Lorenz Equations

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Abstract

In this report, we analyze the Lorenz Equations. Lorenz equations are used to model the convection currents in a fluid where there is a change in the vertical and horizontal temperature. Using Lorenz equations, we model the Chua's electronic circuit and extend our understanding to its chaotic nature.

1 Introduction

The Lorenz equations are a system of ordinary differential equations first studied by Edward Lorenz. It is known for having chaotic solutions for certain parameter values and initial conditions. The equations relate the properties of a two-dimensional fluid layer uniformly warmed from below and cooled from above. In particular, the equations describe the rate of change of three quantities with respect to time: x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation. These equations have a lot of applications in the area of weather modelling, circuit modelling, traffic modelling, chemical reactions and many more.

The report is organised in the following way, section 2 describes the preliminary mathematical definitions required to understand the theoretical analysis down forward, section 3 states the Lorenz equations and the variable definitions, section 4 analysis the Lorenz equations, it has various subsections on critical points, phase analysis, bifurcations and confined space, section 5 explains chaos theory and its properties and introduces attractor, section 6 explains Lorenz attractor in detail, section 7 has the derivation of Lorenz equation from a fluid convection model, section 8 explains in detail Chua's circuit equations and random number generator using Chua's equations, section 9 contains the link to the GitHub repository which has the code for our experiments and section 10 concludes our report.

2 Preliminaries

Before delving into the dynamics of the wonderful and mind-bending Lorenz equations, cover some basics of differential equations whose tools we will be using in later sections to analyze the Lorenz equations.

For a differential equation $\frac{dx}{dt} = f(x)$, the point where $\frac{dx}{dt} = 0$ is called a **critical point**. These points can be stable , unstable, semi-stable, and more depending on the dynamics of the system.

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2.1 Bifurcations

Bifurcations are the change in dynamics of a system on the variation of a parameter. In simple terms, how do the number of critical points and their stability vary with a change in the parameter.

2.1.1 Pitchfork bifurcation

Let $\frac{dx}{dt} = ax - x^3$ be a non-linear differential equation. As we vary the parameter a this equation displays different characteristics. $\frac{dx}{dt} = x(a - x^2)$ has three solutions from which one is always $x = 0$. For $a < 0$, $x = \pm\sqrt[2]{|a|}i$ and does not exist on the real line. For $a > 0$, we have 2 new solutions $x = \pm\sqrt[2]{a}$. By varying the parameter a we changed the dynamics of a system from one critical point to three critical points. Note, as $a > 0$, the critical point $x = 0$ becomes unstable and the critical point $x = \pm\sqrt[2]{a}$ are stable.

3 Lorenz Equations

Lorenz equations are coupled non-linear autonomous differential equations that are used to model fluid convection. These differential equations define the time derivatives of three quantities x, y, and z. Here, x, y, and z represent the rate of convection, variation in horizontal temperature, and variation in vertical temperature respectively. The Lorenz equations are defined as follows:

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$
$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$
$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

σ, ρ, and β are constants that represent the Prandtl number, Rayleigh number, and a parameter respectively. Figure 1 displays a plot of Lorenz's equation.

4 Analysis

4.1 Symmetric along x, y direction

Lorenz equations are symmetric along x and y. In the most simplest sense, substituting $-x$ in the place of x and $-y$ in place of y gives the same equations.

$$\frac{d(-x)}{dt} = \sigma((-y) - (-x)) \Rightarrow \frac{d(x)}{dt} = \sigma((y) - (x)) \tag{4}$$
$$\frac{d(-y)}{dt} = (-x)(\rho - z) - (-y) \Rightarrow \frac{d(y)}{dt} = (x)(\rho - z) - (y) \tag{5}$$
$$\frac{d(z)}{dt} = (-x)(-y) - \beta z \Rightarrow \frac{d(z)}{dt} = (x)(y) - \beta z \tag{6}$$

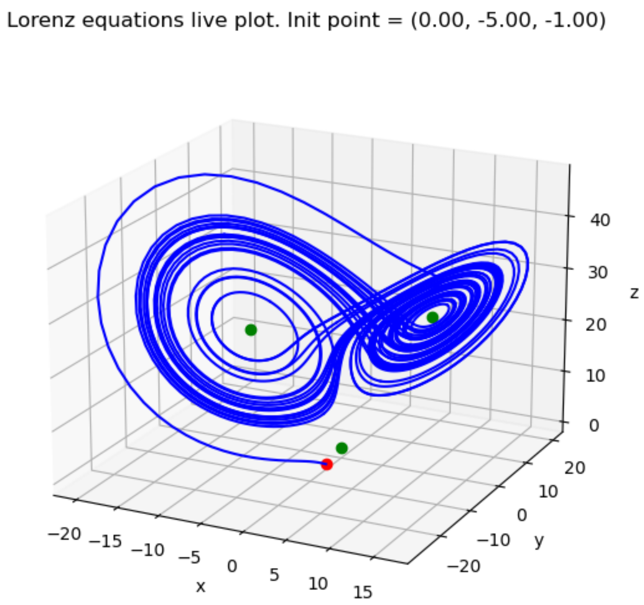


Figure 1: Lorenz equation numerically plotted with initial point = $(0, -5, -1)$ at $\sigma = 10, \rho = 25, \beta = \frac{8}{3}$. The green points display the critical points of the Lorenz equations and the red point is the initial value.

4.2 Critical Points

The Critical points of the differential equation occurs when its time derivative is 0.

$$\frac{dx}{dt} = \sigma(y - x) = 0 \tag{7}$$
$$\frac{dy}{dt} = x(\rho - z) - y = 0 \tag{8}$$
$$\frac{dz}{dt} = xy - \beta z = 0 \tag{9}$$

By inspection, we know that $x = y = z = 0$ is a critical point of the Lorenz equation.

