Maxima & Minima

Sunday, September 19, 2021 10:10 AM

What's the difference between maxima (plural) & maximum (singular)
b/w minima & minimum

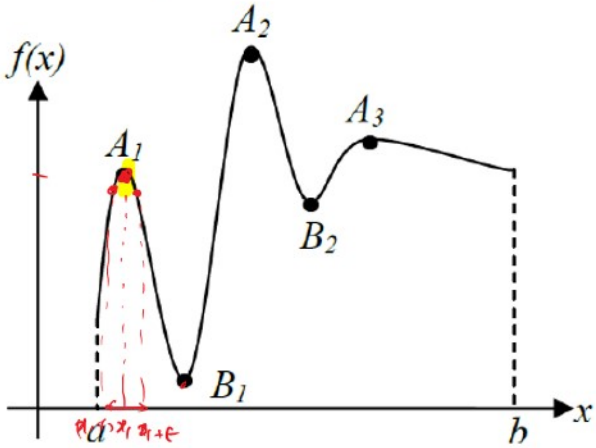
- local / Relative
- Global

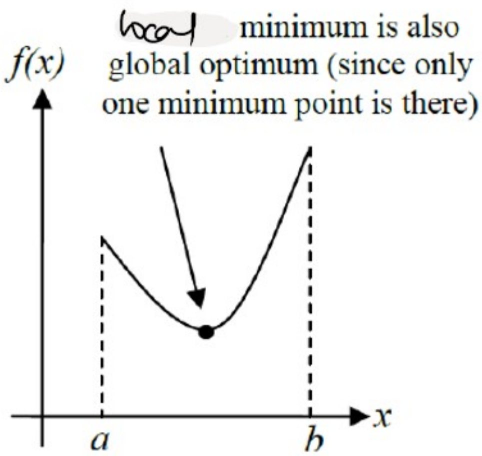
A_1, A_2, A_3 = local maxima

A_2 = Global maximum

B_1, B_2 = local minima

B_1 = Global minimum



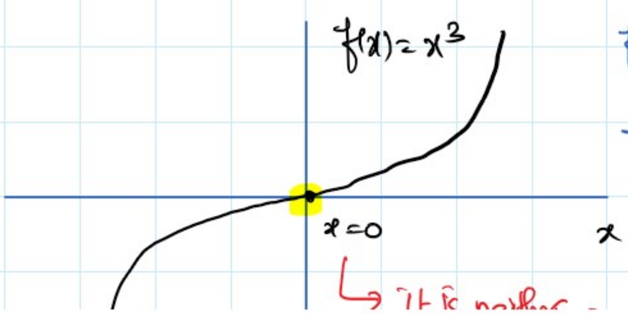


Note: A global maximum/minimum is also a local maximum/minimum respectively but vice-versa is not TRUE.

Critical Point:

A critical point of a function $f(x)$ is such that either $f'(c)=0$ or $f'(c)$ doesn't exist in the domain of $f(x)$

Note #① If a function $f(x)$ has a point 'c' where $f'(c)=0$, it doesn't imply that function has a local maximum/minimum at point $x=c$.

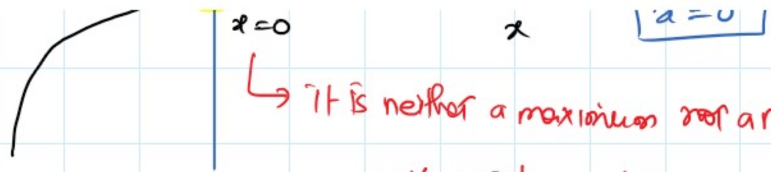


$f(x) = x^3$

$f'(x) = 3x^2 = 0$

$\boxed{x = 0}$

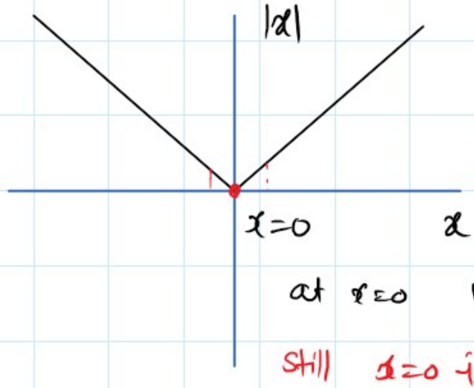
↳ It is neither



↳ it is neither a maximum nor a minimum
rather it's a point of inflection

Note # ② A function may have local maximum or minimum at a point where derivative doesn't exist.

