## MA503: Homework 4

**Exercise 4.1.** [20pts] Consider a Cartesian product  $G = \mathbb{Z} \times \mathbb{Z} = \{ (\alpha, x) \mid \alpha, x \in \mathbb{Z} \}$  and a binary operation  $\cdot$  on G defined as follows:

$$(\alpha_1, x_1) \cdot (\alpha_2, x_2) = (\alpha_1 + \alpha_2, (-1)^{\alpha_2} x_1 + x_2)$$

- (1) [8pts] Prove that  $(G, \cdot)$  is a group.
- (2) [2pts] Is  $(G, \cdot)$  abelian?
- (3) [2pts] Is  $(G, \cdot)$  finite?
- (4) [2pts] Prove that every cyclic group is abelian. Then use (2) to prove that  $(G,\cdot)$  is not cyclic.
- (5) [2pts] Does  $(G, \cdot)$  have torsion?
- (6) [2pts] Is  $\pi_1: G \to \mathbb{Z}$  defined by  $(\alpha, x) \stackrel{\pi_1}{\mapsto} \alpha$  a homomorphism? [I want to emphasize that G is not the direct product of  $\mathbb{Z}$  and  $\mathbb{Z}$ .]
- (7) [2pts] Is  $\pi_2 : G \to \mathbb{Z}$  defined by  $(\alpha, x) \stackrel{\pi_2}{\mapsto} x$  a homomorphism?

Exercise 4.2. [5pts] Find |2| in  $U_{67}$ .

Exercise 4.3. [5pts] Is 2 a primitive root modulo 31?

**Exercise 4.4.** [10pts] Consider a set  $G = \{x_1, x_2, \dots, x_8\}$  of eight elements equipped with a binary operation  $\cdot$  defined by the multiplication table shown below.  $(G, \cdot)$  is a group.

	$x_4$	$x_3$	$x_7$	$x_1$	$x_2$	$x_6$	$x_5$	$x_8$
$x_4$	$x_2$	$x_6$	$x_5$	$x_8$	$x_4$	$x_3$	$x_7$	$x_1$
$x_3$	$x_6$	$x_4$	$x_8$	$x_7$	$x_3$	$x_2$	$x_1$	$x_5$
$x_7$	$x_5$	$x_1$	$x_4$	$x_6$	$x_7$	$x_8$	$x_2$	$x_3$
$x_1$	$x_8$	$x_5$	$x_3$	$x_4$	$x_1$	$x_7$	$x_6$	$x_2$
$x_2$	$x_4$	$x_3$	$x_7$	$x_1$	$x_2$	$x_6$	$x_5$	$x_8$
$x_6$	$x_3$	$x_2$	$x_1$	$x_5$	$x_6$	$x_4$	$x_8$	$x_7$
$\overline{x_5}$	$x_7$	$x_8$	$x_2$	$x_3$	$x_5$	$x_1$	$x_4$	$x_6$
$\overline{x_8}$	$x_1$	$x_7$	$x_6$	$x_2$	$x_8$	$x_5$	$x_3$	$x_4$

- (1) Which element is the identity of G?
- (2) Is G abelian? Why?
- (3) Find  $|x_3|$ .
- (4) Find  $\langle x_4 \rangle$ .
- (5) Find the coset  $x_6 \cdot \langle x_4 \rangle$ .
- (6) Find  $x_5^{-1}$ .
- (7) Is  $x_7$  a primitive element?
- (8) [3pts] Is G cyclic?