From equation 7, we know that $x = y$. Substituting $y = x$ in equation 8, 9, we get the following:

$$x(\rho - 1 - z) = 0 \tag{10}$$
$$x^2 - \beta z = 0 \tag{11}$$

Equation 10 has two cases which will satisfy its equation. In the first case, $x = 0$ but this will give back the same solution $x = y = z = 0$. In the second case, $z = \rho - 1$ is a solution to equation 11. This gives us the following:

$$x^2 - \beta(\rho - 1) = 0 \tag{12}$$
$$x = \pm\sqrt[2]{\beta(\rho - 1)} \tag{13}$$

For Lorenz equations, we have $(0,0,0)$, $(\sqrt[2]{\beta(\rho-1)}, \sqrt[2]{\beta(\rho-1)}, (\rho - 1))$, and $(-\sqrt[2]{\beta(\rho-1)}, -\sqrt[2]{\beta(\rho-1)}, (\rho-1))$ as the three critical points.

4.3 Phase Analysis at Critical Points

To perform the phase plane analysis of the Lorenz equations at the critical points, we employ the 1^{st} order Taylor series expansion. Let $\mathbf{x}^*$ be the critical point, we linearize the Lorenz equation.

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{bmatrix}^T \tag{14}$$
$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) \tag{15}$$
$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla_{\mathbf{x}^*} f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = 0 + \nabla_{\mathbf{x}^*} f(\mathbf{x}^*)\Delta\mathbf{x} \tag{16}$$
$$\nabla_{\mathbf{x}^*} f(\mathbf{x}^*) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z^* & -1 & -x^* \\ y^* & x^* & -\beta \end{bmatrix} \tag{17}$$

By finding the eigenvalues and eigenvectors of $\nabla_{\mathbf{x}^*} f(\mathbf{x}^*)$ we can obtain the stability and the vector field of that point.

4.3.1 Critical point 1

We first delve into the phase analysis of critical point 1. For critical point 1 ($\mathbf{x}_1^*$), $x^* = y^* = z^* = 0$. We obtain the eigenvalues and their corresponding eigenvectors for $\nabla_{\mathbf{x}_1^*} f(\mathbf{x}^*)$.

$$\nabla_{\mathbf{x}_1^*} f(\mathbf{x}^*) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \tag{18}$$
$$(-\beta - \lambda)((-\rho - \lambda)(-1 - \lambda) - \rho\sigma) = (-\beta - \lambda)(\lambda^2 + (\rho + 1)\lambda + (1 - \sigma)\rho) = 0 \tag{19}$$
$$\lambda_1 = -\beta \tag{20}$$
$$\lambda_{2,3} = \frac{-(\rho + 1) \pm \sqrt[2]{(\rho + 1)^2 - 4 * (1 - \sigma)\rho}}{2} \tag{21}$$
$$\lambda_1 = 10.63, \lambda_2 = -\frac{8}{3}, \lambda_3 = -21.63 \tag{22}$$

Here, equation 22 uses the values $\beta = \frac{8}{3}$, $\rho = 24$, and $\sigma = 10$. Two out of the three eigenvalues are real-negative numbers. The vector fields proceed towards this point along two of the directions and are pushed away across one direction.

4.3.2 Critical points 2 and 3

For the critical points $\mathbf{x}_2^*$, $\mathbf{x}_3^*$ of the Lorenz equation, we obtain the matrix $\nabla_{\mathbf{x}_2^*} f(\mathbf{x}^*)$ and $\nabla_{\mathbf{x}_3^*} f(\mathbf{x}^*)$. Computing the eigenvalues for $\nabla_{\mathbf{x}_2^*} f(\mathbf{x}^*)$ as follows:

$$\nabla_{\mathbf{x}_2^*} f(\mathbf{x}^*) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - 23 & -1 & -7.83 \\ 7.83 & 7.83 & -\beta \end{bmatrix} \tag{23}$$

$$det\left(\begin{bmatrix} -10-\lambda & 10 & 0 \\ 1 & -1-\lambda & -7.83 \\ 7.83 & 7.83 & -\frac{8}{3}-\lambda \end{bmatrix}\right) = 0 \tag{24}$$

$$\lambda_1 = -13.62, \lambda_2 = -0.023 + 9.49j, \lambda_3 = -0.023 - 9.49j \tag{25}$$

Note, we have two complex eigen values and one real negative eigen value. Along one dimension the critical point is attracting and along the other two dimensions it is a stable spiral. Similar analysis can be done for $\nabla_{\mathbf{x}_3^*} f(\mathbf{x}^*)$.

4.4 Bifurcations

We know that Lorenz equations are symmetric along the x and y direction, does this cue hint towards something? Bifurcations!!! Majorly Pitchfork bifurcation. We cover two bifurcations witnessed in Lorenz's equations.

4.4.1 Pitchfork Bifurcation

Lorenz equations display a Pitchfork bifurcation at $\rho = 1$. At $\rho = 1$, all the critical points are overlapping at the origin. Note that as $\rho < 1$, critical points $\mathbf{x}_2^x$ and $\mathbf{x}_3^x$ are complex and do not exist (section 4.3.2). As the value of $\rho > 1$, we see 2 more critical points emerge. Graphically this looks like a pitchfork and is termed as *Pitchfork* bifurcation. This can be visualized in Figure 2 and 3.