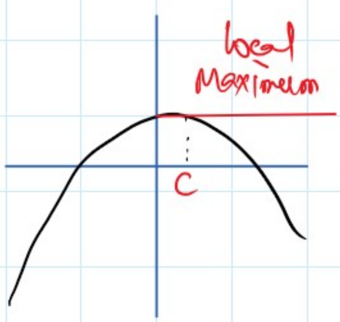
GATE

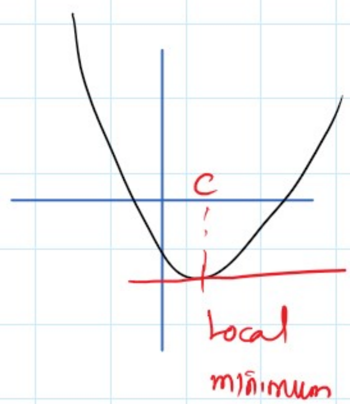


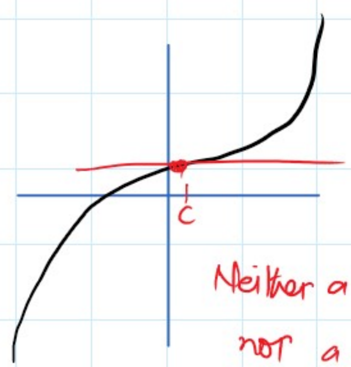
at $x=0$ $|x|$ is non-differentiable.
Still $x=0$ is a local minimum

Examples of critical point:

① $f'(c)=0$

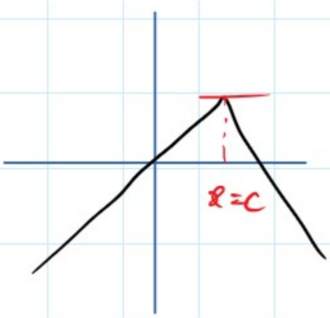




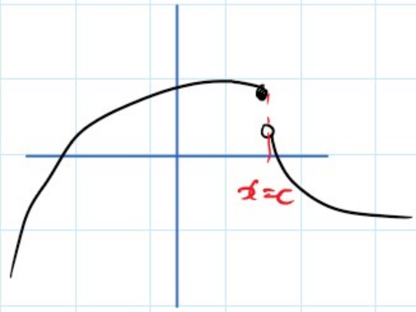


Neither a maximum nor a minimum

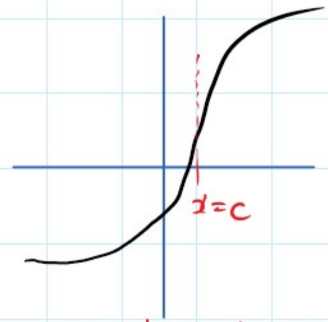
② $f'(c)$ doesn't exist



sharp error



Discontinuous



vertical tangent (slope is ∞)

Finding the maximum and minimum of a continuous function on a closed interval $[a, b]$.

To find the maximum and minimum values of a continuous function f on a closed interval $[a, b]$;

(absolute)

1. Find all of the critical points of f in the interval $[a, b]$.
2. Evaluate f at all of the critical numbers in the interval $[a, b]$.
3. Evaluate f at the endpoints of the interval, (calculate $f(a)$ and $f(b)$.) (boundary points)
4. The largest of the values from steps 2 and 3 is the absolute maximum of the function on the interval $[a, b]$ and the smallest of the values from steps 2 and 3 is the absolute minimum (global) of the function on the interval $[a, b]$.

Q# 24 The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ is ______.

(A) 21 (B) 25 (C) 41 (D) 46 [GATE 13]

Sol

step #1 $f'(x) = 0$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow x^2 - 4x - 2x + 8 = 0$

$\Rightarrow x(x-4) - 2(x-4) = 0$

$\Rightarrow x = 2,\ x = 4$

step #2 f(x) value at critical points

$f(2) = 25$

$f(4) = 21$

step #3 f(x) value at boundary points

$f(1) = 21$

$f(6) = 41$

Let's draw the graph:

$f(x) = x^3 - 9x^2 + 24x + 5$ $\Rightarrow$ no. of roots $= 3$

2

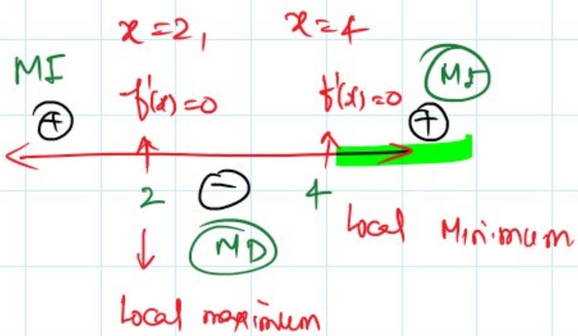
→ all 3 roots are real.
⇓
3 cuts on the x-axis

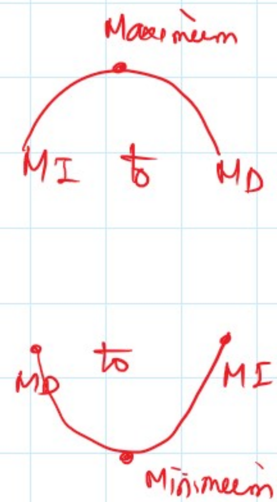
→ one real root and a pair of complex conjugate roots
⇓
only 1 cut on the x-axis.

