University of Engineering & Management, Kolkata

End Semester Examination, February 2021

Course: B.Tech Semester: 1st

Paper Name: Mathematics and Statistics - I

Paper Code: BSC103

Full Marks: 100 Time: 3 hours

Answer all the questions. Each question is of 10 marks.

1. A. i) Verify Lagrange's Mean value theorem for the function

$$f(x) = \begin{cases} x\cos\frac{1}{x}, & for\ x \neq 0 \\ 0, & x = 0 \end{cases} \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

ii) Expand the following function in powers of x by the use of Maclaurin's theorem:

$(a + x)^n$, n being an integer or a fraction.

OR

B. i) If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ is finite, find the value of a and the limit.

ii) Prove that $\frac{a-b}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{b}$ for $0 < a < b$. Hence show that $\frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$.

2. A. i) Evaluate : $\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+3)^3}} + \frac{\sqrt{n}}{\sqrt{(n+6)^3}} + \cdots + \frac{\sqrt{n}}{\sqrt{(n+3(n-1))^3}}\right]$

ii) Prove that : $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx = \frac{\pi^2}{2ab}$

OR

B. i) Show that $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ is convergent. Hence find its value.

ii) Evaluate, if exists, the following improper integral. If it do not exist, go for Principal value: $\int_{-1}^{1} \frac{dx}{x^2}$.

3. A. i) Find the value of $\int_0^1 (\log\frac{1}{t})^{n-1}\,dt$ (if exists).

ii) Show that $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}B\left(\frac{2}{5}, \frac{1}{2}\right)$.

OR

B. i) Prove that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$.

ii) Evaluate $\int_0^2 x(8-x^3)^{1/3}dx$.

4. A. i) Find the surface area of the solid generated by the revolution of the asteroid $x = a\cos^3 t$ and $y = a\sin^3 t$ about the axis of x.

ii) Find the perimeter of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

OR

B. i) Compute the volume of the solid bounded by the infinite spindle shaped surface generated by revolving the curve $y = \frac{1}{1+x^2}$ about its asymptote.

ii) Show that the length of the parabola $y^2 = 4ax$ cut off by its latus rectum is $2a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$.

5. A. i) Show that $\cos x\,(\sin x)^3$ is maximum at $x = \frac{\pi}{3}$.

ii) Find $f_y(0,0)$ for the function $f(x,y) = \begin{cases} \frac{x^4y^4}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

OR

B. Find the maximum value of p^3q^2 subject to the constraint $p + q = 1$, using the method of Lagrange's multiplier.

6. A. i) If $\theta = t^n e^{-\frac{r^2}{4t}}$, find what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$.

ii) If $u = (x^2+y^2)^{\frac{1}{3}}$, then show that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{4}{9}u$.

OR

B. i) If $x = u - v, y = u^2 - v^2$, find $\frac{\partial(x,y)}{\partial(u,v)}$

ii) If $u = \cot^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then show that $xu_x + yu_y + \frac{1}{4}\sin 2u = 0$.

7. A. i) Find the direction from the point (0,1,1) in which the directional derivative of

ii) Find the perimeter of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

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OR

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6. A. i) If $\theta=t^n e^{-\frac{r^2}{4t}}$, find what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$.

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B. i) If $x=u-v, y=u^2-v^2$, find $\frac{\partial(x,y)}{\partial(u,v)}$

ii) If $u=\cot^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then show that $xu_x+yu_y+\frac{1}{4}\sin 2u=0$.

7. A. i) Find the direction from the point (0,-1,-1) in which the directional derivative of $\phi=x^4yz^3$ is maximum? What is the magnitude of the maximum?

ii) If $\vec{A}=x^2y\,\hat{i}-2xz\hat{j}+2yz\hat{k}$, find $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$.

OR

B. i) Find a and b such that the surface $ax^2-byz=(a+2)x$ and $4x^2y+z^3=4$ cut orthogonally at $(1,-1,2)$.

ii) A vector field is given by $\vec{A}=(x^2+xy^2)\hat{i}+(y^2+x^2y)\hat{j}$. Show that the field is irrotational and find the scalar potential.

8. A. i) Show that the following series is absolutely convergent.

$$\sum_{n=1}^{\infty}\frac{x^n}{n+1}\quad 0<x<1.$$

ii) Check whether the following sequence is bounded, monotonic and convergent:

$$\{u_n\},\text{ where } u_n=\frac{3.5.7.....(2n+1)}{2.4.6.....2n}.$$

OR

B. i) Test the convergence of the series: $\sum_1^{\infty}\frac{500}{n^{500}}$.

ii) Test the convergence of the series: $1+\frac{x}{4}+\frac{x^2}{9}+\frac{x^3}{16}+\cdots$

9. A. i) A bag contains 15white and 10 black balls. If 4 balls are drawn at random, what are the probabilities of the following:

a) 2 of them are black

b) At most one of them is black

ii) There are two identical urns containing respectively 5 white, 4 red balls and 6 white, 9 red balls. An urn is selected at random and a ball is drawn from it. Find the probability that the ball is white. If the ball drawn is white, what is the probability that it is from the second urn?

OR

B. i) What is the probability that

a) a leap year selected at random will contain 53 Monday?

b) a non-leap year selected at random will contain 53 Monday?

ii) In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total of their output. 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

10. A. Find mean, median and mode from following distribution of marks obtained by 50 students:

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No. of Students	5	9	6	11	5	6	8

OR

B. Find the mean, median, mode and standard deviation of the following data relating to weight of 90 articles:

Weight (in gm)	10	20	30	40	50	60
No. of articles	14	2	22	11	23	18