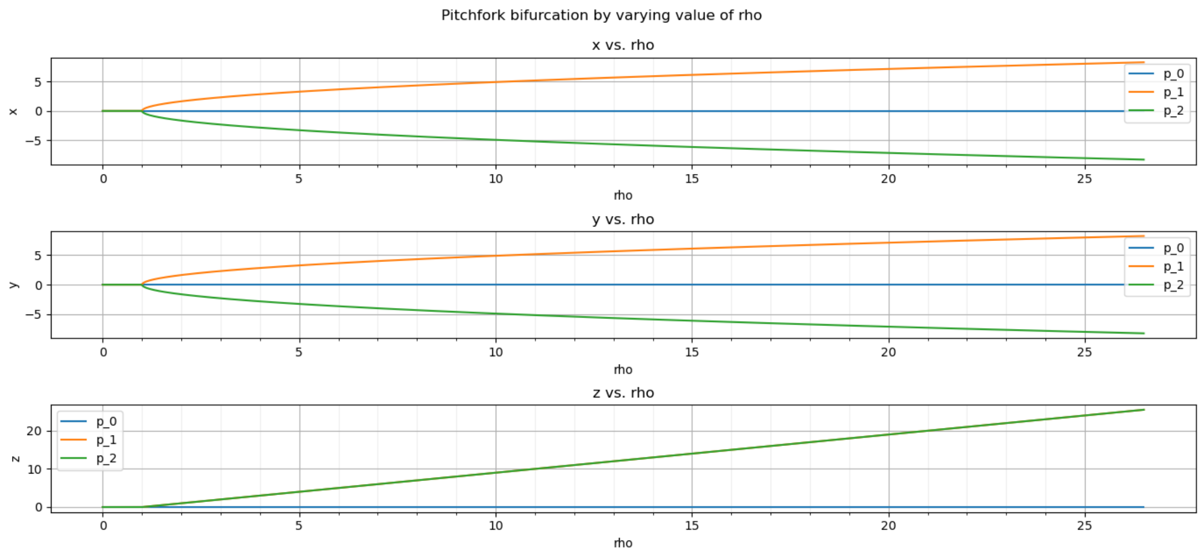


Figure 2: Pitchfork bifurcation at $\rho = 1$. Note that x and y display Pitchfork bifurcation (symmetric). Initial value = $(0, -5, -1)$, $\sigma = 10$, $\beta = \frac{8}{3}$

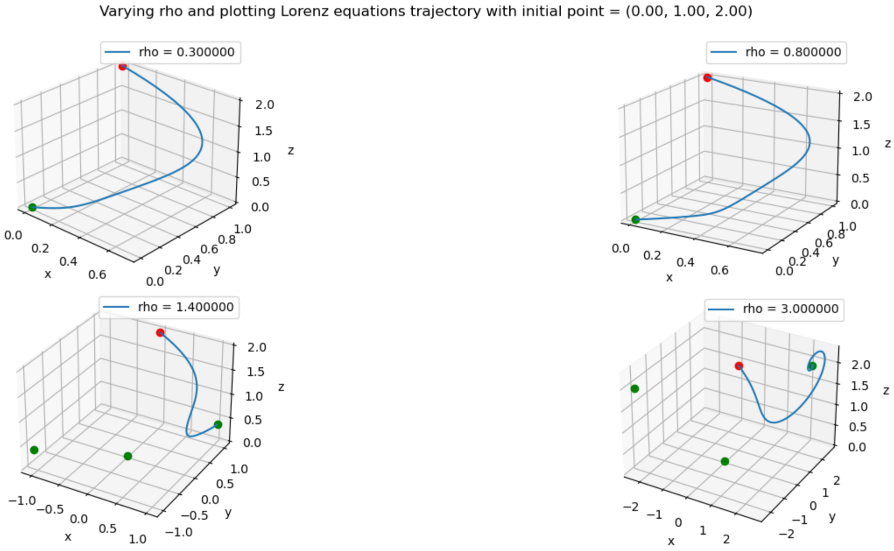


Figure 3: Pitchfork bifurcation at $\rho = 1$. We witness this in the Lorenz trajectory. From one stable critical point when $\rho < 1$, we get two stable critical points. Critical points are marked with green and initial point is marked with red.

4.4.2 Hopf Bifurcation

Hopf bifurcation characterizes the change in stability as ρ varies. As ρ increases the critical points $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ form a spiral where all the solutions converge to circular solutions. This occurs when the eigenvalues are entirely complex. This can be visualized in figure 4 and 5. From Figure 5, we see that the bifurcation occurs at $\rho_h \approx 24.74$. For all the plots below, we use $\beta = \frac{8}{3}$ and $\sigma = 10$

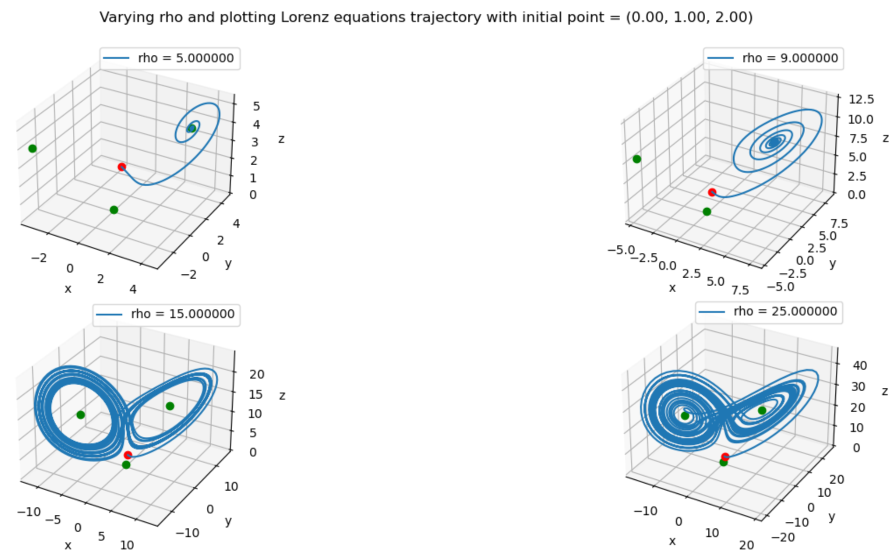


Figure 4: Hopf bifurcation. The spiral from the two critical points turn into circular spiral where the trajectories start to converge. Critical points are marked with green and initial point is marked with red.

