Indian Institute of Technology, Gandhinagar

MA 202 Project Report

Numerical Analysis of Gravitational Lensing

Authors
Saniya Patwardhan 20110135
Varad Sardeshpande 20110220
Rahul Lalani 20110154
Rahul Chembakasseril 20110158

25*th* April, 2022

Under the guidance of

Prof. Uddipta Ghosh
Prof. Tanya Srivastava
Prof. Satyajit Pramanik

Contents

Chapter 1

Introduction to the Problem

Investigating the behaviour of particle-natured light in the vicinity of N massive objects, and subsequent consequences on Gravitational Lensing.

It is known that the motion of two bodies under their mutual Gravitational Interaction can be analyzed and solved analytically. When a 3rd body is added to this system, no such analytical solution exists.

In this project, we have attempted at demonstrating the behaviour of light around super-massive celestial objects like a black-hole. This demonstration combines the use of Physical Laws, approximations to the objects in the problem and the numerical methods to analyze motion of the objects.

Chapter 2

Mathematical Formulation of the Problem

2.1 Universal Law of Gravitation

Newton's law of gravitation is one of the most fundamental laws of nature. The universal law of gravitation states that an attractive force particle is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. It is given by the following equation:

$$F = \frac{Gm_1 m_2}{r^2} \tag{2.1}$$

This theory was developed by Isaac Newton based on empirical observations. The Law of gravitation is applied in the scenarios where mass products of the interacting bodies are substantial. The attractive forces between the sun and the earth are tremendous, but due to the centrifugal force due to the earth's velocity, it is countered. The gravitational acceleration is defined using the same formula. In the case of a black hole, the gravitational pull is so strong that even electromagnetic radiations such as light cannot escape it. The theory of general relativity predicts that a sufficiently compact mass can deform space-time to form a black hole.

2.2 2-Body Problem

Consider a three-dimensional cartesian plane wherein two bodies lie. If the forces on both the bodies are only due to the interaction between them, it can be considered a two-body problem.

For example, consider only the sun and earth in the solar system; the law of gravitation states that there will be a certain attractive force between them along the line joining the two centers. This example is a two-body problem as only gravitational interactions are present, and no other type of force influences this system.

$$F_A = \frac{Gm_E m_S}{r^2} \tag{2.2}$$

Using the Newton's second law we can define the following equation:

$$\frac{d^2x}{dt^2} = \frac{F_{Ax}}{m_E} \tag{2.3}$$

$$\frac{d^2y}{dt^2} = \frac{F_{Ay}}{m_E} \tag{2.4}$$

$$\frac{d^2z}{dt^2} = \frac{F_{Az}}{m_E} \tag{2.5}$$

On solving the above four differential equations we can obtain the solution for the 2-body problem.

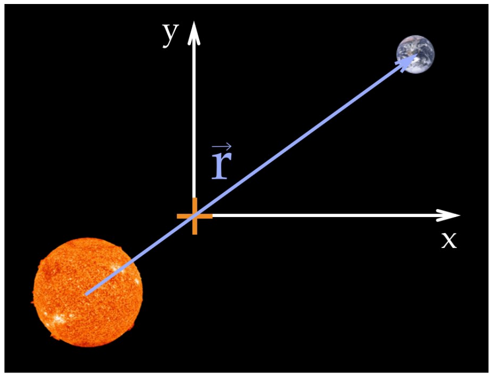


Figure 2.1: Two-body representation

2.3 The N-Body Problem

As for the case of a 2-body problem, similar conditions can be applied to the n-body problem. In the n-body situation, the motion of the 'n' number of bodies is due to the interaction only through gravitational forces. The numerical approach is taken to find the solution for the n-body problem. The following equations can be written for i,jth body in a n-body problem:

$$F_x = \sum_{i,j=1,i\neq j}^{i,j=N} \frac{-Gm_i m_j (x_i - x_j)}{r_{i_j}^3} \tag{2.6}$$

$$F_y = \sum_{i,j=1,i\neq j}^{i,j=N} \frac{-Gm_i m_j (y_i - y_j)}{r_{i_j}^3} \tag{2.7}$$

$$F_z = \sum_{i,j=1,i\neq j}^{i,j=N} \frac{-Gm_i m_j (z_i - z_j)}{r_{i_j}^3} \tag{2.8}$$

Representing differential form of acceleration in terms of force and mass:

$$\frac{d^2x_i}{dt^2} = \frac{F_{ix}}{m_i} \tag{2.9}$$

$$\frac{d^2y_i}{dt^2} = \frac{F_{iy}}{m_i} \tag{2.10}$$

$$\frac{d^2z_i}{dt^2} = \frac{F_{iz}}{m_i} \tag{2.11}$$

Chapter 3

Laws Governing Massive Body (Black-hole) - Light Interactions

For a very long time Newton's Universal Law of Gravitation explained the interaction force between the interaction any two bodies in the universe, until one day, it did not. When Albert Einstein came up his his theory of relativity, the picture changed. There were some limitations in the Universal law of gravitation. In this chapter we will try to understand the research done by various scientists after the Newton 's era and apply the required modifications, assumptions and approximations to our Mathematical model.

3.1 What are black holes?

- A Black hole is a voluminous space that has a singularity at its centre.
- A Singularity is an object or a mass which was once a gigantic star, and which later collapsed its matter into a single point of zero radius and thus infinite density.
- Black-holes have extremely strong gravitational field with escape velocity so high that even light cannot escape a Black-hole, once entered. Essentially, as nothing can have speed greater than light, nothing can escape a black hole.

