chi_1(2)\omega(x)^2\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{2-|\alpha|}\frac{1}{t^p}\\&\le C_{\alpha\beta}\chi_1(2)\omega(x)^{1+1/\sigma}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1+1/\sigma-|\alpha|}\big(\Phi(x)\langle\xi\rangle_k\big)^{1-1/\sigma}\frac{1}{t^p}\\&\le C_{\alpha\beta}\chi_1(2)\omega(x)^{1+1/\sigma}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1+1/\sigma-|\alpha|}\frac{1}{t^{1-\delta-p}}\frac{1}{t^p}\\&\le C_{\alpha\beta}\chi_1(2)\omega(x)^{1+1/\sigma}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{1+1/\sigma-|\alpha|}\frac{1}{t^{1-\delta}}.\end{aligned}$$

The second and third claims follow directly from the definitions of $\tilde{a}(t,x,D_x)$ and $b(t,x,D_x)$. □

Let us define $\delta^* > 0$ as
$$\delta^* = \min\{\delta, 1-r, 1-p\}. \tag{6.4.5}$$
We are interested in the factorization of the operator $P(t,x,\partial_t,D_x)$. This leads to

$$P = (\partial_t - i\tau(t,x,D_x))(\partial_t + i\tau(t,x,D_x)) + b_0(t,x)\partial_t + (a - \tilde{a} + a_1)(t,x,D_x)$$

where the operator $a_1(t,x,D_x)$ is such that, for $t \in [0,T]$,

$$a_1 = -i[\partial_t, \tau] + \tilde{a} - \tau^2 + b \quad \text{and} \quad t^{1-\delta^*} a_1(t,x,D_x) \in OPAG_\sigma^{1+1/\sigma,1}(\omega\Phi^{1/\sigma}, \tilde{g}^{(1)}_{\Phi,k}).$$

6.4.2 Reduction to First Order Pseudodifferential System

We will now reduce the operator P to an equivalent first order 2×2 pseudodifferential system. The procedure is similar to the one used in Chapter 3. To achieve this, we introduce the change of variables $U = U(t,x) = (u_1(t,x), u_2(t,x))^T$, where

$$\begin{cases} u_1(t,x) = (\partial_t + i\tau(t,x,D_x))u(t,x), \\ u_2(t,x) = \omega(x)\langle D_x\rangle_k u(t,x) - H(t,x,D_x)u_1, \end{cases} \tag{6.4.6}$$

and the operator H with the symbol $\sigma(H)(t,x,\xi)$ is such that

$$\sigma(H)(t,x,\xi) = -\frac{i}{2}\omega(x)\langle\xi\rangle_k \frac{\Big(1 - \varphi\Big(t\Phi(x)\langle\xi\rangle_k/3\Big)\Big)}{\tau(t,x,\xi)}.$$

Note that by the definition of H, $\operatorname{supp}\sigma(H) \cap \operatorname{supp}\sigma(a-\tilde{a}) = \emptyset$ and we have

$$\begin{aligned} &\sigma(2iH(t,x,D_x)\circ\tau(t,x,D_x)) \sim 0, \qquad \text{in } Z_{int}(3), \\ &\sigma(2iH(t,x,D_x)\circ\tau(t,x,D_x)) = \omega(x)\langle\xi\rangle_k(1+\sigma(K_1)), \qquad \text{in } Z_{ext}(3), \end{aligned}$$

where $\sigma(K_1) \in AG_\sigma^{-1,-1}\{0;p\}_6^{(1)}(\omega, g_{\Phi,k}) + AG_\sigma^{-1,-1}\{2,1,0;p,q\}_3^{(2)}(\omega, g_{\Phi,k})$. Then, the equation $Pu = f$ is equivalent to the first order 2×2 system :

$$\begin{aligned} LU &= (\partial_t - \mathcal{D} + A_0 + A_1)U = F, \\ U(0,x) &= (f_2 + i\tau(0,x,D_x)f_1, \Phi(x)\langle D_x\rangle f_1)^T, \end{aligned} \tag{6.4.7}$$

where

$$\begin{aligned} F &= (f(t,x), -H(t,x,D_x)f(t,x))^T, \\ \mathcal{D} &= \operatorname{diag}(i\tau(t,x,D_x), -i\tau(t,x,D_x)), \\ A_0 &= \begin{pmatrix} B_0H & B_0 \\ -HB_0H & HB_0 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_1 & B_0 \\ -\mathcal{R}_3 & \mathcal{R}_2 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} B_1H + B_3 & B_1 + B_4 \\ B_2 - HB_3 & i[M,\tau]M^{-1} - H(B_1+B_4) \end{pmatrix}. \end{aligned}$$

The operators M, M^{-1}, B_0, B_1 and B_2 are as follows

$$\begin{aligned}
&M = \omega(x)\langle D_x\rangle_k, \quad M^{-1} = \langle D_x\rangle_k^{-1}\omega(x)^{-1},\\
&B_0 = (a(t,x,D_x) - \tilde{a}(t,x,D_x))\langle D_x\rangle_k^{-1}\omega(x)^{-1},\\
&B_1 = (-i\partial_t\tau(t,x,D_x) + \tilde{a}(t,x,D_x) - \tau(t,x,D_x)^2 + b(t,x,D_x))\langle D_x\rangle_k^{-1}\omega(x)^{-1},\\
&B_2 = 2iH\tau - M + i[M,\tau]M^{-1}H + i[\tau,H] - HB_1H + \partial_t H\\
&B_3 = b_0(1 - i\lambda M^{-1}H), \qquad B_4 = ib_0\lambda M^{-1}.
\end{aligned}$$

Here $b_0 = b_0(t,x)$ is as in (6.1.6). By the definition of operator H, we have $B_0H = \mathcal{R}_1, HB_0 = \mathcal{R}_2,\ HB_0H = \mathcal{R}_3$ for $\mathcal{R}_j \in G^{-\infty,-\infty}(\omega, g_{\Phi,k}), j = 1,2,3,$ and the operator $2iH\tau - M$ is such that

$$\sigma(2iH\tau - M) = \begin{cases} -\omega(x)\langle\xi\rangle_k, & \text{in } Z_{int}(3),\\ \omega(x)\langle\xi\rangle_k\sigma(K_1), & \text{in } Z_{ext}(3). \end{cases}$$

Since $2p \le q$, we have

$$AG^{0,0}_{\sigma}\{2,1,0;p,q\}^{(2)}_{N}(\omega, g_{\Phi,k}) \subset AG^{0,0}_{\sigma}\{0,1,1;p,q\}^{(2)}_{N}(\omega, g_{\Phi,k}).$$

The symbols of operators $\mathcal{D}, A_0$ and A_1 are in the following symbol classes

$$\left.\begin{aligned}
\sigma(\mathcal{D}) &\in AG^{1,1}_{\sigma}\{0;0\}^{(1)}_{2}(\omega, g_{\Phi,k}) + AG^{1,1}_{\sigma}\{1,1,0;p,0\}^{(2)}_{1}(\omega, g_{\Phi,k})\\
\sigma(A_0) &\in AG^{1,1}_{\sigma}\{1;p\}^{(1)}_{2}(\omega, g_{\Phi,k}) + AG^{-\infty,-\infty}_{\sigma}\{0,0,0;0,0\}^{(2)}_{3}(\omega, g_{\Phi,k}),\\
\sigma(A_1) &\in AG^{1,1}_{\sigma}\{0;0\}^{(1)}_{6}(\omega, g_{\Phi,k}) + AG^{0,0}_{\sigma}\{1;r\}^{(1)}_{1}(\omega, g_{\Phi,k})\\
&\quad + AG^{0,0}_{\sigma}\{0,1,1;p,q\}^{(2)}_{1}(\omega, g_{\Phi,k})
\end{aligned}\right\} \tag{6.4.8}$$

and thus, by Lemmas 6.4.1 - 6.4.2 and the choice of δ^* as in (6.4.5),

$$\left.\begin{aligned}
t^{1-\delta^*}\sigma(A_0(t)) &\in C\left([0,T]; AG^{1/\sigma,1/\sigma}_{\sigma}(\omega, g_{\Phi,k})\right),\\
t^{1-\delta^*}\sigma(A_1(t)) &\in C\left([0,T]; AG^{1/\sigma,1/\sigma}_{\sigma}(\omega, \tilde{g}^{(1)}_{\Phi,k}\right).
\end{aligned}\right\} \tag{6.4.9}$$

As $g_{\Phi,k}) \le \tilde{g}^{(1)}_{\Phi,k}) \le \tilde{g}^{(2)}_{\Phi,k})$, from (6.4.9) we have

$$t^{1-\delta^*}\sigma(A_0(t)),\ t^{1-\delta^*}\sigma(A_1(t)) \in C\left([0,T]; AG^{1/\sigma,1/\sigma}_{\sigma}(\omega, \tilde{g}^{(2)}_{\Phi,k}\right). \tag{6.4.10}$$

Let us choose $\lambda > 0$ as large as possible so that

$$|\sigma(A_0(t))| + |\sigma(A_1(t))| \le \frac{\lambda}{t^{1-\delta^*}}(\Phi(x)\langle\xi\rangle_k)^{1/\sigma}. \tag{6.4.11}$$

6.4.3 Energy Estimate

In this section, we prove the estimate (6.1.9). Note that it is sufficient to consider the case $s = (0,0)$ as the operator $\Phi(x)^{s_2}\langle D\rangle^{s_1}L\langle D\rangle^{-s_1}\Phi(x)^{-s_2}$, where $s = (s_1,s_2)$ is the index of the weighted Sobolev space, satisfies the same hypotheses as L.

In the following, we establish some lower bounds for the operator $\mathcal{D} - A_0 - A_1$. The symbol $d(t, x, \xi)$ of the operator $\mathcal{D}(t) + \mathcal{D}^*(t)$ is such that

$$d \in AG^{0,0}_\sigma\{0; 0\}^{(1)}_2(\omega, g_{\Phi,k}) + AG^{0,0}_\sigma\{1/2, 1, 0; p, q\}^{(2)}_1(\omega, g_{\Phi,k}).$$

It follows from the definition of δ^* and Lemma 6.4.1 that

$$t^{1-\delta^*} d \in C([0, T]; AG^{0,0}_\sigma(\omega, \tilde{g}^{(1)}_{\Phi,k}).$$

Thus

$$2\,\mathrm{Re}\langle \mathcal{D}U, U\rangle_{L^2} \geq -\frac{C_1}{t^{1-\delta^*}}\langle U, U\rangle_{L^2}, \quad C_1 > 0. \tag{6.4.12}$$

To control lower order terms, we make the following change of variable

$$V(t, x) = e^{\Lambda(t)\Theta(x,D_x)}U(t, x), \tag{6.4.13}$$

where $\Lambda(t) = \frac{\lambda}{\delta^*}(T^{\delta^*} - t^{\delta^*})$ with λ as in (6.4.11), and the operator $\Theta(x, D_x)$ is as in (6.1.3). From Corollary 5.3.4,

$$e^{\pm\Lambda(t)\Theta(x,D_x)}e^{\mp\Lambda(t)\Theta(x,D_x)} = I + R^{(\pm)}(t, x, D_x),$$

where for sufficiently large k the operators $I + R^{(\pm)}(t, x, D_x)$ are invertible. Let us denote the operators $I + R^{(+)}(t, x, D_x)$, $I + R^{(-)}(t, x, D_x)$ and $e^{\pm\Lambda(t)\Theta(x,D_x)}$ by $\mathcal{R}(t, x, D_x)$, $\tilde{\mathcal{R}}(t, x, D_x)$ and $E^{(\pm)}(t, x, D_x)$, respectively. Then, from (6.4.13),

$$U(t, x) = \mathcal{R}^{(-)}(t, x, D_x)e^{-\Lambda(t)\Theta(x,D_x)}V(t, x) \text{ and } \|U(t)\|_{\Phi,k;s,\Lambda(t),\sigma} = \|V(t)\|_{L^2}.$$

Then $Pu = f$ is equivalent to $L_1V = F_1$ where

$$L_1 = \partial_t - \mathcal{D} + \left(B + \frac{\lambda}{t^{1-\delta}}\Theta(x, D_x)\right)$$

$F_1(t, x) = \mathcal{R}^{-1}E^{(+)}\tilde{\mathcal{R}}F(t, x)$ and the operator $B\,(t, x, D_x)$ is given by

$$B = \mathcal{R}^{-1}E^{(+)}\left(\tilde{\mathcal{R}}\left(\partial_t\tilde{\mathcal{R}}^{-1}\right) + \tilde{\mathcal{R}}A\tilde{\mathcal{R}}^{-1}\right)E^{(-)} - \left(\mathcal{R}^{-1}E^{(+)}\tilde{\mathcal{R}}\mathcal{D}\tilde{\mathcal{R}}^{-1}E^{(-)} - \mathcal{D}\right)$$

Observe that from Proposition 6.3.1 and from the Cauchy data given in conditions of Theorem 6.1.1, we need $\Lambda(t) < \Lambda^* = \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$. This implies $T < \left(\frac{\delta^*}{\lambda}\Lambda^*\right)^{1/\delta^*}$. Then, we have $t^{1-\delta^*}B \in C\left([0, T]; OPAG^{\frac{1}{\sigma}e}_\sigma\{\omega, \tilde{g}^{(2)}_{\Phi,k}\}\right)$. Choosing λ sufficiently large, we obtain

$$\mathrm{Re}\left\langle\left(\frac{\lambda}{t^{1-\delta^*}}\Theta(x, D_x) + B\right)V, V\right\rangle_{L^2} \geq -C_2\|V\|_{L^2}, \quad C_2 > 0. \tag{6.4.14}$$

The above estimate is the result of application of sharp Gårding inequality, see [43, Theorem 18.6.14] for the metric $\tilde{g}^{(2)}_{\Phi,k}$ with the Planck function $(\Phi(x)\langle\xi\rangle_k)^{\frac{1}{\sigma}-\gamma}$. It is important to note that the application of sharp Gårding inequality requires $\sigma \geq 3$.

The estimate (6.1.9) on the solution u can be established by proving that the function $V(t,x)$ satisfies the a priori estimate

$$\|V(t)\|_{L^2}^2 \leq C\left(\|V(0)\|_{L^2}^2 + \int_0^t \|F_1(\tau,\cdot)\|_{L^2}\,d\tau\right), \quad t\in[0,T], C>0. \tag{6.4.15}$$

The Cauchy problem for the operator L_1 is given by

$$\left.\begin{aligned} \partial_t V(t,x) &= \left(\mathcal{D} - \frac{\lambda}{t^{1-\delta}}\Theta(x,D_x) - B\right)V(t,x) + F_1(t,x), \\ V(0,x) &= e^{\Lambda(0)\Theta(x,D_x)}U(0,x) \end{aligned}\right\} \tag{6.4.16}$$

Observe that

$$\begin{aligned} \partial_t\|V(t)\|_{L^2}^2 =& 2\operatorname{Re}\langle \partial_t V, V\rangle_{L^2} \\ =& 2\operatorname{Re}\langle \mathcal{D}V, V\rangle_{L^2} - 2\operatorname{Re}\left\langle \left(\frac{\lambda}{t^{1-\delta}}\left(\Phi(x)\left\langle D_x\right\rangle_k\right)^{1/\sigma} + B\right)V, V\right\rangle_{L^2} \\ &+ 2\operatorname{Re}\langle F_1, V\rangle. \end{aligned}$$

From (6.4.12) and (6.4.14) we have

$$\frac{d}{dt}\|V(t)\|_{L^2}^2 \leq \frac{C}{t^{1-\delta^*}}\|V(t)\|_{L^2} + C\,\|F_1(t,\cdot)\|_{L^2}$$

Considering the above inequality as a differential inequality, we apply Gronwall's lemma and obtain that

$$\|V(t)\|_{L^2}^2 \leq C' e^{\frac{T^{\delta^*}}{\delta^*}}\left(\|V(0)\|_{L^2}^2 + \int_0^t \|F_1(\tau,\cdot)\|_{L^2}^2\,d\tau\right)$$

This proves well-posedness of the Cauchy problem (6.4.16). Note that the solution U to (6.4.7) belongs to $C\left([0,T]; H^{s,\Lambda(t),\sigma}_{\Phi,k}\right)$. Returning to our original solution $u = u(t,x)$ we obtain the estimate (6.1.9) with

$$u \in C([0,T]; H^{s+e,\Lambda(t),\sigma}_{\Phi,k}) \cap C^1([0,T]; H^{s,\Lambda(t)\sigma}_{\Phi,k}).$$

This concludes the proof. □

6.5 Anisotropic Cone Condition

Existence and uniqueness follow from the a priori estimate established in the previous section. It now remains to prove the existence of cone of dependence. Cone condition in the case of very strong blow-up follows in similar lines to the one in Section 3.5.

Let us define a constant c^* such that the quantity $c^*\omega(x)t^{-\frac{p}{2}}$ dominates the characteristic roots, i.e.,

$$c^* = \sup\left\{\sqrt{a(t,x,\xi)}\,\omega(x)^{-1}t^{\frac{p}{2}} : (t,x,\xi)\in[0,T]\times\mathbb{R}^n_x\times\mathbb{R}^n_\xi,\ |\xi|=1\right\}, \tag{6.5.1}$$

where $a(t,x,\xi)$ is as in (6.1.7).

In the following we prove the cone condition for the Cauchy problem (6.1.6). Let $K(x^0,t^0)$ denote the cone with the vertex (x^0,t^0):

$$K(x^0,t^0)=\{(t,x)\in[0,T]\times\mathbb{R}^n : |x-x^0|\le c^*\omega(x)(t^0-t)^{1-\frac{p}{2}}\}.$$

Observe that the slope of the cone is anisotropic, that is, it varies with both x and t.