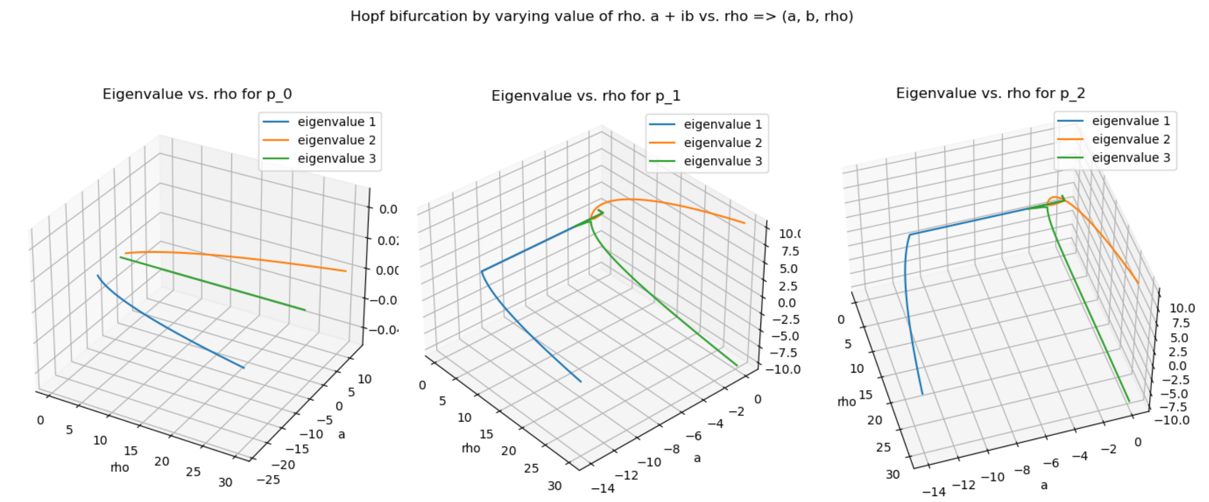


Figure 5: Lorenz eigenvalues plotted vs. ρ. a represents real part and b represents the imaginary part. In the the second and third graph $a \approx 0$ when $\rho \approx 24.74$.

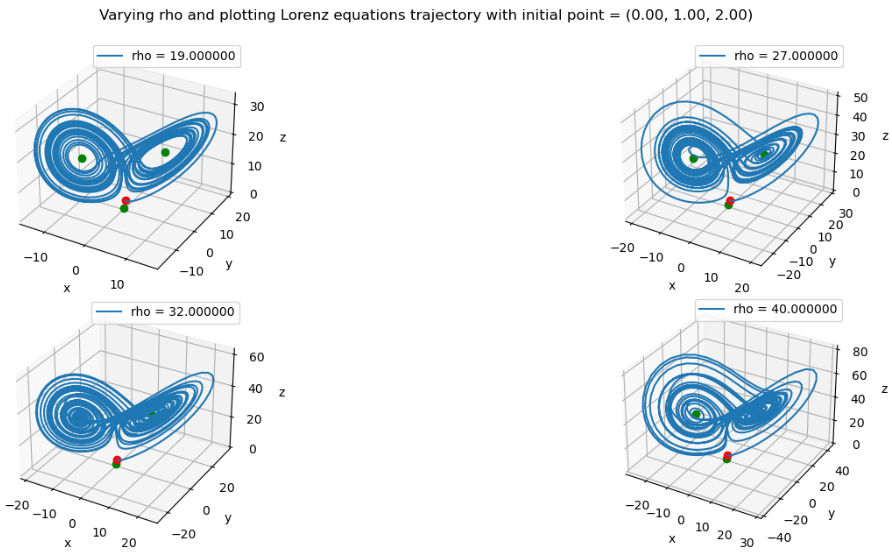


Figure 6: Varying ρ and plotting trajectories.

5 Chaos Theory

Chaos theory deals with the deterministic systems whose behaviour can be predicted in theory. Small differences in initial conditions, like errors in measurements or rounding errors in numerical computation, can produce widely diverging outcomes for such dynamical systems, which makes long term prediction of their behaviour impossible. This can happen even though these systems are deterministic. The deterministic nature of these systems does not make them predictable. This behavior is known as deterministic chaos, or simply chaos. Edward Lorenz summarized chaos as,

"When the present determines the future, but the approximate present does not approximately determine the future".[12]

The word "Chaos" means "a state of disorder". Though no definition of the term Chaos is universally accepted, it can be formulated as, aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions. In order to understand this definition, we need to understand some characteristics of chaotic system.

5.1 Characteristics of Chaotic System

Chaotic systems have a many of distinctive characteristics which are used explain how they evolve dynamically. These characteristics include the following.

5.1.1 Aperiodic long-term behaviour

Aperiodic long-term behaviour suggests that there are trajectories which do not settle down to the fixed point, periodic or quasi-periodic orbits at $t \to \infty$.