The researchers used Newton's laws of gravitation equation,($F = \frac{GM_1M_2}{r_2}$), the mass-energy equivalence relation ($E = mc^2$), and quantum theory of radiation ($E = hV$) to derive a formula for gravitational force acting between a black hole and a light particle passing near the radius of the event horizon of black holes. The work was then extended to calculate the values of different test black holes existing only in X-ray binaries (XRBs).

3.2 Schwarzschild Formulation for Non-Spinning and Spinning Black-holes

The Scwharzschild Radius and corresponding Force exerted by a non-spinning blackhole is given by-

$$R_{bh} = \frac{2GM}{c^2} \tag{3.1}$$

$$F_{non-spinning} = \frac{hc^3}{4GM\lambda} \tag{3.2}$$

For Non Spinning black holes the radius of event horizon and table containing the data of gravitational force vs light particle

In case of a spinning-black hole, the Schwarzschild formulation for the Radius and Force exerted are as follows-

$$R_{bh} = \frac{GM}{c^2} \tag{3.3}$$

$$F_{spinning} = \frac{hc^3}{GM\lambda} \tag{3.4}$$

For Spinning black holes the radius of event horizon and table containing the data of gravitational force vs light particle

A few observations about the above formulations follow-

- For constant surface gravity, the gravitational force between black holes and light particles is inversely proportional to the wavelength of radiation.
- The gravitational force of a spinning black hole is greater than that of a non-spinning black hole of the same mass.
- For constant surface gravity, the shorter wavelength light particle (light wave) has attracted more than the longer wavelength light particle (light wave).

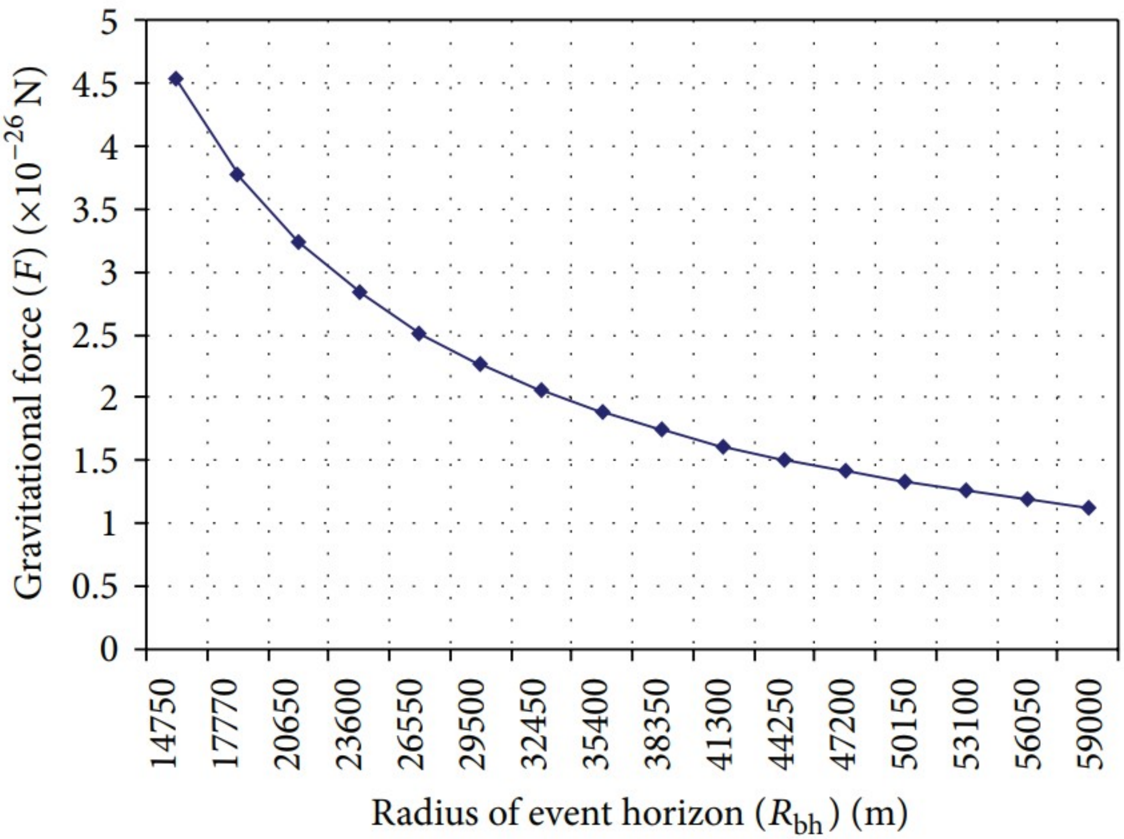


Figure 3.1: Gravitational force vs Radius of event horizon graph for non spinning black holes

TABLE 1: Gravitational force between non-spinning black holes and light particles in XRBs. Data: mass of sun ($M_\odot$) = 1.99×10^{30} kg.

Sl. number	Mass of BHs (M)	($R_s = 2950\ M/M_\odot$) (in metre)	Wavelength of light (λ)	Gravitational force (F) in Newton
1	$5\ M_\odot$	14750	555×10^{-9} m	4.527×10^{-26}
2	$6\ M_\odot$	17700	555×10^{-9} m	3.773×10^{-26}
3	$7\ M_\odot$	20650	555×10^{-9} m	3.234×10^{-26}
4	$8\ M_\odot$	23600	555×10^{-9} m	2.829×10^{-26}
5	$9\ M_\odot$	26550	555×10^{-9} m	2.515×10^{-26}
6	$10\ M_\odot$	29500	555×10^{-9} m	2.263×10^{-26}
7	$11\ M_\odot$	32450	555×10^{-9} m	2.058×10^{-26}
8	$12\ M_\odot$	35400	555×10^{-9} m	1.886×10^{-26}
9	$13\ M_\odot$	38350	555×10^{-9} m	1.741×10^{-26}
10	$14\ M_\odot$	41300	555×10^{-9} m	1.617×10^{-26}
11	$15\ M_\odot$	44250	555×10^{-9} m	1.509×10^{-26}
12	$16\ M_\odot$	47200	555×10^{-9} m	1.414×10^{-26}
13	$17\ M_\odot$	50150	555×10^{-9} m	1.331×10^{-26}
14	$18\ M_\odot$	53100	555×10^{-9} m	1.257×10^{-26}
15	$19\ M_\odot$	56050	555×10^{-9} m	1.191×10^{-26}
16	$20\ M_\odot$	59000	555×10^{-9} m	1.131×10^{-26}

