EXACT D.E

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O.D.E. - First order & first degree :-

★ Order of a D.E. :- The order of a differential equation (i.e, D.E.) is the order of the highest order derivative involve in that D.E.

Ex (1) - $y = x\frac{dy}{dx} + c\frac{dx}{dy} \longrightarrow 1^{st}$ order.

Ex (2) - $\left\{1 + y''\right\}^{3/2} = p \cdot \frac{d^2y}{dx^2} \longrightarrow 2^{nd}$ order.

☆ Degree of a D.E. :- The degree of a D.E. is the power of the highest ordered derivative which involved in that D.E. after making it rational & integral such that derivatives are concerned.

Otherwise, say that, the degree of D.E., which can be written as a polynomial in the derivative, is the degree of the highest order derivative which involved in that D.E.

Ex (1) :- $x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) + c = 0 \longrightarrow 2^{nd}$ deg.

Ex (2) :- $x^2 \frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 + 10 = 0 \longrightarrow 1^{st}$ deg.

Ex (3) :- $(1 + y'')^{3/2} = 10\, y''' \longrightarrow 3^{rd}$ deg.

Ex (4) :- $t \frac{d^2y}{dt^2} + t^2\left(\frac{dy}{dt}\right)^{1/3} - (\cos t)\sqrt{y'} = 2t^3 - 3t + 4$

$\longrightarrow$ ~~[illegible]~~

Ex (5) :- $x e^{y''} + 2x^2 y = 5x + 7 \longrightarrow$ no. deg.

Ex (6) :- $x^2 \frac{d^3y}{dx^3} + 3\left(\frac{d^2y}{dx^2}\right)^3 = 5x^{3/2} + 7$

$\longrightarrow$ deg. = 1 ?

★ **Linear D.E :-** A general form of a linier ODE of order n in the unknown function y and independent variable x is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

Where $a_0, a_1, \cdots, a_n$ are either given functions of x or constants.

Otherwise, the aforesaid D.E. is called non-linear.

N.B(1)- There cannot be any term involving the product of y & its derivatives.

N.B.(2) :- It should be observed that a linear D.E is always of 1st degree but not conversely.

Like — $\frac{dy}{dx} = 1+x^2y^2 \longrightarrow$ 1st deg. but non-linear

as y^2 is present.

★ ODEs of First order & First Degree :

An O.D.E of 1^{st} order & 1^{st} degree can be written as :

$$\frac{dy}{dx} = f(x,y) \quad \text{-------} (1)$$

It can also be written as :

$$M\,dx + N\,dy = 0 \quad \text{-------} (2)$$

Where $M \equiv M(x,y)$ & $N \equiv N(x,y)$.

One important thing must remember that it is not possible to solve all kinds of equations of type (1) or (2).

There are some particulars process for getting G.S. of the D.E. (1) or (2).

These process are:

A) Equations Solvable by separation of variables.

B) Homogeneous Equations.

C) Exact Equation

D) Linear equations of 1^{st} order.

c) Exact Equations :-

A D.E. of the form $M\,dx + N\,dy = 0$ is said to be exact, if $\exists$ (i.e, there exists) a func. $\phi(x,y)$ such that $M\,dx + N\,dy = d\phi$. ——— (2)

(1) is $M\,dx + N\,dy = 0$.

Then (1) becomes: $d\phi = 0$, if we take integration on both sides of $d\phi = 0$, then we get

$\phi(x,y) = c$; $c \equiv$ arbitrary constants.

For ex., $x\,dy + y\,dx = 0$ is an exact D.E., since

$$x\,dy + y\,dx = d(xy)$$

Its g.s. is $xy = c$. $[c \equiv A.C.]$

⊠ Theorem :- The necessary & sufficient condition that $Mdx + Ndy$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working rules for solving $M\,dx + N\,dy = 0$ when it is exact.

Step-1: Integrate $M\,dx$, assuming y as constant.

Step-2: Integrate $N\,dy$, assuming x as constant.

Step-3: Summing the results in step-1 & 2, but terms common to both is written once only.

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Step-4:- Equate the sum to a constant. This will be the solution of the given D.E.

Problem :- Solve : $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xy\,e^{xy^2} - 3y^2\right) dy = 0$

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* Remarks :- $\displaystyle\int_{y=\text{Const.}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c.$

SOLVE $(3X+4Y+5)DX+(4X-3Y+3)DY=0$

Problem: show that $(x^3 - 3x^2y + 2xy^2)dx - (x^3 - 2x^2y + y^3)dy = 0$ is exact & find the solution (when) if $y = 1$ when $x = 1$.

$$\left(xy^2 - e^{1/x^2}\right)dx - x^2y\,dy = 0$$