Proposition 6.5.1. *The Cauchy problem (6.1.6) has a cone dependence, that is, if*

$$f\big|_{K(x^0,t^0)}=0,\quad f_i\big|_{K(x^0,t^0)\cap\{t=0\}}=0,\ i=1,2, \tag{6.5.2}$$

then

$$u\big|_{K(x^0,t^0)}=0. \tag{6.5.3}$$

Proof. Consider $t^0>0$, $C^*>0$ and assume that (6.5.2) holds. We define a set of operators $P_\varepsilon(t,x,\partial_t,D_x), 0\le\varepsilon\le\varepsilon_0$ by means of the operator $P(t,x,\partial_t,D_x)$ in (6.1.6) as follows

$$P_\varepsilon(t,x,\partial_t,D_x)=P(t+\varepsilon,x,\partial_t,D_x),\ t\in[0,T-\varepsilon_0],x\in\mathbb{R}^n,$$

and $\varepsilon_0<T-t^0$, for a fixed and sufficiently small ε_0. For these operators we consider Cauchy problems

$$\begin{aligned}
P_\varepsilon v_\varepsilon&=f, && t\in[0,T-\varepsilon_0],\ x\in\mathbb{R}^n,\\
\partial_t^{k-1}v_\varepsilon(0,x)&=f_k(x), && k=1,2.
\end{aligned}$$

Note that $v_\varepsilon(t,x)=0$ in $K(x^0,t^0)$ and v_ε satisfies an a priori estimate (6.1.9) for all $t\in[0,T-\varepsilon_0]$. Further, we have

$$\begin{aligned}
P_{\varepsilon_1}(v_{\varepsilon_1}-v_{\varepsilon_2})&=(P_{\varepsilon_2}-P_{\varepsilon_1})v_{\varepsilon_2}, && t\in[0,T-\varepsilon_0],\ x\in\mathbb{R}^n,\\
\partial_t^{k-1}(v_{\varepsilon_1}-v_{\varepsilon_2})(0,x)&=0, && k=1,2.
\end{aligned}$$

Since our operator is of second order, for the sake of simplicity we denote $b_j(t,x)$, the coefficients of lower order terms, as $a_{0,j}(t,x), 1\le j\le n$, while $b_0(t,x)$ and $b_{n+1}(t,x)$ are denoted as $a_{1,0}(t,x)$ and $a_{0,0}(t,x)$, respectively. Let $a_{i,0}(t,x)=0,\ 2\le i\le n$. Substituting $s-e$ for s in the a priori estimate, we obtain

$$\begin{aligned}
&\sum_{j=0}^{1}\|\partial_t^j(v_{\varepsilon_1}-v_{\varepsilon_2})(t,\cdot)\|_{\Phi,k;s-je,\Lambda(t),\sigma}\\
&\le C\int_0^t\|(P_{\varepsilon_2}-P_{\varepsilon_1})v_{\varepsilon_2}(\tau,\cdot)\|_{\Phi,k;s-e,\Lambda(\tau),\sigma}\,d\tau\\
&\le C\int_0^t\sum_{i,j=0}^{n}\|(a_{i,j}(\tau+\varepsilon_1,x)-a_{i,j}(\tau+\varepsilon_2,x))D_{ij}v_{\varepsilon_2}(\tau,\cdot)\|_{\Phi,k;s-e,\Lambda(\tau),\sigma}\,d\tau,
\end{aligned} \tag{6.5.4}$$

where $D_{00} = I, D_{10} = \partial_t, D_{i0} = 0, i \geq 2, D_{0j} = \partial_{x_j}, j \neq 0$ and $D_{ij} = \partial_{x_i}\partial_{x_j}, i, j \neq 0$. Using the Taylor series approximation in τ variable, we have

$$\begin{aligned} |a_{i,j}(\tau+\varepsilon_1, x) - a_{i,j}(\tau+\varepsilon_2, x)| &= \Big| \int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} (\partial_t a_{i,j})(r,x) dr \Big| \\ &\leq \omega(x)^2 \Big| \int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} \frac{dr}{r^q} \Big| \\ &\leq \omega(x)^2 |E(\tau, \varepsilon_1, \varepsilon_2)|, \end{aligned}$$

where

$$E(\tau, \varepsilon_1, \varepsilon_2) = \frac{1}{q-1} \left((\tau+\varepsilon_1)^{-q+1} - (\tau+\varepsilon_2)^{-q+1} \right).$$

Note that $\omega(x) \lesssim \Phi(x)$ and $E(\tau, \varepsilon, \varepsilon) = 0$. Then right-hand side of the inequality in (6.5.4) is dominated by

$$C \int_0^t |E(\tau, \varepsilon_1, \varepsilon_2)| \|v_{\varepsilon_2}(\tau, \cdot)\|_{\Phi,k;s+e,\Lambda(\tau),\sigma} \, d\tau,$$

where C is independent of ε. By definition, E is L_1-integrable in τ.

The sequence v_{ε_k}, $k = 1, 2, \ldots$ corresponding to the sequence $\varepsilon_k \to 0$ is in the space

$$C\Big([0,T^*]; H_{\Phi,k}^{s,\Lambda(t),\sigma}\Big) \bigcap C^1\Big([0,T^*]; H_{\Phi,k}^{s-e,\Lambda(t),\sigma}\Big), \quad T^* > 0,$$

and $u = \lim_{k\to\infty} v_{\varepsilon_k}$ in the above space and hence, in $\mathcal{D}'(K(x^0, t^0))$. In particular,

$$\langle u, \varphi \rangle = \lim_{k\to\infty} \langle v_{\varepsilon_k}, \varphi \rangle = 0, \ \forall \varphi \in \mathcal{D}(K(x^0, t^0))$$

gives (6.5.3) and completes the theorem. □

6.6 Discussion

In this chapter we have shown that, in general, the solution to a Cauchy problem with coefficients showcasing very strong blow-up in t experience infinite loss of regulaty index in relation to the Cauchy data defined in a Sobolev space. In the following we discuss related issues.

How optimal is the L^1 integrability condition on the lower order terms? The violation of the condition may lead to nonuniqueness as demonstrated in Example 1.3.4. When we have no singularity on top order terms and $O(t^{-1})$ singularity on the lower terms, we may encounter finite loss as seen Example 1.3.1 or even no loss as seen in Example 1.3.2.

Chapter 7

Discussion

> "Bah! Do you know", the Devil confided, "not even the best mathematicians on other planets - all far ahead of yours - have solved it? Why, there is a chap on Saturn - he looks something like a mushroom on stilts - who solves partial differential equations mentally; and even he's given up."
>
> — Arthur Porges, The Devil and Simon Flagg

In this chapter we give an overview of the main contributions of the this thesis, and identify some interesting questions for future work.

7.1 Looking Back

Let us begin by summarizing the contributions and results of this thesis.

- In Chapter 1, we have come up with a scale for the blow-up. This scale is inspired from the the classification of oscillations provided by Reissig [69, 50].
- In Chapter 2, we have outlined the methodology that we employ in our work. An important feature of our methods (conjugation and subdivision of the extended phase space) is their explicit dependence on the geometry of the phase space. The methods rely the Planck function associated to the metric. This, from our limited knowledge, is new in the literature.
- In chapters 3 - 6, we have addressed well-posedness and regularity results for varied singular behavior in time and plolynomial growth in space.
- In Theorems 4.4.1 and 5.3.1, we establish that the conjugation changes the metric governing the symbols arising after conjugation
- In Sections 1.3, 3.6 and 4.7, we have come up with some interesting examples of singular hyperbolic Cauchy problems displaying zero, finite and infinite loss, in fact, we also give an example for nonuniqueness.

7.2 Looking Ahead

Let us now summarise some interesting open issues which are related to the work presented in this thesis:

- In Section 5.1.1, we have defined the spaces $\mathcal{M}^{\sigma}_{\Phi,k}(\mathbb{R}^n), \sigma \geq 3$. It would be interesting to carry out a topological study these function space.
- We have studied linear strictly hyperbolic equations in this thesis. One can also pose the problem for nonlinear equation and study interaction between the singular behavior in time and the nonlinearity in the global setting.
- In Chapter 1 we have outlined various approaches available in the literature to characterize non-Lipschitzness. In this thesis, we have focused on the singular behavior. It would be interesting to study singular hyperbolic Cauchy problems whose coefficients are also log-Lipschitz or Hölder continuous. For example, see the recent work of Del Santo and Prizzi [27].

Appendix A

Pseudodifferential Operator Calculus

"But the whole wondrous complications of interference, waves, and all, result from the little fact that $\hat{x}\hat{p} - \hat{p}\hat{x}$ is not quite zero."

— Richard Feynman

In this appendix we give pseudodiffential operator calculi for the symbol classes arising in the thesis.

A.1 Calculus for the Symbol Classes in Chapter 3

We give a calculus for the parameter dependent global symbol classes defined in Section 3.3. The following two propositions give their relations to the symbol classes $G^{m_1,m_2}(\omega, g^{\rho,r}_{\Phi,k})$. Let N_1 and N_2 be positive real numbers such that $N_1 \geq N_2$.

Proposition A.1.1. *Let $a = a(t,x,\xi)$ be a symbol with*

$$a \in G^{m_1',m_2'}\{0,0;\tilde{\gamma},0\}_{int,N_1}(\omega, g_{\Phi,k}) + G^{m_1,m_2}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta\}_{ext,N_2}(\omega, g_{\Phi,k}).$$

Then, for $\tilde{m}_1 = \max\{m_1', m_1\}$, $\tilde{m}_2 = \max\{m_2', m_2\}$ and for any $\varepsilon > 0$,

$$t^{l_1+\delta l_2} a \in C\left([0,T]; G^{\tilde{m}_1+\varepsilon,1}(\omega, g^{\rho,\tilde{\rho}}_{\Phi,k})\right).$$

Proof. The hypothesis of the proposition implies that

$$\begin{aligned}\left|D_x^\beta \partial_\xi^\alpha a(t,x,\xi)\right| \leq C_{\alpha,\beta}\langle\xi\rangle_k^{m_1'-|\alpha|}\omega(x)^{m_2'}\Phi(x)^{-|\beta|} + C_{\alpha,\beta}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|} \\ \times t^{-(l_1+\delta(l_2+|\beta|))}\tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)}.\end{aligned} \tag{A.1.1}$$

From the definition of the regions, one can observe that for any $\varepsilon > 0$

$$\begin{aligned}\tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)} &\leq (\Phi(x)\langle\xi\rangle_k)^\varepsilon, \\ t^{-\delta|\beta|} &\leq (\Phi(x)\langle\xi\rangle_k)^{\delta|\beta|}.\end{aligned} \tag{A.1.2}$$

Hence,
$$t^{l_1+\delta l_2}\left|D_x^\beta\partial_\xi^\alpha a(t,x,\xi)\right| \leq C_{\alpha,\beta}\langle\xi\rangle_k^{\tilde{m}_1+\varepsilon-|\alpha|+\delta|\beta|}\omega(x)^{\tilde{m}_2}\Phi(x)^{\varepsilon-(1-\delta)|\beta|}.$$

$\square$

Remark A.1.1. *Consider $\varepsilon,\varepsilon'>0$ such that $\varepsilon<\varepsilon'<1-\delta$. Let*
$$\begin{aligned} a &\in G^{0,0}\{0,0;\tilde{\gamma},0\}_{int,N}(\omega,g_{\Phi,k})+G^{0,0}\{1,0,l_3,l_4;\tilde{\gamma},\delta\}_{ext,N}(\omega,g_{\Phi,k}),\\ b &\in G^{0,0}\{0,0;\tilde{\gamma},0\}_{int,N}(\omega,g_{\Phi,k})+G^{0,0}\{0,1,l_3,l_4;\tilde{\gamma},\delta\}_{ext,N}(\omega,g_{\Phi,k}).\end{aligned}$$
Then, from (A.1.2), we have
$$\begin{aligned} a &\in C\left([0,T];G^{\varepsilon',1}(\Phi^{\varepsilon'},g_{\Phi,k}^{(1,\delta),(1-\delta,0)})\right),\\ b &\in C\left([0,T];G^{\varepsilon,1}(\Phi^{\varepsilon},g_{\Phi,k}^{(1,\delta),(1-\delta,0)})\right).\end{aligned}$$

Let us consider an indicator function $\mathbf{I}_r:[0,\infty)\to\{0,1\}$ defined as
$$\mathbf{I}_r=\begin{cases}0, & \text{if } r=0\\ 1, & \text{otherwise}\end{cases}$$
and denote $1-\mathbf{I}_r$ as $\mathbf{I}_r^c$.

Proposition A.1.2. *Let $a=a(t,x,\xi)$ be a symbol with*
$$a\in G^{m_1',m_2'}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta\}_{int,N_1}(\omega,g_{\Phi,k})+G^{m_1,m_2}\{0,0,0,0;\tilde{\gamma},0\}_{ext,N_2}(\omega,g_{\Phi,k}),$$
and let
$$\tilde{l}=\begin{cases} l_2\delta & \text{if } \tilde{l}_2>0\\ \varepsilon & \text{if } \tilde{l}_2\leq 0 \text{ and } \tilde{l}_1>0\\ 0 & \text{if } \tilde{l}_1\leq 0 \text{ and } \tilde{l}_2\leq 0.\end{cases}$$
Then we have $t^{\tilde{l}}a\in C\left([0,T];G^{\tilde{m}_1,\tilde{m}_2}(\omega,g_{\Phi,k})\right)$ for $\tilde{m}_1=\max\{m_1',m_1\}$, $\tilde{m}_2=\max\{m_2',m_2\}$ and for any $\varepsilon>0$.

Proof. The proposition follows by observing the following estimate
$$\begin{aligned}\left|D_x^\beta\partial_\xi^\alpha a(t,x,\xi)\right| \leq{}& C_{\alpha,\beta}\langle\xi\rangle_k^{m_1'-|\alpha|}\omega(x)^{m_2'}\Phi(x)^{-|\beta|}\tilde{\theta}(t)^{\tilde{l}_1\mathbf{I}^c_{|\beta|}}t^{-\tilde{l}_2\delta\mathbf{I}_{|\beta|}}\\ &+C_{\alpha,\beta}\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}.\end{aligned} \tag{A.1.3}$$

$\square$

Remark A.1.2. *Suppose $a\in G^{1,1}\{1,1;\tilde{\gamma},\delta\}_{int,N_1}+G^{0,0}\{0,0,0,0;\tilde{\gamma},0\}_{ext,N_2}(\omega,g_{\Phi,k})$. Then for any $\varepsilon>0$ satisfying $\varepsilon<1-\delta$, we have $t^{1-\varepsilon}a\in C\left([0,T];G^{1,1}(\omega,g_{\Phi,k})\right)$.*

Remark A.1.3. *Suppose $a\in G^{1,1}\{0,0;\tilde{\gamma},0\}_{int,6}(\omega,g_{\Phi,k})+G^{0,0}\{1,0,1,1;\tilde{\gamma},\delta\}_{ext,1}(\omega,g_{\Phi,k})+G^{0,0}\{0,1,2,1;\tilde{\gamma},\delta\}_{ext,3}(\omega,g_{\Phi,k})$ (as in (3.4.7)). Let $0<\varepsilon<\varepsilon'<1-\delta$. In $Z_{int}(6)$,*
$$\omega(x)\langle\xi\rangle_k\leq\Phi(x)\langle\xi\rangle_k\leq\left(\Phi(x)\langle\xi\rangle_k\right)^{\varepsilon'}\left(\frac{6}{t}\right)^{1-\varepsilon'}.$$

Note that $t^{1-\varepsilon}\left(\frac{1}{t}\tilde{\theta}(t)^{1+|\alpha|+|\beta|}\right) \leq \frac{1}{t^{\varepsilon'}} \leq \left(\Phi(x)\langle\xi\rangle_k\right)^{\varepsilon'}$ *in* $Z_{ext}(1)$ *while in* $Z_{ext}(3)$*, we have* $t^{1-\varepsilon}\left(\frac{1}{t^{\delta}}\tilde{\theta}(t)^{2+(|\alpha|+|\beta|)}\right) \leq \frac{1}{t^{\varepsilon'}} \leq \left(\Phi(x)\langle\xi\rangle_k/3\right)^{\varepsilon'}$*. Hence, by Proposition A.1.1,* $t^{1-\varepsilon}a \in C\left([0,T]; G^{\varepsilon',\varepsilon'}(\omega, g_{\Phi,k}^{(1,\delta),(1-\delta,0)})\right)$.

For $\mu > 0$ and $r \geq 2$, we set

$$Q_{r,\mu} = \{(x,\xi) \in \mathbb{R}^{2n} : \Phi(x)^{\mu} < r, \langle\xi\rangle_k^{\mu} < r\}, \qquad Q_{r,\mu}^c = \mathbb{R}^{2n} \setminus Q_{r,\mu}.$$

Proposition A.1.3. *(Asymptotic expansion) Let* $\{a_j\}, j \geq 0$ *be a sequence of symbols with*

$$\begin{aligned} a_j \in\, & G^{\tilde{m}_1-j,1}\{\tilde{l}_1, \tilde{l}_2; \tilde{\gamma}, \delta_1\}_{int,N_1}(\omega^{\tilde{m}_2}\Phi^{-j}, g_{\Phi,k}) \\ & + G^{m_1-j,1}\{l_1+\delta_2 j, l_2, l_3+2l_4 j, l_4; \tilde{\gamma}, \delta_2\}_{ext,N_2}(\omega^{m_2}\Phi^{-j}, g_{\Phi,k}). \end{aligned}$$

Then, there is a symbol

$$a \in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1, \tilde{l}_2; \tilde{\gamma}, \delta_1\}_{int,N_1}(\omega, g_{\Phi,k}) + G^{m_1,m_2}\{l_1, l_2, l_3, l_4; \tilde{\gamma}, \delta_2\}_{ext,N_2}(\omega, g_{\Phi,k})$$

such that

$$a(t,x,\xi) \sim \sum_{j=0}^{\infty} a_j(t,x,\xi)$$

that is for all $j_0 \geq 1$, $a(t,x,\xi) - \sum_{j=0}^{j_0-1} a_j(t,x,\xi)$ *is in*

$$\begin{aligned} & G^{\tilde{m}_1-j_0,1}\{\tilde{l}_1, \tilde{l}_2; \tilde{\gamma}, \delta_1\}_{int,N_1}(\omega^{\tilde{m}_2}\Phi^{-j_0}, g_{\Phi,k}) \\ & + G^{m_1-(1-\delta_2)j_0+\varepsilon,1}\{l_1, l_2, l_3, l_4; \tilde{\gamma}, \delta_2\}_{ext,N_2}(\omega^{m_2}\Phi^{-(1-\delta_2)j_0+\varepsilon}, g_{\Phi,k}), \end{aligned}$$

where $\varepsilon \ll 1-\delta_2$*. The symbol is uniquely determined modulo* $C\left((0,T]; \mathcal{S}(\mathbb{R}^{2n})\right)$.

Proof. Let us fix $\varepsilon \ll 1-\delta_2$ and set $\mu = 1-\delta_2-\varepsilon$. Consider a C^{∞} cut-off function, χ defined by

$$\chi(x,\xi) = \begin{cases} 1, & (x,\xi) \in Q_{2k,\mu} \\ 0, & (x,\xi) \in Q_{4k,\mu}^c \end{cases}$$

and $0 \leq \chi \leq 1$. For a sequence of positive numbers $\varepsilon_j \to 0$, we define

$$\begin{aligned} & \gamma_0(x,\xi) \equiv 1, \\ & \gamma_j(x,\xi) = 1 - \chi\left(\varepsilon_j x, \varepsilon_j \xi\right), \quad j \geq 1. \end{aligned}$$