Figure 3.2: Gravitational force between non-spinning black holes and light particles

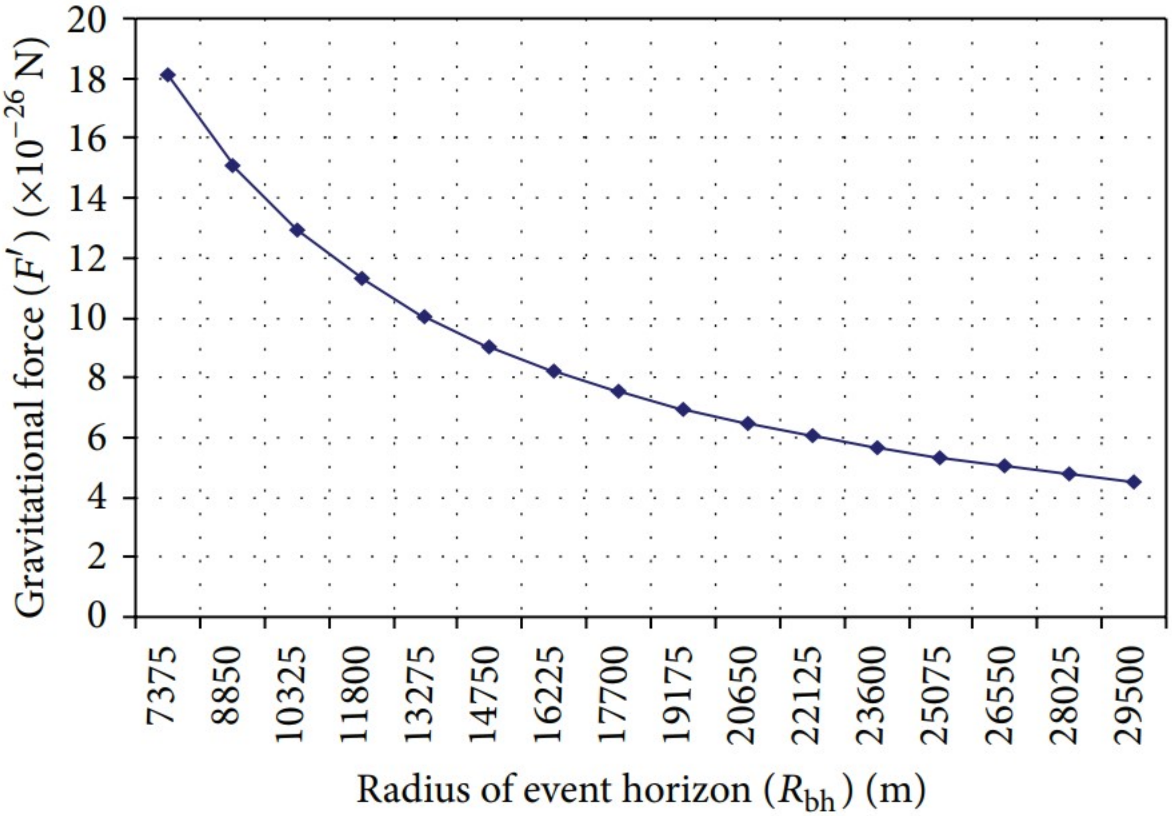


Figure 3.3: Gravitational force vs Radius of event horizon graph for spinning black holes

Table 2: Gravitational force between spinning black holes and light particles in XRBs.

Sl. Number	Mass of BHs (M)	(R_s = 1475 M/$M_\odot$) (in metre)	Wavelength of light (λ)	Gravitational force (F) in Newton
1	5 $M_\odot$	7375	555×10^{-9} m	18.108×10^{-26}
2	6 $M_\odot$	8850	555×10^{-9} m	15.092×10^{-26}
3	7 $M_\odot$	10325	555×10^{-9} m	12.936×10^{-26}
4	8 $M_\odot$	11800	555×10^{-9} m	11.316×10^{-26}
5	9 $M_\odot$	13275	555×10^{-9} m	10.060×10^{-26}
6	10 $M_\odot$	14750	555×10^{-9} m	9.052×10^{-26}
7	11 $M_\odot$	16225	555×10^{-9} m	8.232×10^{-26}
8	12 $M_\odot$	17700	555×10^{-9} m	7.544×10^{-26}
9	13 $M_\odot$	19175	555×10^{-9} m	6.964×10^{-26}
10	14 $M_\odot$	20650	555×10^{-9} m	6.464×10^{-26}
11	15 $M_\odot$	22125	555×10^{-9} m	6.036×10^{-26}
12	16 $M_\odot$	23600	555×10^{-9} m	5.656×10^{-26}
13	17 $M_\odot$	25075	555×10^{-9} m	5.324×10^{-26}
14	18 $M_\odot$	26550	555×10^{-9} m	5.028×10^{-26}
15	19 $M_\odot$	28025	555×10^{-9} m	4.764×10^{-26}
16	20 $M_\odot$	29500	555×10^{-9} m	4.524×10^{-26}

Figure 3.4: Gravitational force between spinning black holes and light particles.