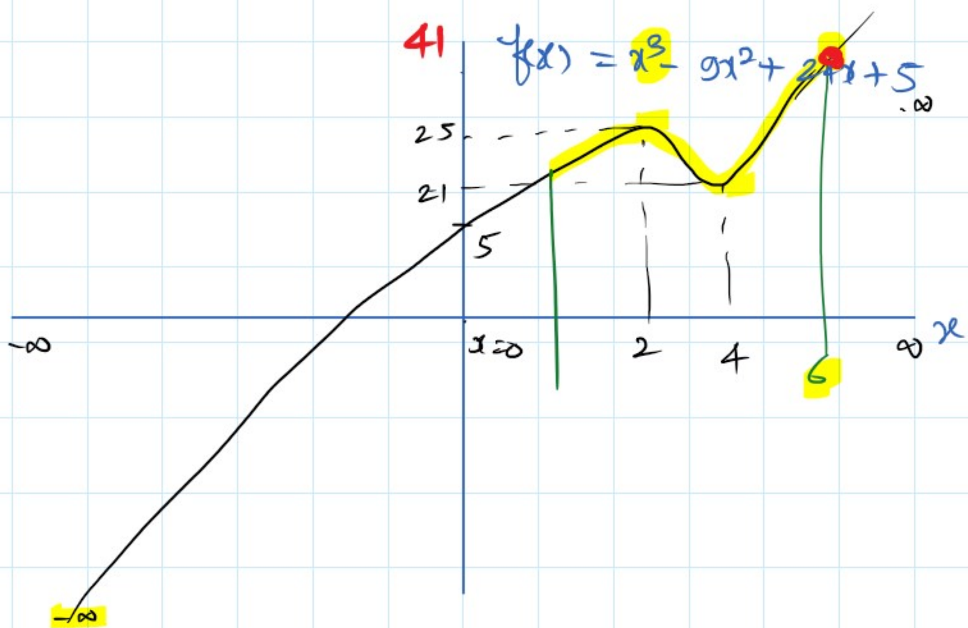
1
T#1 $f(0) = ?$ 25

T#2 $\lim_{x \to \infty} f(x) = ?$ ∞ T#3 $\lim_{x \to -\infty} f(x) = ?$ $-\infty$