We note that $\gamma_j(x,\xi) = 0$ in $Q_{2k,\mu}$ for $j \geq 1$. We choose ε_j such that

$$\varepsilon_j \leq 2^{-j}$$

and set

$$a(t,x,\xi) = \sum_{j=0}^{\infty} \gamma_j(x,\xi) a_j(t,x,\xi).$$

We note that $a(t,x,\xi)$ exists (i.e. the series converges point-wise), since for any fixed point (t,x,ξ) only a finite number of summands contribute to $a(t,x,\xi)$. Indeed, for fixed (t,x,ξ) we can always find a j_0 such that $\Phi(x)^\mu < \frac{1}{\varepsilon_{j_0}}$, $\langle\xi\rangle_k^\mu < \frac{1}{\varepsilon_{j_0}}$ and hence

$$a(t,x,\xi) = \sum_{j=0}^{j_0-1} \gamma_j(x,\xi) a_j(t,x,\xi).$$

Observe that

$$\begin{aligned}
|D_x^\beta \partial_\xi^\alpha \left(\gamma_j(x,\xi)a_j(t,x,\xi)\right)| &\leq \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} |\partial_\xi^{\alpha'} D_x^{\beta'} \gamma_j(x,\xi) D_x^{\beta''} \partial_\xi^{\alpha''} a_j(t,x,\xi)| \\
&\leq | \, \gamma_j(x,\xi) D_x^\beta \partial_\xi^\alpha a_j(t,x,\xi) \\
&\qquad + \sum_{\substack{\alpha'+\alpha''=\alpha, |\alpha'|>0 \\ \beta'+\beta''=\beta, |\beta'|>0}} C_{\alpha'\beta'} \frac{\tilde{\chi}_j(x,\xi)}{\Phi(x)^{\mu|\beta'|}\langle\xi\rangle_k^{\mu|\alpha'|}} D_x^{\beta''} \partial_\xi^{\alpha''} a_j(t,x,\xi) \, |,
\end{aligned}$$

where $\tilde{\chi}_j(x,\xi)$ is a smooth cut-off function supported in $Q_{2k,\mu}^c \cap Q_{4k,\mu}$. This new cut-off function describes the support of the derivatives of $\gamma_j(x,\xi)$. In the last estimate, we also used that $\frac{1}{\varepsilon_j} \sim \langle\xi\rangle_k^\mu$ and $\frac{1}{\varepsilon_j} \sim \Phi(x)^\mu$ if $\tilde{\chi}_j(x,\xi) \neq 0$. We conclude that

$$\begin{aligned}
&|D_x^\beta \partial_\xi^\alpha \gamma_j(x,\xi) a_j(t,x,\xi)| \\
&\leq \frac{1}{2^j}\Big[\langle\xi\rangle_k^{\tilde{m}_1+\mu-j-|\alpha|} \omega(x)^{\tilde{m}_2} \Phi(x)^{\mu-j-|\beta|} \tilde{\theta}(t)^{\tilde{l}_1 \mathbf{I}_{|\beta|}^c} (1/t)^{\delta_1 \tilde{l}_2 \mathbf{I}_{|\beta|}} \chi(t\Phi(x)\langle\xi\rangle_k/N_1) \\
&\quad + \langle\xi\rangle_k^{m_1+\mu-j-|\alpha|} \omega(x)^{m_2} \Phi(x)^{\mu-j-|\beta|} (1/t)^{l_1+\delta_2(l_2+|\beta|+j)} \tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|+2j)} \\
&\qquad \times (1-\chi(t\Phi(x)\langle\xi\rangle_k/N_2))\Big] \\
&\leq \frac{1}{2^j}\Big[\langle\xi\rangle_k^{\tilde{m}_1+\mu-j-|\alpha|} \omega(x)^{\tilde{m}_2} \Phi(x)^{\mu-j-|\beta|} \tilde{\theta}(t)^{\tilde{l}_1 \mathbf{I}_{|\beta|}^c} (1/t)^{\delta_1 \tilde{l}_2 \mathbf{I}_{|\beta|}} \chi(t\Phi(x)\langle\xi\rangle_k/N_1) \\
&\quad + \langle\xi\rangle_k^{m_1+\varepsilon\mathbf{I}_j+\mu-(1-\delta_2)j-|\alpha|} \omega(x)^{m_2} \Phi(x)^{\mu-(1-\delta_2)j-|\beta|+\varepsilon\mathbf{I}_j} (1/t)^{l_1+\delta_2(l_2+|\beta|)} \\
&\qquad \times \tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)} (1-\chi(t\Phi(x)\langle\xi\rangle_k/N_2))\Big],
\end{aligned}$$

where we have estimated $\frac{\Phi(x)^\mu}{2^j} \geq 1$and $\frac{\langle\xi\rangle_k^\mu}{2^j} \geq 1$ (due to the support of cut-off functions) once in each summand and noted that in $Z_{ext}(N_2)$ for every $\varepsilon \ll 1$,

$$\begin{aligned}
\tilde{\theta}(t)^{2l_4 j} &\leq (\Phi(x)\langle\xi\rangle_k/N_2)^{\varepsilon\mathbf{I}_j}, \\
(1/t)^{\delta_2 j} &\leq (\Phi(x)\langle\xi\rangle_k/N_2)^{\delta_2 j}.
\end{aligned}$$

Using this relation, we obtain

$$
\begin{aligned}
&|D_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \\
&\quad \leq \left|D_x^\beta \partial_\xi^\alpha \left(\gamma_0(x,\xi) a_0(t,x,\xi)\right)\right| + \sum_{j=1}^{j_0-1} \left|D_x^\beta \partial_\xi^\alpha \left(\gamma_j(x,\xi) a_j(t,x,\xi)\right)\right| \\
&\quad \leq C_{\alpha\beta}\Big[\langle\xi\rangle_k^{\tilde{m}_1-|\alpha|} \omega(x)^{\tilde{m}_2} \Phi(x)^{-|\beta|} \tilde{\theta}(t)^{\tilde{l}_1 \mathbf{I}^c_{|\beta|}} (1/t)^{\delta_1 \tilde{l}_2 \mathbf{I}_{|\beta|}} \chi(t\Phi(x)\langle\xi\rangle_k/N_1) \\
&\qquad + \langle\xi\rangle_k^{m_1-|\alpha|} \omega(x)^{m_2} \Phi(x)^{-|\beta|} (1/t)^{l_1+\delta_2(l_2+|\beta|)} \tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)} \\
&\qquad\qquad \times (1-\chi(t\Phi(x)\langle\xi\rangle_k/N_2)) \\
&\qquad + \sum_{j=1}^{j_0-1} \frac{1}{2^j} \Big[\langle\xi\rangle_k^{\tilde{m}_1+\mu-j-|\alpha|} \omega(x)^{\tilde{m}_2} \Phi(x)^{\mu-j-|\beta|} \tilde{\theta}(t)^{\tilde{l}_1 \mathbf{I}^c_{|\beta|}} (1/t)^{\delta_1 \tilde{l}_2 \mathbf{I}_{|\beta|}} \\
&\qquad\qquad \times \chi(t\Phi(x)\langle\xi\rangle_k/N_1) + \langle\xi\rangle_k^{m_1+\mu-(1-\delta_2)j+\varepsilon-|\alpha|} \omega(x)^{m_2} \Phi(x)^{\mu-(1-\delta_2)j+\varepsilon-|\beta|} \\
&\qquad\qquad \times (1/t)^{l_1+\delta_2(l_2+|\beta|)} \tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)} (1-\chi(t\Phi(x)\langle\xi\rangle_k/N_2))\Big] \\
&\quad \leq C_{\alpha\beta}\Big[\langle\xi\rangle_k^{\tilde{m}_1-|\alpha|} \omega(x)^{\tilde{m}_2} \Phi(x)^{-|\beta|} \tilde{\theta}(t)^{\tilde{l}_1 \mathbf{I}^c_{|\beta|}} (1/t)^{\delta_1 \tilde{l}_2 \mathbf{I}_{|\beta|}} \chi(t\Phi(x)\langle\xi\rangle_k/N_1) \\
&\qquad + \langle\xi\rangle_k^{m_1-|\alpha|} \omega(x)^{m_2} \Phi(x)^{-|\beta|} (1/t)^{l_1+\delta_2(l_2+|\beta|)} \tilde{\theta}(t)^{l_3+l_4(|\alpha|+|\beta|)} \\
&\qquad\qquad \times (1-\chi(t\Phi(x)\langle\xi\rangle_k/N_2))\Big],
\end{aligned}
$$

where the last inequality holds by the choice μ. Thus, we have

$$
a \in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta_1\}_{int,N_1}(\omega, g_{\Phi,k}) + G^{m_1,m_2}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta_2\}_{ext,N_2}(\omega, g_{\Phi,k}).
$$

Arguing as above, we have

$$
\begin{aligned}
\sum_{j=j_0}^{\infty} \gamma_j a_j \in{}& G^{\tilde{m}_1-j_0,1}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta_1\}_{int,N_1}(\omega^{\tilde{m}_2}\Phi^{-j_0}, g_{\Phi,k}) \\
&+ G^{m_1-(1-\delta_2)j_0+\varepsilon,1}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta_2\}_{ext,N_2}(\omega^{m_2}\Phi^{-(1-\delta_2)j_0+\varepsilon}, g_{\Phi,k}),
\end{aligned}
$$

and thus, $a(t,x,\xi) - \sum_{j=0}^{j_0-1} a_j(t,x,\xi)$ is in

$$
\begin{aligned}
&G^{\tilde{m}_1-j_0,1}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta_1\}_{int,N_1}(\omega^{\tilde{m}_2}\Phi^{-j_0}, g_{\Phi,k}) \\
&+ G^{m_1-(1-\delta_2)j_0+\varepsilon,1}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta_2\}_{ext,N_2}(\omega^{m_2}\Phi^{-(1-\delta_2)j_0+\varepsilon}, g_{\Phi,k}).
\end{aligned}
$$

Lastly, we use Proposition A.1.1 and A.1.2 to conclude that

$$
t^l a_j \in C\left([0,T]; G^{m_1^*-(1-\delta_2)j+\varepsilon\mathbf{I}_j,1}(\omega^{m_2^*}\Phi^{-(1-\delta_2)j+\varepsilon\mathbf{I}_j}, g_{\Phi,k}^{(1,\delta),(1-\delta,0)})\right)
$$

for $m_i^* = \max\{m_i, \tilde{m}_i\}, i = 1,2,\ \delta = \max\{\delta_1,\delta_2\}$ and $l = \max\{l_1 + \delta l_2, \tilde{l}\}$. As j tends to $+\infty$, the intersection of all those spaces belongs to the space $C\left((0,T]; \mathcal{S}(\mathbb{R}^{2n})\right)$. This completes the proof. $\square$

Lemma A.1.4. *Let N_j and $N'_j, j = 1, 2$ be positive real numbers such that $N_1 \geq N_2$ and $N'_1 \geq N'_2$. Suppose*

$$\begin{aligned} &a \in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1, \tilde{l}_2; \tilde{\gamma}, \delta_1\}_{int,N_1}(\omega, g_{\Phi,k}) + G^{m_1,m_2}\{l_1, l_2, l_3, l_4; \tilde{\gamma}, \delta_2\}_{ext,N_2}(\omega, g_{\Phi,k}) \text{ and} \\ &b \in G^{\tilde{m}'_1,\tilde{m}'_2}\{\tilde{l}'_1, \tilde{l}'_2; \tilde{\gamma}, \delta_1\}_{int,N'_1}(\omega, g_{\Phi,k}) + G^{m'_1,m'_2}\{l'_1, l'_2, l'_3, l'_4; \tilde{\gamma}, \delta_2\}_{ext,N'_2}(\omega, g_{\Phi,k}). \end{aligned}$$

Then, for $\tilde{N}_1 = \max\{N_1, N'_1\}$ and $\tilde{N}_2 = \min\{N_2, N'_2\}$,

$$\begin{aligned} ab \in\ & G^{\tilde{m}_1+\tilde{m}'_1,\tilde{m}_2+\tilde{m}'_2}\{\tilde{l}_1 + \tilde{l}'_1, \tilde{l}_2 + \tilde{l}'_2; \tilde{\gamma}, \delta_1\}_{int,\tilde{N}_1}(\omega, g_{\Phi,k}) \\ &+ G^{m_1+m'_1,m_2+m'_2}\{l_1 + l'_1, l_2 + l'_2, l_3 + l'_1, l_4 + l'_1; \tilde{\gamma}, \delta_2\}_{ext,\tilde{N}_2}(\omega, g_{\Phi,k}) \\ &+ G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1, \tilde{l}_2; \tilde{\gamma}, \delta_1\}_{int,N_1}(\omega, g_{\Phi,k}) \cap G^{m'_1,m'_2}\{l'_1, l'_2, l'_3, l'_4; \tilde{\gamma}, \delta_2\}_{ext,N'_2}(\omega, g_{\Phi,k}) \\ &+ G^{\tilde{m}'_1,\tilde{m}'_2}\{\tilde{l}'_1, \tilde{l}'_2; \tilde{\gamma}, \delta_1\}_{int,N'_1}(\omega, g_{\Phi,k}) \cap G^{m_1,m_2}\{l_1, l_2, l_3, l_4; \tilde{\gamma}, \delta_2\}_{ext,N_2}(\omega, g_{\Phi,k}). \end{aligned}$$

Notice that the symbols corresponding to the third summand of the above expression is non-zero only if $N'_2 < t\Phi(x)\langle\xi\rangle_k < N_1$ i.e., in $Z_{int}(N_1) \cap Z_{ext}(N'_2)$. This requires $N'_2 < N_1$. Similarly, the fourth summand is nonvanishing in $Z_{int}(N'_1) \cap Z_{ext}(N_2)$ which requires $N_2 < N'_1$. A straightforward computation proves the above lemma.

Lemma A.1.5. *Let A and B be pseudodifferential operators with the respective symbols $a = \sigma(A)$ and $b = \sigma(B)$ as in Lemma A.1.4. Then, the pseudodifferential operator $C = A \circ B$ has a symbol*

$$\begin{aligned} c = \sigma(C) \in\ & G^{\tilde{m}_1+\tilde{m}'_1,\tilde{m}_2+\tilde{m}'_2}\{\tilde{l}_1 + \tilde{l}'_1, \tilde{l}_2 + \tilde{l}'_2; \tilde{\gamma}, \delta_1\}_{int,\tilde{N}_1}(\omega, g_{\Phi,k}) \\ &+ G^{m_1+m'_1,m_2+m'_2}\{l_1 + l'_1, l_2 + l'_2, l_3 + l'_1, l_4 + l'_1; \tilde{\gamma}, \delta_2\}_{ext,\tilde{N}_2}(\omega, g_{\Phi,k}) \\ &+ G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1, \tilde{l}_2; \tilde{\gamma}, \delta_1\}_{int,N_1}(\omega, g_{\Phi,k}) \cap G^{m'_1,m'_2}\{l'_1, l'_2, l'_3, l'_4; \tilde{\gamma}, \delta_2\}_{ext,N'_2}(\omega, g_{\Phi,k}) \\ &+ G^{\tilde{m}'_1,\tilde{m}'_2}\{\tilde{l}'_1, \tilde{l}'_2; \tilde{\gamma}, \delta_1\}_{int,N'_1}(\omega, g_{\Phi,k}) \cap G^{m_1,m_2}\{l_1, l_2, l_3, l_4; \tilde{\gamma}, \delta_2\}_{ext,N_2}(\omega, g_{\Phi,k}) \end{aligned}$$

for $\tilde{N}_1 = \max\{N_1, N'_1\}$ and $\tilde{N}_2 = \min\{N_2, N'_2\}$ and satisfies

$$c(t, x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha a(t, x, \xi) D_x^\alpha b(t, x, \xi). \tag{A.1.4}$$

The operator C is uniquely determined modulo an operator with a symbol from $C\left((0, T]; \mathcal{S}(\mathbb{R}^{2n})\right)$.

Proof. In view of Propositions A.1.1 - A.1.3 and Lemma A.1.4, it is clear that the operator C is a well-defined pseudodifferential operator. Relation (A.1.4) is a direct consequence of the standard composition rules (see [58, Section 1.2]). □

Lemma A.1.6. *Let A be a pseudodifferential operator with an invertible symbol*

$$a = \sigma(A) \in G^{0,0}\{0, 0; \tilde{\gamma}, 0\}_{int,N_1}(\omega, g_{\Phi,k}) + G^{0,0}\{0, 0, 0, 0; \tilde{\gamma}, 0\}_{ext,N_2}(\omega, g_{\Phi,k}).$$

Then, there exists a parametrix $A^\#$ with symbol

$$a^\# = \sigma\left(A^\#\right) \in G^{0,0}\{0, 0; \tilde{\gamma}, 0\}_{int,N_1}(\omega, g_{\Phi,k}) + G^{0,0}\{0, 0, 0, 0; \tilde{\gamma}, 0\}_{ext,N_2}(\omega, g_{\Phi,k}).$$

Proof. We use the existence of the inverse of a and set

$$a_0^{\#}(t,x,\xi) = a(t,x,\xi)^{-1} \in G^{0,0}\{0,0;\tilde{\gamma},0\}_{int,N_1}(\omega,g_{\Phi,k}) \\ + G^{0,0}\{0,0,0,0;\tilde{\gamma},0\}_{ext,N_2}(\omega,g_{\Phi,k}).$$

In view of Propositions A.1.1 - A.1.2, one can define a sequence $a_j^{\#}(t,x,\xi)$ recursively by

$$\sum_{1\leq|\alpha|\leq j} \frac{1}{\alpha!}\partial_\xi^\alpha a(t,x,\xi) D_x^\alpha a_{j-|\alpha|}^{\#}(t,x,\xi) = -a(t,x,\xi)a_j^{\#}(t,x,\xi)$$

with

$$a_j^{\#} \in G^{-j,1}\{0,0;\tilde{\gamma},0\}_{int,N_1}(\Phi^{-j},g_{\Phi,k}) + G^{-j,1}\{0,0,0,0;\tilde{\gamma},0\}_{ext,N_2}(\Phi^{-j},g_{\Phi,k}).$$

Proposition A.1.3 then yields the existence of a symbol

$$a_R^{\#} \in G^{0,0}\{0,0;\tilde{\gamma},0\}_{int,N_1}(\omega,g_{\Phi,k}) + G^{0,0}\{0,0,0,0;\tilde{\gamma},0\}_{ext,N_2}(\omega,g_{\Phi,k})$$

and a right parametrix $A_R^{\#}(t,x,\xi)$ with symbol $\sigma\left(A_R^{\#}\right) = a_R^{\#}$. We have

$$AA_R^{\#} - I \in C\left([0,T]; G^{-\infty,-\infty}(\omega,g_{\Phi,k})\right).$$

The existence of a left parametrix follows in similar lines. One can also prove the existence of a parametrix $A^{\#}$ by showing that right and left parametrix coincide up to a regularizing operator. □

Now, we perform a conjugation of an operator A where its symbol a is such that

$$a \in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta_1\}_{int,N_1}(\omega,g_{\Phi,k}) + G^{m_1,m_2}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta_2\}_{ext,N_2}(\omega,g_{\Phi,k}) \tag{A.1.5}$$

by the operator $\exp\{E(t,x,D_x)\}$ where the operator E is such that

$$E(t,x,\xi) = \int_0^t \tilde{\psi}(r,x,\xi)dr \tag{A.1.6}$$

for $\tilde{\psi}$ as defined in (3.4.9) and (3.4.10). For the sake of generic presentation, one can replace the factors $\varphi(t\Phi(x)\langle\xi\rangle_k/3)$ and $1-\varphi(t\Phi(x)\langle\xi\rangle_k)$ in (3.4.9) by $\varphi(2t\Phi(x)\langle\xi\rangle_k/N_1')$ and $1-\varphi(t\Phi(x)\langle\xi\rangle_k/N_2')$, respectively, for $N_1' \geq N_2'$. The conjugation operation is given by

$$A_E(t,x,D_x) = e^{-E(t,x,D_x)}A(t,x,D_x)e^{E(t,x,D_x)}.$$

Notice that the operator $\exp\{\pm E(t,x,D_x)\}$ is a pseudodifferential operator of arbitrarily small order if $\tilde{\gamma}\in(0,1)$ and finite order if $\tilde{\gamma}=1$. When $|\alpha|+|\beta|>0$, we have

$$\partial_x^\beta\partial_\xi^\alpha e^{\pm E(t,x,\xi)} \leq C'_{\alpha\beta}e^{\pm E(t,x,\xi)}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{\tilde{\gamma}(|\alpha|+|\beta|)}\chi(Z_{int}(N_1')) \tag{A.1.7}$$

By successive composition of the operators while performing conjugation and using Proposition A.1.3 and Lemma A.1.4, one can prove the following lemma.

Lemma A.1.7. *Let the operators A and E be as in (A.1.5) and (A.1.6). Then*

$$A_E(t,x,D_x) = A(t,x,D) + R(t,x,D_x), \tag{A.1.8}$$

where $(\ln(1+\Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}R(t,x,\xi)$ *belongs to*

$$G^{\tilde{m}_1-1+\tilde{\varepsilon},1}\{\tilde{l}_1,\tilde{l}_2;\tilde{\gamma},\delta_1\}_{int,\tilde{N}_1}(\omega^{\tilde{m}_2}\Phi^{-1+\tilde{\varepsilon}},g_{\Phi,k}) \\ + G^{m_1-1,1}\{l_1,l_2,l_3,l_4;\tilde{\gamma},\delta_2\}_{ext,\tilde{N}_2}(\omega^{m_2}\Phi^{-1},g_{\Phi,k}),$$

for every $\tilde{\varepsilon} \ll 1$ *and* $\tilde{N}_1 = \max\{N_1, N_1'\}$ *and* $\tilde{N}_2 = \min\{N_2, N_2'\}$.