Chapter 4

Physical Assumptions

To best simulate the behaviour of light and super-massive objects, we have made the following assumptions while making the simulations-

- Light is made up of discrete particles called photons whose mass is finite and extremely small compared to the mass of the black-hole.
- Photons are travelling at high speeds but they are lesser than the speed of light.
- We have assumed the black-holes to be perfectly spherical in shape.
- Since we have scaled down both the mass of the black-holes and the speed of the photons, we have also assumed $G = 1$.
- Blackhole-photon interaction and in the case of multiple black-holes, blackhole-blackhole interaction dominates as compared to photon-photon interaction.
- As such, our models and simulations contain a certain degree of physical and theoretical inconsistency which we have tried to

minimize by using numerical approximations and methods.

Chapter 5

Approximations and Numerical Solution Methodology

In the case of a two-body system, both second-order differential equations may be classified in to two first-order differential equations and with the help of these relationship we can write-

$$\frac{\partial x_e}{\partial t^2} = \frac{\partial v_{e,x}}{\partial t} \tag{5.1}$$

$$\frac{\partial y_e}{\partial t^2} = \frac{\partial v_{e,y}}{\partial t} \tag{5.2}$$

From these equation the first-order differential order equations is-

$$\frac{\partial v_{e,x}}{\partial t} = \frac{-GM_S x_e}{r_{S,E}^3} \tag{5.3}$$

$$\frac{\partial x_e}{\partial t} = v_{e,x} \tag{5.4}$$

$$\frac{\partial v_{e,y}}{\partial t} = \frac{-GM_S y_e}{r_{S,E}^3} \tag{5.5}$$

$$\frac{\partial y_e}{\partial t} = v_{e,y} \tag{5.6}$$

For 3 Body and N Body systems, we can get a system of first order equations in the same way.

5.1 Euler-Cromer Method

By using the Taylor series expansion of the velocity, we can write:

$$v(t+h) = v(t) + h\dot{v(t)} + O(h^2) \tag{5.7}$$

$$v(t+h) = v(t) + a(t)h + O(h^2) \tag{5.8}$$

Now similiar condtion for position vector

$$x(t+h) = v(t) + h\dot{x(t)} + O(h^2) \tag{5.9}$$

Here we see that the Euler Cromer approach differs from the Normal Euler method at this point. The concepts space (position) and momentum (velocity) should not be modified at the same time since the system's energy must be preserved. As a result, we must first update velocity while assuming the prior value of position, and then update position using the new velocity value. So, we obtain the following equation-

$$x(t+h) = v(t) + v(t+h)h + O(h^2) \tag{5.10}$$

5.2 Velocity Verlet Method

Verlet integration is a method used to solve the Newton's equations of motion. It is an algorithm to perform the integration for solving the Newton's equations. Velocity verlet method is an algorithm which has similarities to the leapfrog method. The only difference is that the velocity and position are calculated at the same value fo the time variable. The algorithm consists of clear steps to find the solution. This method uses the approach where it explicitly incorporates the velocity, solving the problem of the first time step in the basic Verlet algorithm.

The differential equation for the motion can be given by:

$$\frac{d^2x}{dt^2} = \frac{F(x)}{m} \tag{5.11}$$

After using the Taylor series expansion, a finite difference algorithm is derived and shown below:

$$x(t+b) = x(t) + h\frac{dx}{dt} + \frac{h^2}{2}\frac{d^2x}{dt^2} + O(b^3) \tag{5.12}$$

The above equation can be condensed into the following form:

$$x(t+b) = x(t) + hv(t) + \frac{h^2}{2}a(t) + O(b^3) \tag{5.13}$$

Further using the Taylor Series Expansion, the following expression can be obtained:

$$v(t+b) = v(t) + h\frac{dv}{dt} + \frac{h^2}{2}\frac{d^2v}{dt^2} + O(b^3) \tag{5.14}$$

To calculate $\frac{d^2v}{dt^2}$, we can achieve this by expanding $\frac{dv}{dt}$:

$$\frac{dv(t+b)}{dt} = \frac{dv}{dt} + \frac{h^2}{2}\frac{d^2v}{dt^2} + O(b^2) \tag{5.15}$$

Multiplying the above equation by $\frac{h}{2}$:

$$\frac{h^2}{2}\frac{d^2v(t)}{dt^2} = \frac{h}{2}(\frac{dv(t+b)}{dt} - \frac{dv(t)}{dt}) + O(b^3) \tag{5.16}$$

On solving the equations (5.14) and (5.16) :

$$v(t+b) = v(t) + h\frac{dv}{dt} + \frac{h}{2}(\frac{dv(t+b)}{dt} - \frac{dv(t)}{dt}) + O(b^3) \tag{5.17}$$

The final result of the equation of motion involving the positions, velocity, and Force:

$$v(t+b) = v(t) + \frac{h}{2}(a(t+h) + a(t)) + O(b^3) \tag{5.18}$$

Chapter 6

Algorithm and Python Program

6.1 Glowscript and VPython

- We have programmed the simulation in the VPython programming language. It is the Python programming language along with a visual module for 3D graphics rendering.

- The platform used for testing the simulation with different initial conditions and setups is Glowscript (https://glowscript.org/).

- In order to fully observe the simulations, we recommend that each of the 6 simulations be run on Glowscript which can be run in the web browser itself.