5.1.2 Determinism

Chaotic systems are strictly deterministic i.e. the system has no random or noisy inputs or parameters. Irregular behaviour emerge only from the non-linearity of the system.

5.1.3 Sensitive dependence on initial conditions

Sensitivity to initial conditions is known as the "butterfly effect". It is explained metaphorically as, "a butterfly flapping its wings in Texas can cause a hurricane in China". The flapping wing indicates a small change in the initial condition of the system, which causes a chain of events that makes large-scale phenomena unpredictable. Because of the sensitivity to initial conditions, if we start with a limited amount of knowledge about the system (which is normally the case in practice), the system will no longer be predictable after a certain period of time. This is most prevalent in the case of weather. It's generally predictable only about a week ahead.

Mathematically, the Lyapunov exponent measures the sensitivity to initial conditions in the form of rate of exponential divergence from the perturbed initial conditions. Given two starting trajectories in the phase space that are infinitesimally close, with initial separation $\delta\mathbf{Z}_0$ the two trajectories end up diverging at a rate given by

$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t}|\delta\mathbf{Z}_0|, \tag{26}$$

where t is the time and λ is the Lyapunov exponent. Usually maximal Lyapunov exponent (MLE) is used as it determines the overall predictability of the system. A positive MLE implies that the system is chaotic.

We tried to estimate Lyapunov exponent of the Lorenz system. We solved for two trajectories with initial separation 10^{-9}. The distance between two trajectories is plotted on the log scale in Figure 7. The log of distance between trajectories can be well approximated by a straight line with positive slope which implies that the MLE is positive. Hence, Lorenz system is chaotic.

6 Lorenz Attractor

Attractor defines the equilibrium level of a system. It is a set of numerical values toward which a system tends to evolve, for a broad range of initial conditions of the system. System values that get close enough to the attractor values remain close even if initial conditions are slightly disturbed.

Being a chaotic system, Lorenz equations are sensitive to initial condition. A small variation amplifies exponentially with time. Conversely, there is also order in the system. Numerical

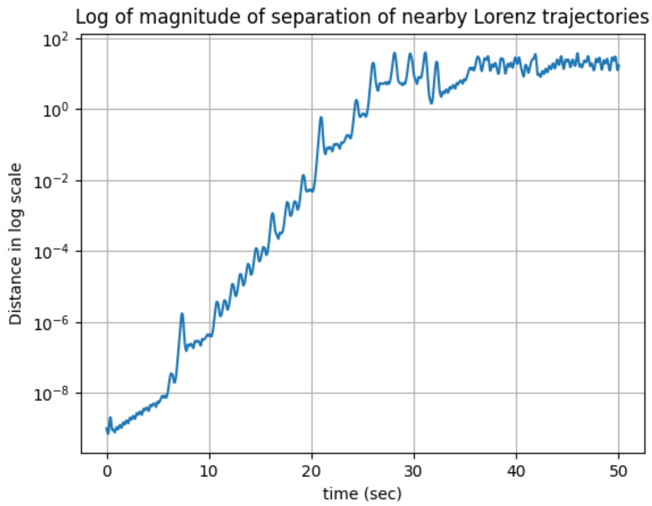


Figure 7: Logarithmic plot for separation of two trajectories with initial separation 10^{-9}, initial point = $(1, 3, 5)$ and $(1, 3, 5 + 10^{-9})$, $\sigma = 10, \rho = 25, \beta = 25$.

solution of the equation when plotted in three dimensions, appears to settle onto a thin set that looks like pair of butterfly wings. This attractor is known as Lorenz attractor or Strange attractor.

Let $\Lambda = (\rho, \alpha, \beta)$ be the space of parameters, for $\rho, \alpha, \beta > 0$. The flow of vector field F,

$$\nabla . < \dot{x}, \dot{y}, \dot{z} >= \nabla . F = -(\alpha + 1 + \beta) < 0, \tag{27}$$

Let $S(t)$ be the closed surface in the phase space with volume $V(t)$.

$$\dot{V}(t) = \int_{S(t)} F. ds = \int_{V(t)} \nabla . F \, dV = -(\sigma + 1 + \beta) V(t) \tag{28}$$

so,

$$V(t) = \exp(-(\sigma + 1 + \beta)t) V(0) \tag{29}$$

This implies that the volumes in phase space shrink exponentially fast to a set of measure zero i.e. given a set of initial conditions with positive measure, all the trajectories will converge to set of measure zero. So some attracting set exists for any set of parameters, $\gamma \subset \Lambda$.

In Figure 7, we notice that after a specific time, the curve levels off and can not be approximated as a straight line anymore. That is because all trajectories of the Lorenz system wind up in its strange attractor. Since trajectories are bounded, they can only get so far apart.

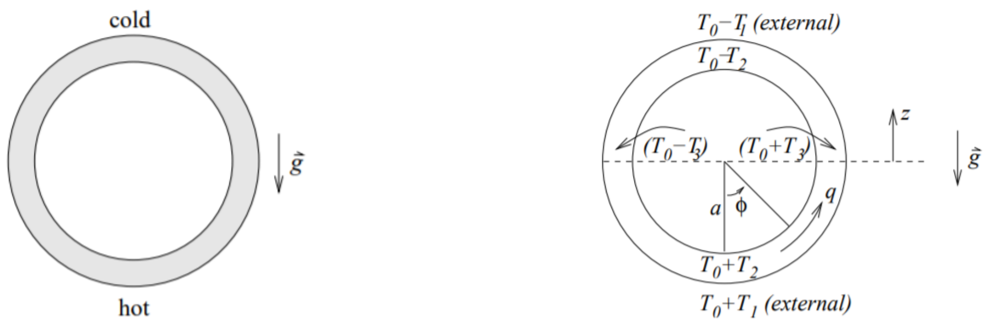
7 Derivation of Lorenz Equations

The Lorenz equations are derived by considering a physical setting of thermal convection (Figure 8a).