3 Find the interval where $f(x)$ is MI and MD.
Also find the critical points.

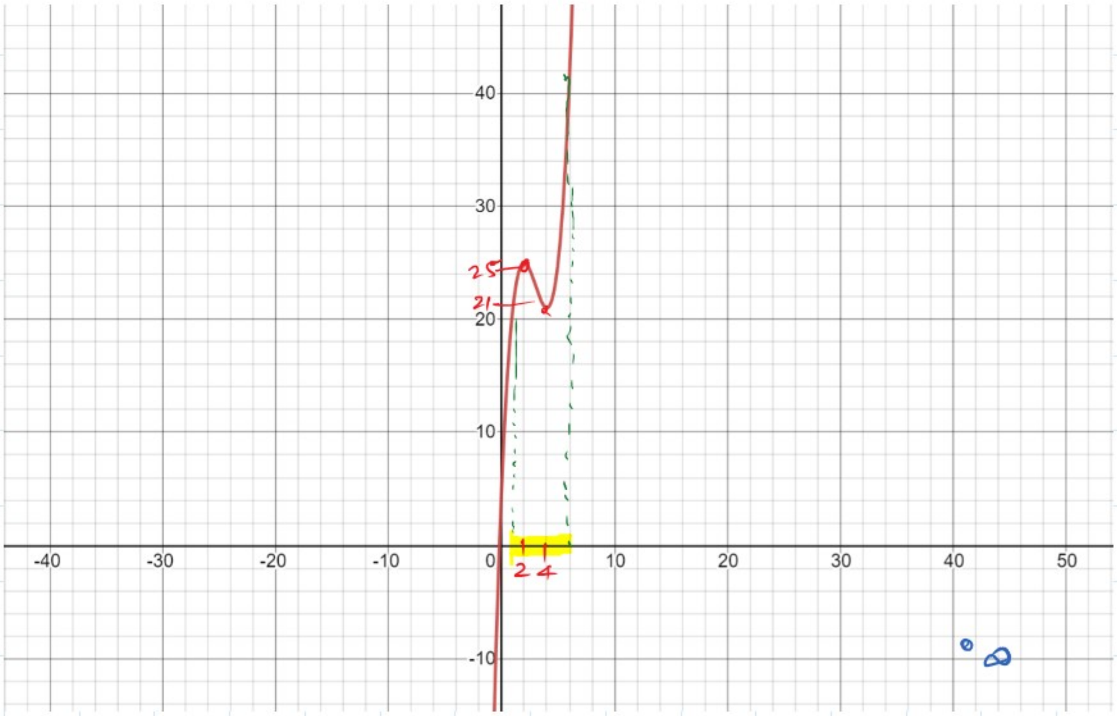






$f(2) = 25$

$f(4) = 21$



Second derivative Test:

① $f(c)$ is a minimum value of function $f(x)$ if $f'(c)=0$ and $f''(c)>0$

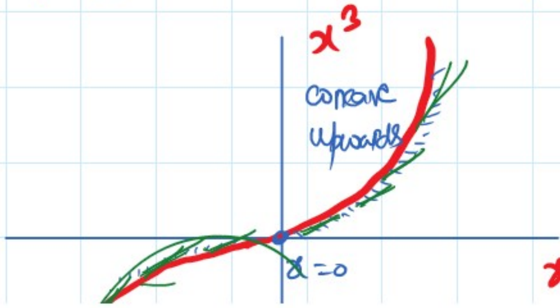
② $f(c)$ is a maximum value of function $f(x)$ if $f'(c)=0$ and $f''(c)<0$

③ Both $f'(c)=0$ and $f''(c)=0$ then second derivative test fails and we need to investigate further.

Concept of concavity:

① If $f''(x)>0$ it implies $f(x)$ is concave upwards

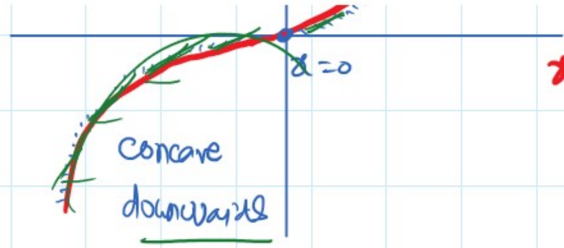
② If $f''(x)<0$ it implies $f(x)$ is concave downwards.



$f(x)=x^3$

$f'(x)=3x^2=0$ $\quad x=0$

$\boxed{f''(x)=6x}$



x.

$$\boxed{f''(x) = 6x}$$

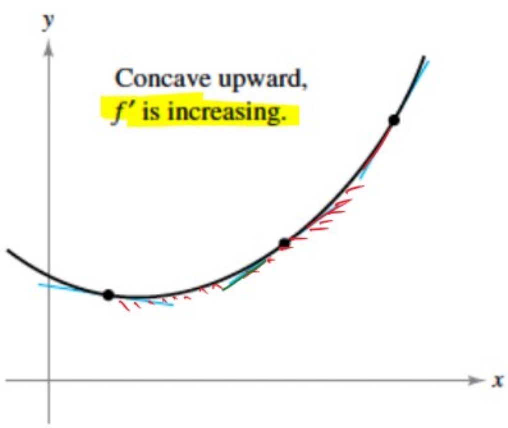
$f'(0)=0$ and $f''(0)=0$
and

For any point in $x>0$ region,

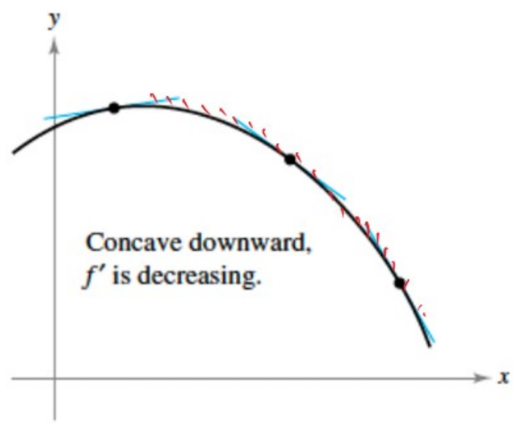
$$\boxed{f''(x) > 0} \Rightarrow \text{CU}$$

lly for any point in $x<0$ region.

$$\boxed{f''(x) < 0} \Rightarrow \text{CD}$$



(a) The graph of f lies above its tangent lines.



