GNR_638_Theoretical_Assignment

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Difference of Two Convex Functions is Not Necessarily Convex

A twice differentiable function $h(x)$ is convex if its curvature is always nonnegative—that is, if its second derivative (or Hessian, in higher dimensions) never dips below zero. This means the graph of $h(x)$ always curves upward (or is flat).

Now, if we have two convex functions f and g, each has this property of upward curvature. However, when you subtract one from the other to form $h(x) = f(x) - g(x)$, the result might lose that consistent upward curvature. Even though both f and g are convex, their difference can behave differently and may not be convex.

- f is convex if its Hessian $H_f(x) \succeq 0$ for all x,
- g is convex if its Hessian $H_g(x) \succeq 0$ for all x.

Their difference is defined as

$$h(x) = f(x) - g(x).$$

For $h(x)$ to be convex, the Hessian of h must satisfy

$$H_h(x) = H_f(x) - H_g(x) \succeq 0 \quad \text{for all } x.$$

In many cases—especially when g has a stronger curvature than f—this condition fails.

A Generalized Quadratic Example in One Dimension

Consider the functions, for any constant $\varepsilon > 0$,

$$f(x) = x^2 \quad \text{and} \quad g(x) = (1+\varepsilon)x^2.$$

Both functions are convex because

$$f''(x) = 2 \geq 0, \quad g''(x) = 2(1+\varepsilon) \geq 0.$$

Their difference is

$$h(x) = f(x) - g(x) = x^2 - (1+\varepsilon)x^2 = -\varepsilon x^2.$$

The second derivative of $h(x)$ is

$$h''(x) = -2\varepsilon,$$

which is negative for $\varepsilon > 0$. Hence, $h(x)$ is concave rather than convex.

A Generalized Quadratic Example in $\mathbb{R}^n$

More generally, consider functions on $\mathbb{R}^n$:

$$f(x) = x^T A x \quad \text{and} \quad g(x) = x^T B x,$$

where A and B are symmetric matrices that are positive semidefinite (ensuring the convexity of f and g). The Hessians are:

$$H_f(x) = 2A \quad \text{and} \quad H_g(x) = 2B.$$

The difference is:

$$h(x) = f(x) - g(x) = x^T(A - B)x,$$

with Hessian:

$$H_h(x) = 2(A - B).$$

For $h(x)$ to be convex, $A-B$ must be positive semidefinite. However, if B dominates A in the sense that $B - A$ is positive definite (or even positive semidefinite with a nonzero part), then $A - B$ will have negative eigenvalues. For example, if we take $A = I$ (the identity matrix) and $B = (1+\varepsilon)I$ with $\varepsilon > 0$, then:

$$h(x) = x^T\left(I - (1+\varepsilon)I\right)x = -\varepsilon\, x^T x = -\varepsilon\|x\|^2,$$

which is concave because its Hessian is $-2\varepsilon I$, a negative definite matrix.

Summary

In both one-dimensional and multi-dimensional settings, the difference of two convex functions

$$h(x) = f(x) - g(x)$$

is not necessarily convex unless the curvature of f dominates that of g in every direction (i.e., $H_f(x) \succeq H_g(x)$ for all x). In the examples above, by choosing g to have strictly more curvature than f, we ensure that h becomes concave (or at least non-convex).

While convexity is preserved under addition and nonnegative scaling, subtracting one convex function from another can destroy the convexity unless special conditions are met.