Remark A.1.4. *In Lemma A.1.7, one can ensure a compensation for the ε increase in the order of remainder symbol in the interior region by an appropriate choice of the order of singularity in the interior region. For example, the conjugation of the operator $\mathcal{D}$ in (3.4.6) yields*

$$\mathcal{D}_E(t,x,D_x) = \mathcal{D}(t,x,D_x) + R(t,x,D_x)$$

where the operator $R(t,x,D_x)$ is such that its symbol satisfies

$$(\ln(1+\Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}R(t,x,\xi) \in G^{\tilde{\varepsilon},1}\{0,0;\tilde{\gamma},0\}_{int,2}(\omega\Phi^{-1+\tilde{\varepsilon}},g_{\Phi,k}) \\ + G^{0,1}\{0,0,1,1;\tilde{\gamma},\delta_1\}_{ext,1}(\omega\Phi^{-1},g_{\Phi,k}),$$

for an arbitrary small $\tilde{\varepsilon} > 0$. By the definition of region $Z_{int}(2)$, we have the estimate

$$(\Phi(x)\langle\xi\rangle_k)^{\tilde{\varepsilon}} \leq \frac{2}{t^{\tilde{\varepsilon}}}.$$

Hence, we have

$$t^{\tilde{\varepsilon}}(\ln(1+\Phi(x)\langle\xi\rangle_k))^{-\tilde{\gamma}}R(t,x,\xi) \in G^{0,1}\{0,0;\tilde{\gamma},0\}_{int,2}(\omega\Phi^{-1},g_{\Phi,k}) \\ + G^{0,1}\{0,0,1,1;\tilde{\gamma},\delta_1\}_{ext,1}(\omega\Phi^{-1},g_{\Phi,k}).$$

A.2 Calculus for the Symbol Classes in Chapter 4

In this section we develop a calculus for the operators with symbols in additive form given in (4.3.5). The following two propositions give their relations to the symbol classes $G^{m_1,m_2}_{\Phi,k}(m_1,m_2)$. Let $\chi_N^{(1)}, \chi_N^{(2)}$ and $\chi_N^{(3)}$ denote the indicator functions for the regions $Z_{int}(N)$, $Z_{mid}(N)$ and $Z_{ext}(N)$, respectively.

Proposition A.2.1. *Let $a = a(t,x,\xi)$ be a symbol with*

$$a \in G^{\tilde{m}_1,\tilde{m}_2}\{0\}(\omega,g_{\Phi,k})_N^{(1)} \cap G^{m_1',m_2'}\{0,0,0\}(\omega,g_{\Phi,k})_N^{(2)} \\ \cap G^{m_1,m_2}\{m_3,m_4,m_5,m_6\}(\omega,g_{\Phi,k})_N^{(3)},$$

for $m_3 \geq 0$ and $m_3 \geq m_4$. Then, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} a &\in L^\infty\left([0,T];G^{m_1^*,m_2^*}(\Phi,g_{\Phi,k})\right), && \textit{if } m_5 \leq m_3 \textit{ and } m_6 \leq 0 \\ t^{1-\varepsilon}a &\in C\left([0,T];G^{m_1^*,m_2^*}(\Phi,g_{\Phi,k})\right), && \textit{otherwise} \end{aligned}$$

where $m_i^* = \max\{\tilde{m}_i, m_i', m_i+m_3\}$, $i=1,2$.

Proof. The definition of the regions and straightforward calculations yield

$$\begin{aligned}
&|D_x^\beta \partial_\xi^\alpha a(t,x,\xi)\ | \\
&\le C_{\alpha,\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\Big(\chi_N^{(1)}\langle\xi\rangle_k^{\tilde m_1}\omega(x)^{\tilde m_2}+\chi_N^{(2)}\langle\xi\rangle_k^{m_1'}\omega(x)^{m_2'} \\
&\qquad + \chi_N^{(3)}\langle\xi\rangle_k^{m_1}\omega(x)^{m_2}\left(\frac{\theta(t)}{t}\right)^{m_3} e^{m_4\psi(t)}\tilde\theta(t)^{m_5+m_6(|\alpha|+|\beta|)}\Big) \\
&\le C_{\alpha,\beta}\Phi(x)^{-|\beta|}\langle\xi\rangle_k^{-|\alpha|}\Big(\chi_N^{(1)}\langle\xi\rangle_k^{\tilde m_1}\omega(x)^{\tilde m_2}+\chi_N^{(2)}\langle\xi\rangle_k^{m_1'}\omega(x)^{m_2'} \\
&\qquad + \chi_N^{(3)}\langle\xi\rangle_k^{m_1}\omega(x)^{m_2}\left(\frac{\theta(h)}{t_{x,\xi}}\right)^{m_3} e^{m_4\psi(h)}\tilde\theta(h)^{m_3}\tilde\theta(t)^{m_5-m_3+m_6(|\alpha|+|\beta|)}\Big) \\
&\le C_{\alpha,\beta}\Phi(x)^{m_2^*-|\beta|}\langle\xi\rangle_k^{m_1^*-|\alpha|}t^{-1+\varepsilon}.
\end{aligned}$$

The last estimate follows from the fact that $\omega(x)\lesssim\Phi(x)$ and

$$\tilde\theta(t)^{m_5-m_3+m_6(|\alpha|+|\beta|)}\le t^{-1+\varepsilon},\quad \varepsilon\in(0,1),$$

since the singularities in our consideration are of logarithmic type. $\square$

Similarly, we can prove the following two propositions.

Proposition A.2.2. *Let $a=a(t,x,\xi)$ be a symbol with*

$$\begin{aligned}
a\in G^{\tilde m_1,\tilde m_2}\{0\}(\omega,g_{\Phi,k})_N^{(1)}&\cap G^{m_1',m_2'}\{m_3',m_4',m_5'\}(\omega,g_{\Phi,k})_N^{(2)} \\
&\cap G^{m_1,m_2}\{0,0,0,0\}(\omega,g_{\Phi,k})_N^{(3)},
\end{aligned}$$

for $m_3'\ge 0$. Then, for any $\varepsilon\in(0,1)$,

$$\begin{aligned}
a&\in L^\infty\left([0,T];G^{m_1^*,m_2^*}(\Phi,g_{\Phi,k})\right), &&\textit{if } m_4',m_5'\le 0\\
t^{1-\varepsilon}a&\in C\left([0,T];G^{m_1^*,m_2^*}(\Phi,g_{\Phi,k})\right), &&\textit{otherwise}
\end{aligned}$$

where $m_i^*=\max\{\tilde m_i, m_i'+m_3', m_i\}$, $i=1,2$.

Proposition A.2.3. *Let $a=a(t,x,\xi)$ be a symbol with*

$$\begin{aligned}
a\in G^{\tilde m_1,\tilde m_2}\{\tilde m_3\}(\omega,g_{\Phi,k})_N^{(1)}&\cap G^{m_1',m_2'}\{0,0,0\}(\omega,g_{\Phi,k})_N^{(2)} \\
&\cap G^{m_1,m_2}\{0,0,0,0\}(\omega,g_{\Phi,k})_N^{(3)},
\end{aligned}$$

for $\tilde m_3\ge 0$. Then, for any $\varepsilon\in(0,1)$,

$$\begin{aligned}
a&\in L^\infty\left([0,T];G^{m_1^*,m_2^*}(\omega,g_{\Phi,k})\right), &&\textit{if } \tilde m_3=0\\
t^{1-\varepsilon}a&\in C\left([0,T];G^{m_1^*,m_2^*}(\omega,g_{\Phi,k})\right), &&\textit{otherwise}
\end{aligned}$$

where $m_i^*=\max\{\tilde m_i, m_i', m_i\}, i=1,2$.

Remark A.2.1. *Since the function $\tilde\theta$ is of logarthmic type, we have*

$$\tilde\theta(\tilde t_{x,\xi})^l\le\tilde\theta(t_{x,\xi})^l\le\tilde\theta(h(x,\xi))^l\le(\Phi(x)\langle\xi\rangle_k)^\varepsilon,$$

for any $l>0$ and $\varepsilon\ll 1$. Similar is the case for the functions θ and ψ.

For $\mu > 0$ and $r \geq 2$, we set

$$Q_{r,\mu} = \{(x,\xi) \in \mathbb{R}^{2n} : \Phi(x) < r, \langle \xi \rangle_k < r\}, \qquad Q^c_{r,\mu} = \mathbb{R}^{2n} \setminus Q_{r,\mu}.$$

Proposition A.2.4 (Asymptotic Expansion)**.** *Let $\{a_j\}, j \geq 0$ be a sequence of symbols with*

$$a_j \in G^{\tilde{m}_1,1}\{\tilde{m}_3\}(\omega^{\tilde{m}_2}\Phi^{-j}, g_{\Phi,k})^{(1)}_N \cap G^{m'_1,1}\{m'_3, m'_4 + 2m'_5 j, m'_5\}(\omega^{m'_2}\Phi^{-j}, g_{\Phi,k})^{(2)}_N$$
$$\cap\, G^{m_1,1}\{m_3, m_4, m_5 + 2m_6 j, m_6\}(\omega^{m_2}\Phi^{-j}, g_{\Phi,k})^{(2)}_N.$$

Then, there is a symbol

$$a \in G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{m}_3\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{m'_1,m'_2}\{m'_3, m'_4, m'_5\}(\omega, g_{\Phi,k})^{(2)}_N$$
$$\cap\, G^{m_1,m_2}\{m_3, m_4, m_5, m_6\}(\omega, g_{\Phi,k})^{(3)}_N$$

such that

$$a(t,x,\xi) \sim \sum_{j=0}^{\infty} a_j(t,x,\xi),$$

that is for all $j_0 \geq 1$, $a(t,x,\xi) - \sum_{j=0}^{j_0-1} a_j(t,x,\xi)$ belongs to

$$G^{\tilde{m}_1-j_0,1}\{\tilde{m}_3\}(\omega^{\tilde{m}_2}\Phi^{-j_0}, g_{\Phi,k})^{(1)}_N \cap G^{m'_1-j_0+\varepsilon,1}\{m'_3, m'_4, m'_5\}(\omega^{m'_2}\Phi^{-j_0+\varepsilon}, g_{\Phi,k})^{(2)}_N$$
$$\cap\, G^{m_1-j_0+\varepsilon,1}\{m_3, m_4, m_5, m_6\}(\omega^{m_2}\Phi^{-j_0+\varepsilon}, g_{\Phi,k})^{(2)}_N,$$

where $\varepsilon \ll 1$. The symbol is uniquely determined modulo $C\left((0,T]; G^{-\infty}\right)$.

Proof. Let us fix $\varepsilon \ll 1$ and set $\mu = 1 - \varepsilon$. Consider a C^∞ cut-off function, χ defined by

$$\chi(x,\xi) = \begin{cases} 1, & (x,\xi) \in Q_{2k,\mu} \\ 0, & (x,\xi) \in Q^c_{4k,\mu} \end{cases}$$

and $0 \leq \chi \leq 1$. For a sequence of positive numbers $\varepsilon_j \to 0$, we define

$$\gamma_0(x,\xi) \equiv 1,$$
$$\gamma_j(x,\xi) = 1 - \chi\left(\varepsilon_j x, \varepsilon_j \xi\right), \quad j \geq 1.$$

We note that $\gamma_j(x,\xi) = 0$ in $Q_{2k,\mu}$ for $j \geq 1$. We choose ε_j such that

$$\varepsilon_j \leq 2^{-j}$$

and set

$$a(t,x,\xi) = \sum_{j=0}^{\infty} \gamma_j(x,\xi) a_j(t,x,\xi).$$

We note that $a(t,x,\xi)$ exists (i.e. the series converges point-wise), since for any fixed point (t,x,ξ) only a finite number of summands contribute to $a(t,x,\xi)$. Indeed, for fixed (t,x,ξ) we can always find a j_0 such that $\Phi(x) < \frac{1}{\varepsilon_{j_0}}$, $\langle \xi \rangle_k < \frac{1}{\varepsilon_{j_0}}$ and hence

$$a(t,x,\xi) = \sum_{j=0}^{j_0-1} \gamma_j(x,\xi) a_j(t,x,\xi).$$

Observe that

$$\begin{aligned}|D_x^\beta\partial_\xi^\alpha\left(\gamma_j(x,\xi)a_j(t,x,\xi)\right)| &\leq \sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}|\partial_\xi^{\alpha'}D_x^{\beta'}\gamma_j(x,\xi)D_x^{\beta''}\partial_\xi^{\alpha''}a_j(t,x,\xi)|\\ &\leq|\ \gamma_j(x,\xi)D_x^\beta\partial_\xi^\alpha a_j(t,x,\xi)\\ &\quad+\sum_{\substack{\alpha'+\alpha''=\alpha,|\alpha'|>0\\ \beta'+\beta''=\beta,|\beta'|>0}}C_{\alpha'\beta'}\frac{\tilde{\chi}_j(x,\xi)}{\Phi(x)^{|\beta'|}\langle\xi\rangle_k^{|\alpha'|}}D_x^{\beta''}\partial_\xi^{\alpha''}a_j(t,x,\xi)\ |,\end{aligned}$$

where $\tilde{\chi}_j(x,\xi)$ is a smooth cut-off function supported in $Q^c_{2k,\mu}\cap Q_{4k,\mu}$. This new cut-off function describes the support of the derivatives of $\gamma_j(x,\xi)$. In the last estimate, we also used that $\frac{1}{\varepsilon_j}\sim\langle\xi\rangle_k$ and $\frac{1}{\varepsilon_j}\sim\Phi(x)$ if $\tilde{\chi}_j(x,\xi)\neq 0$. We conclude that

$$\begin{aligned}&|D_x^\beta\partial_\xi^\alpha\gamma_j(x,\xi)a_j(t,x,\xi)|\\ &\leq\frac{1}{2^j}\langle\xi\rangle_k^{\mu-j-|\alpha|}\Phi(x)^{\mu-j-|\beta|}\Bigg[\chi_N^{(1)}\langle\xi\rangle_k^{\tilde{m}_1}\omega(x)^{\tilde{m}_2}\tilde{\theta}(t)\\ &\quad+\chi_N^{(2)}\langle\xi\rangle_k^{m_1'}\omega(x)^{m_2'}\left(\frac{\theta(t)}{t}\right)^{m_3'}\tilde{\theta}(t)^{m_4'+m_5'(|\alpha|+|\beta|+2j)}\\ &\quad+\chi_N^{(3)}\langle\xi\rangle_k^{m_1}\omega(x)^{m_2}\left(\frac{\theta(t)}{t}\right)^{m_3}e^{m_4\psi(t)}\tilde{\theta}(t)^{m_5+m_6(|\alpha|+|\beta|+2j)}\Bigg],\end{aligned}\tag{A.2.1}$$

where we have estimated $\frac{\Phi(x)^\mu}{2^j}\geq 1$and $\frac{\langle\xi\rangle_k^\mu}{2^j}\geq 1$ (due to the support of cut-off functions) once in each summand. In $Z_{mid}(N)$ and $Z_{ext}(N)$, $\tilde{\theta}(t)\leq\tilde{\theta}(h)$. For any $r\geq 0$ and $j\geq 1$,

$$\tilde{\theta}(t)^{rj}\leq\tilde{\theta}(h)^{rj}\lesssim(\Phi(x)\langle\xi\rangle_k)^\varepsilon,\tag{A.2.2}$$

as the singularity in our consideration is of logarithmic type. From (A.2.1) and (A.2.2), we obtain

$$\begin{aligned}&|D_x^\beta\partial_\xi^\alpha\gamma_j(x,\xi)a_j(t,x,\xi)|\\ &\leq\frac{1}{2^j}\langle\xi\rangle_k^{\mu-j-|\alpha|}\Phi(x)^{\mu-j-|\beta|}\Bigg[\chi_N^{(1)}\langle\xi\rangle_k^{\tilde{m}_1}\omega(x)^{\tilde{m}_2}\tilde{\theta}(t)\\ &\quad+\chi_N^{(2)}\langle\xi\rangle_k^{m_1'+\varepsilon}\Phi(x)^\varepsilon\omega(x)^{m_2'}\left(\frac{\theta(t)}{t}\right)^{m_3'}\tilde{\theta}(t)^{m_4'+m_5'(|\alpha|+|\beta|)}\\ &\quad+\chi_N^{(3)}\langle\xi\rangle_k^{m_1+\varepsilon}\Phi(x)^\varepsilon\omega(x)^{m_2}\left(\frac{\theta(t)}{t}\right)^{m_3}e^{m_4\psi(t)}\tilde{\theta}(t)^{m_5+m_6(|\alpha|+|\beta|)}\Bigg]\\ &\leq\frac{1}{2^j}\langle\xi\rangle_k^{1-j-|\alpha|}\Phi(x)^{1-j-|\beta|}\Bigg[\chi_N^{(1)}\langle\xi\rangle_k^{\tilde{m}_1}\omega(x)^{\tilde{m}_2}\tilde{\theta}(t)\\ &\quad+\chi_N^{(2)}\langle\xi\rangle_k^{m_1'}\omega(x)^{m_2'}\left(\frac{\theta(t)}{t}\right)^{m_3'}\tilde{\theta}(t)^{m_4'+m_5'(|\alpha|+|\beta|)}\\ &\quad+\chi_N^{(3)}\langle\xi\rangle_k^{m_1}\omega(x)^{m_2}\left(\frac{\theta(t)}{t}\right)^{m_3}e^{m_4\psi(t)}\tilde{\theta}(t)^{m_5+m_6(|\alpha|+|\beta|)}\Bigg],\end{aligned}\tag{A.2.3}$$

where $j \geq 1$. Now

$$\begin{aligned} &|D_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \\ &\leq \left|D_x^\beta \partial_\xi^\alpha \left(\gamma_0(x,\xi) a_0(t,x,\xi)\right)\right| + \sum_{j=1}^{j_0-1} \left|D_x^\beta \partial_\xi^\alpha \left(\gamma_j(x,\xi) a_j(t,x,\xi)\right)\right|. \end{aligned}$$

Combining the symbol estimate of a_0 and the estimate (A.2.3), we readily obtain

$$\begin{aligned} a \in\, & G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{m}_3\}(\omega, g_{\Phi,k})_N^{(1)} \cap G^{m_1',m_2'}\{m_3', m_4', m_5'\}(\omega, g_{\Phi,k})_N^{(2)} \\ & \cap G^{m_1,m_2}\{m_3, m_4, m_5, m_6\}(\omega, g_{\Phi,k})_N^{(3)} \end{aligned}$$

Arguing as above, we see that $\sum_{j=j_0}^{\infty} \gamma_j a_j$ belongs to

$$\begin{aligned} & G^{\tilde{m}_1-j_0,1}\{\tilde{m}_3\}(\omega^{\tilde{m}_2}\Phi^{-j_0}, g_{\Phi,k})_N^{(1)} \cap G^{m_1'-j_0+\varepsilon,1}\{m_3', m_4', m_5'\}(\omega^{m_2'}\Phi^{-j_0+\varepsilon}, g_{\Phi,k})_N^{(2)} \\ & \cap G^{m_1-j_0+\varepsilon,1}\{m_3, m_4, m_5, m_6\}(\omega^{m_2}\Phi^{-j_0+\varepsilon}, g_{\Phi,k})_N^{(2)}, \end{aligned}$$

and thus, $a(t,x,\xi) - \sum\limits_{j=0}^{j_0-1} a_j(t,x,\xi)$ belongs to

$$\begin{aligned} & G^{\tilde{m}_1-j_0,1}\{\tilde{m}_3\}(\omega^{\tilde{m}_2}\Phi^{-j_0}, g_{\Phi,k})_N^{(1)} \cap G^{m_1'-j_0+\varepsilon,1}\{m_3', m_4', m_5'\}(\omega^{m_2'}\Phi^{-j_0+\varepsilon}, g_{\Phi,k})_N^{(2)} \\ & \cap G^{m_1-j_0+\varepsilon,1}\{m_3, m_4, m_5, m_6\}(\omega^{m_2}\Phi^{-j_0+\varepsilon}, g_{\Phi,k})_N^{(2)}. \end{aligned}$$

Lastly, we use Propositions A.2.1 - A.2.3 to conclude that

$$t^{1-\varepsilon} a_j \in C\left([0,T]; G^{m_1^*-j,1}(\omega^{m_2^*}\Phi^{r^*-j}, g_{\Phi,k})\right)$$

for $m_1^* = \max\{\tilde{m}_1, m_1' + m_3', m_1 + m_3\}$, $m_1^* = \max\{\tilde{m}_2, m_2', m_2\}$ and $r^* = \max\{m_3', m_3\}$. As j tends to $+\infty$, the intersection of all those spaces belongs to the space $C\left((0,T]; G^{-\infty}\right)$. This completes the proof. □

Lemma A.2.5. *Let A and B be pseudodifferential operators with symbols*

$$\begin{aligned} a = \sigma(A) \in\, & G^{\tilde{m}_1,\tilde{m}_2}\{\tilde{m}_3\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{m_1',m_2'}\{m_3', m_4', m_5'\}(\omega, g_{\Phi,k})_N^{(2)} \\ & \cap G^{m_1,m_2}\{m_3, m_4, m_5, m_6\}(\omega, g_{\Phi,k})_N^{(3)} \end{aligned}$$

and

$$\begin{aligned} b = \sigma(B) \in\, & G^{\tilde{l}_1,\tilde{l}_2}\{\tilde{l}_3\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{l_1',l_2'}\{l_3', l_4', l_5'\}(\omega, g_{\Phi,k})_N^{(2)} \\ & \cap G^{l_1,l_2}\{l_3, l_4, l_5, l_6\}(\omega, g_{\Phi,k})_N^{(3)}. \end{aligned}$$

Then, the pseudodifferential operator $C = A \circ B$ has a symbol $c = \sigma(C)$ in

$$\begin{aligned} & G^{\tilde{m}_1+\tilde{l}_1,\tilde{m}_2+\tilde{l}_2}\{\tilde{m}_3+\tilde{l}_3\}(\omega, g_{\Phi,k})_{2N}^{(1)} \cap G^{m_1'+l_1',m_2'+l_2'}\{m_3'+l_3', m_4+l_4', m_5'+l_5'\}(\omega, g_{\Phi,k})_N^{(2)} \\ & \cap G^{m_1+l_1,m_2+l_2}\{m_3+l_3, m_4+l_4, m_5+l_5, m_6+l_6\}(\omega, g_{\Phi,k})_N^{(3)} \end{aligned}$$

and satisfies

$$c(t,x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha a(t,x,\xi) D_x^\alpha b(t,x,\xi). \tag{A.2.4}$$

The operator C is uniquely determined modulo an operator with symbol from $C\left((0,T]; G^{-\infty}\right)$.