(b) The graph of f lies below its tangent lines.

Figure

Concave up

Concave down

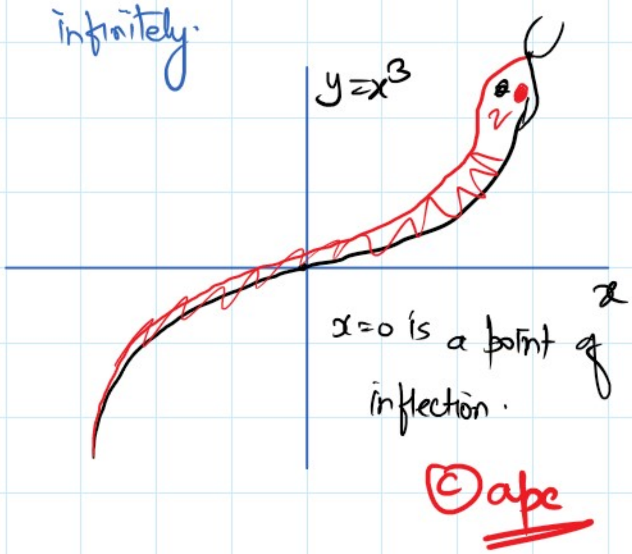
Point of Inflection (POI)

If $f''(x)=0$ or undefined but $f(x)$ is defined at point $x=c$ then that point can be a point of inflection iff:

① $f''(x)$ changes it's sign across the point $x=c$

and

② Tangent exists at point $x=c$ which means $f'(x)$ should exist finitely or infinitely.



$f''(x)=6x$

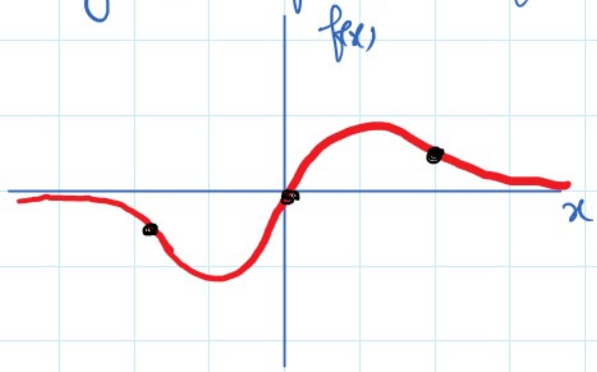
$f'(0)=0 \quad f''(0)=0$

also $f''(x)$ changes it's sign across $x=0$.

for $x>0 \Rightarrow f''(x)>0$ CU

For $x<0 \Rightarrow f''(x)<0$ CD.

Q# 25 How many points of inflection for the given $f(x)$ graph:



Ⓐ 1

Ⓑ 2

Ⓒ 3

Ⓓ 0

Sol:

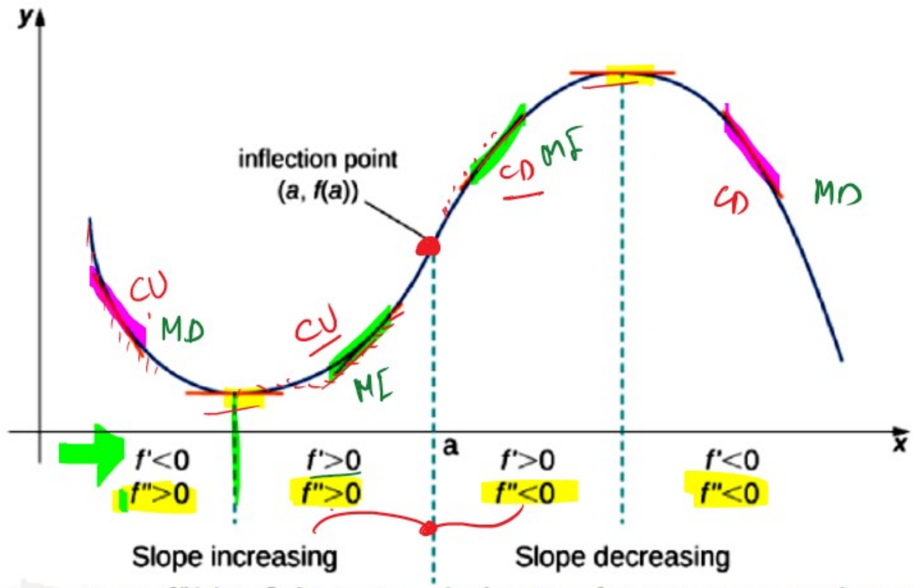


Figure Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.