7.1 Physical Setup and Parameterization

Let the variables used be defined as (diagram in Figure 8b):

- ϕ is position around the loop



(a) An empty circular tube

(b) Tube setup with temperature variations

Figure 8: Physical setup used to derive Lorenz equations. Image courtesy [1]

- a is the radius of the loop (the tube's inner radius is much smaller than a)
- ρ is the density
- External temperature T_E varies linearly with height as:

$$T_E = T_0 - T_1\frac{z}{a} = T_0 + T_1 cos\phi \tag{30}$$

- The quantities inside the loop are average cross-sectionally, and the quantities to be estimated are defined as:

$$\text{momentum} = q = q(\phi, t)$$
$$\text{temperature} = T = T(\phi, t)$$

$$T(t) - T_0 = T_2(t)cos\phi + T_3(t)sin\phi \tag{31}$$

$2T_2(t)$ is the temperature difference between the top and bottom
$2T_3(t)$ is the temperature difference between the sides at mid-height

The assumptions employed to derive the equations are as follows:

- After employing the Boussinesq approximation, in other words employing in compressibility, $\frac{\partial \rho}{\partial t} = 0$, is assumed.
- As it is a closed tube, mass is conserved. Hence, $\frac{\partial q}{\partial \phi} = 0$ is true.

7.2 Momentum Equation

The Navier-Stokes momentum equation for convection is used to derive the momentum equation. The Navier-Stokes momentum equation is

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\frac{1}{\rho}\vec{\nabla}p - \vec{g}\alpha\Delta T + \nu\nabla^2\vec{u}, \tag{32}$$

where p is the pressure, ρ is the density, ν is the kinematic viscosity and $\vec{g}\alpha\Delta T$ represents the external forces. On replacing $\vec{u}$ as q, taking the tangential component of the external force and sign is chosen so that hot fluid rises, the equation can be written for loop as

$$\frac{\partial q}{\partial t} = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} + g\alpha(T(t) - T_0)sin\phi - \Gamma q, \tag{33}$$

where $\vec{u} \cdot \vec{\nabla} \vec{u} = 0$ as $\frac{\partial q}{\partial \phi} = 0$, Γ is a generalized friction coefficient corresponding to viscous resistance proportional to velocity and $\vec{\nabla} p$ in polar coordinates is written $\frac{1}{\rho a} \frac{\partial p}{\partial \phi}$.

On substituting the value of $T(t) - T_0$ from equation 31 in equation 33, the following equation is obtained

$$\frac{\partial q}{\partial t} = -\frac{1}{\rho a} \frac{\partial p}{\partial \phi} + g\alpha(T_2(t)cos\phi + T_3(t)sin\phi)sin\phi - \Gamma q, \tag{34}$$

and after integrating this equation over ϕ from 0 to 2π, the momentum equation is

$$\frac{dq}{dt} = \frac{g\alpha T_3(t)}{2} - \Gamma q. \tag{35}$$

7.3 Temperature Equation

The temperature equation for convection is modelled as

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T, \tag{36}$$

where κ is the heat diffusivity. Converting this equation by considering only cross-sectional averages, the new equation is

$$\frac{\partial T}{\partial t} + \frac{q}{a} \frac{\partial T}{\partial \phi} = \kappa(T_E - T(t)), \tag{37}$$

where it is assumed that the conduction around the loop is negligible.

From equation 30 and equation 31, the different between internal and external temperature is written as

$$T_E - T = (T_1 - T_2(t))cos\phi - T_3(t)sin\phi. \tag{38}$$

After substituting equation 38 in equation 37, the following equation is obtained

$$\frac{dT_2}{dt}cos\phi + \frac{dT_3}{dt}sin\phi - \frac{q}{a}T_2(t)sin\phi + \frac{q}{a}T_3(t)cos\phi = \kappa(T_1 - T_2(t))cos\phi - \kappa T_3(t)sin\phi. \tag{39}$$

After comparing terms with $sin\phi$ and $cos\phi$ and substituting $T_1 - T_2(t)$ as $T_4(t)$, which is the difference between internal and external temperatures at the top and bottom or, the extent to which the system departs from a "conductive equilibrium", the final equations are

$$\frac{dT_3}{dt} = -\kappa T_3(t) + \frac{qT_1}{a} - \frac{qT_4(t)}{a}, \tag{40}$$

and

$$\frac{dT_4}{dt} = -\kappa T_4(t) + \frac{qT_3}{a}. \tag{41}$$

7.4 Final Equations

Defining the following variables, X as velocity, Y as temperature difference between up and down currents and Z as departure from conductive equilibrium

$$x = \frac{q}{a\kappa}, \quad y = \frac{g\alpha T_3(t)}{2a\Gamma\kappa}, \quad z = \frac{g\alpha T_4(t)}{2a\Gamma\kappa}, \quad t' = \kappa t, \tag{42}$$

the final equations are

$$\frac{dx}{dt} = -\sigma x + \sigma y, \tag{43}$$

$$\frac{dy}{dt} = -y + \rho x - xx, \tag{44}$$

and

$$\frac{dz}{dt} = -\beta z + xy, \tag{45}$$

where $\rho = \frac{g\alpha T_1}{2a\Gamma\kappa}$ is "Rayleigh Number", $\sigma = \frac{\Gamma}{\kappa}$ is "Prandtl Number" and the parameter β is related to the horizontal wavenumber of the convective motions.