Proof. In view of Propositions A.2.1 - A.2.4, it is clear that the operator C is a well-defined pseudodifferential operator. Relation (A.2.4) is a direct consequence of the standard composition rules (see [58, Section 1.2]). □

Lemma A.2.6. *Let A be a pseudodifferential operator with an invertible symbol*

$$a = \sigma(A) \in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{0,0}\{0,0,0\}(\omega, g_{\Phi,k})^{(2)}_N \cap G^{0,0}\{0,0,0,0\}(\omega, g_{\Phi,k})^{(3)}_N.$$

Then, there exists a parametrix $A^{\#}$ with symbol $a^{\#}$ in

$$G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{0,0}\{0,0,0\}(\omega, g_{\Phi,k})^{(2)}_N \cap G^{0,0}\{0,0,0,0\}(\omega, g_{\Phi,k})^{(3)}_N.$$

Proof. We use the existence of the inverse of a and set

$$\begin{aligned} a_0^{\#}(t,x,\xi) = a(t,x,\xi)^{-1} \in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{0,0}\{0,0,0\}(\omega, g_{\Phi,k})^{(2)}_N \\ \cap\, G^{0,0}\{0,0,0,0\}(\omega, g_{\Phi,k})^{(3)}_N. \end{aligned}$$

In view of Propositions A.2.1 - A.2.3, one can define a sequence $a_j^{\#}(t,x,\xi)$ recursively by

$$\sum_{1\le|\alpha|\le j} \frac{1}{\alpha!}\partial_\xi^\alpha a(t,x,\xi) D_x^\alpha a^{\#}_{j-|\alpha|}(t,x,\xi) = -a(t,x,\xi)a_j^{\#}(t,x,\xi)$$

with $a_j^{\#}$ in

$$G^{-j,1}\{0\}(\Phi^{-j}, g_{\Phi,k})^{(1)}_N \cap G^{-j,1}\{0,0,0\}(\Phi^{-j}, g_{\Phi,k})^{(2)}_N \cap G^{-j,1}\{0,0,0,0\}(\Phi^{-j}, g_{\Phi,k})^{(3)}_N.$$

Proposition A.2.4 then yields the existence of a symbol

$$a_R^{\#} \in G^{0,0}\{0\}(\omega, g_{\Phi,k})^{(1)}_N \cap G^{0,0}\{0,0,0\}(\omega, g_{\Phi,k})^{(2)}_N \cap G^{0,0}\{0,0,0,0\}(\omega, g_{\Phi,k})^{(3)}_N$$

and a right parametrix $A_R^{\#}(t,x,\xi)$ with symbol $\sigma\left(A_R^{\#}\right) = a_R^{\#}$. We have

$$AA_R^{\#} - I \in C\left([0,T]; G^{-\infty}\right).$$

The existence of a left parametrix follows in similar lines. One can also prove the existence of a parametrix $A^{\#}$ by showing that right and left parametrix coincide up to a regularizing operator. □

A.3 Calculus for the Symbol Classes in Chapter 5

We now discuss in detail the symbol calculus for the class $AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\Phi, g_{\Phi,k})$. The arguments used here are similar to the ones in [58, Section 6.3]. To start with, we prove that $P \in OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ is continuous on $S_\theta^\theta(\mathbb{R}^n), \theta \ge \sigma$. Recall that $S_\theta^\theta(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{2\theta}_{\Phi,k}(\mathbb{R}^n)$ and hence $\mathcal{M}^{2\theta}_{\Phi,k}{}'(\mathbb{R}^n) \hookrightarrow S_\theta^{\theta\prime}(\mathbb{R}^n)$.

Theorem A.3.1. *Let $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ and let $\theta \ge \sigma$. Then, the operator P is a linear and continuous operator from $S_\theta^\theta(\mathbb{R}^n)$ to $S_\theta^\theta(\mathbb{R}^n)$ and it extends to a linear and continuous map from $S_\theta^{\theta\prime}(\mathbb{R}^n)$ to $S_\theta^{\theta\prime}(\mathbb{R}^n)$.*

Proof. Let $u \in S_\theta^\theta\left(\mathbb{R}^d\right)$. Since $\mathcal{F}\left(S_\theta^\theta\left(\mathbb{R}^n\right)\right) = S_\theta^\theta\left(\mathbb{R}^n\right)$, we consider u in a bounded subset F of the Banach space defined by the norm

$$\sup_\beta \sup_{\xi\in\mathbb{R}^n} A^{-|\beta|}(\beta!)^\theta e^{a|\xi|^{1/\theta}} \left|\partial^\beta \widehat{u}(\xi)\right|$$

for some $A > 0, a > 0$. It is sufficient to show that there exist positive constants A_1, B_1, C_1 such that, for every $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha D_x^\beta P u(x)\right| \leq C_1 A_1^{|\alpha|} B_1^{|\beta|} (\alpha!\beta!)^\theta \tag{A.3.1}$$

for all $u \in F$, with A_1, B_1, C_1 independent of $u \in F$. We have, for every $N \in \mathbb{N}$

$$\begin{aligned} x^\alpha D_x^\beta P u(x) &= x^\alpha \sum_{\beta'\leq\beta} \binom{\beta}{\beta'} \int e^{ix\xi} \xi^{\beta'} D_x^{\beta-\beta'} p(x,\xi)\widehat{u}(\xi)\, đ\xi \\ &= x^\alpha \langle x\rangle^{-2N} \sum_{\beta'\leq\beta} \binom{\beta}{\beta'} \int e^{ix\xi} \left(1-\Delta_\xi\right)^N \left[\xi^{\beta'} D_x^{\beta-\beta'} p(x,\xi)\widehat{u}(\xi)\right] đ\xi \end{aligned}$$

Since $\omega(x) \lesssim \Phi(x) \leq \langle x\rangle$, we easily obtain the estimate:

$$\begin{aligned} \left|x^\alpha D_x^\beta P u(x)\right| &\leq C_0 B_0^{|\beta|+2N} (2N!)^\theta \langle x\rangle^{|\alpha|+n-2\gamma N} \\ \times \sum_{\beta'\leq\beta} \binom{\beta}{\beta'} &\left(\beta'!\right)^\theta \left(\beta-\beta'!\right)^\nu \int \langle\xi\rangle_k^m e^{-a|\xi|^{\frac{1}{\theta}}}\, đ\xi \end{aligned}$$

for some B_0, C_0 independent of $u \in F$. Choosing $N \geq (|\alpha|+n)/\gamma$, we obtain that there exist $A_1, B_1, C_1 > 0$ such that (A.3.1) holds for all $u \in F$. Next, observe that, for $u, v \in S_\theta^\theta(\mathbb{R}^n)$

$$\int Pu(x)v(x)dx = \int \widehat{u}(\xi) p_v(\xi)\, đ\xi$$

where

$$p_v(x,\xi) = \int e^{ix\xi} p(x,\xi) v(x) dx$$

By arguing as before, the map $v \mapsto p_v$ is linear and continuous from $S_\theta^\theta(\mathbb{R}^n)$ to itself. Then, we can define, for $u \in S_\theta^{\theta\prime}(\mathbb{R}^n)$

$$Pu(v) = \widehat{u}\left(p_v\right), \quad v \in S_\theta^{\theta\prime}(\mathbb{R}^n)$$

This map extends P and is linear and continuous from $S_\theta^{\theta\prime}(\mathbb{R}^n)$ to itself and it. $\square$

We can associate to P a kernel $K_P \in S_\theta^{\theta\prime}(\mathbb{R}^n)$, given by

$$K_P(x,y) = \int e^{i(x-y)\xi} p(x,\xi)\, đ\xi. \tag{A.3.2}$$

We now prove the following result showing the regularity of the kernel (A.3.2).

Theorem A.3.2. *Let $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$. For $M > 0$, define the region:*

$$\Omega_M = \{(x,y) \in \mathbb{R}^{2n} : |x-y| > M\Phi(x)\}.$$

Then the kernel K_P defined by (A.3.2) is in $C^\infty(\Omega_M)$ and there exist positive constants C, a depending on M such that

$$|D^\beta_x D^\alpha_y K_P(x,y)| \le C^{|\alpha|+|\beta|+1}(\beta!\alpha!)^\theta \exp\Big\{ - a(\Phi(x)\Phi(y))^{\frac{1}{\theta}}\Big\}$$

for all $(x,y) \in \bar{\Omega}_M$ and $\beta, \alpha \in \mathbb{Z}^n_+$.

Proof. As in [58, Lemma 6.3.4], for any given $R > 1$, we can find a sequence $\psi_N(\xi) \in C^\infty_0(\mathbb{R}^n), N = 0, 1, 2, \dots$, such that $\sum_{N=0}^\infty \psi_n = 1$,

$$\text{supp}\ \psi_0 \subset \{\xi : \langle\xi\rangle_k \le 3R\},$$

$$\text{supp}\psi_N \subset \{\xi : 2RN^\theta\langle\xi\rangle_k \le 3R(N+1)^\theta\}, \quad \text{and}$$

$$|D^\alpha_\xi \psi_N(\xi)| \le C^{|\alpha|+1}(\alpha!)^\theta \Big(R\sup\{N^\theta, 1\}\Big)^{-|\alpha|},$$

for $N = 1, 2, \dots$, and $\alpha \in \mathbb{Z}^n_+$. Using this partition of unity, we can decompose K_P as

$$K_P = \sum_{N=0}^\infty K_N,$$

where

$$K_N(x,y) = \int e^{i(x-y)\xi} p(x,\xi)\psi_N(\xi)\,đ\xi.$$

Let $M > 0$ and $(x,y) \in \bar{\Omega}_M$. Let $r \in \{1, 2, \dots, n\}$ such that $|x_r - y_r| \ge \frac{M}{n}\Phi(x)$. Then, for every $\alpha, \beta \in \mathbb{Z}^n$,

$$D^\alpha_x D^\beta_y K_N(x,y) = (-1)^{|\beta|} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \int e^{i(x-y)\xi}\xi^{\beta+\delta}\psi_N(\xi) D^{\alpha-\delta}_x p(x,\xi)\,đ\xi.$$

Given $\lambda > 0$, we consider the operator

$$L = \frac{1}{M_{2\theta,\lambda}(x-y)} \sum_{j=0}^\infty \frac{\lambda^j}{(j!)^{2\theta}}(1-\Delta_\xi)^j, \quad \text{where}$$

$$M_{2\theta,\lambda}(x-y) = \sum_{j=0}^\infty \frac{\lambda^j}{(j!)^{2\theta}}\langle x-y\rangle^j.$$

Since $Le^{i(x-y)\xi} = e^{i(x-y)\xi}$, we can integrate by parts obtaining that

$$\begin{aligned} D^\alpha_x D^\beta_y K_P(x,y) &= \frac{(-1)^{|\beta|}}{M_{2\theta,\lambda}(x-y)} \sum_{\delta\le\alpha}\binom{\alpha}{\delta}\sum_{j=0}^\infty \frac{\lambda^j}{(j!)^{2\theta}}(1-\Delta_\xi)^j \\ &\qquad\times \int e^{i(x-y)\xi}\xi^{\beta+\delta}\psi_N(\xi)D^{\alpha-\delta}_x p(x,\xi)\,đ\xi \\ &= (-1)^{|\beta|+l}\frac{(x_r-y_r)^{-l}}{M_{2\theta,\lambda}(x-y)}\sum_{\delta\le\alpha}\binom{\alpha}{\delta}\sum_{j=0}^\infty\frac{\lambda^j}{(j!)^{2\theta}} \\ &\qquad\times\int e^{i(x-y)\xi}D^l_{\xi_r}(1-\Delta_\xi)^j[\xi^{\beta+\delta}\psi_N(\xi)D^{\alpha-\delta}_x p(x,\xi)]\,đ\xi \end{aligned}$$

where $l \in \mathbb{Z}_+$ is choosen appropriately. Let

$$F_{rljN\alpha\beta\delta} = D^l_{\xi_r}(1-\Delta_\xi)^j[\xi^{\beta+\delta}\psi_N(\xi)D_x^{\alpha-\delta}p(x,\xi)].$$

We denote by e_r the r-th vector of the canonical basis of $\mathbb{R}^n$ and $\langle \beta, e_r\rangle = \beta_r$, $\langle \delta, e_r\rangle = \delta_r$. We see that

$$\begin{aligned} F_{rljN\alpha\beta\delta} = &\sum_{\substack{l_1+l_2+l_3=l \\ l_1 \le \beta_r+\delta_r}} (-i)^{l_1} \frac{l!}{l_1!l_2!l_3!} \frac{(\beta_r+\delta_r)!}{(\beta_r+\delta_r-l_1)!} \\ &\times (1-\Delta_\xi)^j[\xi^{\beta+\delta-l_1e_r}D^{l_2}_{\xi_r}\psi_N(\xi)D^{l_3}_{\xi_r}D_x^{\alpha-\delta}p(x,\xi)]. \end{aligned}$$

Hence

$$\begin{aligned} |F_{rljN\alpha\beta\delta}| = &\sum_{\substack{l_1+l_2+l_3=l \\ l_1 \le \beta_r+\delta_r}} (-i)^{l_1} \frac{l!}{l_1!l_2!l_3!} \frac{(\beta_r+\delta_r)!}{(\beta_r+\delta_r-l_1)!} C_1^{|\alpha-\delta|+l_2+l_3+1} \\ &\times (N_2!)^\theta (N_3!)^\sigma [(\alpha-\delta)!]^\sigma C_2^j (j!)^{2\theta} \left(\frac{1}{RN^\theta}\right)^{l_2} \\ &\times \langle \xi\rangle_k^{m_1-\gamma l_3+|\alpha-\delta|/\sigma+|\beta|+|\delta|-l_1}\omega(x)^{m_2}\Phi(x)^{-\gamma|\alpha-\delta|+(l_3+j)/\sigma}. \end{aligned}$$

Note that on the support of ψ_n, $RN^\theta\langle\xi\rangle_k \le 3R(N+1)^\theta$. Since $\theta \ge \sigma$, it follows that

$$|F_{rljN\alpha\beta\delta}| \le C_1^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta C_2^j(j!)^{2\theta}\left(\frac{C_3}{R}\right)^{\gamma l}\omega(x)^{m_2}\Phi(x)^{-\gamma|\alpha-\delta|+(l_3+j)/\sigma},$$

with C_3 independent of R. We now choose l such that $\gamma l > N$. We observe that for every $c > 1$ there exist positive constants ε, c' such that, for $\tau > 0$,

$$\varepsilon \exp[c'\tau] \le \sum_{j=0}^\infty \left(\frac{\tau^j}{j!}\right)^c. \tag{A.3.3}$$

Setting $c = \theta, \tau = \lambda^{\frac{1}{\theta}}\langle x-y\rangle^{\frac{2}{\theta}}$, we have that

$$|M_{2\theta,\lambda}(x-y)| \ge \varepsilon \exp\{c'\lambda^{\frac{1}{\theta}}|x-y|^{\frac{2}{\theta}}\}.$$

Observe that $|x-y|^2 \ge c''\langle x\rangle\langle y\rangle$. From these estimates, choosing $\lambda < C_2^{-1}$ and R sufficiently large, we see that

$$|D_x^\alpha D_y^\beta K_P(x,y)| \le C_1^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta\left(\frac{C_4}{R}\right)^N \exp\Big\{-\frac{c'}{2}\lambda^{\frac{1}{\theta}}(\Phi(x)\Phi(y))^{\frac{1}{\theta}}\Big\},$$

where C_4 is independent of R. □

Definition A.3.1. *A linear continuous operator from $\mathcal{S}^\theta_\theta(\mathbb{R}^n)$ to $\mathcal{S}^\theta_\theta(\mathbb{R}^n)$ is said to be θ-regularizing if it extends to a linear continuous map from $\mathcal{S}^{\theta'}_\theta(\mathbb{R}^n)$ to $\mathcal{S}^\theta_\theta(\mathbb{R}^n)$.*

Definition A.3.2. *A linear continuous operator from $\mathcal{M}^{\theta}_{\Phi,k}(\mathbb{R}^n)$ to $\mathcal{M}^{\theta}_{\Phi,k}(\mathbb{R}^n)$ is said to be (Φ,θ)-regularizing if it extends to a linear continuous map from ${\mathcal{M}^{\theta}_{\Phi,k}}'(\mathbb{R}^n)$ to $\mathcal{M}^{\theta}_{\Phi,k}(\mathbb{R}^n)$.*

For $t>1$, we set

$$Q_t = \{(x,\xi)\in\mathbb{R}^{2n} : \Phi(x)\langle\xi\rangle_k < t\},$$

and

$$Q^e_t = \mathbb{R}^{2n}\setminus Q_t.$$

Definition A.3.3. *We denote by $FAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega,g_{\Phi,k})$ the space of all formal sums $\sum_{j\geq 0} p_j(x,\xi)$ such that $p_j(x,\xi)\in C^\infty(\mathbb{R}^{2n})$ such that for all $j\geq 0$ and there exists $B,C>0$ such that*

$$\sup_{j\geq 0}\sup_{\alpha,\beta\in\mathbb{Z}^n_+}\sup_{(x,\xi)\in Q^e_{Bj^{2\sigma-2}}} C^{-|\alpha|-|\beta|-2j}(\alpha!)^{-\sigma}(\beta!)^{-\sigma}(j!)^{-2\sigma+2}$$
$$\langle\xi\rangle_k^{-m_1+j+\gamma|\alpha|-|\beta|/\sigma}\omega(x)^{-m_2}\Phi(x)^{j+\gamma|\beta|-|\alpha|/\sigma}|D^\alpha_\xi\partial^\beta_x p_j(x,\xi)| < +\infty.$$

Every symbol $p\in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega,g_{\Phi,k})$ can be identified with an element of $FAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega,g_{\Phi,k})$, by setting $p_0=p$ and $p_j=0$ for $j\geq 1$.