- Gravitational interaction has been simulated by the classical Newton's law of gravitation at large distances from the black hole. On reaching distances within the event horizon of the black-hole (in this case, the radius), we have adopted the Schwarzschild formulation for modelling gravitational interaction.

6.2 Elements of the Simulations

- There are 6 simulations that we have created. Each of the simulations feature "black-holes". These are supermassive objects with small radii. These high-density objects serve as good approximations for the gravitational effect of black-holes.
- We have modelled light as a stream of photons which travel at extremely fast speeds which are good approximations for c.

6.3 Algorithm

- We adopted both the methods- Euler-Cromer and the Velocity Verlet for numerically solving the n-body gravitational interaction problem.
- In the Euler Cromer Method-
  - At every time step i we compute the position (x,y,z) and the velocity along x and y axis (v_x, v_y, v_z)
  - First we calculate the distance r_i from the reference celestial object.
  - Calculate the values of $v_{x,i+1}$, $v_{y,i+1}$ and $v_{z,i+1}$ as $v_{x,i+1} = v_{,}x_i + a_{x,i}\,\Delta t$, $v_{y,i+1} = v_{y,i} + a_{y,i}\,\Delta t$ and $v_{z,i+1} = v_{z,i} + a_{z,i}\,\Delta t$.
  - Now compute x_{i+1},y_{i+1} and z_{i+1} using $v_{x,i+1}$, $v_{y,i+1}$ and $v_{z,i+1}$ as $x_{i+1} = x_i + v_{x,i+1}\,\Delta t$, $y_{i+1} = y_i + v_{y,i+1}\,\Delta t$ and $z_{i+1} = z_i + v_{z,i+1}\,\Delta t$.
  - Now compute $a_{x,i+1}$, $a_{y,i+1}$ and $a_{z,i+1}$ using the new position values x_{i+1}, y_{i+1} and z_{i+1}.
  - Memorize these positions x_{i+1}, y_{i+1} and z_{i+1} and also accelerations $a_{x,i+1}$, $a_{y,i+1}$ and $a_{z,i+1}$ for the next iteration.

- In Velocity Verlet Method-
  - We start by calculating x_{i+1} and y_{i+1} and using $x_{i+1} = x_i + v_{x,i}\,\Delta t + a_{x,i}\,\frac{\Delta t^2}{2}$, $y_{i+1} = y_i + v_{y,i}\,\Delta t + a_{y,i}\,\frac{\Delta t^2}{2}$ and $z_{i+1} = z_i + v_{z,i}\,\Delta t + a_{z,i}\,\frac{\Delta t^2}{2}$
  - We then calculate the updated distance $r_{i+1}^2 = x_{i+1}^2 + y_{i+1}^2 + z_{i+1}^2$ and update the acceleration.
  - We also have to compute the new $a_{x,i+1}$, $a_{y,i+1}$ and $a_{z,i+1}$ from the positions x_{i+1}, y_{i+1} and z_{i+1}.
  - Now we have the updated values for the accelerations, from this we update the values of the velocity. $v_{x,i+1} = v_{x,i} + (a_{x,i} + a_{x,i+1})\frac{\Delta t}{2}$, $v_{y,i+1} = v_{y,i} + (a_{y,i} + a_{y,i+1})\frac{\Delta t}{2}$ and $v_{z,i+1} = v_{z,i} + (a_{z,i} + a_{z,i+1})\frac{\Delta t}{2}$.
  - We then memorize the current position, velocity and acceleration for the next iteration.

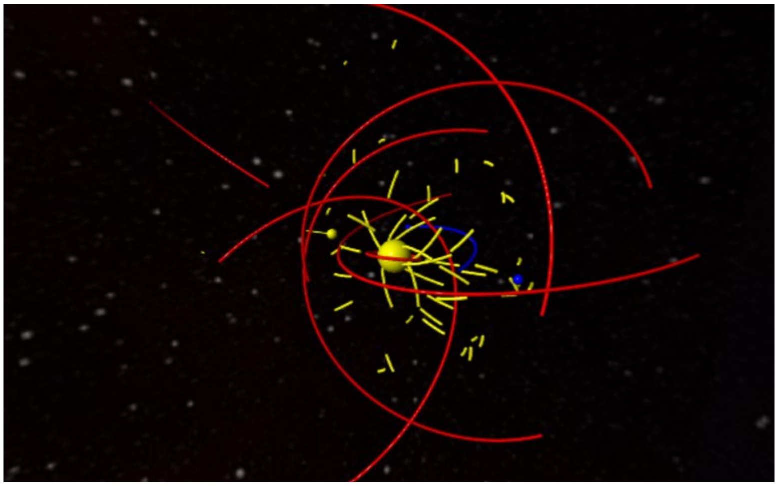
Chapter 7

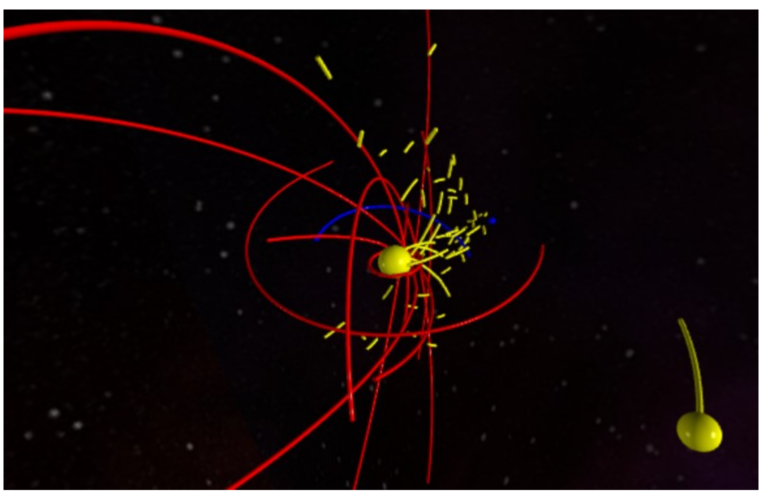
Results and Conclusions

7.1 Solar System Simulation

- In order to test the accuracy of our numerical approximation of the gravitational interaction of n bodies, we first created a simulation of a standard solar system with a central star, orbiting planets like Pluto, Jupiter and Neptune along with an asteroid belt.