8 Applications

This section has some applications which are able to replicate the chaotic behavior of Lorenz equations.

8.1 Chua's Circuit

The Chua's circuit is a physical application of Lorenz attractors. The Chua's circuit was invented in 1983. It consists of five components- one resistor, two capacitors, one inductor, and one Chua's diode as shown in Figure 9.

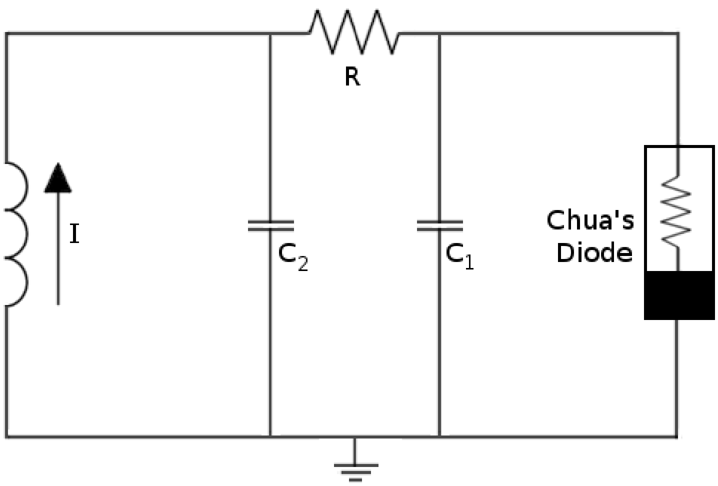


Figure 9: Chua's circuit diagram. Image courtesy [4]

Chua's circuit can be modelled by a set of nonlinear differential equations where x, y and z are plotted against time. The equation are as follows

$$\frac{dx}{dt} = \sigma(y - x - g(x)), \tag{46}$$

$$\frac{dy}{dt} = x - y + z, \tag{47}$$

and

$$\frac{dz}{dt} = -\rho y \tag{48}$$

where α and β depend on the actual circuit components, and x, y, z represent the voltages across capacitors C1, C2 and the current of the inductor respectively, as denoted in Figure 9.

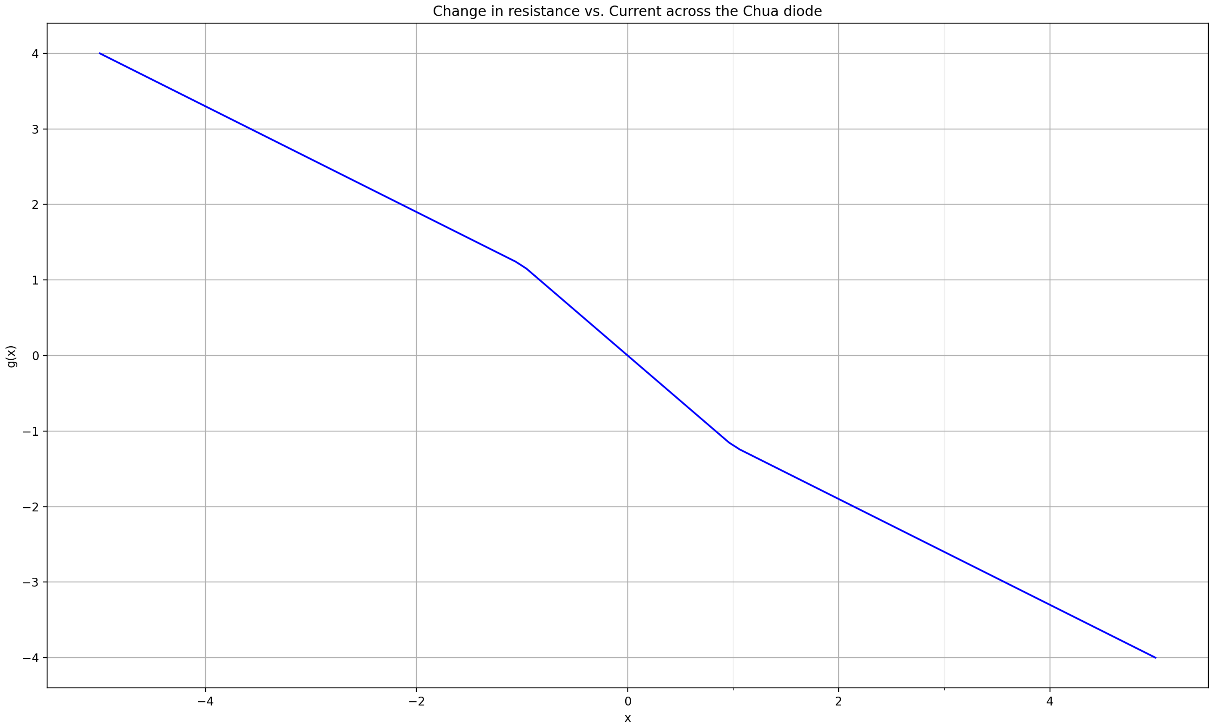


Figure 10: Plot of resistance vs. current across Chua's diode ($n = -0.7$, $m = -1.2$)

$g(x)$ is a pieces-wise linear function representing the change in resistance vs. current across the Chua's Diode. It is given as

$$g(x) = nx + \frac{1}{2}(m - n)(|x + 1| - |x - 1|) \tag{49}$$

where n is the slope of the middle part and m is the slope of the outer parts as shown in Figure 10. The Chua's circuit equations are simulated in Figures 11 and 12, where it is evident that they represent a chaotic system.

8.2 Random Number Generator

Random number generator has very important applications in the field of Information Security. They require random physical process for better randomness and unpredictability. The characteristics of chaos, such as non-periodicity, wide spectrum, unpredictability and sensitivity to initial conditions are good for generating random numbers. Thus, chaotic systems can be used for the generating random numbers.