Definition A.3.4. *Two sums $\sum_{j\geq 0} p_j$, $\sum_{j\geq 0} p'_j$ are said to be equivalent if there exist constants $A,B>0$ such that*

$$\sup_{N\in\mathbb{Z}_+}\sup_{\alpha,\beta\in\mathbb{N}^n}\sup_{(x,\xi)\in Q^e_{BN^{2\sigma-2}}} C^{-|\alpha|-|\beta|-2N}(\alpha!)^{-\sigma}(\beta!)^{-\sigma}(N!)^{-2\sigma+2}$$
$$\langle\xi\rangle_k^{-m_1+N+\gamma|\alpha|-|\beta|/\sigma}\omega(x)^{-m_2}\Phi(x)^{N+\gamma|\beta|-|\alpha|/\sigma}|D^\alpha_\xi\partial^\beta_x\sum_{j<N}(p_j-p'_j)| < +\infty,$$

and we write $\sum_{j\geq 0} p_j\sim\sum_{j\geq 0} p'_j$.

Proposition A.3.3. *Given $\sum_{j\geq 0} p_j \in FAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega,g_{\Phi,k})$, there exists a symbol $p\in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega,g_{\Phi,k})$ such that*

$$p\sim\sum_{j\geq 0} p_j \quad \text{in } FAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega,g_{\Phi,k}).$$

Proof. To construct the symbol p, we consider the excision function $\varphi\in C^\infty(\mathbb{R}^{2n})$ such that $0\leq\varphi\leq 1$ and $\varphi(x,\xi)=0$ if $(x,\xi)\in Q_2$, $\varphi(x,\xi)=1$ if $(x,\xi)\in Q^e_3$ and

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}}\left|D^\alpha_\xi D^\beta_x\varphi(x,\xi)\right| \leq C^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\sigma. \tag{A.3.4}$$

We define for $R>0$

$$\varphi_0(x,\xi)=\varphi(x,\xi) \quad \text{on } \mathbb{R}^{2d}$$
$$\varphi_j(x,\xi)=\varphi\left(\tfrac{x}{Rj^{\sigma-1}},\tfrac{\xi}{Rj^{\sigma-1}}\right), \quad j\geq 1.$$

For sufficiently large R, we will prove that

$$p(x,\xi)=\sum_{j\geq 0}\varphi_j(x,\xi)p_j(x,\xi),$$

is in $AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ and $p \sim \sum_{j\geq 0} p_j$ in $FAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$. From Definition A.3.3, we have

$$
\begin{aligned}
|D^\alpha_\xi D^\beta_x p(x,\xi)| &= \Big|\sum_{j\geq 0}\sum_{\substack{\alpha'\leq\alpha\\ \beta'\leq\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} D^{\alpha-\alpha'}_\xi D^{\beta-\beta'}_x p_j(x,\xi) D^{\alpha'}_\xi D^{\beta'}_x \varphi_j(x,\xi)\Big| \\
&\leq C^{|\alpha|+|\beta|+1}\alpha!\beta!\langle\xi\rangle_k^{m_1-\gamma|\alpha|+|\beta|/\sigma}\omega^{m_2}\Phi^{-\gamma|\beta|+|\alpha|/\sigma}\sum_{j\geq 0} H_{j\alpha\beta}(x,\xi)
\end{aligned}
$$

where

$$
\begin{aligned}
H_{j\alpha\beta}(x,\xi) = \sum_{\substack{\alpha'\leq\alpha\\ \beta'\leq\beta}} &\frac{\big((\alpha-\alpha')!(\beta-\beta')!\big)^{\sigma-1}}{\alpha'!\beta'!} C^{2j-|\alpha'|-|\beta'|}(j!)^{2\sigma-2} \\
&\times \langle\xi\rangle_k^{\gamma|\alpha'|-j}\Phi(x)^{\gamma|\beta'|-j}\left|D^{\alpha'}_\xi D^{\beta'}_x\varphi_j(x,\xi)\right|.
\end{aligned}
$$

From (A.3.4), we have

$$
H_{j\alpha\beta}(x,\xi) \leq C^{|\alpha|+|\beta|+1}(\alpha!)^{\sigma-1}(\beta!)^{\sigma-1}\left(\frac{C_1}{R^2}\right)^j
$$

where $C_1 > 0$ is independent of R. By choosing R sufficiently large, we obtain the required result. We observe that

$$
p(x,\xi) - \sum_{j<N} p_j(x,\xi) = \sum_{j\geq N} p_j(x,\xi)\varphi_j(x,\xi),
$$

for $(x,\xi)\in Q^e_{3RN^{2\sigma-1}}$, $N\in\mathbb{Z}_+$, which we can estimate as before. □

Proposition A.3.4. *Let $p \in AG^{0,0}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ and $\theta \geq 2(\sigma-1)$. If $p \sim 0$ in $FAG^{0,0}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$, then the operator P is (Φ,θ)-regularizing.*

Proof. We will show that the kernel

$$
K_P(x,y) = (2\pi)^{-d}\int e^{i(x-y)\xi}p(x,\xi)d\xi
$$

is in $\mathcal{M}^\theta_{\Phi,k}(\mathbb{R}^{2n})$ impliying that P is (Φ,θ)-regularizing.

There exist $B, C > 0$ such that for every $(x,\xi)\in\mathbb{R}^{2n}$:

$$
\begin{aligned}
\left|D^\alpha_\xi D^\beta_x p(x,\xi)\right| &\leq C_1^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\sigma\langle\xi\rangle_k^{-|\alpha|}\Phi(x)^{-|\beta|} \\
&\times \inf_{0\leq N\leq B_1(\langle\xi\rangle_k\Phi(x))^{\frac{1}{2\sigma-2}}} \frac{C^{2N}(N!)^{2\sigma-2}}{\langle\xi\rangle_k^N\Phi(x)^N}.
\end{aligned}
$$

Using [58, Lemma 6.3.10], we obtain

$$
\left|D^\alpha_\xi D^\beta_x p(x,\xi)\right| \leq C_2^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta \exp\left[-a\Big(\Phi(x)\langle\xi\rangle_k\Big)^{\frac{1}{\theta}}\right]
$$

for some $C_2, a > 0$. □

We now examine the stability of the classes $OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ under transposition, composition and construction of parametrices. Let $u \in \mathcal{M}^{\theta'}_{\Phi,k}(\mathbb{R}^n)$ and $v \in \mathcal{M}^{\theta}_{\Phi,k}(\mathbb{R}^n)$, $\theta \geq 2(\sigma-1)$. To relate with Gelfand-Shilov spaces, one can even consider $u \in \mathcal{S}^{\frac{\theta}{2}\,'}_{\frac{\theta}{2}}(\mathbb{R}^n)$ and $v \in \mathcal{S}^{\frac{\theta}{2}}_{\frac{\theta}{2}}(\mathbb{R}^n)$.

Proposition A.3.5. *Let $P = p(x,D) \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ and let P^t be the transposed operator defined by*

$$\langle P^t u, v\rangle = \langle u, Pv\rangle. \tag{A.3.5}$$

Then, $P^t = Q + R$, where R is (Φ,θ)-regularizing and $Q = q(x,D)$ is in $AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ with

$$q(x,\xi) \sim \sum_{j\geq 0}\sum_{\alpha=j} \frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_x p(x,-\xi) \qquad in\ FAG^{m_1,m_2}_{\Phi,k,\sigma;\sigma,\sigma}.$$

Theorem A.3.6. *Let $P = p(x,D) \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$,$Q = q(x,D) \in AG^{m'_1,m'_2}_{\Phi,\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$. Then $PQ = T + R$ where R is (Φ,θ)-regularizing and $T = t(x,D)$ is in $AG^{m_1+m'_1,m_2+m'_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ with*

$$t(x,\xi) \sim \sum_{j\geq 0}\sum_{\alpha=j} \frac{1}{\alpha!}\partial^\alpha_\xi p(x,\xi) D^\alpha_x q(x,\xi)$$

in $FAG^{m_1+m'_1,m_2+m'_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$.

To prove Proposition A.3.5 and Theorem A.3.6, we introduce more general classes of symbols, called amplitudes. Let $(m_1,m_2,m_3) \in \mathbb{R}^3$.

Definition A.3.5. *We denote by $\Pi^{m_1,m_2,m_3}_{\sigma;\sigma}(\omega, g_{\Phi,k})$ the Banach space of all symbols $a(x,y,\xi) \in C^\infty(\mathbb{R}^3)$ satisfying for some $C>0$ the following estimate*

$$\sup_{\alpha,\beta,\delta\in\mathbb{Z}^n_+}\ \sup_{(x,y,\xi)\in\mathbb{R}^3} C^{-|\alpha|-|\beta|-|\delta|}(\alpha!\beta!\delta!)^{-\sigma}\langle\xi\rangle_k^{-m_1+\gamma|\alpha|-(|\beta|+|\delta|)/\sigma}$$
$$\omega(x)^{-m_2}\Phi(x)^{\gamma|\beta|-|\alpha|/\sigma}\omega(y)^{-m_3}\Phi(y)^{\gamma|\delta|-|\alpha|/\sigma}|D^\alpha_\xi D^\beta_x D^\delta_y a(x,y,\xi)| < +\infty.$$

Given $a \in \Pi^{m_1,m_2,m_3}_{\sigma;\sigma}(\omega, g_{\Phi,k})$, we associate to a the pseudodifferential operator defined by

$$Au(x) = \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)dy\,đ\xi, \quad u \in \mathcal{M}^\theta_{\Phi,k}(\mathbb{R}^n) \tag{A.3.6}$$

Theorem A.3.7. *Let A be an operator defined by an amplitude $a \in \Pi^{m_1,m_2,m_3}_{\sigma;\sigma}(\omega, g_{\Phi,k})$, $(m_1,m_2,m_3) \in \mathbb{R}^3$. Then we may write $A = P + R$, where R is a (Φ,θ)-regularizing operator and $P = p(x,D)$ is in $AG^{m_1,m_2+m_3}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ with $p \sim \sum_{j\geq 0} p_j$ where*

$$p_j(x,\xi) = \sum_{|\alpha|=j}\frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_y a(x,y,\xi)|_{y=x}.$$

The proof of this result uses the similar standard arguments available in the proof of Theorem 6.3.14 in [58]. For the sake of conciseness, we omit the details.

Proof of Theorem A.3.5. From (A.3.5), P^t is defined as

$$P^t u(x) = \int e^{i(x-y)\xi} p(y, -\xi) u(y) dy đ\xi, \quad u \in \mathcal{M}^{\theta}_{\Phi,k}(\mathbb{R}^n)$$

Observe that P^t is an operator of the form (A.3.6) with amplitude $p(y, -\xi)$. By Theorem (A.3.7), $P^t = Q + R$ where R is (Φ, θ)-regularizing and $Q = q(x, D) \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ with

$$q(x, \xi) \sim \sum_{j \geq 0} \sum_{|\alpha|=j} (\alpha!)^{-1} \partial^{\alpha}_{\xi} D^{\alpha}_{x} p(x, -\xi).$$

□

Proof of Theorem A.3.6. We can write $Q = (Q^t)^t$. Then, by Theorem (A.3.7) and Proposition (A.3.5), $Q = Q_1 + R_1$, where R_1 is (Φ, θ)-regularizing and

$$Q_1 u(x) = \int e^{i(x-y)\xi} q_1(y, \xi) u(y) dy đ\xi \tag{A.3.7}$$

with $q_1(y, \xi) \in AG^{m_1',m_2'}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$, $q_1(y, \xi) \sim \sum_{\alpha} (\alpha!)^{-1} \partial^{\alpha}_{\xi} D^{\alpha}_{y} q(y, -\xi)$. From (A.3.7) it follows that

$$\widehat{Q_1 u}(\xi) = \int e^{-iy\xi} q_1(y, \xi) u(y) dy, \quad u \in \mathcal{M}^{\theta}_{\Phi,k}(\mathbb{R}^n)$$

from which we deduce that

$$PQu(x) = \int e^{i(x-y)\xi} p(x, \xi) q_1(y, \xi) u(y) dy đ\xi + PR_1 u(x).$$

We observe that $p(x, \xi) q_1(y, \xi) \in \Pi^{m_1+m_1', m_2, m_2'}_{\sigma;\sigma}(\omega, g_{\Phi,k})$. Applying Theorem (A.3.7), we obtain that

$$PQu(x) = Tu(x) + Ru(x)$$

where R is (Φ, θ)-regularizing and $T = t(x, D) \in OPAG^{m_1+m_1', m_2+m_2'}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ with

$$t(x, \xi) \sim \sum_{j \geq 0} \sum_{|\alpha|=j} (\alpha!)^{-1} \partial^{\alpha}_{\xi} p(x, \xi) D^{\alpha}_{x} q(x, \xi)$$

in $FAG^{m_1+m_1', m_2+m_2'}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$. □

We now state the notion of ellipticity for elements of $OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$.

Definition A.3.6. *A symbol $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ is said to be G_{Φ}-elliptic if there exist $B, C = C(k) > 0$ such that*

$$|p(x, \xi)| \geq C \langle \xi \rangle_k^{m_1} \omega(x)^{m_2}, \quad \forall (x, \xi) \in Q^e_B.$$

Theorem A.3.8. *If $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ is G_{Φ}-elliptic and $P = p(x, D)$, then there exists $E \in AG^{-m_1,-m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ such that $EP = I + R_1, PE = I + R_2$, where R_1, R_2 are (Φ, θ)-regularizing operators.*

The above theorem can be easily proved using Proposition A.3.3 and Proposition A.3.6. For the sake of conciseness, we omit the proof and refer the reader to Theorem 6.3.16 in [58] for the details.

As an immediate consequence of Theorem *A.3.8* we obtain the following result of global regularity.

Corollary A.3.1. *Let $p \in AG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ be G_Φ-elliptic and let $f \in \mathcal{M}^\theta_{\Phi,k}(\mathbb{R}^n)$. If $u \in {\mathcal{M}^\theta_{\Phi,k}}'(\mathbb{R}^n)$ is a solution of the equation*

$$Pu = f,$$

then $u \in \mathcal{M}^\theta_{\Phi,k}(\mathbb{R}^n)$.

This completes the calculus for the class $OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$.

Using similar techniques as in the case of calculus of $OPAG^{m_1,m_2}_{\sigma;\sigma,\sigma}(\omega, g_{\Phi,k})$ and referring to Appendix A in [2], one can also develop a calculus for the class $OPAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$. We start with defining the notion of formal sums in this context.

Definition A.3.7. *We denote by $FAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ the space of all formal sums $\sum_{j\geq 0} p_j(x,\xi)$ such that $p_j(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that for all $j \geq 0$ and following conditions hold*

i) There exist $B, C > 0$ such that

$$\sup_{j\geq 0}\ \sup_{\alpha,\beta\in\mathbb{Z}^n_+}\ \sup_{(x,\xi)\in Q^e_{B(j+|\alpha|)^\sigma}} C^{-|\alpha|-|\beta|-2j}(\alpha!)^{-1}(\beta!j!)^{-\sigma} \langle\xi\rangle_k^{-m_1+j+|\alpha|}\omega(x)^{-m_2}\Phi(x)^{j+|\beta|}|D^\alpha_\xi\partial^\beta_x p_j(x,\xi)| < +\infty. \tag{A.3.8}$$

ii) For every $B_0 > 0$, there exists $C > 0$ such that

$$\sup_{j\geq 0}\ \sup_{\alpha,\beta\in\mathbb{Z}^n_+}\ \sup_{(x,\xi)\in Q_{B_0(j+|\alpha|)^\sigma}} C^{-|\alpha|-|\beta|-2j}(\alpha!\beta!j!)^{-\sigma}|D^\alpha_\xi\partial^\beta_x p_j(x,\xi)| < +\infty. \tag{A.3.9}$$

Every symbol $p \in AG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ can be identified with an element of $FAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$, by setting $p_0 = p$ and $p_j = 0$ for $j \geq 1$.

Definition A.3.8. *Two sums $\sum_{j\geq 0} p_j$, $\sum_{j\geq 0} p'_j$ in $FAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ are said to be equivalent if there exist constants $A, B > 0$ such that*

$$\sup_{N\in\mathbb{Z}_+}\ \sup_{\alpha,\beta\in\mathbb{N}^n}\ \sup_{(x,\xi)\in Q^e_{B(N+|\alpha|)^\sigma}} C^{-|\alpha|-|\beta|-2N}(\alpha!)^{-1}(\beta!N!)^{-\sigma} \langle\xi\rangle_k^{-m_1+N+|\alpha|}\omega(x)^{-m_2}\Phi(x)^{N+|\beta|}|D^\alpha_\xi\partial^\beta_x \sum_{j<N}(p_j - p'_j)| < +\infty,$$

and we write $\sum_{j\geq 0} p_j \sim \sum_{j\geq 0} p'_j$.

Proposition A.3.9. *Given $\sum_{j\geq 0} p_j \in FAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$, there exists a symbol $p \in AG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k})$ such that*

$$p \sim \sum_{j\geq 0} p_j \quad \textit{in } FAG^{m_1,m_2}_{\sigma}(\omega, g_{\Phi,k}).$$

Proof. Following [2], to construct the symbol p, we consider two types of cut-off functions. For a fixed $R > 0$, we can find a sequence of functions $\psi_j(\xi), j = 0, 1, 2, \dots,$ such that $0 \leqslant \psi_j(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^n, \psi_j(\xi) = 1$ if $\langle\xi\rangle_k \leqslant 2R \sup\left(j^{\frac{\sigma}{2}}, 1\right), \psi_j(\xi) = 0$ if $\langle\xi\rangle_k \geqslant 4R \sup\left(j^{\frac{\sigma}{2}}, 1\right)$ and satisfying the following estimates:

$$\left|\partial_\xi^\alpha \psi_j(\xi)\right| \leqslant C_1^{|\alpha|}\left(R \sup\left(j^{\sigma-1}, 1\right)\right)^{-|\alpha|} \qquad \text{if } |\alpha| \leqslant 4j,$$

and

$$\left|\partial_\xi^\alpha \psi_j(\xi)\right| \leqslant C_2^{|\alpha|+1}(\alpha!)^\sigma\left(R \sup\left(j^{\sigma}, 1\right)\right)^{-|\alpha|} \qquad \text{if } |\alpha| > 4j,$$

for some positive constants C_1, C_2 independent of α, R, j.

Similarly, we can choose a sequence of functions $\tilde{\psi}_j(x) \in G_0^\sigma(\mathbb{R}^n), j = 0, 1, 2, \dots,$ supported for $\Phi(x) \leqslant 4R \sup\left(j^{\frac{\sigma}{2}}, 1\right), \psi_j(x) = 1$ for $\Phi(x) \leqslant 2R \sup\left(j^{\frac{\sigma}{2}}, 1\right)$ and

$$|\partial_x^\beta \tilde{\psi}_j(x)| \leqslant C_3^{|\beta|+1}(\beta!)^\sigma\left(R \sup\left(j^{\sigma}, 1\right)\right)^{-|\beta|} \qquad \text{for all } x \in \mathbb{R}^n, \beta \in \mathbb{Z}_+^n.$$

Let us now define $\varphi_j(x,\xi) = (1-\psi_j(\xi))(1-\tilde{\psi}_j(x)), j = 0, 1, 2, \dots$. By the properties listed above, we deduce that the functions φ_j are smooth on $\mathbb{R}^{2n}$, supported in $Q^e_{4R^2 \sup(j^\sigma,1)}$ and $\varphi_j(x,\xi) = 1$ in $Q^e_{16R^2 \sup(j^\sigma,1)}$. Moreover

$$\left|D_\xi^\alpha D_x^\beta \varphi_0(x,\xi)\right| \leqslant A\left(\frac{C}{R}\right)^{|\alpha|+|\beta|}$$

whereas, for $j \geqslant 1$ the functions φ_j satisfy the following estimates:

$$\left|D_\xi^\alpha D_x^\beta \varphi_j(x,\xi)\right| \leqslant A\left(\frac{C}{R}\right)^{|\alpha|+|\beta|}(\beta!)^\sigma\left(j^{\sigma-1}\right)^{-|\alpha|} j^{-\sigma|\beta|} \tag{A.3.10}$$

for $|\alpha| \leqslant 3j, \beta \in \mathbb{Z}_+^n$ and

$$\left|D_\xi^\alpha D_x^\beta \varphi_j(x,\xi)\right| \leqslant A\left(\frac{C}{R}\right)^{|\alpha|+|\beta|}(\alpha!\beta!)^\sigma j^{-\sigma(|\alpha|+|\beta|)} \tag{A.3.11}$$

for $|\alpha| > 3j, \beta \in \mathbb{Z}_+^n$, with A, C positive constants independent of α, β, R, j.