- The simulation correctly predicted the shift in plane of orbit of Neptune and Pluto. It also predicted the elliptical shapes of planetary orbits correctly.

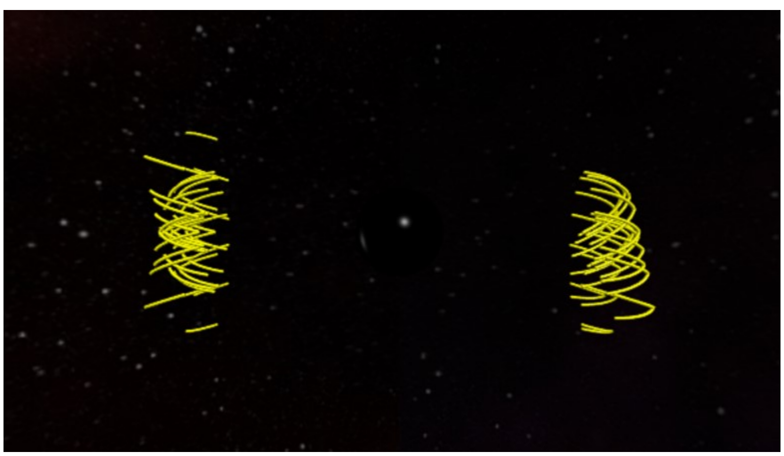


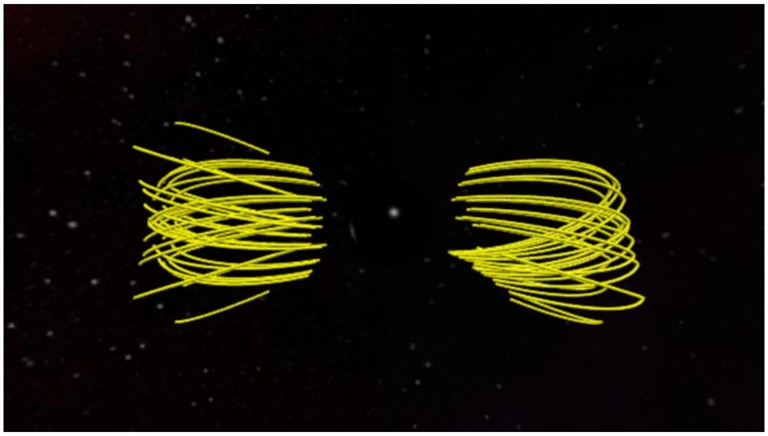


7.2 One Massive Body (Blackhole)

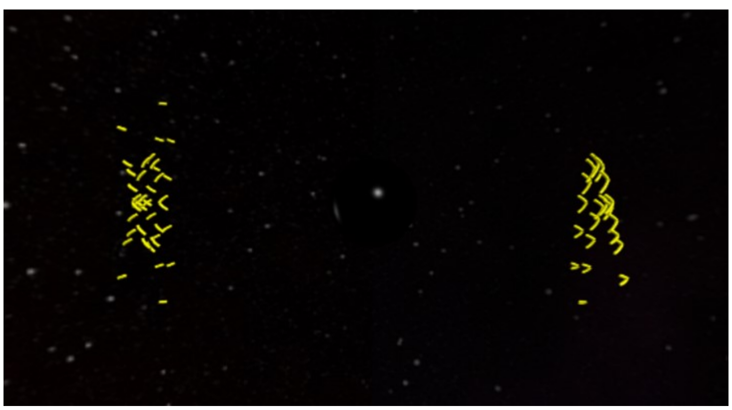
- This simulation features one super-massive body with two local light sources emitting light rays towards it.

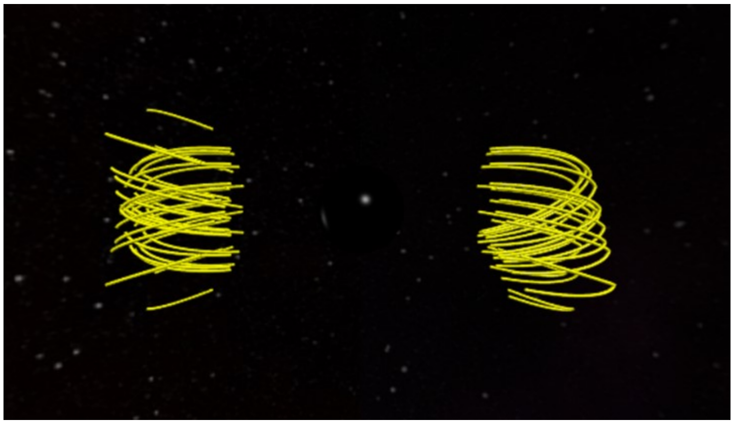
- The simulation correctly demonstrates the bending of light rays towards the black-hole.