Example: Consider the Chua's circuit as the chaotic system. If we look at the z state variable waveform (Figure 11), we can see that it has variant signs in different scrolls. Due to the chaotic behaviour of Chua's circuit, after a long enough time, it is not possible to tell in which scroll the trajectory will be located. So if we sample the z signal at low rate, and set output as 0 or 1 depending on the sign, we obtain a random bit sequence. With the use of extra hardware, we can generate random numbers from random bit sequence.

9 Code

The source code of our experiments and theoretical analysis can be found in the following GitHub repository [Link]. Please check *README.md* to run the code and experiments.

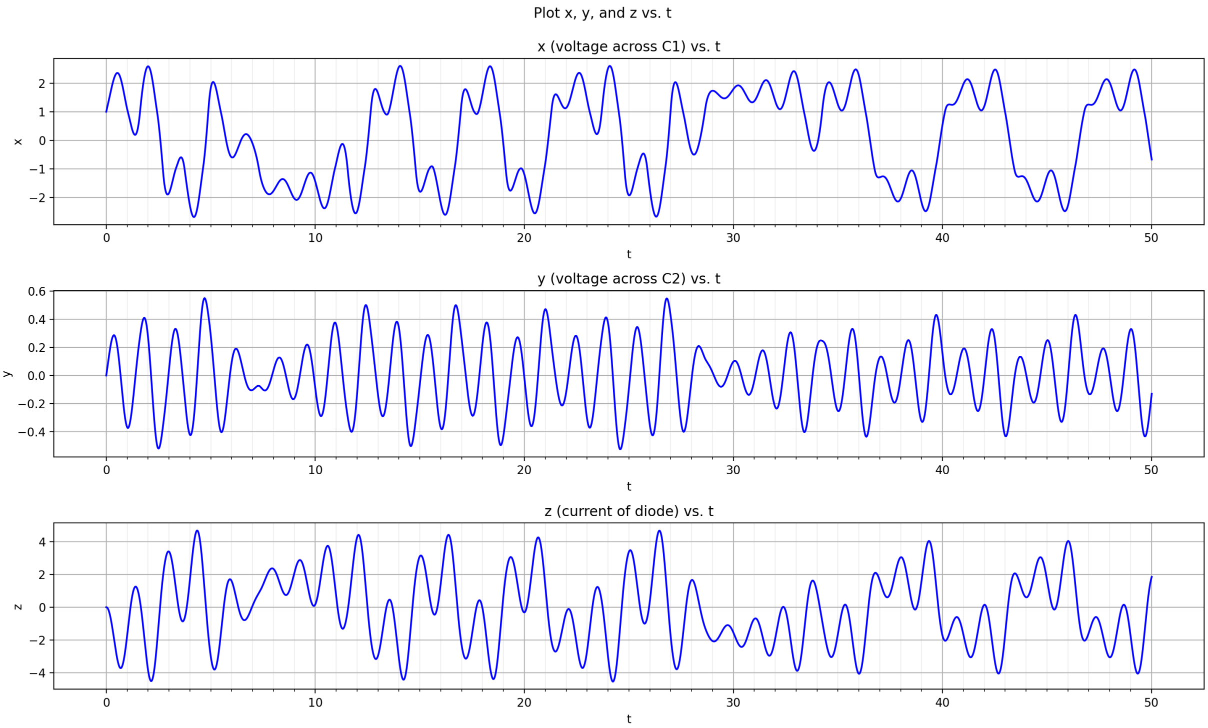


Figure 11: Chua's circuit equation numerically plotted in 2D with initial point = $(0, 0, 1)$ at $\sigma = 16, \rho = 30, \beta = \frac{8}{3}$.

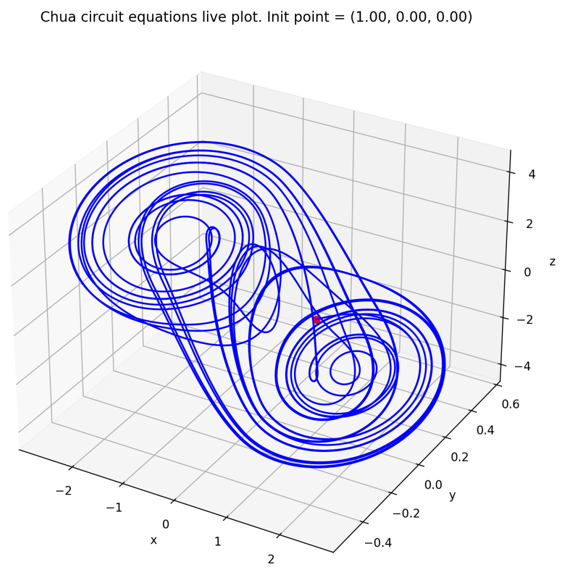


Figure 12: Chua's circuit equation numerically plotted in 3D with initial point = $(0, 0, 1)$ at $\sigma = 16, \rho = 30, \beta = \frac{8}{3}$. The red point is the initial value.

10 Conclusion

In this report, we have described Lorenz equations and concepts related to them like the chaos theory and attractors. We have simulated the solutions of the equations and analysed the nature of the solutions through critical points, stable points and bifurcations. We have also extensively analysed Chua's equations and were able to relate it to the chaotic or butterfly effect of the Lorenz Equations.

11 Acknowledgement

The project for our course *Differential Equations* has been a great learning experience for us and has taught us great about how to do systematic literature survey. At last, we want to thank *Professor Dr. Lakshmi Burra* for her guidance in the course.

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