We now define

$$p(x,\xi) = \sum_{j \geqslant 0} \varphi_j(x,\xi) p_j(x,\xi).$$

Let us first prove that $p \in AG_\sigma^{m_1,m_2}(\omega, g_{\Phi,k})$. We estimate the derivatives of p in the region $Q^e_{4R^2|\alpha|^\sigma}$. On the support of φ_j we have $\Phi(x)\langle\xi\rangle_k \geqslant 4R^2 \sup(j^\sigma, 1)$. Choosing $R^2 \geqslant 2^{\sigma-2}B$ where B is the same constant appearing in Definition A.3.7, for $(x,\xi) \in Q^e_{4R^2|\alpha|^\sigma}$, we have $(x,\xi) \in Q^e_{B(j+|\alpha|)^\sigma}$, then the estimates (A.3.8) on the p_j hold true. Moreover, if $\alpha \neq 0$ and $\beta \neq 0$ $D_\xi^\alpha D_x^\beta \varphi_j(x,\xi)$ is supported in $Q_{16R^2 \sup(j^\sigma,1)}$, then $4R^2|\alpha|^\sigma \leqslant \Phi(x)\langle\xi\rangle_k \leqslant 16R^2 j^\sigma$, and this implies $|\alpha| \leqslant 4j$. Then (A.3.8), (A.3.10) and Leibniz formula give

$$
\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta (p_j(x,\xi)\varphi_j(x,\xi))| \\
&\qquad \leqslant \sum_{\substack{\alpha'\leqslant\alpha \\ \beta'\leqslant\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \left|\partial_\xi^{\alpha'}\partial_x^{\beta'}\varphi_j(x,\xi)\right|\left|\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}p_j(x,\xi)\right| \\
&\qquad \leqslant C_1^{|\alpha|+|\beta|+1}\alpha!(\beta!)^\sigma \langle\xi\rangle_k^{m_1-|\alpha|-j}\omega(x)^{m_2}\Phi(x)^{-|\beta|-j}\left(\frac{C_2}{R}\right)^j (j!)^\sigma \\
&\qquad\qquad \sum_{\substack{\alpha'\leqslant\alpha \\ \beta'\leqslant\beta}} (\alpha'!)^{-1}(\beta'!)^{-\sigma} j^{|\alpha'|} j^{-\sigma(|\alpha'|+|\beta'|)}\langle\xi\rangle_k^{|\alpha'|}\Phi(x)^{|\beta'|}\chi_{\mathrm{supp}(\varphi_j)}
\end{aligned}
$$

where $\chi_{\mathrm{supp}\,(\varphi_j)}$ is the characteristic function of the support of φ_j. Now, if $(x,\xi) \in \mathrm{supp}\,(\varphi_j)$ we have $\langle\xi\rangle_k^{-j}\Phi(x)^{-j} \leqslant (4R^2 j^\sigma)^{-j}$ and $\langle\xi\rangle_k^{|\alpha'|}\Phi(x)^{|\beta'|} \leqslant C^{|\alpha'|+|\beta'|} j^{\sigma(|\alpha'|+|\beta'|)}$, while $(\alpha'!)^{-1} j^{|\alpha'|} \leqslant 2^j C^{|\alpha'|}$. Hence

$$
\left|D_\xi^\alpha D_x^\beta (p_j(x,\xi)\varphi_j(x,\xi))\right| \leqslant C_3^{|\alpha|+|\beta|+1}\alpha!(\beta!)^\sigma\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}\left(\frac{C_4}{R}\right)^j
$$

where C_4 is a constant independent of R. Then, possibly enlarging R and summing over j we obtain that

$$
\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| \leqslant C_5^{|\alpha|+|\beta|+1}\alpha!(\beta!)^\sigma\langle\xi\rangle_k^{m_1-|\alpha|}\omega(x)^{m_2}\Phi(x)^{-|\beta|}
$$

for some $C_5 > 0$ independent of α, β and for $\Phi(x)\langle\xi\rangle_k \geqslant 4R^2|\alpha|^\sigma$. Similarly, using estimates (A.3.9), (A.3.11), we can prove that $p \in AG_{\sigma,\sigma}^{m_1,m_2}(\omega, g_{\Phi,k})$. Then $p \in AG_\sigma^{m_1,m_2}(\omega, g_{\Phi,k})$. To prove that $p \sim \sum_{j\geqslant 0} p_j$ we observe that for $\Phi(x)\langle\xi\rangle_k \geqslant 16R^2N^\sigma$, we have

$$
p(x,\xi) - \sum_{j<N} p_j(x,\xi) = \sum_{j\geqslant N}\varphi_j(x,\xi)p_j(x,\xi)
$$

which we can estimate as before. □

Proposition A.3.10. *Let $p \in AG_\sigma^{0,0}(\omega, g_{\Phi,k})$ and $\theta \geq \sigma$. If $p \sim 0$ in $FAG_\sigma^{0,0}(\omega, g_{\Phi,k})$, then the operator P is (Φ,θ)-regularizing.*

The proof is similar to that of Proposition A.3.4. Using the techniques used in the calculus of $OPAG_{\sigma;\sigma,\sigma}^{m_1,m_2}(\omega, g_{\Phi,k})$, one can examine the stability of the classes $OPAG_\sigma^{m_1,m_2}(\omega, g_{\Phi,k})$ under transposition, composition and construction of parametrices. One can consider $\theta \geq \sigma$ and prove Proposition A.3.5, Theorem A.3.6 and Theorem A.3.8 for the class $OPAG_\sigma^{m_1,m_2}(\omega, g_{\Phi,k})$.

Appendix B

Sharp Gårding Inequality for a Parameter Dependent Matrix

> I love inequalities. So if somebody shows me a new inequality, I say: "Oh, thats beautiful, let me think about it," and I may have some ideas connected to it. The point of view is that inequalities are more interesting than equalities.
>
> — Louis Nirenberg

In this appendix we prove the sharp Gårding inequality for a matrix pseudodifferential operator with symbol $a(t,x,\xi)$ in

$$G^{1,1}\{1\}(\omega, g_{\Phi,k})_N^{(1)} \cap G^{0,0}\{1,0,1\}(\omega, g_{\Phi,k})_N^{(2)} \cap G^{-1,-1}\{2,1,2,1\}(\omega, g_{\Phi,k})_N^{(3)}. \tag{B.0.1}$$

This is used Sections 4.5.5 and 4.5.4 to arrive at an energy estimate. By Propositions A.2.1 and A.2.2, for any $\varepsilon \in (0,1)$,

$$a \in L^\infty\left([0,1]; G^{1,1}(\Phi, g_{\Phi,k})\right), \qquad \text{if } \tilde{\theta} \text{ is bounded,} \tag{B.0.2}$$

$$t^{1-\varepsilon} a \in C\left([0,1]; G^{1,1}(\Phi, g_{\Phi,k})\right), \qquad \text{otherwise.} \tag{B.0.3}$$

We give below the sharp Gårding inequality when $\tilde{\theta}$ is unbounded. The case for bounded $\tilde{\theta}$ follows in similar lines by taking $\varepsilon = 1$ in the proof.

Theorem B.0.1. *Let $a(t,x,\xi)$ be 2×2 positive semi-definite matrix belonging to the symbol class given in (B.0.1). Then, for each $t \in [0,T]$ and $\varepsilon \in (0,1)$ we have*

$$\left(t^{1-\varepsilon} a^w(t,x,D)u, u\right) \geqq -C\|u\|^2, \quad u \in \mathcal{S}\left(\mathbb{R}^n; \mathbb{R}^2\right). \tag{B.0.4}$$

Proof. We have the following region-wise seminorms for $a(t,x,\xi)$:

$$\text{in } Z_{int}(N), \qquad |a|_j^g(w) \le h^{-1}\tilde{\theta},$$

$$\text{in } Z_{mid}(N), \qquad |a|_j^g(w) \le \frac{\theta(t)}{t}\tilde{\theta}^j,$$

$$\text{in } Z_{ext}(N), \qquad |a|_j^g(w) \le h\left(\frac{\theta(t)}{t}\right)^2 e^{\psi(t)}\tilde{\theta}^{2+j}.$$

Here $w \in \mathbb{R}^{2n}$, $g = g_{\Phi,k}$ and the norm of the j^{th} differential at x given by

$$|f|_j^g := \sup_{y_l \in \mathbb{R}^{2n}} |f^{(j)}(x; y_1, \ldots, y_j)| \Big/ \prod_1^j g(y_l)^{\frac{1}{2}}.$$

Using an appropriate affine symplectic transformation one can assume that $g_{\Phi,k} = he$ where e is the Euclidean metric form. Then

$$\text{in } Z_{int}(N), \qquad |a|_j^e(w) \le h^{(j-2)/2}\tilde{\theta}, \tag{B.0.5}$$

$$\text{in } Z_{mid}(N), \qquad |a|_j^e(w) \le h^{j/2}\frac{\omega(t)}{t}\tilde{\theta}^j, \tag{B.0.6}$$

$$\text{in } Z_{ext}(N), \qquad |a|_k^e(w) \le h^{(j+2)/2}\left(\frac{\omega(t)}{t}\right)^2 e^{\psi(t)}\tilde{\theta}^{2+j}. \tag{B.0.7}$$

Let $a_t^{(0)} + a_t^{(1)}(x,\xi)$ be the first order Taylor expansion of a at $(x,\xi) = 0$. Let $v \in \mathbb{R}^2$. By the above semi-norms we have

$$\text{in } Z_{int}(N), \quad \begin{cases} \left(a_t^{(0)}v, v\right) + \left(a_t^{(1)}(x,0)v, v\right) + \frac{|x|^2}{2}(\tilde{\theta}v, v) \geqq 0, \\ \left(a_t^{(0)}v, v\right) + \left(a_t^{(1)}(0,\xi)v, v\right) + \frac{|\xi|^2}{2}(\tilde{\theta}v, v) \geqq 0, \end{cases} \tag{B.0.8}$$

$$\text{in } Z_{mid}(N), \quad \begin{cases} \left(a_t^{(0)}v, v\right) + \left(a_t^{(1)}(x,0)v, v\right) + \frac{|x|^2}{2}\left(\frac{\theta(t)}{t}\tilde{\theta}^2 hv, v\right) \geqq 0, \\ \left(a_t^{(0)}v, v\right) + \left(a_t^{(1)}(0,\xi)v, v\right) + \frac{|\xi|^2}{2}\left(\frac{\theta(t)}{t}\tilde{\theta}^2 hv, v\right) \geqq 0, \end{cases} \tag{B.0.9}$$

$$\text{in } Z_{ext}(N), \quad \begin{cases} \left(a_t^{(0)}v, v\right) + \left(a_t^{(1)}(x,0)v, v\right) + \frac{|x|^2}{2}\left(\frac{\theta(t)^2}{t^2}e^{\psi(t)}\tilde{\theta}^4 h^2 v, v\right) \geqq 0, \\ \left(a_t^{(0)}v, v\right) + \left(a_t^{(1)}(0,\xi)v, v\right) + \frac{|\xi|^2}{2}\left(\frac{\theta(t)^2}{t^2}e^{\psi(t)}\tilde{\theta}^4 h^2 v, v\right) \geqq 0. \end{cases} \tag{B.0.10}$$

Let us fix $0 < \varepsilon < \varepsilon' < 1$. Using the definition of the regions, we see that in the whole of extended phase space

$$\begin{cases} \left(t^{1-\varepsilon'}a_t^{(0)}v, v\right) + \left(t^{1-\varepsilon'}a_t^{(1)}(x,0)v, v\right) + \frac{|x|^2}{2}(t^{1-\varepsilon'}\tilde{\theta}(t)^q v, v) \geqq 0, \\ \left(t^{1-\varepsilon'}a_t^{(0)}v, v\right) + \left(t^{1-\varepsilon'}a_t^{(1)}(0,\xi)v, v\right) + \frac{|\xi|^2}{2}(t^{1-\varepsilon'}\tilde{\theta}(t)^q v, v) \geqq 0, \end{cases}$$

where $q = 1$ in $Z_{int}(N)$ while in $Z_{mid}(N)$ and $Z_{ext}(N)$ $q = 2$. Since the function $\tilde{\theta}(t)$ is of logarithmic type, in the whole of extended phase space, we have

$$\begin{cases} t^{1-\varepsilon}\left((a_t^{(0)}v, v) + (a_t^{(1)}(x,0)v, v) + \frac{|x|^2}{2}\|v\|^2\right) \geqq 0, \\ t^{1-\varepsilon}\left((a_t^{(0)}v, v) + (a_t^{(1)}(0,\xi)v, v) + \frac{|\xi|^2}{2}\|v\|^2\right) \geqq 0. \end{cases}$$

From here on we proceed as in [43, Theorem 18.6.14] to obtain the result. □

Appendix C

Certain Examples of Singular Functions

> The art of doing mathematics consists in finding that special case which contains all the germs of generality.
>
> — David Hilbert

In this chapter, we show that the singular behavior specified at a single point (say at $t = 0$) is independent from the regularity of the coefficients on the whole $[0, T]$. More precisely, it is possible to construct a function $f_1 \in C^1(0, T]$ and log-Lipschitz continuous on $[0, T]$, such that

$$\limsup_{t \to 0^+} \, t^q \, |f_1'(t)| = +\infty \tag{C.0.1}$$

for all $q \geq 1$. Conversely, it is easy to find a function $f_2 \in C([0, T]) \cap C^1(0, T]$ but Hölder-continuous on $[0, T]$ for no $\alpha < 1$, such that

$$\limsup_{t \to 0^+} \, t \, |f_2'(t)| < +\infty. \tag{C.0.2}$$

It easy to come up with an example for the latter case compared to the former. For example,

$$f_2(t) = \begin{cases} 0, & \text{if } t = 0, \\ e^{-\sqrt{|\ln t|}}, & \text{if } t \in (0, 1]. \end{cases}$$

Note that the above function satisfies (C.0.2) but it is not Hölder continuous at $t = 0$ because for any $\alpha \in (0, 1)$, $t^\alpha = o(f_2(t))$ as $t \to 0$.

We now construct a function f_1 satisfying the estimate C.0.1. Let us begin with the following lemma.

Lemma C.0.1. *Suppose μ is a modulus of continuity with $\mu'(0) = \infty$. There is a μ-continuous smooth function f on $\mathbb{R}_+$, with prescribed derivative $p_k \in \mathbb{R}$ at points of a prescribed discrete subset $(t_k)_{k \geq 1}$ of $\mathbb{R}_+$ such that $f'(t_k) = p_k$, for all $k \geq 1$.*

Proof. We may assume without loss of generality that $\frac{\mu(t)}{t}$ is decreasing with $\sup_{t>0} \frac{\mu(t)}{t} = +\infty$. Let us fix a smooth function ϕ with support in $[-1/2, 1/2]$ and with $\phi'(0) = 1 =$

$||\phi||_\infty$. Define positive numbers δ_k such that $2^k|p_k| \leq \frac{\mu(\delta_k)}{\delta_k}$. Let $I_k, k \geq 1$ be pairwise disjoint intervals of length δ_k centered at t_k.

For any $k \geq 1$ consider

$$\phi_k(x) := \delta_k p_k \phi\left(\frac{t - t_k}{\delta_k}\right) \tag{C.0.3}$$

a smooth function supported in I_k such that $\phi_k'(t_k) = p_k$ and $||\phi_k'||_\infty = |p_k|$. It satisfies $|\phi_k(t) - \phi_k(\tau)| \leq 2^{-k}\mu(|t-\tau|)$ for all t and τ in $\mathbb{R}$: indeed, to check the latter, it is sufficient to look at points t and τ both in I_k, and for these points $|t-\tau| \leq \delta_k$, hence

$$|\phi_k(t) - \phi_k(\tau)| \leq |p_k||t-\tau| \leq 2^{-k}\frac{\mu(\delta_k)}{\delta_k}|t-\tau| \leq 2^{-k}\mu(|t-\tau|). \tag{C.0.4}$$

Let $f(t) = \sum_{k=1}^\infty \phi_k$. Note that $f(t)$ satisfies the required properties and hence the lemma is proved. □

To obtain a function satisfying the estimate (C.0.1), we define $f(t) = \sum_{k=1}^\infty \phi_k$ with $t_k = \frac{1}{k}$ and $p_k = e^k$.

Appendix D

Applications of Singular Hyperbolic Equations

If you are acquainted with the principle, what do you care for a myriad instances and applications?

— Henry David Thoreau, Walden

In this chapter we describe certain applications of singular hyperbolic equations to cosmology, transonic gas dynamics (Tricomi equation) and transversal vibrations of elastic string (Kirchhoff equation).

D.1 Cosmology

Singular hyperbolic equations arise naturally in the study of waves propagating in the universe modeled by the cosmological models with expansion, in particular, the Einstein-de Sitter (EdeS) spacetime. The EdeS model was jointly proposed by Einstein and de Sitter [30]. Recently it was used in [71] to study cosmological black holes. In EdeS spacetime, the wave equation with source term f written in coordinate form is

$$\left(\partial_t^2 - \frac{1}{t^{4/3}}\Delta_x + \frac{2}{t}\partial_t\right)u(t,x) = f, \quad t > 0,\ x \in \mathbb{R}^3.$$

By imposing certain *weighted initial conditions*, the existence of solution to the above singular hyperbolic equation is proved in [33].

D.2 Transonic Gas Dynamics

Consider the Tricomi equation

$$(\partial_t^2 - t\partial_x^2)u(t,x) = 0. \tag{D.2.1}$$

Th equation is used to describe the transonic gas dynamics [55]. Our interest is mainly the case $t > 0$, i.e., when the equation (D.2.1) is hyperbolic. Under the change of variable $\tau = \frac{2}{3}t^{3/2}$, this yields

$$\left(\partial_\tau^2 - \partial_x^2 + \frac{1}{3\tau}\partial_\tau\right)u(\tau,x) = 0,$$

which is Euler-Poison-Darboux (EPD) equation, an example of singular hyperbolic equation. In fact, the tricomi equation is studied as a special case of EPD equation in [55].

One can generalize the above observations to study the degenerate equations

$$(\partial_t^2 - t^{2p}\partial_x^2)u(t,x) = 0, \quad p > 0.$$

With the change of variable $\tau = \frac{t^{p+1}}{p+1}$, the above equation corresponds to the following singular hyperbolic equation

$$\left(\partial_\tau^2 - \partial_x^2 + \frac{p}{p+1}\frac{1}{\tau}\partial_\tau\right)u(\tau,x) = 0,$$

which is again the EPD type equation.

D.3 Kirchhoff Type Equations

Let us consider the linear strictly hyperbolic Cauchy problem

$$\begin{cases}(\partial_t^2 - a(t,x)\Delta_x)u(t,x) = 0, & \text{in } (0,T]\times\mathbb{R}^n,\\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x). & \end{cases} \tag{D.3.1}$$

Suppose that $a \in C^1((0,T];\mathcal{B}(\mathbb{R}^n))$ and for $t > 0,\ x \in \mathbb{R}^n$,

$$\left.\begin{aligned}&|\partial_t a(t,x)| \leq Ct^{-1},\\ &|\partial_x^\beta a(t,x)| \leq C_\beta t^{-p}, \quad p \in [0,1),\ |\beta| > 0.\end{aligned}\right\} \tag{D.3.2}$$

We have the following result by Cicognani [9].