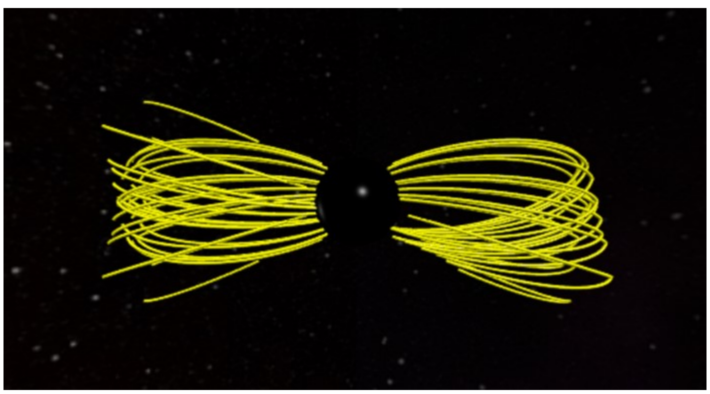




- The following three screenshots are from the Velocity-Verlet implementation of the same simulation.



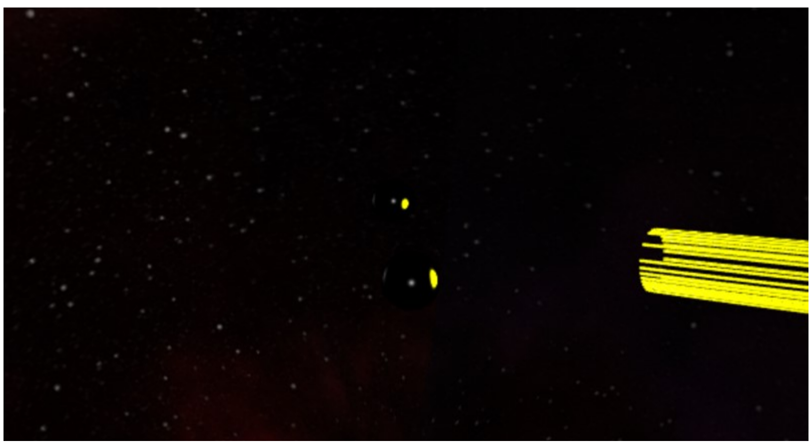


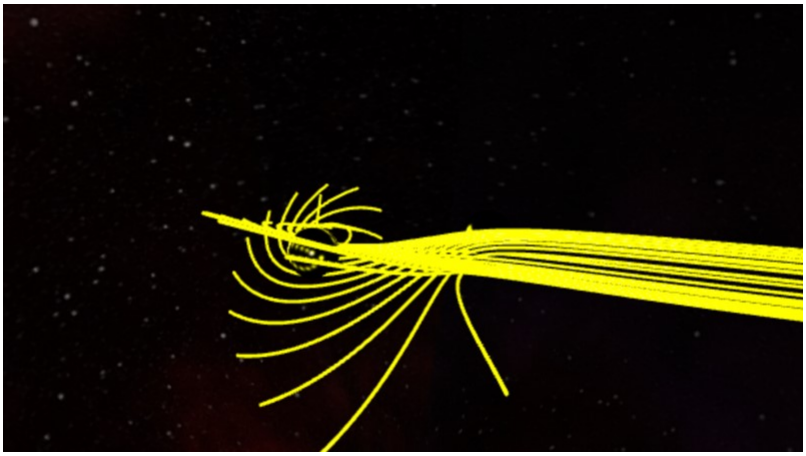


- We made this simulation in two ways according to the algorithms specified in 6.3. For the set of initial conditions that we employed, we obtained almost identical results for both methods.
- A noteworthy point is that both these methods are energy-conserving methods.

7.3 Two Revolving Massive Bodies

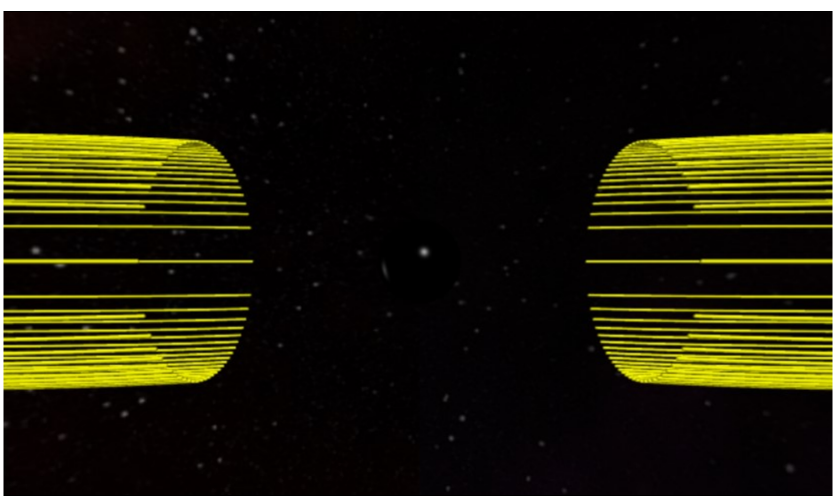
- Here, we simulated two super-massive bodies to revolve in orbits around their common centre of mass.
- Then we introduced light rays approaching from $infinity$ into the system.
- The simulation correctly displayed the bending of light rays towards both the black-holes.