Theorem D.3.1. *(Cicognani [9]) For the Cauchy problem (D.3.1) satisfying the conditions (D.3.2), There are positive constants C, κ such that for every $u \in C([0,T];H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T];H^s(\mathbb{R}^n))$ satisfying (D.3.1) we have*

$$\|u(t)\|_{s-\kappa} + \|\partial_t u(t)\|_{s-1-\kappa} \leq C(\|u(0)\|_s + \|\partial_t u(0)\|_{s-1}), \quad 0 \leq t \leq T. \tag{D.3.3}$$

In (D.3.2) one can replace t^{-1} and t^{-p} with $|T_0 - t|^{-1}$ and $|T_0 - t|^{-p}$, respectively, $T_0 \in [0,T], t \neq T_0$. One can apply the inequality (D.3.3) to study of the blowup rate in some nonlinear equations. For instance, consider a smooth solution u for $t < T$ of (D.3.1) for $n = 1$ and

$$a(t,x,u) := a\left(\int_0^t \partial_x u(s,x)ds\right), \quad a(y) \geq a_0 > 0,$$

such that

$$\left|\partial_x^\beta u(t,x)\right| \leq C_\beta (T-t)^{-1}, \quad t < T.$$

If a' is bounded and $|a^{(j)}(y)| \leq A_j e^{\mu|y|}$ for $\mu < 1/C_1$ and $j \geq 2$, then $a(t,x)$ satisfies (D.3.2) with $p \in (\mu C_1, 1)$, and t^{-1}, t^{-p} replaced with $(T-t)^{-1}, (T-t)^{-p}$, respectively. So the inequality (D.3.3) implies $u \in C^\infty$ even for $t = T$. This suggests that $(T-t)^{-1}$ is not a sufficient breakdown rate of the derivatives $\partial_x^\beta u$ to have blow-up of u at $t = T$. This can be compared with the results in [1].

Next, consider the Cauchy problem (D.3.1) where the coefficient a is a function of t alone, i.e., $a := a(t) \in C^2([0,T))$ and satisfies following conditions

$$\left.\begin{array}{r} a_0 \leq a(t) \leq a_1, \\ |a^{(j)}(t)| \leq (C(T-t)^{-1})^{-j}, \quad j=1,2, \end{array}\right\} \tag{D.3.4}$$

for some positive constants a_0, a_1 and C. We have the following result by Hirosawa [39].

Theorem D.3.2. *(Hirosawa [39]) The Cauchy problem (D.3.1) with the coefficient as in (D.3.4) is L^2 well-posed, i.e., there is no loss of regularity at $t = T$.*

We now apply Theorem D.3.2 to estimate the existence time for a nonlinear wave equation of Kirchhoff type given by

$$\begin{cases} (\partial_t^2 - (1 + \|\nabla u(t,\cdot)\|^2)\Delta_x)u(t,x) = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \end{cases} \tag{D.3.5}$$

where $\|\nabla u(t,\cdot)\|^2 = \sum_{j=1}^n \int_{\mathbb{R}^n} \left|\partial_{x_j} u(t,x)\right|^2 \mathrm{d}x$.

Kirchhoff [49] proposed the Kirchhoff equation to study small transversal vibrations of an elastic string with fixed ends which is composed of a non-homogeneous material. The distinct points can have distinct densities and tensions. This equation is deduced in [31, Section 2].

Now we shall prove the local solvability of (D.3.5). One of characteristics of Kirchhoff equation is that the solution satisfies an energy conservation law, that is, the following total energy $E(t)$

$$E(t) := \frac{1}{2}\left(\|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|^4\right)$$

is conserved with respect to t, where $\|u(\cdot)\|$ denotes the usual $L^2(\mathbb{R}^n)$ norm. Namely, the first order derivatives of the solution are uniformly bounded with respect to t in $L^2(\mathbb{R}^n)$. However, the boundedness of the second-order derivatives are not clear. The most essential problem is how one can show it. Let us estimate the usual second order hyperbolic energy $E_1(t)$ for (D.3.5)

$$E_1(t) := \frac{1}{2}\left(\|u_t(t,\cdot)\|_1^2 + a(t)\|\nabla u(t,\cdot)\|_1^2\right),$$

where

$$a(t) = 1 + \|\nabla u(t,\cdot)\|^2,$$

and $\|u(\cdot)\|_1 = \|\nabla u(\cdot)\|$. Using the energy conservation law and Cauchy-Schwarz inequality we have

$$E_1'(t) = \frac{1}{2}a'(t)\|\nabla u(t,\cdot)\|_1^2 \leqslant |a'(t)|\, E_1(t),$$

and

$$|a'(t)| = 2\,|\langle \nabla u(t,\cdot), \nabla u_t(t,\cdot)\rangle| \leqslant 4E(0)^{1/2}E_1(t)^{1/2}, \tag{D.3.6}$$

where $\langle u(\cdot), v(\cdot)\rangle$ denotes the usual $L^2(\mathbb{R}^n)$ inner product. Therefore, we obtain

$$E_1(t) \leqslant \frac{1}{4E(0)\left(T_1-t\right)^2} \quad \text{with } T_1=\frac{1}{2E(0)^{1/2}E_1(0)^{1/2}}. \tag{D.3.7}$$

It follows that $E_1(t)$ is bounded just before $t=T_1$, but $E_1(t)$ is estimated from above only by an unbounded function near $t=T_1$. Therefore, $E_1(t)$ may blow up at $t=T_1$ at the worst. That is all which has been already known about the local solvability under the assumptions $E(0)<\infty$ and $E_1(0)<\infty$.

However, we can show that the time T_1 is not a really critical time by applying Theorem D.3.2. Without loss of generality let $E(0) \geqslant \frac{1}{16}$. From (24) and (25) we have

$$|a'(t)| \leqslant 2\left(T_1-t\right)^{-1}$$

and

$$|a''(t)| = 2\left|-a(t)\|\Delta u(t,\cdot)\|^2+\|\nabla u_t(t,\cdot)\|^2\right| \leqslant 4E_1(t) \leqslant 4\left(T_1-t\right)^{-2}.$$

Therefore, by Theorem D.3.2 with $T=T_1$ from (D.3.7), the solution u of (D.3.5) exists without loss of regularity at $t=T_1$. More precisely, Theorem D.3.2 ensures that there exists a constant $\mathscr{E}_0>1$ which depends only on $E(0)$ such that the a priori estimate $E_1(t) \leqslant \mathscr{E}_0 E_1(0)$ holds for any $t \in [0,T_1)$. It follows from an usual argument that we can prove the existence of a unique solution of (D.3.5) at $t=T_1$ satisfying $E_1\left(T_1\right) \leqslant \mathscr{E}_0 E_1(0)$.

Bibliography

[1] Alinhac, S. (2000) Remarks on the blowup rate of classical solutions to quasilinear multidimensional hyperbolic systems. *Journal d e Mathématiques Pures et Appliquées*, **79**, 839–854.

[2] Ascanelli, A. and Cappiello, M. Hölder continuity in time for SG hyperbolic systems. *Journal of Differential Equations*.

[3] Ascanelli, A. and Cappiello, M. Log-Lipschitz regularity for SG hyperbolic systems. *Journal of Differential Equations*.

[4] Beals, R. (1974) Spatially inhomogeneous pseudodifferential operators, I. *Communications on Pure and Applied Mathematics*, **27**, 1–24.

[5] Beals, R. (1974) Spatially inhomogeneous pseudodifferential operators, II. *Communications on Pure and Applied Mathematics*, **27**, 161–205.

[6] Bernardi, E., Bove, A., and Petkov, V. (2015) Cauchy problem for effectively hyperbolic operators with triple characteristics of variable multiplicity. *Journal of Hyperbolic Differential Equations*, **12**, 535–579.

[7] Carroll, R. and Showalter, R. (1977) *Singular and Degenerate Cauchy Problems*. ISSN, Elsevier Science.

[8] Cicognani, M. (1999) Strictly hyperbolic equations with non regular coefficients with respect to time. *Annali dell' Università di Ferrara*, **45**, 45–58.

[9] Cicognani, M. (2003) The Cauchy problem for strictly hyperbolic operators with non-absolutely continuous coefficients. *Tsukuba J. Math.*, **27**, 1–12.

[10] Cicognani, M. (2004) Coefficients with unbounded derivatives in hyperbolic equations. *Mathematische Nachrichten*, **276**, 31–46.

[11] Cicognani, M. and Colombini, F. (2006) Modulus of continuity of the coefficients and loss of derivatives in the strictly hyperbolic Cauchy problem. *Journal of Differential Equations*, **221**, 143–157.

[12] Cicognani, M. and Lorenz, D. (2018) Strictly hyperbolic equations with coefficients low-regular in time and smooth in space. *J. Pseudo Differ. Oper. Appl.*, **9**, 643–675.

[13] Colombini, F., De Giorgi, E., and Spagnolo, S. (1979) Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, **6**, 511–559.

[14] Colombini, F., Del Santo, D., Fanelli, F., and Métivier, G. (2013) A well-posedness result for hyperbolic operators with Zygmund coefficients. *Journal de Mathématiques Pures et Appliquées*, **100**, 455–475.

[15] Colombini, F., Del Santo, D., and Kinoshita, T. (2002) Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients. *Ann. Scoula. Norm.-Sci.*, **Ser. 5, 1**, 327–358.

[16] Colombini, F., Del Santo, D., and Reissig, M. (2003) On the optimal regularity of coefficients in hyperbolic Cauchy problems. *Bulletin des Sciences Mathématiques*, **127**, 328–347.

[17] Colombini, F. and Lerner, N. (1995) Hyperbolic operators with non-Lipschitz coefficients. *Duke Mathematical Journal*, **77**, 657 – 698.

[18] Colombini, F., Santo, D. D., Fanelli, F., and Métivier, G. (2013) Time-dependent loss of derivatives for hyperbolic operators with non regular coefficients. *Communications in Partial Differential Equations*, **38**, 1791–1817.

[19] Colombini, F., Santo, D. D., Fanelli, F., and Métivier, G. (2015) The well-posedness issue in Sobolev spaces for hyperbolic systems with Zygmund-type coefficients. *Commun. Part. Diff. Eq.*, **40**, 2082–2121.

[20] Comtet, L. (1974) *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. Springer Netherlands.

[21] Cordes, H. O. (1995) *The Technique of Pseudodifferential Operators*. Cambridge Univ. Press.

[22] Coriasco, S. (1998) Fourier integral operators in SG classes II: application to SG hyperbolic Cauchy problems. *Annali dell' Universita di Ferrara*, **44**, 81–122.

[23] Coriasco, S. (1999) Fourier integral operators in SG classes I: composition theorems and action on SG Sobolev spaces. *Rend. Sem. Mat. Univ. Pol. Torino*, **57**, 249–302.

[24] Coriasco, S. and Toft, J. (2014) Asymptotic expansions for Hörmander symbol classes in the calculus of pseudo-differential operators. *Journal of Pseudo-Differential Operators and Applications*, **5**, 27–41.

[25] da Silva, A. (2001) *Lectures on Symplectic Geometry*. No. 1764 in Lecture Notes in Mathematics, Springer.

[26] de Hoop, M., Hörmann, G., and Oberguggenberger, M. (2008) Evolution systems for paraxial wave equations of Schrödinger-type with non-smooth coefficients. *Journal of Differential Equations*, **245**, 1413–1432.

[27] Del Santo, D. and Prizzi, M. (2021) Well-posedness for hyperbolic equations whose coefficients lose regularity at one point. *Monatshefte für Mathematik*, pp. 1–11.

[28] Dencker, N. (2006) The resolution of the Nirenberg-Trèves conjecture. *Annals of mathematics*, pp. 405–444.

[29] Egorov, Y. and Shubin, M. (1993) *Partial Differential Equations IV: Microlocal Analysis and Hyperbolic Equations*. Encyclopaedia of mathematical sciences, Springer-Verlag.

[30] Einstein, A. and de Sitter, W. (1932) On the relation between the expansion and the mean density of the universe. *Proceedings of the National Academy of Sciences*, **18**, 213–214.

[31] Figueiredo, G. M., Morales-Rodrigo, C., Santos Júnior, J. R., and Suárez, A. (2014) Study of a nonlinear Kirchhoff equation with non-homogeneous material. *Journal of Mathematical Analysis and Applications*, **416**, 597–608.

[32] Folland, G. (1995) *Introduction to Partial Differential Equations*. Mathematical Notes, Princeton University Press.

[33] Galstian, A. and Yagdjian, K. (2014) Microlocal Analysis for Waves Propagating in Einstein & de Sitter Spacetime. *Mathematical Physics, Analysis and Geometry*, **17**, 223–246.

[34] Gel'fand, I. and Shilov, G. (2016) *Generalized Functions, Volume 2: Spaces of Fundamental and Generalized Functions*. AMS Chelsea Publishing, American Mathematical Society.

[35] Ghisi, M. and Gobbino, M. (2021) Finite vs infinite derivative loss for abstract wave equations with singular time-dependent propagation speed. *Bulletin des Sciences Mathématiques*, **166**, 102918.

[36] Gårding, L. (1957) *Hyperbolic differential equations*. University Chicago.

[37] Grigis, A., Sjöstrand, J., Cassels, J., Society, L. M., and Hitchin, N. (1994) *Microlocal Analysis for Differential Operators: An Introduction*. Lecture note series / London mathematical society, Cambridge University Press.

[38] Hadamard, J. (1932) *Le probléme de Cauchy et les équations aux dérivées partielles linéaires hyperboliques: lecons professées á l'Université Yale*. Mrs. Hepsa Ely Silliman memorial lectures, Hermann et cie.

[39] Hirosawa, F. (2003) Loss of regularity for the solutions to hyperbolic equations with non-regular coefficients-an application to Kirchhoff equation. *Mathematical Methods in the Applied Sciences*, **26**, 783–799.

[40] Hörmander, L. (1963) *Linear Partial Differential Operators*. Springer-Verlag, Berlin.

[41] Hörmander, L. (1983) *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag.

[42] Hörmander, L. (1983) *The Analysis of Linear Partial Differential Operators II*. Springer-Verlag.

[43] Hörmander, L. (1983) *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag.

[44] Hörmander, L. (1983) *The Analysis of Linear Partial Differential Operators IV*. Springer-Verlag.

[45] Hörmann, G. and de Hoop, M. (2001) Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients. *Acta Applicandae Mathematicae*, **67**, 173–224.

[46] Jannelli, E. (1985) Regularly hyperbolic systems and Gevrey classes. *Annali di Matematica Pura ed Applicata*, **140**, 133–145.

[47] Kajitani, K. (1983) Cauchy problem for nonstrictly hyperbolic systems in Gevrey classes. *Journal of Mathematics of Kyoto University*, **23**, 599–616.

[48] Kajitani, K. and Nishitani, T. (2006) *The Hyperbolic Cauchy Problem*. Lecture Notes in Mathematics, Springer Berlin Heidelberg.

[49] Kirchhoff, G. (1897) *Vorlesungen über mechanik*. Leipzig, B. G. Teubner.

[50] Kubo, A. and Reissig, M. (2003) Construction of parametrix to strictly hyperbolic Cauchy problems with fast oscillations in nonLipschitz coefficients. *Commun. Part. Diff. Eq.*, **28**, 1471–1502.

[51] Kumano-go, H., Vaillancourt, R., and Nagase, M. (1982) *Pseudo-differential Operators*. MIT Press.

[52] Lax, P. D. (1957) Asymptotic solutions of oscillatory initial value problems. *Duke Mathematical Journal*, **24**, 627 – 646.

[53] Leray, J. (1953) *Hyperbolic differential equations*. Inst. Adv. Study, Princeton.

[54] Lerner, N. (2010) *Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators*. Birkhäuser Basel.

[55] Manwell, A. (1979) *The Tricomi Equation with Applications to the Theory of Plane Transonic Flow*. Research notes in mathematics, Pitman Advanced Pub. Program.

[56] Métivier, G. (2008) *Para-Differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems*. Publications of the Scuola Normale Superiore, Scuola Normale Superiore.

[57] Mizohata, S. (1961) Some remarks on the Cauchy problem. *Journal of Mathematics of Kyoto University*, **1**, 109 – 127.

[58] Nicola, F. and Rodino, L. (2011) *Global Pseudo-differential Calculus on Euclidean Spaces*. Birkhäuser Basel.

[59] Nirenberg, L. (1973) *Lectures on Linear Partial Differential Equations*. Regional conference series in mathematics, Conference Board of the Mathematical Sciences.

[60] Nishitani, T. (1983) Sur les équations hyperboliquesá coefficients höldériens en t et de classe de gevrey en x. *Bull. Sci. Math.*, **107**, 113–138.

[61] Nishitani, T. (2019) *Cauchy Problem for Differential Operators with Double Characteristics*. Springer.

[62] Pattar, R. R. and Kiran, N. U. (2021) Global well-posedness of a class of strictly hyperbolic Cauchy problems with coefficients non-absolutely continuous in time. *Bulletin des Sciences Mathématiques*, **171**, 103037.

[63] Pattar, R. R. and Kiran, N. U. (2021) Strictly hyperbolic Cauchy problems on $\mathbb{R}^n$ with unbounded and singular coeicients. *Annali dell Universitá di Ferrara*.

[64] Pattar, R. R. and Kiran, N. U. (2021) Strictly hyperbolic equations with coefficients sublogarithmic in time. *arXiv:2111.11701*.

[65] Pattar, R. R. and Kiran, N. U. (2022) Energy estimates and global well-posedness for a broad class of strictly hyperbolic Cauchy problems with coefficients singular in time. *Journal of Pseudo-Differential Operators and Applications*, **13**, 9.

[66] Pattar, R. R. and Kiran, N. U. (2022) Global well-posedness of a class of singular hyperbolic Cauchy problems. *Monatshefte für Mathematik*.

[67] Petrovskij, I. G. (1938) On the Cauchy problem for systems of linear partial differential equations. *Bull. Univ. Mosk. Ser. Int. Mat. Mekh.*, **1**, 1 – 74.

[68] Rauch, J. (2012) *Hyperbolic Partial Differential Equations and Geometric Optics*. Graduate studies in mathematics, American Mathematical Society.

[69] Reissig, M. (2003) Hyperbolic equations with non-Lipschitz coefficients. *Rend. Sem. Mat. Univ. Politec. Torino*, **61**, 135–181.

[70] Schulze, B. (1998) *Boundary Value Problems and Singular Pseudo-Differential Operators*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts, Wiley.

[71] Sultana, J. and Dyer, C. C. (2005) Cosmological black holes: A black hole in the Einstein-de Sitter universe. *General Relativity and Gravitation*, **37**, 1347–1370.

[72] Taylor, M. (2017) *Pseudodifferential Operators (PMS-34)*. Princeton Mathematical Series, Princeton University Press.

[73] Tréves, F. (1980) *Introduction to Pseudodifferential and Fourier Integral Operators Volume 2: Fourier Integral Operators*. Introduction to Pseudodifferential and Fourier Integral Operators, Plenum Press.

[74] Tréves, J. (2013) *Introduction to Pseudodifferential and Fourier Integral Operators: Pseudodifferential Operators*. University Series in Mathematics, Springer US.

[75] Uday Kiran, N., Coriasco, S., and Battisti, U. (2016) Hyperbolic operators with non-Lipschitz coefficients. *Recent Advances in Theoretical & Computational Partial Differential Equations with Applications*, Panjab University, Chandigarh.

[76] Yagdjian, K. (1997) *The Cauchy Problem for Hyperbolic Operators: Multiple Characteristics, Micro-local Approach*. Mathematical Topics, Vol 12, John Wiley & Son Limited.

[77] Yamazaki, T. (1990) On the $L^2(\mathbb{R}^n)$ well-posedness of some singular or degenerate partial differential equations of hyperbolic type. *Communications in Partial Differential Equations*, **15**, 1029–1078.