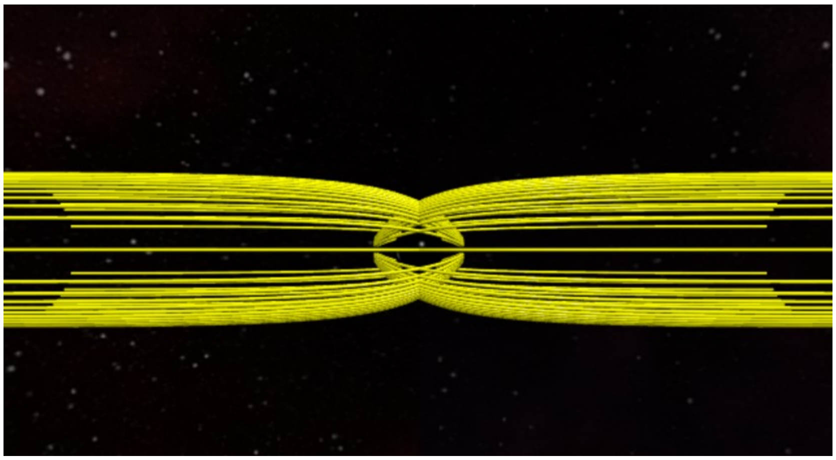




7.4 Light Source at Infinity

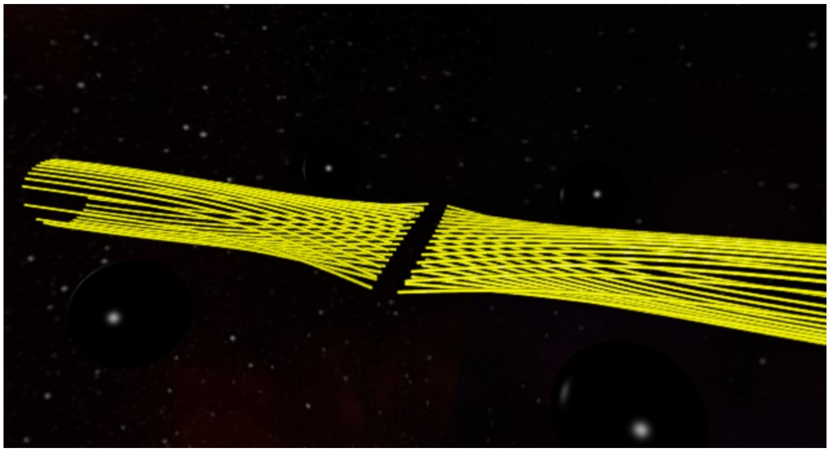
- This simulation was done to get an estimate of the Focal Length of the Gravitational Lens- the black-hole.
- The simulation correctly captures the approaching beam of light and it's focus to a point beyond the black-hole.

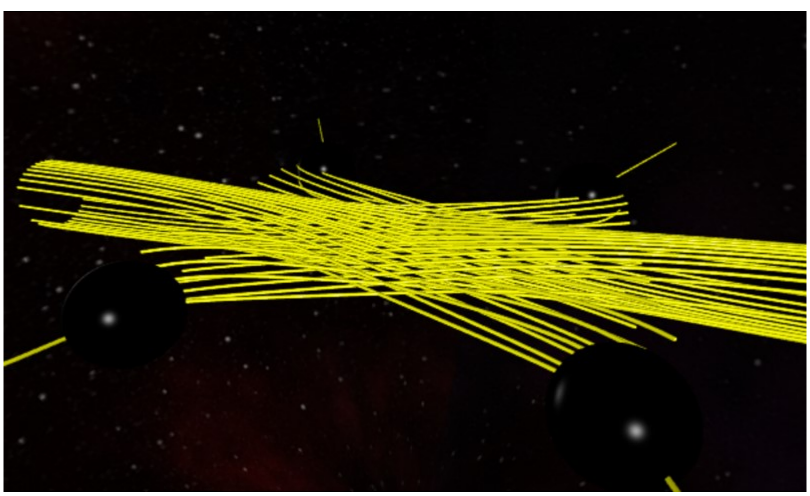


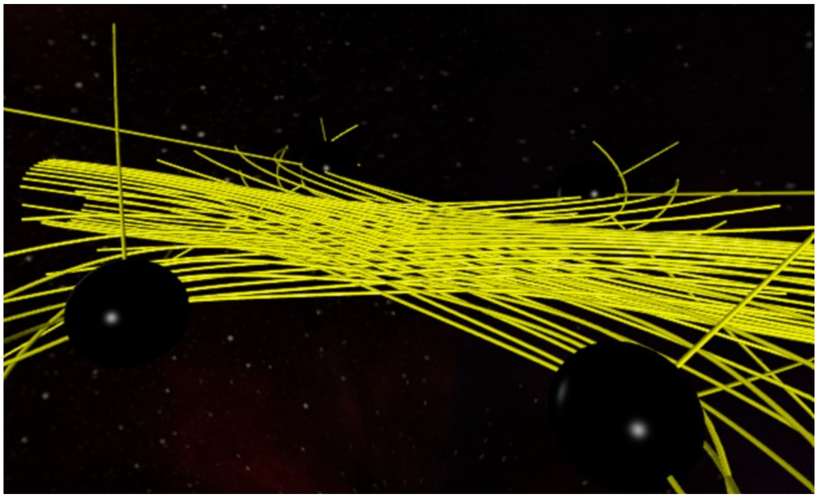


7.5 Galaxy Cluster

- This simulation displays the behaviour of light around a system of supermassive objects. Here too, the source is at infinity with the objects placed at close proximity to the origin.
- We can observe different results by varying the initial conditions such as the position of the light source and the starting positions of the black-holes.







Chapter 8

Github Repository

- The source files for the simulations written in VPython script are uploaded on Github at https://github.com/RahulVC02/MA2O2-Course-Project .

- The README.md file contains instructions to run the simulations on the web browser on https://glowscript.org/ .

Chapter 9

References

- https://www.hindawi.com/journals/jgrav/2013/232676
- https://www.youtube.com/channel/UCWBTKIyw-zX-2k63cB6qciQ
- https://github.com/plugyawn
- http://www.scholarpedia.org/article/N-bodysimulations(gravitational)
- https://arxiv.org/pdf/0912.0490.pdf